

# A TREE-VALUED MARKOV PROCESS ASSOCIATED WITH AN ADMISSIBLE FAMILY OF BRANCHING MECHANISMS

HONGWEI BI AND HUI HE

ABSTRACT. By studying an admissible family of branching mechanisms introduced in Li (2014), we obtain a pruning procedure on Lévy trees. Then we construct a decreasing Lévy-CRT-valued process  $\{\mathcal{T}_t\}$  by pruning Lévy trees and an analogous process  $\{\mathcal{T}_t^*\}$  by pruning a critical Lévy tree conditioned to be infinite. Under a regular condition on the admissible family of branching mechanisms, we show that the law of  $\{\mathcal{T}_t\}$  at the ascension time can be represented by  $\{\mathcal{T}_t^*\}$ . The results generalize those studied in Abraham and Delmas (2012).

## 1. INTRODUCTION

A general pruning procedure was introduced in Abraham et al.[7] on Lévy trees and was further explored by Abraham and Delmas [1]. In particular, a decreasing continuum-tree-valued process was constructed and studied in [1] which is associated with a family of branching mechanisms obtained by shifting a branching mechanism. More precisely, let  $\psi$  be a branching mechanism defined by

$$(1) \quad \psi(\lambda) = b\lambda + c\lambda^2 + \int_{(0,\infty)} \left( e^{-\lambda z} - 1 + \lambda z \right) m(dz), \quad \lambda \geq 0,$$

where  $b \in \mathbb{R}$ ,  $c \geq 0$  and  $m$  is a  $\sigma$ -finite measure on  $(0, +\infty)$  such that  $\int_0^\infty (z \wedge z^2)m(dz) < +\infty$ . Define  $\psi^\theta(\lambda) = \psi(\theta + \lambda) - \psi(\theta)$ . Denote by  $\Theta^\psi$  the set of  $\theta$  such that  $\int_1^\infty e^{-\theta z} m(dz) < \infty$ . The family of branching mechanisms  $\{\psi^\theta, \theta \in \Theta^\psi\}$  was considered in [1].

Li [21] introduced the *admissible family* of branching mechanisms which generalized those used in [1]. Roughly, the model is described as follows: Given a time interval  $\mathfrak{T} \subset \mathbb{R}$ , let  $(\theta, \lambda) \mapsto \zeta_\theta(\lambda)$  be a continuous function on  $\mathfrak{T} \times [0, \infty)$  with representation

$$\zeta_\theta(\lambda) = \beta_\theta \lambda + \int_{(0,\infty)} (1 - e^{-z\lambda}) n_\theta(dz), \quad \theta \in \mathfrak{T}, \lambda \geq 0,$$

where  $\beta_\theta \geq 0$  and  $(1 \wedge z)n_\theta(dz)$  is a finite kernel from  $\mathfrak{T}$  to  $(0, \infty)$ . Then  $\{\psi_\theta, \theta \in \mathfrak{T}\}$  is called an admissible family if

$$\psi_q(\lambda) = \psi_t(\lambda) + \int_t^q \zeta_\theta(\lambda) d\theta, \quad q \geq t \in \mathfrak{T}, \lambda \geq 0.$$

In particular,  $\{\psi^\theta, \theta \in \Theta^\psi\}$  considered in [1] is an admissible family with

$$\zeta_\theta(\lambda) = 2c\lambda + \int_{(0,\infty)} (1 - e^{-z\lambda}) z e^{-z\theta} m(dz).$$

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By using the techniques of stochastic equations and measure-valued processes, Li [21] studied a class of increasing path-valued Markov processes associated with the admissible family. Those path-valued processes can be regarded as counterparts of the tree-valued processes constructed in [1] (However, to the best of our knowledge, no link is actually pointed out between tree-valued processes and path-valued branching processes). It is natural to ask whether there exists a continuum-tree-valued process associated with a given admissible family by pruning Lévy trees.

The second motivation of the present work is the study of the so-called *ascension process*. It was first introduced in the pioneer work of Aldous and Pitman [10], where they constructed a tree-valued Markov process  $\{\mathcal{G}(u)\}$  by pruning Galton-Watson trees (edge percolation on trees) and an analogous process  $\{\mathcal{G}^*(u)\}$  by pruning a critical or subcritical Galton-Watson tree conditioned to be infinite. It was shown in [10] that the process  $\{\mathcal{G}(u)\}$  run until its ascension time (the first time for which the total mass is finite) has a representation in terms of  $\{\mathcal{G}^*(u)\}$  in the special case of Poisson offspring distributions. By using the pruning procedure defined in [7] and exploration processes introduced in [18], Abraham and Delmas [1] extended the above results to Lévy trees, where a decreasing Lévy-tree-valued process  $\{\mathcal{T}_\theta, \theta \in \Theta^\psi\}$  was constructed such that  $\mathcal{T}_\theta$  is a  $\psi^\theta$ -Lévy tree. They also showed that  $\{\mathcal{T}_\theta\}$  run until its ascension time can be represented in terms of another tree-valued process obtained by applying the same pruning procedure to a Lévy tree conditioned on non-extinction. Similar results can be found in [3] for Galton-Watson trees where the trees are pruned based on bond percolation. The cases for sub-trees of Lévy trees were also studied in [4].

In this paper the framework is locally compact measured rooted real tree  $(\mathcal{T}, d, \emptyset, \mathbf{m})$ . The collection is denoted by  $\mathbb{T}$ . Based on the pruning procedure of [1, 7], we introduce a more general pruning mechanism as follows. Let  $\mathcal{T} \in \mathbb{T}, t \in \mathfrak{T}$  and  $q \in \mathfrak{T}_t = \mathfrak{T} \cap [t, \infty)$ . Put marks on  $\mathfrak{T}$  with a Poisson point measure  $M_t^\mathfrak{T}([t, q], dy)$  as follows:

- (1) Assign marks to the skeleton of  $\mathcal{T}$  according to a Poisson point measure with intensity  $\int_t^q \beta_\theta d\theta \ell^\mathfrak{T}(dy)$ , where  $\ell^\mathfrak{T}$  is the length measure of  $\mathfrak{T}$ ;
- (2) Assign marks to each node  $y \in \mathcal{T}$  of infinite degree with probability  $1 - m_{\Delta_y}(t, q)$  (see (23)).

We prune  $\mathcal{T}$  according to the marks and consider the pruned tree  $\mathcal{T}_q^t$  containing the root. Theorem 4.2 then gives the connection between the pruning mechanism and the family of admissible branching mechanisms. More precisely, denote  $\mathbb{N}^{\psi_t}$  the excursion measure induced by  $\psi_t$ . Then the process  $\{\mathcal{T}_q^t, q \in \mathfrak{T}_t\}$  is Markovian under  $\mathbb{N}^{\psi_t}$  and  $\mathbb{N}^{\psi_t}(\mathcal{T}_q^t \in d\mathcal{T}) = \mathbb{N}^{\psi_q}(d\mathcal{T})$ . In addition, the special Markov property holds. Roughly speaking, it gives the conditional distribution of the tree of individuals with marked ancestors with respect to the tree of individuals with no marked ancestor.

Due to the consistency property, there is a decreasing tree-valued Markov process  $\{\mathcal{T}_q, q \in \mathfrak{T}\}$  such that  $\mathcal{T}_q$  is a  $\psi_q$ -Lévy tree. Let  $\mathbf{N}^\Psi$  be the law of  $\{\mathcal{T}_q, q \in \mathfrak{T}\}$ . Define the total mass of  $\mathcal{T}_q$  and the ascension time by  $\sigma_q = \mathbf{m}^{\mathcal{T}_q}(\mathcal{T}_q)$  and  $A = \inf\{q \in \mathfrak{T}; \sigma_q < +\infty\}$ . The distribution of  $\sigma_t$  condition on  $\mathcal{T}_q$  and of  $A$  under  $\mathbf{N}^\Psi$  are respectively given in Lemmas 5.2 and 5.5. Conditional on  $A$ , the distribution of the functionals of the tree at the ascension time  $\mathcal{T}_A$  is considered in Theorem 5.7 and Proposition 5.8. This generalizes results in [1] to admissible branching mechanisms. An expression for the distribution of the height of the tree is also given in Proposition 5.10 which is a direct extension of that in [5].

Under a regular condition on the admissible family, we prove that the law of the tree-valued process at its ascension time can be represented in terms of another tree-valued process obtained by pruning a critical Lévy tree conditioned to be infinite; see Theorem 6.1 and Corollary 6.4.

We remark that all the results in this paper are stated using the framework of real trees but not exploration processes. However the proof of Theorem 4.3 relies on explosion process, since Theorem 0.1 and 3.2 of [7] are used explicitly there.

Let us mention that the study of theory of continuum random trees(CRT) was initiated by Aldous [8, 9]. Lévy trees, also known as Lévy CRT, were first studied by Le Gall and Le Jan [18, 19], where it was shown that Lévy trees code the genealogy of continuous state branching processes (CSBP). Later, in [11], it was shown that Galton-Watson trees which code the genealogy of Galton-Watson processes, suitably rescaled, converge to Lévy trees, as rescaled Galton-Watson processes converge to CSBP. Then based on [21] and the present work, one may expect to introduce the notation “admissible family” to study the Galton-Watson processes and Galton-Watson trees. A general pruning procedure on Galton-Watson trees may be developed, which is possibly a combination of Aldous and Pitman’s pruning procedure in [10] and Abraham et al.’s pruning procedure in [3]. This gives the third motivation of the present work. We will explore these questions in the future.

The rest of the paper is organized as follows. In Section 2, we introduce and study the admissible family of branching mechanisms. We recall some notations and results on real trees and Lévy trees in Section 3. Based on the study of admissible family, in Section 4, the pruning procedure will be given and the marginal distributions of the pruning process are studied. The evolution of the tree-valued process will be explored in Section 5. Finally, in the last section, we construct a tree-valued process by pruning a critical Lévy tree conditioned to be infinite and get the representation of the tree at the ascension time.

## 2. ADMISSIBLE FAMILY OF BRANCHING MECHANISMS

Throughout the paper, for  $-\infty \leq a \leq b \leq +\infty$ , we make the convention  $\int_a^b = \int_{(a,b)}$ .

The admissible family of branching mechanisms was first introduced by Li [21]. Suppose that  $\mathfrak{T} \subset \mathbb{R}$  is an interval and  $\Psi = \{\psi_q, q \in \mathfrak{T}\}$  is a family of branching mechanisms, where  $\psi_q$  is given by

$$\psi_q(\lambda) = b_q \lambda + c \lambda^2 + \int_0^\infty (e^{-\lambda z} - 1 + \lambda z) m_q(dz), \quad \lambda \geq 0$$

with parameters  $(b, m) = (b_q, m_q)$  for  $q \in \mathfrak{T}$  such that  $b_q \in \mathbb{R}$  and  $\int (z \wedge z^2) m_q(dz) < \infty$ .

**Definition 2.1.** [Li(2014)] We call  $\{\psi_q, q \in \mathfrak{T}\}$  an admissible family if for each  $\lambda > 0$  the function  $q \mapsto \psi_q(\lambda)$  is increasing and continuously differentiable with

$$(2) \quad \zeta_q(\lambda) := \frac{\partial}{\partial q} \psi_q(\lambda) = \beta_q \lambda + \int_0^\infty (1 - e^{-z\lambda}) n_q(dz), \quad q \in \mathfrak{T}, \quad \lambda > 0,$$

where  $\beta_q \geq 0$  and  $(1 \wedge z) n_q(dz)$  is a finite kernel from  $\mathfrak{T}$  to  $(0, \infty)$  satisfying

$$(3) \quad \int_t^q \beta_\theta d\theta + \int_t^q d\theta \int_0^\infty z n_\theta(dz) < \infty, \quad q \geq t \in \mathfrak{T}.$$

*Remark 2.2.* In fact, it is assumed in [21] that  $q \mapsto \psi_q(\lambda)$  is decreasing and  $\zeta_q(\lambda) = -\frac{\partial}{\partial q} \psi_q(\lambda)$ . In that case, we will get an increasing tree-valued process.

*Remark 2.3.* For the purpose of this work, we also weaken the assumptions on  $\beta_q$  and  $n_q(dz)$ . In [21], it is assumed that

$$(4) \quad \sup_{t \leq \theta \leq q} \left( \beta_\theta + \int_0^\infty z n_\theta(dz) \right) < \infty, \quad q \geq t \in \mathfrak{T},$$

which is essential there. If we assume (4), some interesting cases of pruning Lévy trees may be excluded. See Example 2.5 below as an example.

*Remark 2.4.* It is also possible to assume that

$$\psi_q(\lambda) = b_q\lambda + c\lambda^2 + \int_0^\infty (e^{-\lambda z} - 1 + \lambda z \mathbf{1}_{\{z \leq 1\}}) m_q(dz)$$

with parameters  $(b, m) = (b_q, m_q)$  for  $q \in \mathfrak{T}$  such that  $b_q \in \mathbb{R}$  and  $\int (1 \wedge z^2) m_q(dz) < \infty$ . Then (3) would be replaced by

$$\int_t^q \beta_\theta d\theta + \int_t^q d\theta \int_{(0,1]} z n_\theta(dz) < \infty, \quad q \geq t \in \mathfrak{T}.$$

We assume further that  $\psi_q$  is conservative; i.e.  $\int_{(0,\epsilon]} \frac{d\lambda}{|\psi_q(\lambda)|} = +\infty$  for all  $\epsilon > 0$ . We conjecture that all results in this work can be deduced in this framework.

In the following we give some examples of the admissible family of branching mechanisms.

*Example 2.5.* Let  $\psi$  be defined in (1). Abraham and Delmas [1] considered  $\psi_q(\lambda) = \psi(q + \lambda) - \psi(q)$ ,  $q \in \Theta^\psi$ , where  $\Theta^\psi$  is the set of  $\theta \in \mathbb{R}$  such that  $\int_1^\infty e^{-\theta z} m(dz) < \infty$ . Then  $\{\psi_q, q \in \Theta^\psi\}$  is an admissible family with

$$b_q = b + 2cq + \int_0^\infty z(1 - e^{-zq})m(dz), \quad m_q(dz) = e^{-zq} m(dz),$$

and

$$\beta_q = 2c, \quad n_q(dz) = z e^{-zq} m(dz).$$

Note that  $\Theta^\psi = [\theta_\infty, +\infty)$  or  $(\theta_\infty, +\infty)$  for some  $\theta_\infty \in [-\infty, 0]$ . However, in the case of  $\Theta^\psi = [\theta_\infty, +\infty)$ ,  $n_{\theta_\infty}(dz)$  may fail to satisfy (4). A sufficient condition that (4) holds is  $\int_1^\infty z^2 e^{-\theta_\infty z} m(dz) < \infty$ . We remark here that for the study of the ascension process, we always exclude the case of  $\Theta^\psi = [\theta_\infty, +\infty)$ ; see Remark 5.4 in Section 5 below.

*Example 2.6.* Let  $\psi$  be defined in (1). Let  $f \geq 0$  be a bounded decreasing function on  $\mathbb{R}$  with bounded derivative and  $\sup_{x \geq 0} |xf'(x)| < +\infty$ . Let  $g$  be a differentiable increasing function on  $\mathbb{R}$ . For  $q \in \mathbb{R}$ , let  $\psi_q$  be a branching mechanism with parameters  $(b_q, m_q)$  defined by

$$b_q = b + g(q) + \int_0^\infty (f(0) - f(qz))zm(dz), \quad m_q(dz) = f(qz)m(dz).$$

Then one can check that  $\{\psi_q, q \in \mathbb{R}\}$  is an admissible family of branching mechanisms with

$$\frac{\partial}{\partial q} \psi_q(\lambda) = g'(q)\lambda - \int_0^\infty (1 - e^{-z\lambda})zf'(qz)m(dz), \quad q \in \mathbb{R}, \lambda \geq 0,$$

and

$$\beta_q = g'(q), \quad n_q(dz) = -zf'(qz)m(dz).$$

In particular, if  $m = 0$ , then  $\psi_q(\lambda) = (b + g(q))\lambda + c\lambda^2$ . If  $f \equiv 1$ , then  $\psi_q(\lambda) = \psi(\lambda) + g(q)\lambda$ .

