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Branching Particle Systems in Spectrally One-sided Lévy Processes

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Abstract We investigate the branching structure coded by the excursion above zero of a spectrally positive Lévy process. The main idea is to identify the level of the Lévy excursion as the time and count the number of jumps upcrossing the level. By regarding the size of a jump as the birth site of a particle, we construct a branching particle system in which the particles undergo nonlocal branchings and deterministic spatial motions to the left on the positive half line. A particle is removed from the system as soon as it reaches the origin. Then a measure-valued Borel right Markov process can be defined as the counting measures of the particle system. Its total mass evolves according to a Crump-Mode-Jagers branching process and its support represents the residual life times of those existing particles. A similar result for spectrally negative Lévy process is established by a time reversal approach. Properties of the measure-valued processes can be studied via the excursions for the corresponding Lévy processes.

Keywords Lévy process, spectrally one-sided, subordinator, branching particle system, non-local branching, Crump-Mode-Jagers branching process.
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1 Introduction

Branching processes embedded in processes with independent increments have been studied by many authors. The study yields detailed information and understandings in the two classes of processes. In particular, Dwass [7] constructed branching processes from simple random walks. To study random walks in random environment Kesten et al [11] constructed a Galton-Watson

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process with geometric offspring law from a simple random walk. Multitype branching processes have also been introduced in the study of random walks in random environment; see [9, 10, 12] and the references therein. Since continuous state branching processes and Brownian motions arise as the scaling limits of Galton-Watson processes and simple random walks, respectively, we may naturally expect some branching structures embedded in a Brownian motion. The well-known Knight-Ray theorem brings an answer to this question; see also [16, 21].

Le Gall and Le Jan [18, 19] recovered a deep connection between general continuous state branching processes and spectrally positive Lévy processes. Furthermore, Duquesne and Le Gall [4, 5] showed that the branching points of a Lévy tree constructed in [18] are of two types: binary nodes (i.e. vertex of degree three), which are given by the Brownian part of the Lévy process, and infinite nodes (i.e. vertex of infinite degrees), which are given by the jumps of the Lévy process. The size of the jump is also called the size of the corresponding infinite node (or the mass of the forest attached to the node).

In the interesting recent work [14], Lambert used spectrally positive Lévy processes for the first time to code random splitting trees. In the population dynamics represented by the splitting tree, the number of individuals evolves according to a binary Crump-Mode-Jagers process. It was proved in [14] that the contour process of the splitting tree truncated up to a certain level is a spectrally positive Lévy process reflected below this level and killed upon hitting zero. From this result Lambert derived a number of properties of the splitting tree and the Crump-Mode-Jagers process.

The purpose of this paper is to give a formulation of the branching structures of spectrally one-sided Lévy processes in terms of measure-valued processes, which we call *single-birth branching particle systems*. Those structures are undoubtedly conveyed by the random splitting trees, so we could have derived the results from those of Lambert [14]. However, we think a simple construction of the branching particle systems directly from the Lévy process is of interest. In addition, we show that the branching systems are Borel right Markov processes in a suitable state space and characterize their transition semigroups using some simple quasi-linear integral equations. Those properties make the branching systems easier to handle than the Crump-Mode-Jagers processes. A more precise description of the branching structures is given in the next paragraph.

Let us consider a typical trajectory of the spectrally positive Lévy process with negative drift $\{S_t : t \geq 0\}$ started from $a > 0$ and killed upon hitting zero; see Figure 1. Let $\{y_i : i = 1, 2, 3\}$ denote the sizes of jumps. Then the sample path of a branching particle system can be obtained in the following way: At time zero, an ancestor starts off from $a > 0$ and moves toward the left at the unit speed. At times z_1 and z_3, it gives birth to two children at positions y_1 and y_3, respectively. At time z_2, the first child of the ancestor gives birth to a child at position y_2. Once an individual hits zero, it is removed from the system. So the ancestor dies at time a and its two children die at times $z_1 + y_1$ and $z_3 + y_3$, respectively.

From the structures described above, we use a time reversal to derive a similar result for spectrally negative Lévy processes with positive drift. We will see that the branching systems we encounter here are actually very special cases of the models studied in [3, 20]. Unfortunately, by now we can only treat Lévy processes with bounded variations as in [14]. An interesting open question is to give a description of the branching structures of general spectrally one-sided Lévy processes in terms of measure-valued branching processes. We hope to see the precise formulation of such structures in the future.

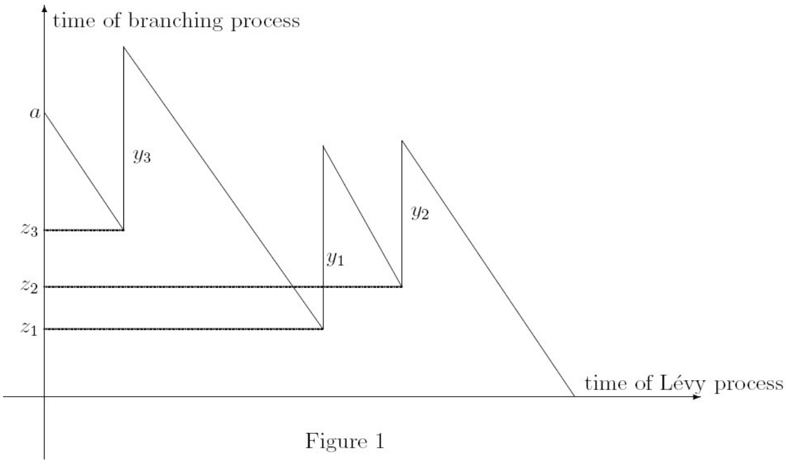

Figure 1

The rest of this paper is arranged as follows. In Section 2, we introduce some branching particle systems on the positive half line involving nonlocal branching structures. In Section 3, we extend the model to the case with infinite branching rates. In Section 4 the result on the branching structures in spectrally positive Lévy processes with negative drift is established. In Section 5, we derive the branching structures for spectrally negative Lévy processes with positive drift by a time reversal approach. Some properties of our branching systems are studied in Section 6. In Section 7, we discusses briefly the connection of the branching systems with the Crump-Mode-Jagers models.

Notations. Write $\mathbb{R}_+ = [0, \infty)$. Given a metric space E, we denote by $B(E)$ the Banach space of bounded Borel functions on E endowed with the supremum/uniform norm "$\|\cdot\|$". Let $C(E)$ be the subspace of $B(E)$ consisting of bounded continuous functions on E. We use the superscript "+" to denote the subset of positive elements of the function spaces, e.g., $B^+(\mathbb{R}_+)$ and $C^+(0, \infty)$. Let $M(E)$ denote the space of finite Borel measures on E endowed with the topology of weak convergence. Let $N(E)$ be the set of integer-valued measures in $M(E)$. For a measure μ and a function f on E write $\langle \mu, f\rangle = \int f d\mu$ if the integral exists. Other notations will be explained when they first appear.

2 Branching systems on the positive half line

We begin with the description of a branching system of particles on $\mathbb{R}_+$.

Suppose that $\alpha > 0$ is a constant, $\eta = \eta(dx)$ is a probability measure on $(0, \infty)$ and $g = g(z)$ is a probability generating function with $g'(1) < \infty$. Let $\{\xi_t : t \geq 0\}$ be the $\mathbb{R}_+$-valued Markov process defined by $\xi_t := (\xi_0 - t) \vee 0$. Let $F(0, \cdot)$ be the unit mass at $\delta_0 \in N(\mathbb{R}_+)$. For $x > 0$, let $F(x, \cdot)$ be the distribution on $N(\mathbb{R}_+)$ of the random measure

$$\delta_x + \sum_{i=1}^{Z} \delta_{Y_i},$$

where Z is an integer-valued random variable with distribution determined by $g = g(z)$ and $\{Y_1, Y_2, \cdots\}$ are i.i.d random variables on $(0, \infty)$ with distribution $\eta(dx)$. Here we assumed that Z and $\{Y_1, Y_2, \cdots\}$ are independent.

Suppose that we have a set of particles on $\mathbb{R}_+$ moving independently according to the law of $\{\xi_t : t \geq 0\}$. A particle is frozen as soon as it reaches zero. Before that at each α-exponentially distributed random time, the particle gives birth to a random number of offspring according to the law specified by the generating function $g = g(z)$, and those offspring are scattered over $\mathbb{R}_+$ independently according to the distribution $\eta(dx)$. It is assumed as usual that the reproduction of different particles are independent of each other. Let $\bar{X}_t(B)$ denote the number of particles in the set $B \in \mathscr{B}(\mathbb{R}_+)$ at time $t \geq 0$. By Dawson et al [3, p.103] one can see that $\{\bar{X}_t : t \geq 0\}$ is a Markov process on $N(\mathbb{R}_+)$ with transition semigroup $(\bar{Q}_t)_{t \geq 0}$ defined by

$$\int_{N(0,\infty)} e^{-\langle \nu, f \rangle} \bar{Q}_t(\mu, d\nu) = \exp\{-\langle \mu, \bar{U}_t f \rangle\}, \qquad f \in B^+(\mathbb{R}_+), \tag{2.1}$$

where $(t, x) \mapsto \bar{U}_t f(x)$ is the unique positive solution of

$$\begin{aligned} e^{-\bar{U}_t f(x)} &= e^{-f((x-t)\vee 0)} - \alpha \int_0^t e^{-\bar{U}_{t-s} f((x-s)\vee 0)} ds + \alpha \int_0^t e^{-\bar{U}_{t-s} f(0)} 1_{\{x \leq s\}} ds \\ &\quad + \alpha \int_0^t e^{-\bar{U}_{t-s} f(x-s)} 1_{\{x > s\}} g(\langle \eta, e^{-\bar{U}_{t-s} f} \rangle) ds; \end{aligned}$$

see also Dawson et al [3, pp.95-96] and Li [20, p.98]. By Proposition 2.9 of [20], the above equation can be rewritten as

$$e^{-\bar{U}_t f(x)} = e^{-f((x-t)\vee 0)} - \alpha \int_0^t e^{-\bar{U}_{t-s} f(x-s)} 1_{\{x > s\}} \big[1 - g(\langle \eta, e^{-\bar{U}_{t-s} f} \rangle)\big] ds. \tag{2.2}$$

By Proposition A.49 of [20], for $f \in B(\mathbb{R}_+)$ there is a unique locally bounded solution $(t, x) \mapsto \bar{\pi}_t f(x)$ to the equation

$$\bar{\pi}_t f(x) = f((x-t) \vee 0) + \alpha g'(1) \int_0^t 1_{\{x > s\}} \langle \eta, \bar{\pi}_{t-s} f \rangle ds. \tag{2.3}$$

Moreover, the linear operators $(\bar{\pi}_t)_{t \geq 0}$ on $B(\mathbb{R}_+)$ form a semigroup and

$$\|\bar{\pi}_t f\| \leq \|f\| e^{\alpha g'(1) t}, \qquad t \geq 0. \tag{2.4}$$

Proposition 2.1. *For $t \geq 0$ and $f \in B^+(\mathbb{R}_+)$ we have $\bar{U}_t f \leq \bar{\pi}_t f$ and for $t \geq 0$ and $f \in B^+(\mathbb{R}_+)$ we have*

$$\int_{N(\mathbb{R}_+)} \langle \nu, f\rangle \bar{Q}_t(\mu, d\nu) = \langle \mu, \bar{\pi}_t f\rangle. \tag{2.5}$$

Proof. For $t \geq 0$ and $f \in B^+(\mathbb{R}_+)$ one can use (2.2) and (2.3) to see

$$\bar{\pi}_t f(x) = \frac{\partial}{\partial\theta}\bar{U}_t(\theta f)(x)\Big|_{\theta=0}.$$

Then (2.5) follows by differentiating both sides of (2.1). By (2.1), (2.5) and Jensen's inequality it is clear that $\bar{U}_t f(x) \leq \bar{\pi}_t f(x)$ for $x \geq 0$. By linearity we also have (2.3) and (2.5) for $f \in B(\mathbb{R}_+)$. □

Proposition 2.2. *For any $f \in B^+(\mathbb{R}_+)$ the mapping $t \mapsto \bar{U}_t f(\cdot + t)$ from $[0,\infty)$ to $B^+(\mathbb{R}_+)$ is increasing and locally Lipschitz in the supremum norm. Moreover, for any $t \geq r \geq 0$ we have*

$$0 \leq e^{-\bar{U}_r f(x+r)} - e^{-\bar{U}_t f(x+t)} \leq \alpha(t-r). \tag{2.6}$$

Proof. For any $t, x \geq 0$ one can use (2.2) to see

$$e^{-\bar{U}_t f(x+t)} = e^{-f(x\vee 0)} - \alpha\int_0^t e^{-\bar{U}_s f(x+s)} 1_{\{x+s>0\}}\big[1 - g(\langle \eta, e^{-\bar{U}_s f}\rangle)\big]ds.$$

Then we have (2.6). Since $t \mapsto \bar{U}_t f$ is locally bounded by Proposition 2.1, we see $t \mapsto \bar{U}_t f(\cdot + t)$ is increasing and locally Lipschitz in the supremum norm. □

Proposition 2.3. *For any $f \in B^+(\mathbb{R}_+)$ the function $(t,x) \mapsto \bar{U}_t f(x)$ is the unique locally bounded positive solution of*

$$\bar{U}_t f(x) = f((x-t)\vee 0) + \alpha\int_0^t 1_{\{x>s\}}\big[1 - g(\langle \eta, e^{-\bar{U}_{t-s} f}\rangle)\big]ds. \tag{2.7}$$

Proof. For notational convenience, in this proof we set $f(x) = f(0)$ and $\bar{U}_t f(x) = \bar{U}_t f(0)$ for all $x \leq 0$ and $t \geq 0$. Let $0 = t_0 < t_1 < \cdots < t_n = t$ be a partition of $[0,t]$. For $x \in \mathbb{R}_+$, we can write

$$\bar{U}_t f(x) = f(x-t) + \sum_{i=1}^n \Big[\bar{U}_{t-t_{i-1}} f(x - t_{i-1}) - \bar{U}_{t-t_i} f(x-t_i)\Big]. \tag{2.8}$$