*Example 2.7.* Let  $\mathfrak{T}_- \subset (-\infty, 0]$  be an interval and let  $\{\psi_q, q \in \mathfrak{T}_-\}$  be an admissible family of branching mechanisms with parameters  $(b_q, m_q)$ . Assume that  $0 \in \mathfrak{T}_-$  and  $\psi_0$  is critical. Note  $\eta_q$  the largest root of  $\psi_q(s) = 0$ . For  $q \in -\mathfrak{T}_- := \{-t, t \in \mathfrak{T}_-\}$ , define  $\psi_q(\lambda) = \psi_{-q}(\lambda + \eta_{-q})$ . Then we have  $\{\psi_q, q \in \mathfrak{T}_- \cup (-\mathfrak{T}_-)\}$  is an admissible family of branching mechanisms such that for  $q \in -\mathfrak{T}_-$ ,

$$b_q = b_{-q} + 2c\eta_{-q} + \int_0^\infty (1 - e^{-z\eta_{-q}})zm_{-q}(dz), \quad m_q = e^{-z\eta_{-q}} m_{-q}(dz).$$

Now we show how to get pruning parameters for a given admissible family of branching mechanisms. Without loss of generality, we always assume that  $\psi_t \neq \psi_q$  for  $t \neq q \in \mathfrak{T}$ . It follows from Definition 2.1 that for  $q \geq t \in \mathfrak{T}$ ,

$$(5) \quad b_q = b_t + \int_t^q \beta_\theta d\theta + \int_t^q d\theta \int_0^\infty z n_\theta(dz)$$

and

$$(6) \quad m_t(dz) = m_q(dz) + \int_{\{t \leq \theta < q\}} n_\theta(dz) d\theta.$$

*Remark 2.8.* By (5) one can see  $q \mapsto b_q$  is a continuous increasing function on  $\mathfrak{T}$ . In particular,  $b_t = b_q$  implies  $\psi_t = \psi_q$  and vice versa.

For  $t \in \mathfrak{T}$ , note  $\mathfrak{T}_t = \mathfrak{T} \cap [t, +\infty)$ . Using (6), we get for any  $q \in \mathfrak{T}_t$ ,  $m_q(dz) \ll m_t(dz)$  on  $(0, \infty)$ . Denote by  $m_z(t, q)$  the corresponding Radon-Nikodym derivative; i.e.  $m_q(dz) = m_z(t, q)m_t(dz)$ . Then we have

$$(7) \quad \int_{t \leq \theta < q} n_\theta(dz) d\theta = (1 - m_z(t, q)) m_t(dz), \quad q \in \mathfrak{T}_t,$$

which implies  $m_t(dz)$ -a.e.,

$$(8) \quad m_z(t, q) \leq 1 \text{ and } q \mapsto m_z(t, q) \text{ is decreasing.}$$

Furthermore, we yield for  $t \leq \theta \leq q$ ,  $m_t(dz)$ -a.e.,

$$(9) \quad m_z(t, q) = m_z(t, \theta)m_z(\theta, q).$$

Equation (8),(9) hold  $m_t(dz)$ -a.e. Since the uncountable union of the null set is not necessarily a null set, then we make the following assumptions:

(H1) For every  $z \in (0, \infty)$  and  $t \in \mathfrak{T}$ ,

$$m_z(t, q) \leq 1 \text{ and } q \mapsto m_z(t, q) \text{ is decreasing.}$$

(H2) For every  $z \in (0, \infty)$  and  $t \leq \theta \leq q \in \mathfrak{T}$ ,

$$m_z(t, q) = m_z(t, \theta)m_z(\theta, q).$$

(H3) For every  $q \in \mathfrak{T}$ ,

$$\int_0^\infty \frac{d\lambda}{\psi_q(\lambda)} < +\infty.$$

By (H1), we can define a measure  $m_z(t, dq)$  on  $\mathfrak{T}_t$  by

$$(10) \quad m_z(t, [t, q]) = -\ln(m_z(t, q)),$$

which induces the pruning measure on branching nodes of infinite degree of a  $\psi_t$ -Lévy tree. By (H2) we have a tree-valued Markov processes on  $\mathfrak{T}$ . (H3) is used to ensure that all trees are locally compact.

*From now on, we assume that (H1-3) are in force.*

### 3. REAL TREES AND LÉVY TREES

In the section, we recall some basic notations and facts on real trees and Lévy trees. We mainly follow from Section 2 in [2] or [5].

**3.1. Notations.** Let  $(E, d)$  be a metric Polish space. We denote by  $M_f(E)$  (resp.  $M_f^{\text{loc}}(E)$ ) the space of all finite (resp. locally finite) Borel measures on  $E$ . For  $x \in E$ , let  $\delta_x$  denote the Dirac measure at point  $x$ . For  $\mu \in M_f^{\text{loc}}(E)$  and  $f$  a non-negative measurable function, we set  $\langle \mu, f \rangle = \int f(x) \mu(dx) = \mu(f)$ .

**3.2. Real trees.** We refer to [14] or [17] for a general presentation of random real trees. A metric space  $(\mathcal{T}, d)$  is a *real tree* if the following properties are satisfied: for every  $s, t \in \mathcal{T}$ ,

- (i) there is a unique isometric map  $f_{s,t} : [0, d(s, t)] \rightarrow \mathcal{T}$  such that  $f_{s,t}(0) = s$  and  $f_{s,t}(d(s, t)) = t$ .
- (ii) if  $q$  is a continuous injective map,  $q : [0, 1] \rightarrow \mathcal{T}$  such that  $q(0) = s$  and  $q(1) = t$ , then  $q([0, 1]) = f_{s,t}([0, d(s, t)])$ .

If  $s, t \in \mathcal{T}$ , we will note  $\llbracket s, t \rrbracket$  the range of the isometric map  $f_{s,t}$  and  $\llbracket s, t \rrbracket \setminus \{t\}$ .

$(\mathcal{T}, d, \emptyset)$  is called a *rooted real tree* with root  $\emptyset$  if  $(\mathcal{T}, d)$  is a real tree and  $\emptyset \in \mathcal{T}$  is a distinguished vertex. For every  $x \in \mathcal{T}$ ,  $\llbracket \emptyset, x \rrbracket$  is interpreted as the ancestral line of vertex  $x$ . The degree  $n(x)$  is the number of connected components of  $\mathcal{T} \setminus \{x\}$  and the number of children of  $x \neq \emptyset$  is  $\kappa_x = n(x) - 1$  and of the root is  $\kappa_\emptyset = n(\emptyset)$ . Note  $\text{Lf}(\mathcal{T}) = \{x \in \mathcal{T}, \kappa_x = 0\}$ ,  $\text{Br}(\mathcal{T}) = \{x \in \mathcal{T}, \kappa_x \geq 2\}$ ,  $\text{Br}_\infty(\mathcal{T}) = \{x \in \mathcal{T}, \kappa_x = \infty\}$  respectively the set of leaves, branching points and infinite branching points. The skeleton of  $\mathcal{T}$  is the set of points in the tree that aren't leaves:  $\text{Sk}(\mathcal{T}) = \mathcal{T} \setminus \text{Lf}(\mathcal{T})$ . The trace of the Borel  $\sigma$ -field of  $\mathcal{T}$  restricted to  $\text{Sk}(\mathcal{T})$  is generated by the sets  $\llbracket s, s' \rrbracket$ ;  $s, s' \in \text{Sk}(\mathcal{T})$ . One defines uniquely a  $\sigma$ -finite Borel measure  $\ell^\mathcal{T}$  on  $\mathcal{T}$ , called the length measure of  $\mathcal{T}$ , such that  $\ell^\mathcal{T}(\text{Lf}(\mathcal{T})) = 0$  and  $\ell^\mathcal{T}(\llbracket s, s' \rrbracket) = d(s, s')$ .

**3.3. Measured rooted real trees.** We briefly review the Gromov-Hausdorff-Prohorov metric on rooted measured metric space presented in [6]; see also [13] and [15] for some related works.

Let  $(X, d)$  be a Polish metric space. Recall  $d_H(A, B)$  the Hausdorff distance between  $A$  and  $B$  for  $A, B \in \mathcal{B}(X)$  and  $d_P(\mu, \nu)$  the Prohorov distance between  $\mu$  and  $\nu$  for  $\mu, \nu \in M_f(X)$ . A rooted measured metric space  $\mathcal{X} = (X, d, \emptyset, \mu)$  is a metric space  $(X, d)$  with a distinguished element  $\emptyset \in X$  and a locally finite Borel measure  $\mu \in M_f^{\text{loc}}(X)$ . Let  $\mathcal{X}$  and  $\mathcal{X}'$  be two compact rooted measured metric spaces, and define

$$d_{\text{GHP}}^c(\mathcal{X}, \mathcal{X}') = \inf_{\Phi, \Phi', Z} (d_H^Z(\Phi(X), \Phi'(X')) + d^Z(\Phi(\emptyset), \Phi'(\emptyset')) + d_P^Z(\Phi_*\mu, \Phi'_*\mu')),$$

where the infimum is taken over all isometric embedding  $\Phi : X \rightarrow Z$  and  $\Phi' : X' \rightarrow Z$  into some common Polish metric space  $(Z, d^Z)$  and  $\Phi_*\mu$  is the measure  $\mu$  transported by  $\Phi$ .

If  $\mathcal{X}$  is a rooted measured metric space, then for  $r \geq 0$  we will consider its restriction to the ball of radius  $r$  centered at  $\emptyset$ ,  $\mathcal{X}^{(r)} = (X^{(r)}, d^{(r)}, \emptyset, \mu^{(r)})$ , where  $X^{(r)} = \{x \in X, d(\emptyset, x) \leq r\}$  with  $d^{(r)}$  and  $\mu^{(r)}$  defined in an obvious way.

By a measured rooted real tree (MRRT)  $(\mathcal{T}, d, \emptyset, \mathbf{m})$ , we mean  $(\mathcal{T}, d, \emptyset)$  is a locally compact rooted real tree and  $\mathbf{m} \in M_f^{\text{loc}}(\mathcal{T})$  is a locally finite measure on  $\mathcal{T}$ . When there is no confusion, we will simply write  $\mathcal{T}$  for  $(\mathcal{T}, d, \emptyset, \mathbf{m})$ . We define for two MRRTs  $\mathcal{T}_1, \mathcal{T}_2$  the Gromov-Hausdorff-Prohorov (GHP) metric as follows:

$$d_{\text{GHP}}(\mathcal{T}_1, \mathcal{T}_2) = \int_0^\infty e^{-r} \left( 1 \wedge d_{\text{GHP}}^c(\mathcal{T}_1^{(r)}, \mathcal{T}_2^{(r)}) \right) dr.$$

$\mathcal{T}_1$  and  $\mathcal{T}_2$  are said GHP-isometric if  $d_{\text{GHP}}(\mathcal{T}_1, \mathcal{T}_2) = 0$ . Denote by  $\mathbb{T}$  the set of (GHP-isometry classes of) MRRTs  $(\mathcal{T}, d, \emptyset, \mathbf{m})$ . According to Corollary 2.8 in [6],  $(\mathbb{T}, d_{\text{GHP}})$  is a Polish metric space.

**3.4. Grafting procedure.** Let  $\mathcal{T}$  be a measured rooted real tree and let  $((\mathcal{T}_i, x_i), i \in I)$  be a finite or countable family of elements of  $\mathbb{T} \times \mathcal{T}$ . We define the real tree obtained by grafting the trees  $\mathcal{T}_i$  on  $\mathcal{T}$  at point  $x_i$ . We set  $\hat{\mathcal{T}} = \mathcal{T} \sqcup (\bigsqcup_{i \in I} \mathcal{T}_i \setminus \{\emptyset^{\mathcal{T}_i}\})$  where the symbol  $\sqcup$  means that we choose for the sets  $(\mathcal{T}_i)_{i \in I}$  representatives of GHP-isometry classes in  $\mathbb{T}$  which are disjoint subsets of some common set and that we perform the disjoint union of all these sets. We set  $\emptyset^{\hat{\mathcal{T}}} = \emptyset^{\mathcal{T}}$ . The set  $\hat{\mathcal{T}}$  is endowed with the following metric  $d^{\hat{\mathcal{T}}}$ : if  $s, t \in \hat{\mathcal{T}}$ ,

$$d^{\hat{\mathcal{T}}}(s, t) = \begin{cases} d^{\mathcal{T}}(s, t) & \text{if } s, t \in \mathcal{T}, \\ d^{\mathcal{T}}(s, x_i) + d^{\mathcal{T}_i}(\emptyset^{\mathcal{T}_i}, t) & \text{if } s \in \mathcal{T}, t \in \mathcal{T}_i \setminus \{\emptyset^{\mathcal{T}_i}\}, \\ d^{\mathcal{T}_i}(s, t) & \text{if } s, t \in \mathcal{T}_i \setminus \{\emptyset^{\mathcal{T}_i}\}, \\ d^{\mathcal{T}}(x_i, x_j) + d^{\mathcal{T}_j}(\emptyset^{\mathcal{T}_j}, s) + d^{\mathcal{T}_i}(\emptyset^{\mathcal{T}_i}, t) & \text{if } i \neq j \text{ and } s \in \mathcal{T}_j \setminus \{\emptyset^{\mathcal{T}_j}\}, t \in \mathcal{T}_i \setminus \{\emptyset^{\mathcal{T}_i}\}. \end{cases}$$

We define the mass measure on  $\hat{\mathcal{T}}$  by

$$\mathbf{m}^{\hat{\mathcal{T}}} = \mathbf{m}^{\mathcal{T}} + \sum_{i \in I} \left( \mathbf{1}_{\mathcal{T}_i \setminus \{\emptyset^{\mathcal{T}_i}\}} \mathbf{m}^{\mathcal{T}_i} + \mathbf{m}^{\mathcal{T}_i}(\{\emptyset^{\mathcal{T}_i}\}) \delta_{x_i} \right).$$

Then  $(\hat{\mathcal{T}}, d^{\hat{\mathcal{T}}}, \emptyset^{\hat{\mathcal{T}}})$  is still a complete rooted real tree (Notice that it is not always true that  $\hat{\mathcal{T}}$  remains locally compact or that  $\mathbf{m}^{\hat{\mathcal{T}}}$  is a locally finite measure on  $\hat{\mathcal{T}}$ ). We use  $\mathcal{T} \otimes_{i \in I} (\mathcal{T}_i, x_i) = (\hat{\mathcal{T}}, d^{\hat{\mathcal{T}}}, \emptyset^{\hat{\mathcal{T}}}, \mathbf{m}^{\hat{\mathcal{T}}})$  for the grafted tree with the convention that  $\mathcal{T} \otimes_{i \in I} (\mathcal{T}_i, x_i) = \mathcal{T}$  for  $I = \emptyset$ . If  $\varphi$  is an isometry from  $\mathcal{T}$  onto  $\mathcal{T}'$ , then  $\mathcal{T} \otimes_{i \in I} (\mathcal{T}_i, x_i)$  and  $\mathcal{T}' \otimes_{i \in I} (\mathcal{T}_i, \varphi(x_i))$  are also isometric. Therefore, the grafting procedure is well defined on  $\mathbb{T}$ .

**3.5. Sub-trees above a given level.** For  $\mathcal{T} \in \mathbb{T}$ , define  $H_{\max}(\mathcal{T}) = \sup_{x \in \mathcal{T}} d^{\mathcal{T}}(\emptyset^{\mathcal{T}}, x)$  the height of  $\mathcal{T}$  and for  $a \geq 0$ ,

$$\mathcal{T}^{(a)} = \{x \in \mathcal{T}, d(\emptyset, x) \leq a\} \quad \text{and} \quad \mathcal{T}(a) = \{x \in \mathcal{T}, d(\emptyset, x) = a\}$$

the restriction of the tree  $\mathcal{T}$  under level  $a$  and the set of vertices of  $\mathcal{T}$  at level  $a$  respectively. We denote by  $(\mathcal{T}^{i, \circ}, i \in I)$  the connected components of  $\mathcal{T} \setminus \mathcal{T}^{(a)}$ . Let  $\emptyset_i$  be the most recent common ancestor of all the vertices of  $\mathcal{T}^{i, \circ}$ . We consider the real tree  $\mathcal{T}^i = \mathcal{T}^{i, \circ} \cup \{\emptyset_i\}$  rooted at point  $\emptyset_i$  with mass measure  $\mathbf{m}^{\mathcal{T}^i}$  defined as the restriction of  $\mathbf{m}^{\mathcal{T}}$  to  $\mathcal{T}^{i, \circ}$  and  $\mathbf{m}^{\mathcal{T}^i}(\emptyset_i) = 0$ . Notice that  $\mathcal{T} = \mathcal{T}^{(a)} \otimes_{i \in I} (\mathcal{T}^i, \emptyset_i)$ . We will consider the point measure on  $\mathcal{T} \times \mathbb{T}$ :

$$(11) \quad \mathcal{N}_a^{\mathcal{T}} = \sum_{i \in I} \delta_{(\emptyset_i, \mathcal{T}^i)}.$$

**3.6. Excursion measure of a Lévy tree.** Recall (1). We say  $\psi$  is subcritical, critical or super-critical if  $b > 0$ ,  $b = 0$  or  $b < 0$ , respectively. In particular, we say  $\psi$  is (sub)critical, if  $b \geq 0$ . We assume the Grey condition holds:

$$(12) \quad \int^{+\infty} \frac{d\lambda}{\psi(\lambda)} < +\infty.$$

*Remark 3.1.* The Grey condition is used to ensure that the corresponding Lévy tree is locally compact and also implies  $c > 0$  or  $\int_{(0,1)} \ell m(d\ell) = +\infty$  which is equivalent to the fact that the Lévy process with index  $\psi$  is of infinite variation.

Let  $v^\psi$  be the unique non-negative solution of the equation

$$(13) \quad \int_{v^\psi(a)}^{+\infty} \frac{d\lambda}{\psi(\lambda)} = a.$$

Results from [12] in the (sub)critical cases, where height functions are introduced to code the compact real trees, can be extended to the super-critical cases; see [5]. We recall the results as follows. Note  $\mathbb{N}^\psi[d\mathcal{T}]$  on  $\mathbb{T}$  the excursion measure of a Lévy tree. A  $\psi$ -Lévy tree is a “random” tree with law  $\mathbb{N}^\psi$  and the following properties:

- (i) **Height.** For all  $a > 0$ ,  $\mathbb{N}^\psi[H_{\max}(\mathcal{T}) > a] = v^\psi(a)$ .
- (ii) **Mass measure.** The mass measure  $\mathbf{m}^\mathcal{T}$  is supported on  $\text{Lf}(\mathcal{T})$ ,  $\mathbb{N}^\psi[d\mathcal{T}]$ -a.e.
- (iii) **Local time.** There exists a process  $\{\ell^a, a \geq 0\}$  supported on  $\mathcal{T}(a)$  which is càdlàg for the weak topology on the set of finite measures on  $\mathcal{T}$  such that  $\mathbb{N}^\psi[d\mathcal{T}]$ -a.e.  $\mathbf{m}^\mathcal{T}(dx) = \int_0^\infty \ell^a(dx) da$ , and  $\ell^0 = 0$ ,  $\inf\{a > 0; \ell^a = 0\} = \sup\{a \geq 0; \ell^a \neq 0\} = H_{\max}(\mathcal{T})$ .
- (iv) **Branching property.** Given  $\mathcal{T}^{(a)}$  for any  $a > 0$ ,  $\mathcal{N}_a^\mathcal{T}(dx, d\mathcal{T}')$  under  $\mathbb{N}^\psi[d\mathcal{T}|H_{\max}(\mathcal{T}) > a]$  is a Poisson point measure on  $\mathcal{T}(a) \times \mathbb{T}$  with intensity  $\ell^a(dx)\mathbb{N}^\psi[d\mathcal{T}']$ .
- (v) **Branching points.**  $\mathbb{N}^\psi[d\mathcal{T}]$ -a.e., the branching points of  $\mathcal{T}$  have 2 children or an infinite number of children.
  - The set of binary branching points (i.e. with 2 children) is empty  $\mathbb{N}^\psi$ -a.e. if  $c = 0$  and is a countable dense subset of  $\mathcal{T}$  if  $c > 0$ ;
  - The set  $\text{Br}_\infty(\mathcal{T})$  of infinite branching points is nonempty with  $\mathbb{N}^\psi$ -positive measure if and only if  $m \neq 0$ . If  $\langle m, 1 \rangle = +\infty$ , the set  $\text{Br}_\infty(\mathcal{T})$  is  $\mathbb{N}^\psi$ -a.e. a countable dense subset of  $\mathcal{T}$ .
- (vi) **Mass of the nodes.** The set  $\{d(\emptyset, x), x \in \text{Br}_\infty(\mathcal{T})\}$  coincides  $\mathbb{N}^\psi$ -a.e. with the set of discontinuity times of the mapping  $a \mapsto \ell^a$ . Moreover,  $\mathbb{N}^\psi$ -a.e., for every such discontinuity time  $b$ , there is a unique  $x_b \in \text{Br}_\infty(\mathcal{T})$  with  $d(\emptyset, x_b) = b$  and  $\Delta_b > 0$ , such that  $\ell^b = \ell^{b-} + \Delta_b \delta_{x_b}$ .

In order to stress the dependence on  $\mathcal{T}$ , we may write  $\ell^{a, \mathcal{T}}$  for  $\ell^a$ . We set  $\sigma^\mathcal{T}$  or simply  $\sigma$  when there is no confusion, for the total mass of the mass measure on  $\mathcal{T}$ :

$$(14) \quad \sigma = \mathbf{m}^\mathcal{T}(\mathcal{T}).$$

Notice that  $\mathbf{m}^\mathcal{T}(\{x\}) = 0$  for any  $x \in \mathcal{T}$ .

**3.7. Related measures on Lévy trees.** We define a probability measure on  $\mathbb{T}$  as follows. Let  $r > 0$  and  $\sum_{k \in \mathcal{K}} \delta_{\mathcal{T}^k}$  be a Poisson random measure on  $\mathbb{T}$  with intensity  $r\mathbb{N}^\psi$ . Consider  $\emptyset$  as the trivial MRRT reduced to the root with null mass measure. Note  $\mathcal{T} = \emptyset \oplus_{k \in \mathcal{K}} (\mathcal{T}^k, \emptyset)$ . Using Property (i) as well as (16) below, one easily get that  $\mathcal{T}$  is a locally compact MRRT, and thus belongs to  $\mathbb{T}$ . We denote by  $\mathbb{P}_r^\psi$  its distribution. The corresponding local time, mass measure and total mass are respectively defined by  $\ell^a = \sum_{k \in \mathcal{K}} \ell^{a, \mathcal{T}^k}$ ,  $\mathbf{m}^\mathcal{T} = \sum_{k \in \mathcal{K}} \mathbf{m}^{\mathcal{T}^k}$  and  $\sigma = \sum_{k \in \mathcal{K}} \sigma^{\mathcal{T}^k}$ . By construction, we have  $\mathbb{P}_r^\psi(d\mathcal{T})$ -a.s.  $\emptyset \in \text{Br}_\infty(\mathcal{T})$ ,  $\Delta_\emptyset = r$  and  $\ell^0 = r\delta_\emptyset$ . Under  $\mathbb{P}_r^\psi$  (or  $\mathbb{N}^\psi$ ), we define the process  $\mathcal{Z} = \{\mathcal{Z}_a, a \geq 0\}$  by

$$(15) \quad \mathcal{Z}_a = \langle \ell^a, 1 \rangle.$$

Denote by  $\eta$  the largest root of  $\psi(s) = 0$ . Let  $\psi^{-1} : [0, +\infty) \mapsto [\eta, +\infty)$  be the inverse function of  $\psi$ . Notice that (under  $\mathbb{P}_r^\psi$  or  $\mathbb{N}^\psi$ ):  $\sigma = \int_0^{+\infty} \mathcal{Z}_a da = \mathbf{m}^\mathcal{T}(\mathcal{T})$ . In particular, as  $\sigma$  is distributed as the total mass of a CSBP (accumulated mass over all times) under its canonical measure (intuitively it describes the distribution of CSBP started at an infinitesimal mass), we have that for  $\lambda \geq 0$ ,

$$(16) \quad \mathbb{N}^\psi \left[ 1 - e^{-\lambda\sigma} \right] = \psi^{-1}(\lambda), \quad \mathbb{N}^\psi[1 - e^{-\lambda\mathcal{Z}_a}] = u^\psi(a, \lambda),$$



where  $(u^\psi(a, \lambda), a \geq 0, \lambda > 0)$  is the unique non-negative solution to

$$(17) \quad \int_{u^\psi(a, \lambda)}^\lambda \frac{dr}{\psi(r)} = a; \quad u^\psi(0, \lambda) = \lambda.$$

see e.g. (29) and Lemma 2.4 in [1]. The semigroup property implies

$$(18) \quad u^\psi(a, u^\psi(a', \lambda)) = u^\psi(a + a', \lambda), \quad \lim_{\lambda \rightarrow \infty} u^\psi(a, \lambda) = v^\psi(a).$$

Finally, we recall the Girsanov transformation in [1]. Let  $\theta \in \Theta^\psi$  and  $a > 0$ . We set

$$M_a^{\psi, \theta} = \exp \left\{ \theta \mathcal{Z}_0 - \theta \mathcal{Z}_a - \psi(\theta) \int_0^a \mathcal{Z}_s ds \right\}.$$

Recall that  $\mathcal{Z}_0 = \langle \ell_0, 1 \rangle = 0$  under  $\mathbb{N}^\psi$ . For any non-negative measurable functional  $F$  defined on  $\mathbb{T}$ , we have for  $\theta \in \Theta^\psi$  and  $a \geq 0$ ,

$$(19) \quad \mathbb{E}_r^{\psi, \theta} [F(\mathcal{T}^{(a)})] = \mathbb{E}_r^\psi [F(\mathcal{T}^{(a)}) M_a^{\psi, \theta}] \quad \text{and} \quad \mathbb{N}^{\psi, \theta} [F(\mathcal{T}^{(a)})] = \mathbb{N}^\psi [F(\mathcal{T}^{(a)}) M_a^{\psi, \theta}].$$

In particular,

$$(20) \quad \mathbb{N}^{\psi, \theta} [F(\mathcal{T})] = \mathbb{N}^\psi [F(\mathcal{T}) e^{-\psi(\theta)\sigma} \mathbf{1}_{\{\sigma < +\infty\}}],$$

and by (29) in [5], we also have for  $\theta > 0$  such that  $\psi(\theta) \geq 0$ ,

$$(21) \quad \mathbb{N}^{\psi, \theta} \left[ 1 - \exp \left\{ \theta \mathcal{Z}_a + \psi(\theta) \int_0^a \mathcal{Z}_s ds \right\} \right] = -\theta.$$

#### 4. A GENERAL PRUNING PROCEDURE

In this section we define a pruning procedure on a Lévy tree associated with the admissible family of branching mechanisms.

Recall (2), (10) and  $\mathfrak{T}_t = \mathfrak{T} \cap [t, \infty)$  for  $t \in \mathfrak{T}$ . For  $\mathcal{T} \in \mathbb{T}$ , we consider under probability measure  $\mathbb{Q}$  two Poisson random measures  $M_t^{ske}(d\theta, dy)$  and  $M_t^{nod}(d\theta, dy)$  on the product space  $\mathfrak{T}_t \times \mathcal{T}$  with intensity

$$\beta_\theta d\theta \ell^\mathcal{T}(dy) \quad \text{and} \quad \sum_{x \in \text{Br}_\infty(\mathcal{T}) \setminus \{\emptyset\}} m_{\Delta_x}(t, d\theta) \delta_x(dy),$$

respectively. Then  $M_t^{ske}(d\theta, dy)$  characterizes the marks on the skeleton and  $M_t^{nod}(d\theta, dy)$  describes the marks on the nodes of infinite degree.

We define a new Poisson random measure on  $\mathfrak{T}_t \times \mathcal{T}$  by

$$M_t^\mathcal{T}(d\theta, dy) = M_t^{ske}(d\theta, dy) + M_t^{nod}(d\theta, dy).$$

The pruned tree at time  $q$  for  $q \in \mathfrak{T}_t$  can thus be defined as

$$(22) \quad \mathcal{T}_q^t = \{x \in \mathcal{T}, M_t^\mathcal{T}([t, q] \times \llbracket \emptyset, x \rrbracket) = 0\}$$

with the induced metric, root  $\emptyset$  and mass measure restricted to  $\mathcal{T}_q^t$ .

*Remark 4.1.* Note that  $\mathbb{N}^{\psi_t}$ -a.e.  $n(\emptyset) = 1$  and  $\mathbb{P}_r^{\psi_t}$ -a.s.  $n(\emptyset) = \infty$  with  $\Delta_0 = r$ . The above definition of  $\mathcal{T}_q^t$  indicates that we do not add marks on the root even though  $\emptyset$  is a node of infinite degree with mass  $r$ .

For fixed  $q \in \mathfrak{T}_t$ ,  $M_t^\mathcal{T}([t, q], dy) = M_t^{ske}([t, q], dy) + M_t^{nod}([t, q], dy)$  is also a point measure on tree  $\mathcal{T}$ :

- (i)  $M_t^{ske}([t, q], dy)$  is a Poisson point measure with intensity  $\int_t^q \beta_\theta d\theta \ell^\mathcal{T}(dy)$  on the skeleton of  $\mathcal{T}$ ;
- (ii) The atoms of  $M_t^{nod}([t, q], dy)$  give the marked nodes: each node of infinite degree is marked (or pruned) independently from the others with probability

$$(23) \quad \mathbb{Q} \left( M_t^{nod}([t, q], \{y\}) > 0 \right) = 1 - \exp\{-m_{\Delta_y}(t, [t, q])\} = 1 - m_{\Delta_y}(t, q),$$

where  $\Delta_y$  is the mass associated with the node.

Thus for fixed  $q \in \mathfrak{T}_t$ , there exists a measurable functional  $\mathcal{M}_{\alpha_{t,q}, p_{t,q}}$  on  $\mathbb{T}$  such that

$$(24) \quad \mathcal{T}_q^t = \mathcal{M}_{\alpha_{t,q}, p_{t,q}}(\mathcal{T}),$$

where  $\alpha_{t,q} = \int_t^q \beta_\theta d\theta$ ,  $p_{t,q} = 1 - m_z(t, q)$ . For  $t$  fixed and  $q \in \mathfrak{T}_t$ , if  $\psi_t$  is (sub)critical, then we deduce from Theorems 0.1 and 3.2 in [7] the following result in our setting.