Note that Proposition 2.2 implies $\bar{U}_{t-t_{i-1}} f(x-t_{i-1}) - \bar{U}_{t-t_i} f(x-t_i) \geq 0$. By (2.2), (2.6) and Taylor's formula, as $t_i - t_{i-1} \to 0$,

$$\Big[\bar{U}_{t-t_{i-1}} f(x-t_{i-1}) - \bar{U}_{t-t_i} f(x-t_i)\Big]$$

$$
\begin{aligned}
&= e^{\bar{U}_{t-t_{i-1}}f(x-t_{i-1})}\Big[e^{-\bar{U}_{t-t_i}f(x-t_i)} - e^{-\bar{U}_{t-t_{i-1}}f(x-t_{i-1})}\Big] + o(t_i - t_{i-1}) \\
&= \int_0^{t_i - t_{i-1}} \big[1+\varepsilon_i(s,x)\big]1_{\{x-t_{i-1}>s\}}\big[1 - g(\langle \eta, e^{-\bar{U}_{t-t_{i-1}-s}f}\rangle)\big]ds \\
&\quad + o(t_i - t_{i-1}),
\end{aligned}
$$

where

$$
\varepsilon_i(s,x) = e^{\bar{U}_{t-t_{i-1}}f(x-t_{i-1})}\Big[e^{-\bar{U}_{t-t_{i-1}-s}f(x-t_{i-1}-s)} - e^{-\bar{U}_{t-t_{i-1}}f(x-t_{i-1})}\Big].
$$

By Propositions 2.1 and 2.2 one can see that

$$
0 \le \varepsilon_i(s,x) \le \alpha(t_i - t_{i-1})\exp\big\{\|f\|e^{\alpha g'(1)t}\big\}, \qquad 0 \le s \le t_i - t_{i-1}.
$$

It then follows that

$$
\begin{aligned}
&\Big[\bar{U}_{t-t_{i-1}}f(x-t_{i-1}) - \bar{U}_{t-t_i}f(x-t_i)\Big] \\
&\quad = \int_0^{t_i-t_{i-1}} 1_{\{x-t_{i-1}>s\}}\big[1 - g(\langle \eta,\, e^{-\bar{U}_{t-t_{i-1}-s}f}\rangle)\big]ds + o(t_i - t_{i-1}) \\
&\quad = \int_{t_{i-1}}^{t_i} 1_{\{x>s\}}\big[1 - g(\langle \eta,\, e^{-\bar{U}_{t-s}f}\rangle)\big]ds + o(t_i - t_{i-1}).
\end{aligned}
$$

Substituting this into (2.8) and letting $\max_{1\le i\le n}(t_i - t_{i-1}) \to 0$ we obtain (2.7). The uniqueness of the solution of the equation follows from Proposition 2.18 in [20]. □

Theorem 2.4. *There is a Borel right transition semigroup $(Q_t)_{t\ge 0}$ on $N(0,\infty)$ defined by*

$$
\int_{N(0,\infty)} e^{-\langle \nu, f\rangle} Q_t(\mu, d\nu) = e^{-\langle \mu, U_t f\rangle}, \qquad f \in B^+(0,\infty), \tag{2.9}
$$

where $(t,x) \mapsto U_t f(x)$ is the unique locally bounded positive solution of

$$
\begin{aligned}
U_t f(x) \;=\;& f(x-t)1_{\{x>t\}} \\
&+ \alpha\int_0^t 1_{\{x>t-s\}}[1 - g(\langle \eta, e^{-U_s f}\rangle)]ds, \quad t \ge 0, x > 0.
\end{aligned} \tag{2.10}
$$

Proof. It is not hard to see that (2.10) is a special cases of (2.21) in [20, p.39]. By (2.2) we have $\bar{U}_t f(0) = f(0)$ for all $t \ge 0$. Consequently, if $\{\bar{X}_t : t \ge 0\}$ is a Markov process with transition semigroup $(\bar{Q}_t)_{t\ge 0}$ defined by (2.1) and (2.7), then $\{\bar{X}_t|_{(0,\infty)} : t \ge 0\}$ is a Markov process in $N(0,\infty)$ with transition semigroup $(Q_t)_{t\ge 0}$ defined by (2.9) and (2.10). By Theorem 5.12 of [20], we can extend $(Q_t)_{t\ge 0}$ to a Borel right semigroup on the space of finite measures on $(0,\infty)$. Then $(Q_t)_{t\ge 0}$ itself is a Borel right semigroup. □

By Proposition 2.1 we have the following:

Proposition 2.5. *For every $f \in B(0,\infty)$ there is a unique locally bounded solution $(t,x) \mapsto \pi_t f(x)$ of*

$$\pi_t f(x) = f((x-t) \vee 0) + \alpha g'(1) \int_0^t 1_{\{x>s\}} \langle \eta, \pi_{t-s} f \rangle ds. \tag{2.11}$$

Moreover, the linear operators $(\pi_t)_{t\geq 0}$ on $B(0,\infty)$ form a semigroup and

$$\int_{N(0,\infty)} \langle \nu, f \rangle Q_t(\mu, d\nu) = \langle \mu, \pi_t f \rangle, \qquad t \geq 0, f \in B(0,\infty). \tag{2.12}$$

Proposition 2.6. *We have $U_t f(x) \leq \pi_t f(x) \leq \|f\| e^{\alpha g'(1)t}$ for $t \geq 0, x > 0$ and $f \in B(0,\infty)$.*

A Markov process in $N(0,\infty)$ with transition semigroup $(Q_t)_{t\geq 0}$ defined by (2.9) and (2.10) will be referred to as a *branching system* of particles with parameters (g, α, η), where g is the *generating function*, α is the *branching rate* and η is the *offspring position law*.

3 The system with infinite branching rate

In this section, we consider a system of particles, which can be thought of as a branching system with infinite branching rate. Let $\rho(x) = x$ for $x \in (0,\infty)$. Let $B_\rho(0,\infty)$ be the set of Borel functions on $(0,\infty)$ bounded by $\rho \cdot \text{const}$. Let $C_\rho(0,\infty)$ be the subset of $B_\rho(0,\infty)$ consisting of continuous functions. Let $M_\rho(0,\infty)$ be the set of Borel measures μ on $(0,\infty)$ satisfying $\langle \mu, \rho \rangle < \infty$. Let $N_\rho(0,\infty)$ be the set of integer-valued measures in $M_\rho(0,\infty)$. We endow $M_\rho(0,\infty)$ and $N_\rho(0,\infty)$ with the topology defined by the convention that

$$\mu_n \to \mu \text{ if and only if } \langle \mu_n, f \rangle \to \langle \mu, f \rangle \text{ for all } f \in C_\rho(0,\infty). \tag{3.1}$$

We say a function $(t,x) \mapsto u_t(x)$ on $[0,\infty) \times (0,\infty)$ is *locally ρ-bounded* if

$$\sup_{0\leq s\leq t} \sup_{x\in(0,\infty)} |\rho(x)^{-1} u_s(x)| < \infty, \qquad t \geq 0.$$

Let $c > 0$ be a constant and let $\Pi(dz)$ be a σ-finite measure on $(0,\infty)$ such that $\langle \Pi, \rho \rangle < c$. Given $f \in B_\rho^+(0,\infty)$, we consider the following evolution equation:

$$U_t f(x) = f(x-t) 1_{\{x>t\}} + c^{-1} \int_0^t 1_{\{x>s\}} \langle \Pi, 1 - e^{-U_{t-s} f} \rangle ds. \tag{3.2}$$

Lemma 3.1. *For each $f \in B_\rho^+(0,\infty)$ there is at most one locally ρ-bounded positive solution of (3.2).*

Proof. Suppose that $(t,x)\mapsto U_tf(x)$ and $(t,x)\mapsto V_tf(x)$ are two locally ρ-bounded solutions of (3.2). Let

$$l_T(x)=\sup_{0\le t\le T}|\rho(x)^{-1}(U_tf(x)-V_tf(x))|.$$

Then for any $0\le t\le T$ we have

$$\begin{aligned}|U_tf(x)-V_tf(x)| &\le c^{-1}\int_0^t 1_{\{x>s\}}\langle\Pi,|e^{-U_{t-s}f}-e^{-V_{t-s}f}|\rangle ds\\ &\le c^{-1}\int_0^t 1_{\{x>s\}}\langle\Pi,|U_{t-s}f-V_{t-s}f|\rangle ds\\ &\le c^{-1}\int_0^t 1_{\{x>s\}}ds\|l_T\|\langle\Pi,\rho\rangle\le c^{-1}\rho(x)\|l_T\|\langle\Pi,\rho\rangle,\end{aligned}$$

which implies $\|l_T\|\le c^{-1}\|l_T\|\langle\Pi,\rho\rangle$. Then we have $\|l_T\|=0$ as $\langle\Pi,\rho\rangle<c$. $\square$

Proposition 3.2. *For each $f\in B_\rho^+(0,\infty)$, there is a unique locally ρ-bounded positive solution $(t,x)\mapsto U_tf(x)$ of (3.2) and the solution is increasing in $(\Pi,f)\in M_\rho(0,\infty)\times B_\rho^+(0,\infty)$. Furthermore, the operators $(U_t)_{t\ge0}$ on $B_\rho^+(0,\infty)$ form a semigroup and*

$$\|\rho^{-1}U_tf\|\le(c-\langle\Pi,\rho\rangle)^{-1}\|\rho^{-1}f\|,\qquad t\ge0. \tag{3.3}$$

Proof. *Step 1)* We first assume that $\Pi\in M(0,\infty)$ and $f\in B^+(0,\infty)$. By Theorem 2.4 there is a unique locally bounded positive solution $(t,x)\mapsto U_tf(x)$ of (3.2). This solution can also be constructed by a simple iteration procedure. In fact, if we let $u_0(t,x)=0$ and define $u_n(t,x)=u_n(t,x,f)$ inductively by

$$\begin{aligned}u_n(t,x) &:= f(x-t)1_{\{x>t\}}\\ &\quad +c^{-1}\int_0^t 1_{\{x>s\}}ds\int_0^\infty[1-e^{-u_{n-1}(t-s,z)}]\Pi(dz),\end{aligned} \tag{3.4}$$

then $u_n(t,x)\to U_tf(x)$ increasingly as $n\to\infty$; see Proposition 2.18 of [20]. Using this construction one can see that the solution of (3.2) is increasing in $(\Pi,f)\in M(0,\infty)\times B^+(0,\infty)$.

Step 2) Next, we assume that $\Pi\in M(0,\infty)$ and $f\in B_\rho^+(0,\infty)$. Let $f_k=f\wedge k$ for $k\ge1$. Let $(t,x)\mapsto U_tf_k(x)$ be the unique locally bounded positive solution of (3.2) with f replaced by f_k. According to the argument above the sequence $\{U_tf_k\}$ is increasing in $k\ge1$. By (3.2) and Proposition 2.6 we have

$$\begin{aligned}U_tf_k(x) &\le \|\rho^{-1}f_k\|\rho(x)+c^{-1}\int_0^t 1_{\{x>s\}}ds\int_{\mathbb{R}+}U_{t-s}f_k(z)\Pi(dz)\\ &\le \Big[\|\rho^{-1}f_k\|+c^{-1}\|f_k\|\langle\Pi,1\rangle\exp\{c^{-1}\langle\Pi,1\rangle t\}\Big]\rho(x).\end{aligned}$$

Thus $(t,x) \mapsto U_t f_k(x)$ is locally ρ-bounded. On the other hand, if we set

$$l_k(t,x) := \sup_{0\le s\le t} U_s f_k(x),$$

then

$$\begin{aligned} l_k(t,x) &\le \|\rho^{-1}f_k\|\rho(x) + c^{-1}\rho(x)\sup_{0\le s\le t}\int_{(0,\infty)} U_s f_k(z)\Pi(dz) \\ &\le \Big[\|\rho^{-1}f\| + c^{-1}\|\rho^{-1}l_k(t)\|\langle\Pi,\rho\rangle\Big]\rho(x). \end{aligned}$$

It follows that

$$\rho(x)^{-1}l_k(t,x) \le \|\rho^{-1}f\| + c^{-1}\|\rho^{-1}l_k(t)\|\langle\Pi,\rho\rangle,$$

which implies

$$\|\rho^{-1}l_k(t)\| \le \frac{\|\rho^{-1}f\|}{1-c^{-1}\langle\Pi,\rho\rangle} = \frac{c\|\rho^{-1}f\|}{c-\langle\Pi,\rho\rangle}. \tag{3.5}$$

In particular, we have

$$\|\rho^{-1}U_t f_k\| \le c\|\rho^{-1}f\|(c-\langle\Pi,\rho\rangle)^{-1}, \qquad t\ge 0.$$

Then the limit $U_t f(x) := \lim_{k\to\infty} U_t f_k(x)$ exists. It is easy to see that $(t,x) \mapsto U_t f(x)$ is a locally ρ-bounded positive solution of (3.2) satisfying (3.3).

Step 3) In the general case, let $\Pi_k(dz) = 1_{\{z\ge 1/k\}}\Pi(dz)$ for $k\ge 1$. For $f\in B^+(0,\infty)$ let $(t,x)\mapsto U_t^{(k)}f(x)$ be the unique locally ρ-bounded positive solution of (3.2) with Π replaced by Π_k. By the second step, we can define $U_t^{(k)}f$ by the equation for any $f\in B_\rho^+(0,\infty)$. The sequence $\{U_t^{(k)}f\}$ is increasing by the first and the second steps. As in the second step one can see the limit $U_t f(x) := \lim_{k\to\infty} U_t^{(k)}f(x)$ exists and is a locally ρ-bounded positive solution of (3.2) satisfying (3.3). The uniqueness of the solution follows from Lemma 3.1, which yields the semigroup property of $(U_t)_{t\ge 0}$. □

Proposition 3.3. *For each $f\in B_\rho(0,\infty)$, there is a unique locally ρ-bounded solution $(t,x)\mapsto \pi_t f(x)$ of*

$$\pi_t f(x) = f(x-t)1_{\{x>t\}} + c^{-1}\int_0^t 1_{\{x>t-s\}}\langle\Pi,\pi_s f\rangle ds. \tag{3.6}$$