**Lemma 4.2.** (i) *The distribution of  $\mathcal{T}_q^t$  under  $\mathbb{N}^{\psi_t}$  is  $\mathbb{N}^{\psi_q}$ ;*  
(ii) *Given  $\mathcal{T} \in \mathbb{T}$ , let  $\mathcal{M}(dx, d\mathcal{T}) = \sum_{i \in I} \delta_{(x_i, \mathcal{T}_i)}$  be a Poisson random measure on  $\mathcal{T} \times \mathbb{T}$  with intensity  $\mathbf{m}^\mathcal{T}(dx) \left( \alpha_{t,q} \mathbb{N}^{\psi_t}[d\mathcal{T}] + \int_0^\infty p_{t,q} \mathbb{P}_z^{\psi_t}(d\mathcal{T}) m_t(dz) \right)$ . Then under  $\mathbb{N}^{\psi_q}$ ,  $\mathcal{T} \otimes_{i \in I} (\mathcal{T}_i, x_i)$  has the same distribution as  $\mathcal{T}$  under  $\mathbb{N}^{\psi_t}$ .*

We remark that in the special setting of [1] the pruning procedure performs with  $\beta_q$  a positive constant and  $m_{\Delta_y}(0, q) = e^{-q\Delta_y}$ . Our main result in this section is the following theorem which is a generalization of the above result to the supercritical case.

**Theorem 4.3.** *Assume that  $\{\psi_t, t \in \mathfrak{T}\}$  is an admissible family satisfying (H1-3). Then we have*

- (a) *The tree-valued process  $\{\mathcal{T}_q^t, q \in \mathfrak{T}_t\}$  is a Markov process under  $\mathbb{N}^{\psi_t}$ ;*
- (b) *For fixed  $q \in \mathfrak{T}_t$ , the distribution of  $\mathcal{T}_q^t$  under  $\mathbb{N}^{\psi_t}$  is  $\mathbb{N}^{\psi_q}$ ;*
- (c) *Given  $\mathcal{T} \in \mathbb{T}$ , let  $\mathcal{M}(dx, d\mathcal{T}) = \sum_{i \in I} \delta_{(x_i, \mathcal{T}_i)}$  be a Poisson random measure on  $\mathcal{T} \times \mathbb{T}$  with intensity*

$$(25) \quad \mathbf{m}^\mathcal{T}(dx) \left( \int_t^q \beta_\theta d\theta \mathbb{N}^{\psi_t}[d\mathcal{T}] + \int_t^q d\theta \int_0^\infty n_\theta(dz) \mathbb{P}_z^{\psi_t}(d\mathcal{T}) \right).$$

*Then for  $q \in \mathfrak{T}_t$ ,  $(\mathcal{T}, \mathcal{M}_{\alpha_{t,q}, p_{t,q}}(\mathcal{T}))$  under  $\mathbb{N}^{\psi_t}$  has the same distribution as  $(\tilde{\mathcal{T}}, \mathcal{T})$  under  $\mathbb{N}^{\psi_q}$ , where*

$$(26) \quad \tilde{\mathcal{T}} = \mathcal{T} \otimes_{i \in I} (\mathcal{T}_i, x_i).$$

*Remark 4.4.* (c) in Theorem 4.3 is the so-called special Markov property which describes the two dimensional distribution of the tree-valued process. One may follow the proof in Appendix A in [16] to extend (c) to have pruning times in (sub)critical cases and then follow the arguments in *Step 4* below to extend the result to super-critical cases.

*Proof.* The proof will be divided into five steps:

*Step 1:* We prove (a). It suffices to study the behavior of  $M_t^\mathcal{T}(d\theta, dy)$  under  $\mathbb{Q}$ . Given a branching node  $y \in Br_\infty(\mathcal{T})$ , for  $t \leq \theta \leq q \in \mathfrak{T}$ , we have

$$\begin{aligned} & \mathbb{Q} \left( M_t^{nod}([\theta, q], \{y\}) > 0 \mid M_t^{nod}([t, \theta], \{y\}) = 0 \right) \\ &= \frac{\mathbb{Q} \left( M_t^{nod}([\theta, q], \{y\}) > 0, M_t^{nod}([t, \theta], \{y\}) = 0 \right)}{\mathbb{Q} \left( M_t^{nod}([t, \theta], \{y\}) = 0 \right)} \end{aligned}$$

$$\begin{aligned}
&= \frac{\mathbb{Q}(M_t^{nod}([t, q], \{y\}) > 0) - \mathbb{Q}(M_t^{nod}([t, \theta], \{y\}) > 0)}{\mathbb{Q}(M_t^{nod}([t, \theta], \{y\}) = 0)} \\
&= \frac{m_{\Delta_y}(t, \theta) - m_{\Delta_y}(t, q)}{m_{\Delta_y}(t, \theta)} \\
&= 1 - m_{\Delta_y}(\theta, q) \\
(27) \quad &= \mathbb{Q}(M_\theta^{nod}([\theta, q], \{y\}) > 0),
\end{aligned}$$

where we used (23) for the third equality and assumption (H2) for the fourth. Similarly, one can prove that for  $x \in \mathcal{T}$  and  $t \leq \theta \leq q \in \mathfrak{I}$ ,

$$\mathbb{Q}(M_t^{ske}([\theta, q], [\emptyset, x]) > 0 | M_t^{ske}([t, \theta], [\emptyset, x]) = 0) = \mathbb{Q}(M_\theta^{ske}([\theta, q], [\emptyset, x]) > 0).$$

Then (a) follows readily.

*Step 2:* It remains to study the super-critical case for the second and the third assertions. Without loss of generality, we may assume  $t = 0$ . From now on we shall assume that  $\psi_0$  is super-critical. For this proof only, we set for  $x \in \mathcal{T}$ ,  $|x| = d^{\mathcal{T}}(\emptyset, x)$ . We also write  $\mathcal{T}_q$  for  $\mathcal{T}_q^0$ .

In this step, however, we consider the desired results for a special subcritical case. Let  $\eta_0 > 0$  be the maximum root of  $\psi_0(s) = 0$ . Define

$$(28) \quad \psi_q^{\eta_0}(\lambda) = \psi_q(\lambda + \eta_0) - \psi_q(\eta_0), \quad \lambda \geq 0, \quad q \in \mathfrak{I}_0.$$

One can check that if  $\{\psi_q, q \in \mathfrak{I}_0\}$  is an admissible family satisfying (H1-3), then  $\{\psi_q^{\eta_0}, q \in \mathfrak{I}_0\}$  is also an admissible family with parameters  $((b_q^{\eta_0}, m_q^{\eta_0}), q \in \mathfrak{I}_0)$  satisfying (H1-3) such that

$$(29) \quad b_q^{\eta_0} = b_q + 2c\eta_0 + \int_0^\infty z(1 - e^{-\eta_0 z})m_q(dz), \quad m_q^{\eta_0}(dz) = e^{-\eta_0 z} m_q(dz).$$

In addition, an application of (5) and (6) yields

$$\begin{aligned}
b_q^{\eta_0} &= b_0^{\eta_0} + \int_0^q \beta_\theta d\theta + \int_0^q d\theta \int_0^\infty z n_\theta(dz) - \int_0^q d\theta \int_0^\infty z(1 - e^{-\eta_0 z})n_\theta(dz) \\
&= b_0^{\eta_0} + \int_0^q \beta_\theta d\theta + \int_0^q d\theta \int_0^\infty z e^{-\eta_0 z} n_\theta(dz) \\
&\geq b_0^{\eta_0}.
\end{aligned}$$

Since  $b_0^{\eta_0} = \psi_0'(\eta_0) > 0$ , then  $\psi_q^{\eta_0}$  is subcritical for all  $q \in \mathfrak{I}_0$ . Moreover, by (28) and (29),

$$\begin{aligned}
(30) \quad \frac{\partial}{\partial q} \psi_q^{\eta_0}(\lambda) &= \zeta_q(\lambda + \eta_0) - \zeta_q(\eta_0) \\
&= \beta_q \lambda + \int_0^\infty (1 - e^{-\lambda z}) e^{-\eta_0 z} n_q(dz),
\end{aligned}$$

and

$$\frac{m_q^{\eta_0}(dz)}{m_0^{\eta_0}(dz)} = \frac{m_q(dz)}{m_0(dz)} = m_z(0, q),$$

which implies  $\{\psi_q, q \in \mathfrak{I}_0\}$  and  $\{\psi_q^{\eta_0}, q \in \mathfrak{I}_0\}$  induce the same pruning parameters  $\beta_q$  and  $m_z(0, q)$ . Therefore, using Lemma 4.2 for (sub)critical case, we get

(b')  $\mathcal{T}_q = \mathcal{M}_{\alpha_0, q, p_0, q}(\mathcal{T})$  is a  $\psi_q^{\eta_0}$ -Lévy tree under  $\mathbb{N}^{\psi_0^{\eta_0}}$ ;

(c') Given  $\mathcal{T} \in \mathbb{T}$ , let  $\mathcal{M}^{\eta_0}(dx, d\mathcal{T}) = \sum_{i \in I_{\eta_0}} \delta_{(x_i, \mathcal{T}_i)}$  be a Poisson point measure on  $\mathcal{T} \times \mathbb{T}$  with intensity

$$(31) \quad \mathbf{m}^{\mathcal{T}}(dx) \left( \int_0^q \beta_\theta d\theta \mathbb{N}^{\psi_0^{\eta_0}}[d\mathcal{T}] + \int_0^q d\theta \int_0^\infty e^{-\eta_0 z} n_\theta(dz) \mathbb{P}_z^{\psi_0^{\eta_0}}(d\mathcal{T}) \right).$$

Then for  $q \in \mathfrak{I}_0$ ,  $(\mathcal{T}, \mathcal{M}_{\alpha_0, q, p_0, q}(\mathcal{T}))$  under  $\mathbb{N}^{\psi_0^{\eta_0}}$  has the same distribution as  $(\hat{\mathcal{T}}_{\eta_0}, \mathcal{T})$  under  $\mathbb{N}^{\psi_q^{\eta_0}}$ , where

$$(32) \quad \hat{\mathcal{T}}_{\eta_0} = \mathcal{T} \otimes_{i \in I_{\eta_0}} (\mathcal{T}_i, x_i).$$

*Step 3:* We shall prove (b) when  $\psi_0$  is super-critical. Recall  $\mathcal{T}^{(a)} = \{x \in \mathcal{T}; d^{\mathcal{T}}(\emptyset, x) \leq a\}$ . By Girsanov transformation (19), for any nonnegative function  $F$  on  $\mathbb{T}$ , we have

$$(33) \quad \begin{aligned} \mathbb{N}^{\psi_0}[F(\mathcal{T}_q^{(a)})] &= \mathbb{N}^{\psi_0}[F(\mathcal{M}_{\alpha_0, q, p_0, q}(\mathcal{T}^{(a)}))] \\ &= \mathbb{N}^{\psi_0^{\eta_0}}[e^{\eta_0 \mathcal{Z}_a} F(\mathcal{M}_{\alpha_0, q, p_0, q}(\mathcal{T}^{(a)}))] \\ &= \mathbb{N}^{\psi_q^{\eta_0}}[e^{\eta_0 \hat{\mathcal{Z}}_a} F(\mathcal{T}^{(a)})], \end{aligned}$$

where the last equality follows from Special Markov property (c') and

$$\hat{\mathcal{Z}}_a = \langle \ell^a, \hat{\mathcal{T}}, 1 \rangle = \mathcal{Z}_a + \sum_{i \in I_{\eta_0}} \mathbf{1}_{\{|x_i| \leq a\}} \mathcal{Z}_{a-|x_i|}^{\mathcal{T}_i}$$

with  $\mathcal{Z}_a^{\mathcal{T}_i} = \langle \ell^a, \mathcal{T}_i, 1 \rangle$ . Then by the property of Poisson random measure,

$$(34) \quad \mathbb{N}^{\psi_q^{\eta_0}}[e^{\eta_0 \hat{\mathcal{Z}}_a} F(\mathcal{T}^{(a)})] = \mathbb{N}^{\psi_q^{\eta_0}}[e^{\eta_0 \mathcal{Z}_a} F(\mathcal{T}^{(a)}) H(a, \eta_0)]$$

with

$$\begin{aligned} H(a, \eta_0) &= \mathbb{N}^{\psi_q^{\eta_0}} \left[ e^{\eta_0 \sum_{i \in I_{\eta_0}} \mathbf{1}_{\{|x_i| \leq a\}} \mathcal{Z}_{a-|x_i|}^{\mathcal{T}_i}} \Big| \mathcal{T} \right] \\ &= \exp \left\{ - \int_{\mathcal{T}^{(a)}} \mathbf{m}^{\mathcal{T}}(dx) \left( \int_0^q \beta_{\theta} d\theta \mathbb{N}^{\psi_0^{\eta_0}} [1 - e^{\eta_0 \mathcal{Z}_{a-|x|}}] \right. \right. \\ &\quad \left. \left. + \int_0^q d\theta \int_0^{\infty} e^{-\eta_0 z} n_{\theta}(dz) \left( 1 - e^{-z \mathbb{N}^{\psi_0^{\eta_0}} [1 - e^{\eta_0 \mathcal{Z}_{a-|x|}]} \right) \right) \right\}. \end{aligned}$$

Thanks to (21) and the fact that  $\psi_0(\eta_0) = 0$ , we get

$$(35) \quad \mathbb{N}^{\psi_0^{\eta_0}} [1 - e^{\eta_0 \mathcal{Z}_{a-|x|}}] = -\eta_0,$$

which implies

$$H(a, \eta_0) = \exp \left\{ - \int_{\mathcal{T}^{(a)}} \mathbf{m}^{\mathcal{T}}(dx) \left( - \int_0^q \beta_{\theta} d\theta \eta_0 + \int_0^q d\theta \int_0^{\infty} e^{-\eta_0 z} n_{\theta}(dz) (1 - e^{z\eta_0}) \right) \right\}.$$

Since  $\mathbf{m}^{\mathcal{T}}(\mathcal{T}^{(a)}) = \int_0^a \mathcal{Z}_s ds$  and

$$(36) \quad \begin{aligned} \psi_q(\eta_0) &= \psi_q(\eta_0) - \psi_0(\eta_0) \\ &= \int_0^q \beta_{\theta} d\theta \eta_0 - \int_0^q d\theta \int_0^{\infty} e^{-\eta_0 z} n_{\theta}(dz) (1 - e^{z\eta_0}), \end{aligned}$$

then we have

$$H(a, \eta_0) = \exp \left\{ \psi_q(\eta_0) \mathbf{m}^{\mathcal{T}}(\mathcal{T}^{(a)}) \right\} = \exp \left\{ \psi_q(\eta_0) \int_0^a \mathcal{Z}_s ds \right\}.$$

By (33), (34) and Girsanov transformation (19) again, we obtain

$$\begin{aligned} \mathbb{N}^{\psi_0}[F(\mathcal{T}_q^{(a)})] &= \mathbb{N}^{\psi_q^{\eta_0}} [e^{\eta_0 \mathcal{Z}_a + \psi_q(\eta_0) \int_0^a \mathcal{Z}_s ds} F(\mathcal{T}^{(a)})] \\ &= \mathbb{N}^{\psi_q}[F(\mathcal{T}^{(a)})], \end{aligned}$$

which implies that under  $\mathbb{N}^{\psi_0}$ ,  $\mathcal{T}_q$  is a  $\psi_q$ -Lévy tree. We complete the proof of assertion (b).