Furthermore, the solution is increasing in $(\Pi,f)\in M_\rho(0,\infty)\times B_\rho(0,\infty)$ and $(\pi_t)_{t\ge 0}$ is a semigroup of linear operators on $B_\rho(0,\infty)$ such that

$$\|\rho^{-1}\pi_t f\| \le (c-\langle\Pi,\rho\rangle)^{-1}\|\rho^{-1}f\|, \qquad t\ge 0. \tag{3.7}$$

Proof. For $f \in B_\rho^+(0,\infty)$ one can obtain (3.6) by differentiating both sides of (3.2), and (3.7) follows by (3.3). By the linearity, the equation has a solution for any $f \in B_\rho(0,\infty)$ and (3.7) remains true. By Proposition 3.2 one can see the solution is increasing in $(\Pi, f) \in M_\rho(0,\infty) \times B_\rho(0,\infty)$. The uniqueness of the solution follows by a modification of the proof of Lemma 3.1. □

Theorem 3.4. *There is a Borel right semigroup $(Q_t)_{t\geq 0}$ on $N_\rho(0,\infty)$ defined by*

$$\int_{N_\rho(0,\infty)} e^{-\langle \nu, f\rangle} Q_t(\mu, d\nu) = e^{-\langle \mu, U_t f\rangle}, \quad f \in B_\rho^+(0,\infty), \tag{3.8}$$

where $(t,x) \mapsto U_t f(x)$ is the unique locally ρ-bounded positive solution of (3.2). Furthermore, we have

$$\int_{N_\rho(0,\infty)} \langle \nu, f\rangle Q_t(\mu, d\nu) = \langle \mu, \pi_t f\rangle, \qquad f \in B_\rho(0,\infty), \tag{3.9}$$

where $(t,x) \mapsto \pi_t f(x)$ is the unique locally ρ-bounded solution of

$$\pi_t f(x) = f(x-t)1_{\{x>t\}} + c^{-1}\int_0^t 1_{\{x>t-s\}}\langle \Pi, \pi_s f\rangle ds. \tag{3.10}$$

Proof. Let $(U_t^{(k)})_{t\geq 0}$ be defined as in the last step of the proof of Proposition 3.2. By Theorem 2.1, we can define a Borel right semigroup $(Q_t^{(k)})_{t\geq 0}$ on $N(0,\infty)$ by

$$\int_{N(0,\infty)} e^{-\langle \nu, f\rangle} Q_t^{(k)}(\mu, d\nu) = e^{-\langle \mu, U_t^{(k)} f\rangle}, \qquad f \in B^+(0,\infty). \tag{3.11}$$

In view of (2.12) and (3.7), if $\mu \in N_\rho(0,\infty)$ is a finite measure, we can regard $Q_t^{(k)}(\mu,\cdot)$ as a probability measure on $N_\rho(0,\infty)$. Clearly, $N_\rho(0,\infty)$ is a closed subset of $M_\rho(0,\infty)$ and the latter is an isomorphism of $M(0,\infty)$ under the mapping $\nu(dx) \mapsto x\nu(dx)$. By Theorem 1.20 of [20] and the last step of the proof of Proposition 3.2 one can see (3.8) really defines a probability measure $Q_t(\mu,\cdot)$ on $N_\rho(0,\infty)$ for any finite measure $\mu \in N_\rho(0,\infty)$. By approximating $\mu \in N_\rho(0,\infty)$ with an increasing sequence of finite measures, we infer the formula defines a probability kernel on $N_\rho(0,\infty)$. Here (3.2) can be regarded as a special form of (6.11) in [20]. By Theorem 6.3 in [20], we can extend $(Q_t)_{t\geq 0}$ to a Borel right semigroup on $M_\rho(0,\infty)$. Then we infer that $(Q_t)_{t\geq 0}$ itself is a Borel right semigroup. The moment formula (3.9) can be obtained as in the proof of Proposition 2.1. □

A Markov process in $N_\rho(0,\infty)$ with transition semigroup $(Q_t)_{t\geq 0}$ defined by (3.2) and (3.8) will be referred to as a *single-birth branching system* of particles with *offspring position law* Π. Clearly, when Π is a finite measure on $(0,\infty)$, this reduces to a special case of the model introduced in the last section.

4 Subordinators with negative drift

In this section, we give a description of the branching structures in subordinators with negative drift. Set

$$C^1(\mathbb{R}) = \{f \in C(\mathbb{R}) : f \text{ is differentiable and has bounded derivative.}\}$$

Let $c > 0$ be a constant and let Π be a σ-finite measure on $(0,\infty)$ satisfying $\langle \Pi, \rho\rangle < c$. Suppose that $\{S_t : t \geq 0\}$ is a subordinator with negative drift generated by the operator A given by

$$Af(x) = \int_0^\infty [f(x+z) - f(x)]\Pi(dz) - cf'(x), \qquad f \in C^1(\mathbb{R}). \tag{4.1}$$

We assume $S_0 = a > 0$. Our assumption implies that $S_t \to -\infty$ as $t \to \infty$, so the hitting time

$$\tau_0^- := \inf\{t > 0 : S_t \leq 0\}$$

is a.s. finite. For $t \geq 0$ set

$$J(t) := \{u \in [0, \tau_0^-] : S_{u-} \leq t < S_u\} \tag{4.2}$$

with the convention that $S_{0-} = 0$. Then we define the measure-valued process

$$X_t = \sum_{u \in J(t)} \delta_{S_u - t}, \quad t \geq 0. \tag{4.3}$$

It is easy to see that $X_0 = \delta_a$.

Theorem 4.1. *The process $\{X_t : t \geq 0\}$ is a single-birth branching system in $N_\rho(0,\infty)$ with transition semigroup $(Q_t)_{t\geq 0}$ defined by (3.2) and (3.8).*

Proof. *Step 1)* We first assume $\Pi(dz)$ is a finite measure on $(0,\infty)$. In this case we clearly have $\mathbf{P}\{\#J(t) < \infty \text{ for all } t \geq 0\} = 1$. Let

$$C(t) = \{u \in [0, \tau_0^-] : S_u = S_{u-} = t\} \quad \text{and} \quad \zeta(t) = \#C(t).$$

We can write $C(t) = \{\tau_1(t), \cdots, \tau_{\zeta(t)}(t)\}$ by ranking the elements in increasing order. Let $\tau_0(t) = 0$ and

$$\sigma_i(t) = \inf\{u \geq \tau_{i-1}(t) : S_u > t\}, \quad i = 1, 2, \cdots, \zeta(t).$$

Then it is easy to see that $J(t) = \{\sigma_1(t), \cdots, \sigma_{\zeta(t)}(t)\}$ and

$$X_t = \sum_{i=1}^{\zeta(t)} \delta_{S_{\sigma_i(t)} - t}, \quad t \geq 0. \tag{4.4}$$

In particular, we have $\zeta(t) = \#J(t)$. Write $M_t = \min_{0\leq r\leq t} S_r$ and $L_t = S_t - M_t$. Set $\eta_0 = 0$ and for $k \geq 1$ define inductively

$$\zeta_k = \inf\{t > \eta_{k-1} : S_t \neq M_t\} \quad \text{and} \quad \eta_k = \inf\{t > \zeta_k : S_t = M_t\}.$$

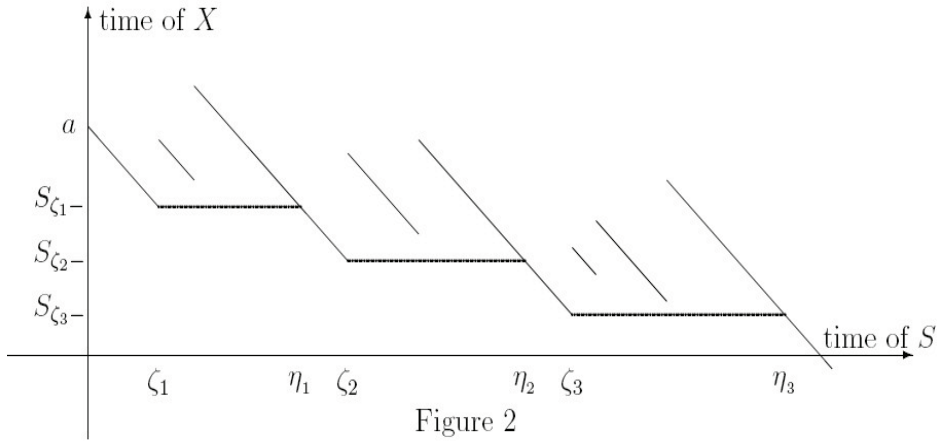

Figure 2

It is clear that $a - S_{\zeta_1-}$ is an exponentially distributed random variable with parameter λ/c, where $\lambda = \Pi(0,\infty)$. By the memoryless property one can see $S_{\eta_{k-1}} - S_{\zeta_k-} = S_{\zeta_{k-1}-} - S_{\zeta_k-}$ is also exponentially distributed with parameter λ/c for each $k \geq 1$. Let $e_k(t) = (L_{\zeta_k+t} - L_{\zeta_k-})1_{\{t<\eta_k-\zeta_k\}}$ and let $F(dw)$ denote the distribution of $\{e_1(t) : t \geq 0\}$ on $D^+[0,\infty)$, the space of positive càdlàg functions on $[0,\infty)$ equipped with the Skorokhod topology. Then

$$(S_{\eta_{k-1}} - S_{\zeta_k-}, \{e_k(t) : t \geq 0\}), \quad k = 1, 2, \cdots \tag{4.5}$$

are i.i.d. random variables in $(0,\infty) \times D^+[0,\infty)$ with

$$\mathbf{P}(S_{\eta_{k-1}} - S_{\zeta_k-} \in dy, e_k \in dw) = \frac{\lambda}{c} e^{-\lambda y/c} dy F(dw), \quad y > 0, w \in D^+[0,\infty) \tag{4.6}$$

It follows that

$$(S_{\zeta_k-}, \{e_k(t) : t \geq 0\}), \quad k = 1, 2, \cdots \tag{4.7}$$

are positioned in $(-\infty, a) \times D^+[0,\infty)$ as the atoms of a Poisson random measure with intensity $c^{-1}\lambda dy F(dw)$. Let $n = \max\{k \geq 0 : \eta_k \leq \tau_0^- < \zeta_{k+1}\}$. Then $S_{\zeta_n-} \overset{(d)}{=} a \wedge \Theta$, where Θ is exponentially distributed with parameter λ/c. It is easy to see from (4.4) that

$$X_t = \begin{cases} \delta_{a-t} & \text{for } 0 \leq t < S_{\zeta_n-}, \\ \delta_{a-S_{\zeta_n-}} + \delta_{S_{\zeta_n} - S_{\zeta_n-}} & \text{for } t = S_{\zeta_n-}. \end{cases}$$

Therefore, the first offspring in the particle system is born at time S_{ζ_n-}. By (4.6) we have

$$\mathbf{P}(S_{\zeta_n} - S_{\zeta_n-} \in dz) = \mathbf{P}(e_n(0) \in dz) = \Pi(dz), \quad z > 0.$$

By the i.i.d. property of the random variables in (4.5) we infer that $\{X_t : t \geq 0\}$ is a branching system with parameters $(g, c^{-1}\lambda, \lambda^{-1}\Pi)$, where $g(z) \equiv z$. In other words, the system have transition semigroup defined by (3.2) and (3.8).

Step 2) In the general case, let us consider an approximation of the subordinator with drift. Let $\{N(ds, dz)\}$ be a Poisson random measure on $(0,\infty)^2$ with intensity $ds\Pi(dz)$. Then a realization of $\{S_t : t \geq 0\}$ is constructed by

$$S_t := a + \int_0^t \int_0^\infty zN(ds, dz) - ct.$$

For each $k \geq 1$ we can define another subordinator with drift $\{S_t^{(k)} : t \geq 0\}$ by

$$S_t^{(k)} := a + \int_0^t \int_{1/k}^\infty zN(ds, dz) - ct.$$

Then $S_t^{(k)} \leq S_t$ and as $k \to \infty$ we have

$$\sup_{0\leq t\leq T} (S_t^{(k)} - S_t) = S_T^{(k)} - S_T \to 0, \qquad T \geq 0. \tag{4.8}$$

Let $\Pi_k(dz) := 1_{\{z\geq 1/k\}}\Pi(dz)$. Let $\{X_t^{(k)} : t \geq 0\}$ be the measure-valued process defined by (4.3) with $\{S_t : t \geq 0\}$ replaced by $\{S_t^{(k)} : t \geq 0\}$. Then the first step implies that $\{X_t^{(k)} : t \geq 0\}$ is a branching system in $N(0,\infty)$ with transition semigroup $(Q_t^{(k)})_{t\geq 0}$ given by (3.11), where $(U_t^{(k)})_{t\geq 0}$ is defined as in the last step of the proof of Proposition 3.2. Then we can also think of $\{X_t^{(k)} : t \geq 0\}$ as a Markov process in $N_\rho(0,\infty)$. For $t > t_n \geq t_{n-1} \geq \cdots \geq t_1 \geq 0$ and $\{f, f_n, \cdots, f_1\} \subset C_\rho^+(0,\infty)$, we have

$$\begin{aligned}
&\mathbf{E}\exp\Big\{ -\sum_{i=1}^n \langle X_{t_i}^{(k)}, f_i\rangle - \langle X_t^{(k)}, f\rangle\Big\} \\
&\quad= \mathbf{E}\exp\Big\{ -\sum_{i=1}^n \langle X_{t_i}^{(k)}, f_i\rangle - \langle X_{t_n}^{(k)}, U_{t-t_n}^{(k)} f\rangle\Big\} \\
&\quad= \mathbf{E}\exp\Big\{ -\sum_{i=1}^n \langle X_{t_i}^{(k)}, f_i\rangle - \langle X_{t_n}^{(k)}, U_{t-t_n} f\rangle\Big\} + \varepsilon_k(f)
\end{aligned} \tag{4.9}$$

with

$$|\varepsilon_k(f)| \leq \mathbf{E}\Big|\exp\big\{ -\langle X_{t_n}^{(k)}, U_{t-t_n} f\rangle\big\} - \exp\big\{ -\langle X_{t_n}^{(k)}, U_{t-t_n}^{(k)} f\rangle\big\}\Big|.$$