*Step 4:* We shall prove (c) when  $\psi_0$  is super-critical. Recall (26) and (32). Note that

$$\tilde{\mathcal{T}}^{(a)} = \mathcal{T}^{(a)} \otimes_{i \in I, |x_i| \leq a} (\mathcal{T}_i^{(a-|x_i|)}, x_i).$$

It suffices to show that for all  $a \geq 0$ ,  $(\mathcal{T}^{(a)}, \mathcal{M}_{\alpha_0, q, p_0, q}(\mathcal{T}^{(a)}))$  under  $\mathbb{N}^{\psi_0}$  has the same distribution as  $(\tilde{\mathcal{T}}^{(a)}, \mathcal{T}^{(a)})$  under  $\mathbb{N}^{\psi_q}$ . By (19), we have for any nonnegative functional  $F$  on  $\mathbb{T}^2$ ,

$$(37) \quad \mathbb{N}^{\psi_0} \left[ F(\mathcal{T}^{(a)}, \mathcal{M}_{\alpha_0, q, p_0, q}(\mathcal{T}^{(a)})) \right] = \mathbb{N}^{\psi_0^{\eta_0}} \left[ e^{\eta_0 \hat{Z}_a} F(\mathcal{T}^{(a)}, \mathcal{M}_{\alpha_0, q, p_0, q}(\mathcal{T}^{(a)})) \right].$$

By (c') in *Step 2*, we deduce that  $(\mathcal{T}^{(a)}, \mathcal{M}_{\alpha_0, q, p_0, q}(\mathcal{T}^{(a)}))$  under  $\mathbb{N}^{\psi_0^{\eta_0}}$  has the same distribution as  $(\hat{\mathcal{T}}^{(a)}, \mathcal{T}^{(a)})$  under  $\mathbb{N}^{\psi_q^{\eta_0}}$ , where

$$\hat{\mathcal{T}}^{(a)} = \mathcal{T}^{(a)} \otimes_{i \in I_{\eta_0}, |x_i| \leq a} (\mathcal{T}_i^{(a-|x_i|)}, x_i).$$

Thus we yield

$$(38) \quad \mathbb{N}^{\psi_0} \left[ F(\mathcal{T}^{(a)}, \mathcal{M}_{\alpha_0, q, p_0, q}(\mathcal{T}^{(a)})) \right] = \mathbb{N}^{\psi_q^{\eta_0}} \left[ e^{\eta_0 \hat{Z}_a} F(\hat{\mathcal{T}}^{(a)}, \mathcal{T}^{(a)}) \right].$$

We CLAIM that for all  $a > 0$  and any nonnegative measurable functional  $\Phi$  on  $\mathcal{T}^{(a)} \times \mathbb{T}$ ,

$$(39) \quad \mathbb{N}^{\psi_q^{\eta_0}} \left[ e^{\eta_0 \hat{Z}_a} F(\mathcal{T}^{(a)}) \exp\{-\langle \mathcal{M}_a^{\eta_0}, \Phi \rangle\} \right] = \mathbb{N}^{\psi_q} \left[ F(\mathcal{T}^{(a)}) \exp\{-\langle \mathcal{M}_a, \Phi \rangle\} \right],$$

where

$$\mathcal{M}_a^{\eta_0}(dx, d\mathcal{T}) = \sum_{i \in I_{\eta_0}} \mathbf{1}_{|x_i| \leq a} \delta_{(x_i, \mathcal{T}_i^{(a-|x_i|)})}(dx, d\mathcal{T}),$$

and

$$\mathcal{M}_a(dx, d\mathcal{T}) = \sum_{i \in I} \mathbf{1}_{|x_i| \leq a} \delta_{(x_i, \mathcal{T}_i^{(a-|x_i|)})}(dx, d\mathcal{T}).$$

Then we deduce from (39) that

$$\begin{aligned} \mathbb{N}^{\psi_q^{\eta_0}} \left[ e^{\eta_0 \hat{Z}_a} F(\hat{\mathcal{T}}^{(a)}, \mathcal{T}^{(a)}) \right] &= \mathbb{N}^{\psi_q^{\eta_0}} \left[ e^{\eta_0 \hat{Z}_a} F(\mathcal{T}^{(a)} \otimes_{i \in I_{\eta_0}, |x_i| \leq a} (\mathcal{T}_i^{(a-|x_i|)}, x_i), \mathcal{T}^{(a)}) \right] \\ &= \mathbb{N}^{\psi_q} \left[ F(\mathcal{T}^{(a)} \otimes_{i \in I, |x_i| \leq a} (\mathcal{T}_i^{(a-|x_i|)}, x_i), \mathcal{T}^{(a)}) \right] \\ &= \mathbb{N}^{\psi_q} \left[ F(\tilde{\mathcal{T}}^{(a)}, \mathcal{T}^{(a)}) \right], \end{aligned}$$

which, together with (38), gives

$$\mathbb{N}^{\psi_0} \left[ F(\mathcal{T}^{(a)}, \mathcal{M}_{\alpha_0, q, p_0, q}(\mathcal{T}^{(a)})) \right] = \mathbb{N}^{\psi_q} \left[ F(\tilde{\mathcal{T}}^{(a)}, \mathcal{T}^{(a)}) \right].$$

Since  $a$  is arbitrary, assertion (c) follows readily.

*Step 5:* The remainder of this proof is devoted to (39). Define

$$g(a, x) = \mathbb{N}^{\psi_0} \left[ 1 - e^{-\Phi(x, \mathcal{T}^{(a-|x|)})} \right].$$

Then we have

$$\mathbb{P}_z^{\psi_0} \left( 1 - e^{-\Phi(x, \mathcal{T}^{(a-|x|)})} \right) = 1 - e^{-zg(a, x)}.$$

First, by the property of Poisson random measure, we get

$$(40) \quad \begin{aligned} &\mathbb{N}^{\psi_q} \left[ F(\mathcal{T}^{(a)}) \exp\{-\langle \mathcal{M}_a, \Phi \rangle\} \right] \\ &= \mathbb{N}^{\psi_q} \left[ F(\mathcal{T}^{(a)}) \exp \left\{ - \int_{\mathcal{T}^{(a)}} \mathbf{m}^{\mathcal{T}^{(a)}}(dx) G(a, x) \right\} \right], \end{aligned}$$

where

$$(41) \quad G(a, x) = \left[ \int_0^q \beta_\theta d\theta g(a, x) + \int_0^q d\theta \int_0^\infty n_\theta(dz) \left( 1 - e^{-zg(a, x)} \right) \right].$$

Then thanks to (19), we obtain

$$(42) \quad \begin{aligned} g(a, x) &= \mathbb{N}^{\psi_0} \left[ 1 - e^{-\Phi(x, \mathcal{T}^{(a-|x|)})} \right] \\ &= \mathbb{N}^{\psi_0^{\eta_0}} \left[ e^{\eta_0 \mathcal{Z}_{a-|x|}} \left( 1 - e^{-\Phi(x, \mathcal{T}^{(a-|x|)})} \right) \right] \\ &= \mathbb{N}^{\psi_0^{\eta_0}} \left[ e^{\eta_0 \mathcal{Z}_{a-|x|}} - 1 + 1 - e^{-\Phi(x, \mathcal{T}^{(a-|x|)}) + \eta_0 \mathcal{Z}_{a-|x|}} \right] \\ &= \eta_0 + \mathbb{N}^{\psi_0^{\eta_0}} \left[ 1 - e^{-\Phi(x, \mathcal{T}^{(a-|x|)}) + \eta_0 \mathcal{Z}_{a-|x|}} \right] =: \eta_0 + g_{\eta_0}(a, x), \end{aligned}$$

where the last equality follows from (35). Using (36) and (42), we have

$$(43) \quad \begin{aligned} G(a, x) &= \int_0^q \beta_\theta d\theta \eta_0 - \int_0^q d\theta \int_0^\infty e^{-z\eta_0} n_\theta(dz) (1 - e^{z\eta_0}) \\ &\quad + \int_0^q \beta_\theta d\theta g_{\eta_0}(a, x) + \int_0^q d\theta \int_0^\infty e^{-z\eta_0} n_\theta(dz) \left( 1 - e^{-zg_{\eta_0}(a, x)} \right) \\ &= \psi_q(\eta_0) + \int_0^q \beta_\theta d\theta g_{\eta_0}(a, x) + \int_0^q d\theta \int_0^\infty e^{-z\eta_0} n_\theta(dz) \left( 1 - e^{-zg_{\eta_0}(a, x)} \right) \\ &=: \psi_q(\eta_0) + G_{\eta_0}(a, x). \end{aligned}$$

Applying (19) to (40) gives

$$(44) \quad \begin{aligned} &\mathbb{N}^{\psi_q} \left[ F(\mathcal{T}^{(a)}) \exp\{-\langle \mathcal{M}_a, \Phi \rangle\} \right] \\ &= \mathbb{N}^{\psi_q^{\eta_0}} \left[ \exp \left\{ \eta_0 \mathcal{Z}_a + \psi_q(\eta_0) \int_0^a \mathcal{Z}_s ds \right\} F(\mathcal{T}^{(a)}) \exp \left\{ - \int_{\mathcal{T}^{(a)}} \mathbf{m}^{\mathcal{T}^{(a)}}(dx) G(a, x) \right\} \right] \\ &= \mathbb{N}^{\psi_q^{\eta_0}} \left[ e^{\eta_0 \mathcal{Z}_a} F(\mathcal{T}^{(a)}) \exp \left\{ - \int_{\mathcal{T}^{(a)}} \mathbf{m}^{\mathcal{T}^{(a)}}(dx) G_{\eta_0}(a, x) \right\} \right], \end{aligned}$$

where the last equality follows from (43) and the fact that  $\mathbf{m}^{\mathcal{T}^{(a)}}(\mathcal{T}^{(a)}) = \int_0^a \mathcal{Z}_s ds$ .

On the other hand, using the property of Poisson point measure again, we obtain

$$\begin{aligned} &\mathbb{N}^{\psi_q^{\eta_0}} \left[ e^{\eta_0 \hat{\mathcal{Z}}_a} F(\mathcal{T}^{(a)}) \exp\{-\langle \Phi, \mathcal{M}_a^{\eta_0} \rangle\} \right] \\ &= \mathbb{N}^{\psi_q^{\eta_0}} \left[ e^{\eta_0 \mathcal{Z}_a} F(\mathcal{T}^{(a)}) \exp \left\{ -\langle \Phi, \mathcal{M}_a^{\eta_0} \rangle + \eta_0 \sum_{i \in I_{\eta_0}} \mathbf{1}_{|x_i| \leq a} \mathcal{Z}_{a-|x_i|}^{\mathcal{T}_i} \right\} \right] \\ &= \mathbb{N}^{\psi_q^{\eta_0}} \left[ e^{\eta_0 \mathcal{Z}_a} F(\mathcal{T}^{(a)}) \exp \left\{ - \int_{\mathcal{T}^{(a)}} \mathbf{m}^{\mathcal{T}^{(a)}}(dx) G_{\eta_0}(a, x) \right\} \right], \end{aligned}$$

which, together with (44), implies (39).  $\square$

A direct consequence of Theorem 4.3 is as follows.

**Corollary 4.5.** *Assume that  $\{\psi_t, t \in \mathfrak{T}\}$  is an admissible family satisfying (H1-3). Then for  $r > 0$  we have the tree-valued process  $\{\mathcal{T}_q^t, q \in \mathfrak{T}_t\}$  is a Markov process under  $\mathbb{P}_r^{\psi_t}$  and for fixed  $q \in \mathfrak{T}_t$ , the distribution of  $\mathcal{T}_q^t$  under  $\mathbb{P}_r^{\psi_t}$  is  $\mathbb{P}_r^{\psi_q}$ .*

## 5. A TREE-VALUED PROCESS

From the construction of  $\{\mathcal{T}_q^t, q \in \mathfrak{T}_t\}$ , we have  $\mathcal{T}_q^t \subset \mathcal{T}_p^t$  for  $p \leq q \in \mathfrak{T}_t$ . The process  $\{\mathcal{T}_q^t, q \in \mathfrak{T}_t\}$  is a non-increasing process (for the inclusion of trees). Corollary 4.5 (rep. Theorem 4.3) implies that for  $t_1 \leq t_2 \in \mathfrak{T}$ ,  $\{\mathcal{T}_q^{t_2}, q \in \mathfrak{T}_{t_2}\}$  under  $\mathbb{P}_r^{\psi^{t_2}}$  (resp.  $\mathbb{N}^{\psi^{t_2}}$ ) has the same distribution as  $\{\mathcal{T}_q^{t_1}, q \in \mathfrak{T}_{t_2}\}$  under  $\mathbb{P}_r^{\psi^{t_1}}$  (resp.  $\mathbb{N}^{\psi^{t_1}}$ ). Then there exists a projective limit  $\{\mathcal{T}_t, t \in \mathfrak{T}\}$  which is a tree-valued process such that  $\{\mathcal{T}_q, q \in \mathfrak{T}_t\}$  has the same finite dimensional distribution as  $\{\mathcal{T}_q^t, q \in \mathfrak{T}_t\}$  under  $\mathbb{N}^{\psi^t}$ . Denote by  $\mathbf{P}^\Psi$  and  $\mathbf{N}^\Psi$  the distribution and excursion law of  $\{\mathcal{T}_t, t \in \mathfrak{T}\}$ . We have for any nonnegative measurable functional  $F$ ,

$$\mathbf{N}^\Psi[F(\mathcal{T}_q)] = \mathbb{N}^{\psi_q}[F(\mathcal{T})].$$

Set

$$\sigma_t = \mathbf{m}^{\mathcal{T}_t}(\mathcal{T}_t), \quad t \in \mathfrak{T}.$$

Then one can check that  $\{\sigma_t, t \in \mathfrak{T}\}$  is a non-increasing  $[0, \infty]$ -Markov process. For  $t \in \mathfrak{T}$ , set  $\Psi_t = \{\psi_q, q \in \mathfrak{T}_t\}$  and  $\Psi_t^{\eta_t} = \{\psi_q^{\eta_t}, q \in \mathfrak{T}_t\}$ . We study in this section the property of the tree-valued process which is direction generalization of Section 6 in [1].

**Proposition 5.1.** *For  $t \in \mathfrak{T}$  and any non-negative measurable functional  $F$ ,*

$$\mathbf{N}^{\Psi_t}[F(\mathcal{T}_q, q \in \mathfrak{T}_t) \mathbf{1}_{\{\sigma_t < \infty\}}] = \mathbf{N}^{\Psi_t^{\eta_t}}[F(\mathcal{T}_q, q \in \mathfrak{T}_t)].$$

*Proof.* Recall that  $\{\psi_q, q \in \mathfrak{T}_t\}$  and  $\{\psi_q^{\eta_t}, q \in \mathfrak{T}_t\}$  induce the same pruning parameters. Then the desired result is a direct consequence of the fact  $\mathbb{N}^{\psi_t}[F(\mathcal{T}) \mathbf{1}_{\{\sigma < \infty\}}] = \mathbb{N}^{\psi_t^{\eta_t}}[F(\mathcal{T})]$ ; see (20).  $\square$

We then study the behavior of  $\{\sigma_t, t \in \mathfrak{T}\}$ .