Let $(t,x) \mapsto \pi_t f(x)$ be the unique locally ρ-bounded solution of (3.6) and let $(t,x) \mapsto \pi_t^{(k)} f(x)$ be the unique locally ρ-bounded solution of the equation with γ replaced by γ_k. By Proposition 3.3 and Theorem 3.4,

$$\varepsilon_k(f) \quad\leq\quad \mathbf{E}\big\langle X_{t_n}^{(k)}, \big|U_{t-t_n} f - U_{t-t_n}^{(k)} f\big|\big\rangle$$

$$\begin{aligned} &= \pi_{t_n}^{(k)}\big|U_{t-t_n}f - U_{t-t_n}^{(k)}f\big|(a) \\ &\le \pi_{t_n}\big|U_{t-t_n}f - U_{t-t_n}^{(k)}f\big|(a). \end{aligned}$$

By the proof of Proposition 3.2 we have $U_t f(x) = \lim_{k\to\infty} U_t^{(k)} f(x)$ increasingly. Then $\varepsilon_k(f) \to 0$ as $k \to \infty$. From (4.9) we get

$$\begin{aligned} &\mathbf{E}\exp\Big\{-\sum_{i=1}^n \langle X_{t_i}, f_i\rangle - \langle X_t, f\rangle\Big\} \\ &\quad = \mathbf{E}\exp\Big\{-\sum_{i=1}^n \langle X_{t_i}, f_i\rangle - \langle X_{t_n}, U_{t-t_n}f\rangle\Big\}. \end{aligned}$$

The above equality can be extended to $\{f, f_n, \cdots, f_1\} \subset B_\rho^+(0,\infty)$ by a monotone class argument. Then $\{X_t : t \ge 0\}$ is a Markov process in $N_\rho(0,\infty)$ with transition semigroup $(Q_t)_{t\ge 0}$ given by (3.2) and (3.8). □

5 Subordinators with negative drift

In this section, we give a characterization of the branching structures in negative subordinators with positive drift. We shall derive the result from the one in the last section by a time reversal approach. Suppose that Π is a σ-finite measure on $(0,\infty)$ with $\int_0^\infty 1\wedge z\Pi(dz) < \infty$. Let $\{S_t^* : t \ge 0\}$ be a Lévy process generated by A^* such that

$$A^* f(x) = \int_0^\infty [f(x-z) - f(x)]\Pi(dz) + cf'(x), \qquad f \in C^1(\mathbb{R}). \tag{5.1}$$

Assume $S_0^* = 0$ and $0 < c < \langle \Pi, \rho\rangle \le \infty$. Then S^* has Laplace exponent

$$\psi(\beta) = c\beta - \int_0^\infty (1 - e^{-\beta z})\Pi(dz), \qquad \beta \ge 0$$

Namely, $\mathbf{E}e^{\beta S_t^*} = e^{t\psi(\beta)}$. For $q \ge 0$ let $\Phi(q) = \sup\{t \ge 0 : \psi(t) = q\}$. Define

$$\tau_0^- := \inf\{t > 0 : S_t^* \le 0\}.$$

We have $\mathbf{P}(0 < \tau_0^- < \infty) = 1$; see Corollary 5 in Section VII.1 of [1]. Then for $t \ge 0$, set

$$J(t) := \{u \in (0, \tau_0^-] : S_u^* \le t < S_{u-}^*\}$$

and define

$$X_t^* := \sum_{u\in J(t)} \delta_{S_{u-}^* - t}. \tag{5.2}$$

with $X_0^* = \delta_{S^*_{\tau_0^- -}}$. Note that $\#J(t) < \infty, a.s.$

Theorem 5.1. *There is a Borel right semigroup $(Q_t)_{t\geq 0}$ on $N_\rho(0,\infty)$ defined by*

$$\int_{N_\rho(0,\infty)} e^{-\langle \nu, f\rangle} Q_t(\mu, d\nu) = e^{-\langle \mu, U_t f\rangle}, \quad f \in B_\rho^+(0,\infty), \tag{5.3}$$

where $(t,x) \mapsto U_t f(x)$ is the unique locally ρ-bounded positive solution of

$$U_t f(x) = f(x-t)1_{\{x>t\}} + c^{-1}\int_0^t 1_{\{x>s\}} ds \int_0^\infty [1 - e^{-U_{t-s}f(z)}]\Pi^+(dz), \tag{5.4}$$

where $\Pi^+(dz) = e^{-\Phi(0)z}\Pi(dz)$. Furthermore, we have

$$\int_{N_\rho(0,\infty)} \langle \nu, f\rangle Q_t(\mu, d\nu) = \langle \mu, \pi_t f\rangle, \qquad f \in B_\rho(0,\infty), \tag{5.5}$$

where $(t,x) \mapsto \pi_t f(x)$ is the unique locally ρ-bounded solution of (3.6).

Proof. Note that $\beta_0 := \Phi(0)$ is the largest solution of $\psi(\beta) = 0$. It follows that

$$c - \int_0^\infty z e^{-z\Phi(0)}\Pi(dz) = \psi'(\Phi(0)) > 0.$$

Then (c, Π^+) satisfies the conditions of Theorem 3.4. □

For reader's convenience, we first present a result on the distribution of time reversed Lévy processes which should be well-known to experts. For $a > 0$, let $\{S_t^\# : t \geq 0\}$ and $\{S_t^- : t \geq 0\}$ be two subordinators with drift starting at $a > 0$ with $\mathbf{E}e^{-\beta S_t^\#} = e^{t\psi(\beta+\Phi(0))-a\beta}$ and $\mathbf{E}e^{-\beta S_t^-} = e^{t\psi(\beta)-a\beta}$, respectively. Define $T^\#(0) = \inf\{t \geq 0 : S_t^\# \leq 0\}$ and $T^-(0) = \inf\{t \geq 0 : S_t^- \leq 0\}$. Note that $\mathbf{P}\{T^\#(0) < \infty\} = 1$.