**Lemma 5.2.** *For  $t \leq q \in \mathfrak{T}$  and  $\lambda \geq 0$ , we have*

$$\mathbf{N}^\Psi[e^{-\lambda \sigma_t} | \mathcal{T}_q] = \exp\{-\psi_q(\psi_t^{-1}(\lambda)) \sigma_q\}$$

and  $\mathbf{N}^\Psi[\sigma_t < +\infty | \mathcal{T}_q] = \exp\{-\psi_q(\psi_t^{-1}(0)) \sigma_q\}$ . Moreover, if  $\psi_t$  is subcritical, then

$$(45) \quad \mathbf{N}^\Psi[\sigma_t | \mathcal{T}_q] = \psi'_q(0) \sigma_q / \psi'_t(0).$$

*Proof.* Recall (2) and (16). Using (c) in Theorem 4.3, we obtain

$$\begin{aligned} \mathbf{N}^\Psi[e^{-\lambda \sigma_t} | \mathcal{T}_q] &= \mathbf{N}^\Psi \left[ e^{-\lambda \sigma_q - \lambda \sum_{i \in I} \sigma_i} | \mathcal{T}_q \right] \\ &= e^{-\lambda \sigma_q} e^{-\int_{\mathcal{T}_q} \mathbf{m}^{\mathcal{T}_q}(dx) G(\lambda)}, \end{aligned}$$

where  $\sigma_i = \mathbf{m}^{\mathcal{T}_i}(\mathcal{T}_i)$  and

$$\begin{aligned} G(\lambda) &= \int_t^q \beta_\theta d\theta \mathbf{N}^{\psi_t} [1 - e^{-\lambda \sigma}] + \int_t^q d\theta \int_0^\infty n_\theta(dz) \mathbb{P}_z^{\psi_t} (1 - e^{-\lambda \sigma}) \\ &= \psi_t^{-1}(\lambda) \int_t^q \beta_\theta d\theta + \int_t^q d\theta \int_0^\infty n_\theta(dz) \left(1 - e^{-z \psi_t^{-1}(\lambda)}\right) \\ &= \int_t^q \zeta_\theta(\psi_t^{-1}(\lambda)) d\theta \\ &= \psi_q(\psi_t^{-1}(\lambda)) - \psi_t(\psi_t^{-1}(\lambda)). \end{aligned}$$

Hence,

$$(46) \quad \mathbf{N}^\Psi[e^{-\lambda \sigma_t} | \mathcal{T}_q] = \exp\{-\psi_q(\psi_t^{-1}(\lambda)) \sigma_q\}.$$

Consequently,

$$\mathbf{N}^\Psi[\sigma_t < +\infty | \mathcal{T}_q] = \lim_{\lambda \rightarrow 0} \mathbf{N}^\Psi[e^{-\lambda\sigma_t} | \mathcal{T}_q] = \exp\{-\psi_q(\psi_t^{-1}(0))\sigma_q\}.$$

If  $\psi_t$  is subcritical, then  $\mathbf{N}^\Psi$ -a.e.  $\sigma_t < \infty$ , so we conclude that

$$\mathbf{N}^\Psi[\sigma_t | \mathcal{T}_q] = \frac{d}{d\lambda} \mathbf{N}^\Psi[e^{-\lambda\sigma_t} | \mathcal{T}_q] \Big|_{\lambda=0} = \psi'_q(0)\sigma_q / \psi'_t(0).$$

□

Recall that  $\eta_q$  is the largest root of  $\psi_q(\lambda) = 0$ . Thus

$$\eta_q = \lim_{\lambda \rightarrow 0^+} \psi_q^{-1}(\lambda) = \psi_q^{-1}(0).$$

Define the *ascension time*

$$A = \inf\{t \in \mathfrak{T}; \sigma_t < +\infty\}$$

with the convention that  $\inf\{\emptyset\} = \inf \mathfrak{T} =: t_\infty$ . Since  $\mathfrak{T}$  is an interval, without loss of generality, we may always assume that

$$0 \in \mathfrak{T}, \quad t_\infty < 0.$$

Recall (2). Let us consider the following condition:

$$(47) \quad \lim_{t \rightarrow t_\infty^+} \int_t^0 \zeta_\theta(\lambda) d\theta = \psi_0(\lambda) - \lim_{t \rightarrow t_\infty^+} \psi_t(\lambda) < +\infty, \quad \text{for some } \lambda > 0.$$

We have

**Proposition 5.3.**  $\lim_{q \rightarrow t_\infty^+} \psi_q^{-1}(0) < \infty$  if and only if (47) holds.

*Proof.* “if” part: Condition (47) implies

$$(48) \quad \lim_{t \rightarrow t_\infty^+} \int_t^0 \beta_\theta d\theta + \lim_{t \rightarrow t_\infty^+} \int_t^0 \int_0^\infty (1 \wedge z) n_\theta(dz) d\theta < +\infty.$$

By (6), (48) and the fact that  $1 \wedge z^2 \leq 1 \wedge z$ , we have  $\sup_{q \in \mathfrak{T}} \int_0^\infty (1 \wedge z^2) m_q(dz) < \infty$ . Furthermore, thanks to monotonicity of  $q \mapsto (1 \wedge z^2) m_q(dz)$ , there exists a  $\sigma$ -finite measure  $m_{t_\infty}(dz)$  on  $(0, \infty)$  such that

$$\int (1 \wedge z^2) m_{t_\infty}(dz) < +\infty,$$

and as  $q \rightarrow t_\infty$ ,  $(1 \wedge z^2) m_q(dz) \rightarrow (1 \wedge z^2) m_{t_\infty}(dz)$  in  $M_f((0, \infty))$ . Therefore, we may define for  $\lambda \geq 0$ ,

$$\psi_{t_\infty}(\lambda) = b_{t_\infty} \lambda + c \lambda^2 + \int_0^\infty (e^{-\lambda z} - 1 + \lambda z \mathbf{1}_{\{z \leq 1\}}) m_{t_\infty}(dz)$$

with

$$b_{t_\infty} = b_0 + \int_1^\infty z m_0(dz) - \lim_{q \rightarrow t_\infty^+} \left( \int_q^0 \beta_\theta d\theta + \int_q^0 \int_0^1 z n_\theta(dz) d\theta \right).$$

We deduce that  $\psi_{t_\infty}(\lambda)$  is a convex function. Since  $\psi_q$  satisfies (H3) which implies  $c > 0$  or  $\int_0^1 z m_q(dz) = \infty$ , then we have  $\lim_{\lambda \rightarrow \infty} \psi_{t_\infty}(\lambda) = \infty$  and  $\psi_{t_\infty}^{-1}(0) < \infty$  is the largest root of  $\psi_{t_\infty}(\lambda) = 0$ . Notice that  $e^{-\lambda z} - 1 + \lambda z \mathbf{1}_{\{z \leq 1\}} \leq 1 \wedge z^2$ . By (5) and (6),

$$\begin{aligned} \psi_q(\lambda) &= \left( b_0 + \int_1^\infty z m_0(dz) - \int_q^0 \beta_\theta d\theta - \int_q^0 \int_0^1 z n_\theta(dz) d\theta \right) \lambda \\ &\quad + c \lambda^2 + \int_0^\infty (e^{-\lambda z} - 1 + \lambda z \mathbf{1}_{\{z \leq 1\}}) m_q(dz) \end{aligned}$$



$$\rightarrow \psi_{t_\infty}(\lambda), \quad \text{as } q \rightarrow t_\infty + .$$

Then we conclude  $\psi_{t_\infty}^{-1}(0) = \lim_{q \rightarrow t_\infty +} \psi_q^{-1}(0) < \infty$ .

“only if” part: If  $\int_{t_\infty +}^0 \zeta_\theta(\lambda) d\theta = +\infty$  for some  $\lambda > 0$  (hence for all  $\lambda > 0$ ), by (5) and (6),

$$\psi_q(\lambda) = \psi_0(\lambda) - \int_q^0 \zeta_\theta(\lambda) d\theta \rightarrow -\infty, \quad \text{as } q \rightarrow t_\infty + .$$

Then we have  $\lim_{q \rightarrow t_\infty +} \psi_q^{-1}(0) = +\infty$ . □

Define  $\psi_{t_\infty}^{-1}(0) = \lim_{q \rightarrow t_\infty} \psi_q^{-1}(0)$  and

$$\mathfrak{T}_\infty = \begin{cases} \mathfrak{T} \cup \{t_\infty\}, & \psi_{t_\infty}^{-1}(0) < +\infty \\ \mathfrak{T}, & \psi_{t_\infty}^{-1}(0) = +\infty. \end{cases}$$

*Remark 5.4.* From the proof of Proposition 5.3 we see that it is possible to extend the definition of a given admissible family to  $\mathfrak{T}_\infty$ . For some results in the sequel of this paper, we need to avoid this case by assuming  $t_\infty \notin \mathfrak{T}_\infty$  (hence  $t_\infty \notin \mathfrak{T}$ ).

Next, we study the distribution of  $A$  and  $\mathcal{T}_A$ .

**Lemma 5.5.** For  $q \in \mathfrak{T} \cup \{t_\infty\}$ ,

$$\mathbf{N}^\Psi[A > q] = \psi_q^{-1}(0),$$

and

$$(49) \quad \mathbf{N}^\Psi[A = t_\infty] = \begin{cases} 0, & t_\infty \notin \mathfrak{T}_\infty \\ \infty, & t_\infty \in \mathfrak{T}_\infty. \end{cases}$$

*Proof.* Recall (16). By Lemma 5.2, for  $q > t_\infty$ ,

$$\begin{aligned} \mathbf{N}^\Psi[A > q] &= \mathbf{N}^\Psi[\sigma_q = +\infty] \\ &= \mathbb{N}^{\psi_q}[\sigma = +\infty] \\ &= \lim_{\lambda \rightarrow 0} \mathbb{N}^{\psi_q} [1 - e^{-\lambda\sigma}] \\ &= \lim_{\lambda \rightarrow 0} \psi_q^{-1}(\lambda) \\ &= \psi_q^{-1}(0). \end{aligned}$$

Letting  $q \rightarrow t_\infty$  gives the case of  $q = t_\infty$ . Using Lemma 5.2 again, we obtain

$$\begin{aligned} \mathbf{N}^\Psi[A = t_\infty | \mathcal{T}_0] &= \mathbf{N}^\Psi[\forall q > t_\infty, \sigma_q < +\infty | \mathcal{T}_0] \\ &= \lim_{q \rightarrow t_\infty} \mathbf{N}^\Psi[\sigma_q < +\infty | \mathcal{T}_0] \\ &= \lim_{q \rightarrow t_\infty} e^{-\psi_0(\psi_q^{-1}(0))\sigma_0} \\ &= \begin{cases} 0, & \text{if } t_\infty \notin \mathfrak{T}_\infty \\ e^{-\sigma_0 \psi_0(\psi_{t_\infty}^{-1}(0))}, & \text{if } t_\infty \in \mathfrak{T}_\infty. \end{cases} \end{aligned}$$

Notice that  $\forall \lambda > 0, \mathbb{N}^{\psi_0}[e^{-\lambda\sigma}] = +\infty$ , the desired follows. □

*Remark 5.6.* (49) implies that if  $t_\infty \in \mathfrak{T}_\infty$ , then  $\mathbf{N}^\Psi[\mathcal{T}_t \text{ is compact for all } t > t_\infty] = +\infty$ . If  $t_\infty \notin \mathfrak{T}_\infty$ , then  $\mathbf{N}^\Psi$ -a.e. there exists  $t \in \mathfrak{T}$  such that  $\mathcal{T}_q$  is not compact ( $\sigma_q = \infty$ ) for  $t > q \in \mathfrak{T}$ .

For the rest of the paper, we focus on the ascension time  $A$  and tree at the ascension time  $\mathcal{T}_A$ . Then it is necessary that there exists some point  $c \in \mathfrak{X}$ , such that for  $q < c$ ,  $\psi_q$  is a supercritical branching mechanism. For this purpose, **from now on, we always assume that  $\psi_0$  is critical and  $t_\infty < 0$ .**

**Theorem 5.7.** *For  $q \in (t_\infty, 0)$  and any nonnegative measurable functional  $F$  on  $\mathbb{T}$ ,*

$$(50) \quad \mathbf{N}^\Psi [F(\mathcal{T}_A) | A = q] = \psi'_q(\eta_q) \mathbb{N}^{\psi_q} [F(\mathcal{T}) \sigma \mathbf{1}_{\{\sigma < \infty\}}]$$

and for  $\lambda \geq 0$ ,

$$(51) \quad \mathbf{N}^\Psi [e^{-\lambda \sigma_A} | A = q] = \frac{\psi'_q(\eta_q)}{\psi'_q(\psi_q^{-1}(\lambda))}.$$

In particular, we have

$$(52) \quad \mathbf{N}^\Psi [\sigma_A < \infty | A = q] = 1.$$

*Proof.* By Lemma 5.2, we have for every  $t_\infty < t \leq q < 0$ ,

$$\begin{aligned} \mathbf{N}^\Psi [F(\mathcal{T}_q) \mathbf{1}_{\{A > t\}}] &= \mathbf{N}^\Psi [F(\mathcal{T}_q) \mathbf{1}_{\{\sigma_t = +\infty\}}] \\ &= \mathbf{N}^\Psi [F(\mathcal{T}_q) \mathbf{N}^\Psi [\sigma_t = +\infty | \mathcal{T}_q]] \\ &= \mathbf{N}^\Psi \left[ F(\mathcal{T}_q) \left( 1 - e^{-\sigma_q \psi_q(\psi_t^{-1}(0))} \right) \right] \\ &= \mathbf{N}^\Psi \left[ F(\mathcal{T}_q) \left( 1 - e^{-\sigma_q \psi_q(\eta_t)} \right) \right]. \end{aligned}$$

Since  $\eta_t$  is the largest root of  $\psi_t(s) = 0$ , we have the mapping  $t \mapsto \eta_t$  is differentiable with

$$(53) \quad \frac{d\eta_t}{dt} = -\frac{\zeta_t(\eta_t)}{\psi'_t(\eta_t)}.$$

Then we get

$$\begin{aligned} \frac{d}{dt} \mathbf{N}^\Psi [F(\mathcal{T}_q) \mathbf{1}_{\{A > t\}}] &= \mathbf{N}^\Psi \left[ F(\mathcal{T}_q) \sigma_q e^{-\sigma_q \psi_q(\eta_t)} \right] \frac{d\psi_q(\eta_t)}{dt} \\ &= -\mathbf{N}^\Psi \left[ F(\mathcal{T}_q) \sigma_q e^{-\sigma_q \psi_q(\eta_t)} \right] \frac{\psi'_q(\eta_t) \zeta_t(\eta_t)}{\psi'_t(\eta_t)}. \end{aligned}$$

So we have

$$\begin{aligned} \frac{\mathbf{N}^\Psi [F(\mathcal{T}_A), A \in dq]}{dq} &= -\frac{d}{dt} (\mathbf{N}^\Psi [F(\mathcal{T}_q) \mathbf{1}_{\{A > t\}}]) \Big|_{t=q} \\ &= \zeta_t(\eta_t) \mathbf{N}^\Psi [F(\mathcal{T}_q) \sigma_q \mathbf{1}_{\{\sigma_q < +\infty\}}]. \end{aligned}$$

Thus

$$(54) \quad \mathbf{N}^\Psi [F(\mathcal{T}_A) | A = q] = \frac{\mathbf{N}^\Psi [F(\mathcal{T}_q) \sigma_q \mathbf{1}_{\{\sigma_q < +\infty\}}]}{\mathbf{N}^\Psi [\sigma_q \mathbf{1}_{\{\sigma_q < +\infty\}}]} = \frac{\mathbb{N}^{\psi_q} [F(\mathcal{T}) \sigma \mathbf{1}_{\{\sigma < +\infty\}}]}{\mathbb{N}^{\psi_q} [\sigma \mathbf{1}_{\{\sigma < +\infty\}}]}.$$

Notice that

$$\mathbb{N}^{\psi_q} [\sigma e^{-r\sigma}] = \frac{d}{dr} \mathbb{N}^{\psi_q} [1 - e^{-r\sigma}] = \frac{d}{dr} \psi_q^{-1}(r) = \frac{1}{\psi'_q(\psi_q^{-1}(r))}.$$

Then we have

$$\mathbb{N}^{\psi_q} [\sigma \mathbf{1}_{\{\sigma < +\infty\}}] = \lim_{r \rightarrow 0} \mathbb{N}^{\psi_q} [\sigma e^{-r\sigma}] = \frac{1}{\psi'_q(\eta_q)},$$

which, together with (54), implies (50). Using (54) again, we deduce

$$(55) \quad \mathbf{N}^\Psi \left[ e^{-\lambda \sigma_A} | A = q \right] = \frac{\mathbb{N}^{\psi_q} \left[ e^{-\lambda \sigma} \sigma \right]}{\mathbb{N}^{\psi_q} \left[ \sigma \mathbf{1}_{\{\sigma < +\infty\}} \right]} = \frac{\psi'_q(\eta_q)}{\psi'_q(\psi_q^{-1}(\lambda))}.$$

Then (52) is a direct consequence of (55) by letting  $\lambda \rightarrow 0$ .  $\square$

**Proposition 5.8.** *Assume  $t_\infty \in \mathfrak{T}$ . Then for any nonnegative measurable functional  $F$  on  $\mathbb{T}$ ,*

$$\mathbf{N}^\Psi \left[ F(\mathcal{T}_A) \mathbf{1}_{\{A=t_\infty\}} \right] = \mathbb{N}^{\psi_{t_\infty}^{\eta_{t_\infty}}} \left[ F(\mathcal{T}) \right],$$

where  $\psi_{t_\infty}^{\eta_{t_\infty}}(\lambda) = \psi_{t_\infty}(\eta_{t_\infty} + \lambda)$ . In particular, for  $\lambda \geq 0$ ,

$$\mathbf{N}^\Psi \left[ (1 - e^{-\lambda \sigma_A}) \mathbf{1}_{\{A=t_\infty\}} \right] = \psi_{t_\infty}^{-1}(\lambda) - \eta_{t_\infty}.$$

*Proof.* First we have

$$(56) \quad \begin{aligned} \mathbf{N}^\Psi \left[ F(\mathcal{T}_A) \mathbf{1}_{\{A=t_\infty\}} \right] &= \mathbf{N}^\Psi \left[ F(\mathcal{T}_{t_\infty}) \mathbf{1}_{\{\sigma_{t_\infty} < +\infty\}} \right] \\ &= \mathbb{N}^{\psi_{t_\infty}} \left[ F(\mathcal{T}) \mathbf{1}_{\{\sigma < +\infty\}} \right] \\ &= \mathbb{N}^{\psi_{t_\infty}^{\eta_{t_\infty}}} \left[ F(\mathcal{T}) \right], \end{aligned}$$

where the last equality follows from (20). Then we deduce from (56) that

$$(57) \quad \begin{aligned} \mathbf{N}^\Psi \left[ (1 - e^{-\lambda \sigma_A}) \mathbf{1}_{\{A=t_\infty\}} \right] &= \mathbb{N}^{\psi_{t_\infty}^{\eta_{t_\infty}}} \left[ 1 - e^{-\lambda \sigma} \right] \\ &= (\psi_{t_\infty}^{\eta_{t_\infty}})^{-1}(\lambda) \\ &= \psi_{t_\infty}^{-1}(\lambda) - \eta_{t_\infty}. \end{aligned}$$

$\square$

Recall that  $\mathfrak{T}_t = \mathfrak{T} \cap [t, \infty)$  for  $t \in \mathfrak{T}$ . According to *Step 2* in the proof of Theorem 4.3, for  $q \in \mathfrak{T}$ ,  $\Psi_q^{\eta_q} = \{\psi_t^{\eta_q}, t \in \mathfrak{T}_q\}$  is also an admissible family satisfying (H1-3), where  $\psi_t^{\eta_q}(\lambda) = \psi_t(\lambda + \eta_q) - \psi_t(\eta_q)$ . Set  $\mathfrak{T}_0^q = \{\theta \geq 0, \theta + q \in \mathfrak{T}_q\}$  and  $\Psi^q = \{\psi_{\theta+q}^{\eta_q}, \theta \in \mathfrak{T}_0^q\}$ . Then we have the following corollary.

**Corollary 5.9.** *For  $q \in (t_\infty, 0)$ , for any nonnegative measurable functional  $F$ ,*

$$\mathbf{N}^\Psi \left[ F(\mathcal{T}_{A+t}, t \in \mathfrak{T}_0^q) | A = q \right] = \psi'_q(\eta_q) \mathbf{N}^{\Psi^q} \left[ F(\mathcal{T}_t, t \in \mathfrak{T}_0^q) \sigma_0 \right].$$

*Proof.* Using (20) and (50), we have for any nonnegative measurable functional  $F$  on  $\mathbb{T}$ ,

$$(58) \quad \mathbf{N}^\Psi \left[ F(\mathcal{T}_A) | A = q \right] = \psi'_q(\eta_q) \mathbb{N}^{\psi_q^{\eta_q}} \left[ F(\mathcal{T}) \sigma \right].$$

Note that  $\mathfrak{T}_0^q = \mathfrak{T}_q - q$ . Then the desired result follows from the fact that  $\{\psi_t, t \in \mathfrak{T}_q\}$  and  $\{\psi_t^{\eta_q}, t \in \mathfrak{T}_q\}$  induce the same pruning parameters.  $\square$

An application of Corollary 5.9 is to study the distribution of exit times. Define

$$A_h = \sup\{t \in \mathfrak{T}; H_{max}(\mathcal{T}_t) > h\}, \quad h > 0$$

with the convention  $\sup \emptyset = t_\infty$ . Then  $A_h$  is the last time that the height of the trees is larger than  $h$ . The next result is a generalization of Proposition 4.3 in [5] which computes the conditional distribution of  $A_h$ , given  $A$ .

**Proposition 5.10.** *For  $t_\infty < q < q_0 < 0$ , we have*

$$\begin{aligned}\mathbf{N}^\Psi[A_h > q_0|A = q] &= \frac{\psi'_{q_0}(\eta_q)}{\psi'_q(\eta_q)} - \psi'_{q_0}(\eta_q)\psi_q^{\eta_q}(v^{\psi_q^{\eta_q}}(h)) \int_{v^{\psi_q^{\eta_q}}(h)}^\infty \frac{dr}{\psi_q^{\eta_q}(r)^2}; \\ \mathbf{N}^\Psi[A_h = A|A = q] &= \psi'_q(\eta_q)\psi_q^{\eta_q}(v^{\psi_q^{\eta_q}}(h)) \int_{v^{\psi_q^{\eta_q}}(h)}^\infty \frac{dr}{\psi_q^{\eta_q}(r)^2}.\end{aligned}$$

*Proof.* The second equality follows from the fact  $\mathbf{N}^\Psi[A_h \geq q|A = q] = 1$  and the first equality as  $q_0 \rightarrow q$ . We only need to prove the first one. Recall (15). We use  $\mathcal{Z}_a(\mathcal{T})$  here to stress the dependence on  $\mathcal{T}$ . Note that

$$\mathbf{N}^\Psi[A_h > q_0|A = q] = \mathbf{N}^\Psi[\mathcal{Z}_h(\mathcal{T}_{q_0}) > 0|A = q] = \mathbf{N}^\Psi[\mathcal{Z}_h(\mathcal{T}_{A+q_0-q}) > 0|A = q],$$

which, by Corollary 5.9, is equal to  $\psi'_q(\eta_q)\mathbf{N}^{\Psi^q}[\mathbf{1}_{\{\mathcal{Z}_h(\mathcal{T}_{q_0-q}) > 0\}}\sigma_0]$ . Since for every  $t \in \mathfrak{T}_q$ ,  $\psi_t^{\eta_q}$  is subcritical, by (45), we have

$$\begin{aligned}\mathbf{N}^\Psi[A_h > q_0|A = q] &= \psi'_q(\eta_q)\mathbf{N}^{\Psi^q} \left[ \mathbf{1}_{\{\mathcal{Z}_h(\mathcal{T}_{q_0-q}) > 0\}}\mathbf{N}^{\Psi^q}[\sigma_0|\mathcal{T}_{q_0-q}] \right] \\ &= \psi'_{q_0}(\eta_q)\mathbf{N}^{\Psi^q} \left[ \mathbf{1}_{\{\mathcal{Z}_h(\mathcal{T}_{q_0-q}) > 0\}}\sigma_{q_0-q} \right] \\ &= \psi'_{q_0}(\eta_q)\mathbf{N}^{\psi_q^{\eta_q}} \left[ \mathbf{1}_{\{\mathcal{Z}_h > 0\}}\sigma \right] \\ &= \psi'_{q_0}(\eta_q)\mathbf{N}^{\psi_q^{\eta_q}}[\sigma] - \psi'_{q_0}(\eta_q)\mathbf{N}^{\psi_q^{\eta_q}} \left[ \mathbf{1}_{\{\mathcal{Z}_h=0\}} \int_0^h \mathcal{Z}_a da \right] \\ &= \frac{\psi'_{q_0}(\eta_q)}{\psi'_q(\eta_q)} - \psi'_{q_0}(\eta_q) \int_0^h da \lim_{\lambda \rightarrow 0} \mathbf{N}^{\psi_q^{\eta_q}} \left[ \mathcal{Z}_a e^{-\lambda \mathcal{Z}_h} \right],\end{aligned}$$

where we used (58) in the last equality.

Recall (13), (17) and (18). Then by (16) and branching property (iv), conditioning on  $\mathcal{Z}_a$ , we yield

$$\lim_{\lambda \rightarrow 0} \mathbf{N}^{\psi_q^{\eta_q}} \left[ \mathcal{Z}_a e^{-\lambda \mathcal{Z}_h} \right] = \lim_{\lambda \rightarrow 0} \mathbf{N}^{\psi_q^{\eta_q}} \left[ \mathcal{Z}_a e^{-\mathcal{Z}_a u^{\psi_q^{\eta_q}}(h-a, \lambda)} \right] = \frac{\partial}{\partial \lambda} u^{\psi_q^{\eta_q}}(a, v^{\psi_q^{\eta_q}}(h-a)).$$

Since

$$\begin{aligned}\frac{\partial}{\partial \lambda} u^{\psi_q^{\eta_q}}(a, v^{\psi_q^{\eta_q}}(h-a)) &= \frac{\psi_q^{\eta_q}(u^{\psi_q^{\eta_q}}(a, v^{\psi_q^{\eta_q}}(h-a)))}{\psi_q^{\eta_q}(v^{\psi_q^{\eta_q}}(h-a))} \\ &= \frac{\psi_q^{\eta_q}(v^{\psi_q^{\eta_q}}(h))}{\psi_q^{\eta_q}(v^{\psi_q^{\eta_q}}(h-a))^2} \frac{\partial}{\partial a} v^{\psi_q^{\eta_q}}(h-a).\end{aligned}$$

Then we conclude that

$$\begin{aligned}\mathbf{N}^\Psi[A_h > q_0|A = q] &= \frac{\psi'_{q_0}(\eta_q)}{\psi'_q(\eta_q)} - \psi'_{q_0}(\eta_q) \int_0^h \frac{\partial}{\partial \lambda} u^{\psi_q^{\eta_q}}(a, v^{\psi_q^{\eta_q}}(h-a)) da \\ &= \frac{\psi'_{q_0}(\eta_q)}{\psi'_q(\eta_q)} - \psi'_{q_0}(\eta_q)\psi_q^{\eta_q}(v^{\psi_q^{\eta_q}}(h)) \int_{v^{\psi_q^{\eta_q}}(h)}^\infty \frac{dr}{\psi_q^{\eta_q}(r)^2}.\end{aligned}$$

□

*Remark 5.11.* It is easy to see that  $\mathbf{N}^\Psi[A_h \geq q] = v^{\psi_q}(h)$ .

*Remark 5.12.* Recall Theorem 4.3 and Remark 4.4, an explicit construction of an increasing tree-valued process as that of [5] may be given which has the same distribution as  $\{\mathcal{T}_q, q \in \mathfrak{T}_t\}$  under  $\mathbf{N}^\Psi$ . Then by similar arguments as in [5] (Theorem 4.6 there), one can derive the joint distribution of  $(\mathcal{T}_{A_h-}, \mathcal{T}_{A_h})$  (and hence  $(\mathcal{T}_{A-}, \mathcal{T}_A)$ ). We left these to the interested readers.

## 6. TREE AT THE ASCENSION TIME

In this section, we study the representation of the tree at the ascension time. Recall that we shall always assume that

$$0 \in \mathfrak{T}, \quad t_\infty < 0, \quad \text{and } \psi_0 \text{ is critical.}$$

We first consider an infinite CRT and its pruning. An infinite CRT was constructed in [1] which is the Lévy CRT conditioned to have infinite height. Before recalling its construction, we stress that under  $\mathbb{P}_r^\psi$ , the root  $\emptyset$  belongs to  $\text{Br}_\infty$  and has mass  $\Delta_0 = r$ . We identify the half real line  $[0, +\infty)$  with a real tree denoted by  $\llbracket 0, \infty \rrbracket$  with the null mass measure. We denote by  $dx$  the length measure on  $\llbracket 0, \infty \rrbracket$ . Let  $\sum_{i \in I_1^*} \delta_{(x_i^*, \mathcal{T}^{*,i})}$  and  $\sum_{i \in I_2^*} \delta_{(x_i^*, \mathcal{T}^{*,i})}$  be two independent Poisson random measures on  $\llbracket 0, \infty \rrbracket \times \mathbb{T}$  with intensities

$$dx \, 2c\mathbb{N}^{\psi_0}[d\mathcal{T}] \quad \text{and} \quad dx \int_0^\infty lm_0(dl) \mathbb{P}_l^{\psi_0}(d\mathcal{T}),$$

respectively. The infinite CRT from [1] is defined as

$$(59) \quad \mathcal{T}^* = \llbracket \emptyset, \infty \rrbracket \otimes_{i \in I_1^* \cup I_2^*} (x^{*,i}, \mathcal{T}^{*,i}).$$

We denote by  $\mathbb{P}^{*,\psi_0}(d\mathcal{T}^*)$  the distribution of  $\mathcal{T}^*$  and  $\mathbb{E}^{*,\psi_0}(d\mathcal{T}^*)$  the corresponding expectation. Similarly to the setting in Section 4, we consider on  $\mathcal{T}^*$  the mark processes  $M_{ske}^{\mathcal{T}^*}(dq, dy)$  and  $M_{node}^{\mathcal{T}^*}(dq, dy)$  which are Poisson random measures on  $\mathfrak{T}_0 \times \mathcal{T}^*$  with intensities

$$\mathbf{1}_{\{q \in \mathfrak{T}_0\}} \beta_q dq \ell^{\mathcal{T}^*}(dy) \quad \text{and} \quad \mathbf{1}_{\{q \in \mathfrak{T}_0\}} \sum_{i \in I_1^* \cup I_2^*} \sum_{x \in \text{Br}_\infty(\mathcal{T}^{*,i})} m_{\Delta_x}(0, dq) \delta_x(dy),$$

respectively. We identify  $x^{*,i}$  as the root of  $\mathcal{T}^{*,i}$ . In particular nodes in  $\llbracket \emptyset, \infty \rrbracket$  with infinite degree will be charged by  $M_{node}^{\mathcal{T}^*}$ . Then set

$$M^{\mathcal{T}^*}(dq, dy) = M_{ske}^{\mathcal{T}^*}(dq, dy) + M_{node}^{\mathcal{T}^*}(dq, dy).$$

For every  $q \in \mathfrak{T}_0$ , the pruned tree at time  $q$  is defined as

$$\mathcal{T}_q^* = \{x \in \mathcal{T}^*; M^{\mathcal{T}^*}([0, q] \times \llbracket \emptyset, x \rrbracket) = 0\},$$

with the induced metric, root  $\emptyset$  and mass measure restricted to  $\mathcal{T}_q^*$ . Our main result in this section is the following theorem and the proof will be postponed to the end of this section.

**Theorem 6.1.** *Given  $q \in (t_\infty, 0)$ , if there exists  $\bar{q} \in \mathfrak{T}_0$  such that  $\psi_{\bar{q}}(\lambda) = \psi_q(\eta_q + \lambda)$ , then conditioned on  $\{A = q\}$ ,  $\mathcal{T}_A$  is distributed as  $\mathcal{T}_{\bar{q}}^*$ .*

We remark here that this result is a generalization of the particular setting in [1] or of the discrete case considered in [10, 3]. Now we give some applications of it. Recall  $\mathfrak{T}_0^q = \{\theta \geq 0, \theta + q \in \mathfrak{T}_q\}$ . A similar reasoning as Corollary 5.9 yields the following corollary.

**Corollary 6.2.** *Given  $q \in (t_\infty, 0)$ , if there exists  $\bar{q} \in \mathfrak{T}_0$  such that  $\psi_{\bar{q}+t}(\lambda) = \psi_{q+t}^{\eta_q}(\lambda)$  for all  $t \in \mathfrak{T}_0^{\bar{q}}$ , then conditioned on  $\{A = q\}$ ,  $\{\mathcal{T}_{A+t}, t \in \mathfrak{T}_0^{\bar{q}}\}$  is distributed as  $\{\mathcal{T}_{\bar{q}+t}^*, t \in \mathfrak{T}_0^{\bar{q}}\}$ .*

**Lemma 6.3.** *Assume that  $t_\infty \notin \mathfrak{T}_\infty$  and for every  $t \in (t_\infty, 0)$ , there exists  $\bar{t} \in \mathfrak{T}$  such that  $\psi_{\bar{t}}(\lambda) = \psi_t(\eta_t + \lambda)$ . Then  $t \rightarrow \bar{t}$  is differentiable and*

$$\frac{d\bar{t}}{dt} = \frac{\zeta_t'(\eta_t) \psi_t'(\eta_t) - \psi_t''(\eta_t) \zeta_t(\eta_t)}{\zeta_{\bar{t}}'(\eta_t) \psi_{\bar{t}}'(\eta_t)} =: \frac{-\gamma_t}{\zeta_{\bar{t}}'(\eta_t)}, \quad t \in (t_\infty, 0).$$

*Proof.* It is obvious that  $t \rightarrow \bar{t}$  is differentiable. By  $\psi_{\bar{t}}(\lambda) = \psi_t(\eta_t + \lambda)$  and (53), we have for all  $\lambda > 0$ ,

$$\frac{d\bar{t}}{d\lambda} = \frac{\zeta_t(\eta_t + \lambda)\psi'_t(\eta_t) - \psi'_t(\eta_t + \lambda)\zeta_t(\eta_t)}{\zeta_{\bar{t}}(\lambda)\psi'_{\bar{t}}(\eta_t)}.$$

The result follows by taking  $\lambda \rightarrow 0$ .  $\square$

Define  $\bar{t}_\infty = \sup\{\bar{t}; t \in \mathfrak{T}, b_t < 0\}$ . For  $t \in (0, \bar{t}_\infty)$ , let  $\hat{t}$  be the unique negative number such that  $\bar{t} = t$ . Let  $U$  be a positive “random” variable with nonnegative “density” with respect to the Lebesgue measure given by

$$\mathbf{1}_{\{t \in (0, \bar{t}_\infty)\}} \frac{\zeta_{\hat{t}}(\eta_{\hat{t}})\zeta'_{\hat{t}}(0)}{\psi'_{\hat{t}}(\eta_{\hat{t}})\gamma_{\hat{t}}}.$$

Assume that  $U$  is independent of  $\mathcal{T}^*$ .

**Corollary 6.4.** *Suppose that all assumptions in Lemma 6.3 hold. Then  $\mathcal{T}_A$  is distributed under  $\mathbf{N}^\Psi$  as  $\mathcal{T}_U^*$ .*

*Remark 6.5.* If  $U$  has the same distribution as  $A$ , then we have  $\mathcal{T}_A$  is distributed as  $\mathcal{T}_U^*$ .

*Proof.* Recall (53). By Lemma 5.5, we have the law of  $A$  under  $\mathbf{N}^\Psi$  has a density with respect to the Lebesgue measure on  $\mathbb{R}$  give by

$$\mathbf{1}_{\{t \in (t_\infty, 0)\}} \frac{\zeta_t(\eta_t)}{\psi'_t(\eta_t)}.$$

Thus for any nonnegative measurable function  $F$  on  $\mathbb{T}$ , we deduce from Theorem 6.1 that

$$\begin{aligned} \mathbf{N}^\Psi[F(\mathcal{T}_A)] &= \int_{(t_\infty, 0)} \mathbb{E}^{*, \psi_0}[F(\mathcal{T}_{\bar{t}}^*)] \frac{\zeta_t(\eta_t)}{\psi'_t(\eta_t)} dt \\ &= \int_{(t_\infty, 0)} \mathbb{E}^{*, \psi_0}[F(\mathcal{T}_{\hat{t}}^*)] \frac{\zeta_{\hat{t}}(\eta_{\hat{t}})}{\psi'_{\hat{t}}(\eta_{\hat{t}})} d\hat{t} \\ &= \int_{(t_\infty, 0)} \mathbb{E}^{*, \psi_0}[F(\mathcal{T}_{\hat{t}}^*)] \frac{\zeta_{\hat{t}}(\eta_{\hat{t}})}{\psi'_{\hat{t}}(\eta_{\hat{t}})} \frac{d\hat{t}}{d\bar{t}} dt \\ &= \int_{(0, \bar{t}_\infty)} \mathbb{E}^{*, \psi_0}[F(\mathcal{T}_{\hat{t}}^*)] \frac{\zeta_{\hat{t}}(\eta_{\hat{t}})}{\psi'_{\hat{t}}(\eta_{\hat{t}})} \frac{\zeta'_{\hat{t}}(0)}{\gamma_{\hat{t}}} dt \\ &= \mathbb{E}^{*, \psi_0}[F(\mathcal{T}_U^*)], \end{aligned}$$

where the fourth equality follows from Lemma 6.3. We have completed the proof.  $\square$

By Corollaries 6.2 and 6.4, we derive the following result which is a generalization of Corollary 8.2 in [1].

**Corollary 6.6.** *Suppose  $t_\infty \notin \mathfrak{T}_\infty$  and  $[0, \infty) \subset \mathfrak{T}$ . If for every  $q \in (t_\infty, 0)$ , there exists  $\bar{q} \in \mathfrak{T}_0$  such that  $\psi_{\bar{q}+t}(\lambda) = \psi_{q+t}^{\eta_q}(\lambda)$  for all  $t \in \mathfrak{T}_0$ , then  $\{\mathcal{T}_{A+t}, t \geq 0\}$  is distributed under  $\mathbf{N}^\Psi$  as  $\{\mathcal{T}_{U+t}^*, t \geq 0\}$ .*

In the following we give some examples.

*Example 6.7.* Recall  $\{\psi^q(\lambda), q \in \Theta^\psi\}$  in Example 2.5. It was assumed in [1] that  $\psi$  is critical,  $\theta_\infty := \inf \Theta^\psi \notin \Theta^\psi$  and  $\theta_\infty < 0$ . So  $\{\psi^q, q \in \Theta^\psi\}$  satisfies assumptions in Corollary 6.6. Then for  $t \in (\theta_\infty, 0)$ ,  $\eta_t = \bar{t} - t$  and

$$\zeta_t(\lambda) = 2c\lambda + \int_0^\infty (1 - e^{-z\lambda}) e^{-zt} zm(dz) = \psi'(t + \lambda) - \psi'(t).$$

Using  $\eta_{\hat{t}} = t - \hat{t}$ , it is easy to see

$$\mathbf{1}_{\{t \in (0, \bar{\theta}_\infty)\}} \frac{\zeta_{\hat{t}}(\eta_{\hat{t}}) \zeta'_{\hat{t}}(0)}{\psi'_{\hat{t}}(\eta_{\hat{t}}) \gamma_{\hat{t}}} = \mathbf{1}_{\{t \in (0, \bar{\theta}_\infty)\}} \left( 1 - \frac{\psi'(t)}{\psi'(\hat{t})} \right).$$

Then we go back to Corollary 8.2 in [1].

*Example 6.8.* Let  $b, c > 0$  be two constants. Define  $\psi_q(\lambda) = qb\lambda + c\lambda^2$  with  $q \in \mathbb{R}$  and  $\lambda \geq 0$ . Then  $\{\psi_q, q \in \mathbb{R}\}$  satisfies assumptions in Corollary 6.6. In particular, if  $b = 2c$ , we have  $\psi_q(\lambda) = \psi_0(q + \lambda) - \psi_0(q)$ .

*Example 6.9.* Recall  $\{\psi_q, q \in \mathfrak{T}_- \cup (-\mathfrak{T}_-)\}$  considered in Example 2.7. It is easy to verify that  $\{\psi_q, q \in \mathfrak{T}_- \cup (-\mathfrak{T}_-)\}$  satisfies assumptions in Corollary 6.4.

The end of this section is devoted to the proof of Theorem 6.1.

**Proposition 6.10.** *For any nonnegative measurable functional  $F$  on  $\mathbb{T}$  and for every  $q \in \mathfrak{T}_0$ ,*

$$(60) \quad \psi'_q(0) \mathbf{N}^\Psi [\sigma_q F(\mathcal{T}_q)] = \mathbb{E}^{*, \psi_0} [F(\mathcal{T}_q^*)].$$

*Proof.* First, recall the Bismut decomposition of Lévy tree  $\mathcal{T}$  along a spine  $[\![\emptyset, x]\!]$  for  $x \in \text{Lf}(\mathcal{T})$ . Take the spine as a subtree, and consider the connected component  $(\mathcal{T}^i, i \in I_x)$  of  $\mathcal{T} \setminus [\![\emptyset, x]\!]$ . Let the branching point  $(x_i, i \in I_x)$  be the root. Then  $\mathcal{T} = [\![\emptyset, x]\!] \otimes_{i \in I_x} (\mathcal{T}^i, x_i)$ . We deduce from Theorem 2.18 in [5] (or Theorem 7.1 in [1], which were originally proposed in [12]) and  $\sigma = m^\mathcal{T}(\mathcal{T})$  that,

$$(61) \quad \begin{aligned} \psi'_q(0) \mathbf{N}^\Psi [\sigma_q F(\mathcal{T}_q)] &= \psi'_q(0) \mathbf{N}^{\psi_q} [\sigma F(\mathcal{T})] \\ &= \psi'_q(0) \mathbf{N}^{\psi_q} \left[ \int m^\mathcal{T}(dx) F([\![\emptyset, x]\!] \otimes_{i \in I_x} (\mathcal{T}^i, x_i)) \right] \\ &= \psi'_q(0) \int_0^\infty da e^{-\psi'_q(0)a} \mathbb{E} \left[ F([\![\emptyset, a]\!] \otimes_{i \in \tilde{I}, z_i \leq a} \tilde{\mathcal{T}}^i) \right], \end{aligned}$$

where under  $\mathbb{E}$ ,  $\sum_{i \in \tilde{I}} \delta_{(z_i, \tilde{\mathcal{T}}^i)}(dz, d\mathcal{T})$  is a Poisson random measure on  $[0, \infty) \times \mathbb{T}$  with intensity

$$dz \left( 2c \mathbf{N}^{\psi_q}[d\mathcal{T}] + \int_0^\infty l m_q(dl) \mathbb{P}_l^{\psi_q}(d\mathcal{T}) \right).$$

For  $i \in I_1^* \cup I_2^*$ , define

$$\mathcal{T}_q^{*,i} = \{x \in \mathcal{T}^{*,i} : M^{\mathcal{T}^*}([0, q] \times [\![\emptyset, x]\!]) = 0\}, \quad q \in \mathfrak{T}_0.$$

With abuse of notation, we have

$$(62) \quad \mathcal{T}_q^* = [\![\emptyset, \xi]\!] \otimes_{i \in I_1^* \cup I_2^*, x^{*,i} < \xi} (x^{*,i}, \mathcal{T}_q^{*,i}),$$

where

$$\begin{aligned} \xi &:= \sup\{x \in [\![\emptyset, +\infty[\!]: M^{\mathcal{T}^*}([0, q] \times [\![\emptyset, x]\!]) = 0\} \\ &= \sup\{x \in [\![\emptyset, +\infty[\!]: M_{skc}^{\mathcal{T}^*}([0, q] \times [\![\emptyset, x]\!]) = 0\} \wedge \inf\{x^{*,i} : M_{node}^{\mathcal{T}^*}([0, q] \times \{x^{*,i}\}) > 0\} \\ &=: \xi_1 \wedge \xi_2. \end{aligned}$$

Thanks to (61) and (62), it suffices to show that  $\xi$  is exponentially distributed with parameter  $\psi'_q(0)$ . Indeed, it is obvious that  $\xi_1$  is exponentially distributed with parameter  $\int_0^q \beta_\theta d\theta$ . By Corollary 4.5 and the property of Poisson random measure, we have

$$\sum_{i \in I_2^*} \mathbf{1}_{\{M_{node}^{\mathcal{T}^*}([0, q] \times \{x^{*,i}\}) > 0\}} \delta_{(x^{*,i}, \mathcal{T}_q^{*,i})}(dx, d\mathcal{T})$$

is a Poisson random measure with intensity  $dx \int_0^q d\theta \int_0^\infty zn_\theta(dz) \mathbb{P}_z^{\psi_q}(d\mathcal{T})$ . So we deduce that  $\xi_2$  is exponentially distributed with parameter  $\int_0^q d\theta \int_0^\infty zn_\theta(dz)$ . Hence  $\xi$  is exponentially distributed with parameter

$$\int_0^q \beta_\theta d\theta + \int_0^q d\theta \int_0^\infty zn_\theta(dz),$$

which, by (5), is just  $b_q = \psi'_q(0)$ . The result follows.  $\square$

Now we are in position to prove Theorem 6.1.

**Proof of Theorem 6.1:** For any nonnegative measurable function  $F$  on  $\mathbb{T}$ , by (54), we have for  $q < 0$ ,

$$\begin{aligned} \mathbf{N}^\Psi[F(\mathcal{T}_A)|A = q] &= \psi'_q(\eta_q) \mathbf{N}^{\psi_q} [F(\mathcal{T}) \sigma \mathbf{1}_{\{\sigma < \infty\}}] \\ (63) \qquad \qquad \qquad &= \psi'_q(\eta_q) \mathbf{N}^{\psi_q^{\eta_q}} [F(\mathcal{T}) \sigma], \end{aligned}$$

where the last equality follows from (20). Since  $\psi_{\bar{q}}(\lambda) = \psi_q(\eta_q + \lambda) = \psi_q^{\eta_q}(\lambda)$  and  $\psi'_q(\eta_q) = \psi'_{\bar{q}}(0)$ , an application of Proposition 6.10 yields

$$\mathbf{N}^\Psi[F(\mathcal{T}_A)|A = q] = \psi'_{\bar{q}}(0) \mathbf{N}^\Psi [\sigma_{\bar{q}} F(\mathcal{T}_{\bar{q}})] = \mathbb{E}^{*, \psi_0} [F(\mathcal{T}_{\bar{q}}^*)].$$

We have completed the proof.  $\square$

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HONGWEI BI, SCHOOL OF INSURANCE AND ECONOMICS, UNIVERSITY OF INTERNATIONAL BUSINESS AND ECONOMICS, BEIJING 100029, P.R.CHINA.

*E-mail address:* bihw@uibe.edu.cn

HUI HE, LABORATORY OF MATHEMATICS AND COMPLEX SYSTEMS, SCHOOL OF MATHEMATICAL SCIENCES, BEIJING NORMAL UNIVERSITY, BEIJING 100875, P.R.CHINA.

*E-mail address:* hehui@bnu.edu.cn