Lemma 5.2. *Given $S^*_{\tau_0^- -} = a$, the time reversed process $\{S^*_{(\tau_0^- - t)-}, 0 \leq t < \tau_0^-\}$ has the same distribution as $\{S_t^\#, 0 \leq t < T^\#(0)\}$.*

Proof. Define $I_t = \inf\{0 \wedge S_s^* : 0 \leq s \leq t\}$ and

$$J_t = \sum_{s \leq t} 1_{\{S_s^* < I_{s-}\}}(S_s^* - S_{s-}^*).$$

For $a > 0$, set $\varsigma(a) = \sup\{t \geq 0 : S_t^* - J_t \leq x\}$. By Lemma 21 and Theorem 17 in Chapter VII of [1], conditioned on $S^*_{\tau_0^- -} = a$,

$$\{S_t^* : 0 \leq t < \tau_0^-\} \stackrel{(d)}{=} \{S_t^* - J_t : 0 \leq t < \varsigma(a)\}.$$

Then by Theorem 18 and Lemma 7 in Chapter VII of [1], under $\mathbf{P}\{\cdot | S^*_{\tau_0^- -} = a\}$, $\{S^*_{(\tau_0^- - t)-} : 0 \leq t < \tau_0^-\}$ has the same law as $\{S_t^- : 0 \leq t < T^-(0)\}$ under $\mathbf{P}\{\cdot | T^-(0) < \infty\}$ which is the same as the law of $\{S_t^\# : 0 \leq t < T^\#(0)\}$. We have completed the proof. □

Theorem 5.3. *The measure-valued process $\{X_t^* : t \geq 0\}$ defined by (5.2) is a single-birth branching system with transition semigroup $(Q_t)_{t\geq 0}$ determined by (5.3) and*

$$\mathbf{P}\{S(X_0^*) \in da\} = c^{-1}e^{-\Phi(0)a}\Pi([a,\infty))da \quad \text{for} \quad a > 0, \tag{5.6}$$

where $S(X_0^*) = S^*_{\tau_0^- -}$ denotes the support for X_0^*.

Proof. (5.6) follows from Theorem 17 in Section VII of [1]. With the convention $S^\#_{0-} = 0$ we let

$$J^\#(t) := \{u \in [0, T^\#(0)] : S^\#_{t-} \leq u < S^\#_t\}.$$

For each $t \geq 0$ define the random measure $X_t^\#$ on $(0,\infty)$ by

$$X_t^\# = \sum_{u \in J^\#(t)} \delta_{S^\#_u - t}.$$

Then by Theorem 4.1, $X^\#$ is a Markov process with transition semigroup $(Q_t)_{t\geq 0}$ given by (5.3).

On the other hand, given $S^*_{\tau_0^- -} = a$, by Lemma 5.2, we have

$$\{S^*_{(\tau_0^- - t)-} : 0 \leq t < \tau_0^-\} \overset{(d)}{=} \{S_t^\# : 0 \leq t < T^\#(0)\}.$$

Thus given $S^*_{\tau_0^- -} = a$

$$\{X_t^* : 0 \leq t < \infty\} \overset{(d)}{=} \{X_t^\# : 0 \leq t < \infty\}.$$

We have completed the proof. $\square$

6 Properties of the branching systems

In this section we discuss the properties of the measure-valued processes via the exit problems for Lévy processes. For a Lévy process S and any $x \geq 0$ let

$$\tau_x^+ = \inf\{t > 0 : S_t > x\}, \quad \tau_x^- = \inf\{t > 0 : S_t \leq x\} \tag{6.1}$$

with the convention $\inf \emptyset = \infty$. Set $\mathbf{P}_x\{\cdot\} = \mathbf{P}\{\cdot | S_0 = x\}$.

6.1 Properties of X

In this subsection we discuss the properties of the measure-valued process X in Theorem 4.1, which is determined by process S which satisfies that $S_0 = a$ and $S_t + ct$ is a subordinator with Lévy measure Π and $\int_0^\infty z\Pi(dz) < c$. Recall $\psi(\lambda) = c\lambda - \int_0^\infty (1 - e^{-\lambda z})\Pi(dz)$ and $\Phi(q) = \sup\{t \geq 0 : \psi(t) = q\}$ for $q \geq 0$. Let W denote the scale function of S, i.e., an increasing and continuous function on $[0, \infty)$ taking values in $[0, \infty)$ with

$$\int_0^\infty e^{-\lambda x} W(x) dx = \frac{1}{\psi(\lambda)},$$

and we make the convention that $W(x) = 0$ for $x < 0$. We will need the following solution to the two-sided exit problems.

Lemma 6.1. *For any $t \geq x, y \geq 0$ and $z > 0$,*

$$\mathbf{E}_x e^{-q\tau_0^-} = e^{-x\Phi(q)}, \qquad \mathbf{P}_x\{\tau_0^- < \tau_t^+\} = \frac{W(t-x)}{W(t)}$$

and

$$\begin{aligned}\mathbf{P}_x &\Big\{ S_{\tau_t^+ -} \in dy, S_{\tau_t^+} - t \in dz, \tau_t^+ < \tau_0^- \Big\} \\ &= \Big(\frac{W(t-x)W(y)}{W(t)} - W(y-x) \Big) dy \Pi(t - y + dz).\end{aligned}$$

Proof. The first identity is from the beginning of page 212 of [13]. The second identity follows by (8.8) of [13] with $q = 0$. The third identity is (8.29) of [13]. □

We first present a representation of X_t for any fixed $t > 0$.

Proposition 6.2. *The random measure X_t has the same distribution as $\sum_{i=0}^{N-1} \delta_{Y_i}$, where N and (Y_i) are independent random variables.*

- *For $a > t$,*

$$\mathbf{P}\{N = n\} = \frac{1}{cW(t)} \left(1 - \frac{1}{cW(t)} \right)^{n-1}, \quad n \geq 1, \tag{6.2}$$

  $Y_0 = a - t$ and $Y_i, i = 1, 2, ...$ are i.i.d. random variables with common distribution

$$\frac{1}{cW(t)} \int_0^t W(y) \Pi(t - y + dz) dy, \quad z > 0. \tag{6.3}$$

- *For $a \leq t$, $\mathbf{P}\{N = 0\} = W(t-a)/W(t)$ and*

$$\mathbf{P}\{N = n\} = \frac{1}{cW(t)} \left(1 - \frac{W(t-a)}{W(t)} \right) \left(1 - \frac{1}{cW(t)} \right)^{n-1}, \quad n \geq 1, \tag{6.4}$$

$(Y_i)_{i\geq 1}$ are i.i.d. random variables with common distribution (6.3) and Y_0 is an independent random variable with distribution

$$\int_0^t \left(\frac{W(t-a)W(y)}{W(t)} - W(y-a)\right)\Pi(t-y+dz)dy, \quad z>0.$$

Proof. Observe that by the construction, the total mass $X_t(0,\infty)$ is exactly the total number of excursions above level t, which is the same as the number of continuous downcrossings of level t. In addition, each excursion of S started with a jump upcrossing level t has to come back to level t due to overall negative drift and lack of negative jumps. Then (6.2) and (6.4) follow easily from the strong Markov property and Lemma 6.1.

For $t<a$, given $N=n\geq 1$, the excursion of S above 0 contains n excursions at level t. The first excursion starts from a and all the excursions end at a. Further, by the strong Markov property the second to the nth excursion starts with i.i.d. initial value $t+Y_1,\dots,t+Y_{n-1}$, respectively. By the construction the support of X_t is $\{a-t,Y_1,\dots,Y_{n-1}\}$. Note that Y_1 is overshoot of the first upward jump across level t. Then by Lemma 6.1

$$\begin{aligned}\mathbf{P}_a\{Y_1\in dz\} &= \mathbf{P}_t\{S_{\tau_t^+}\in t+dz, \tau_t^+<\tau_0^-\}\\ &= \frac{W(0)}{W(t)}\int_0^t W(y)\Pi(t-y+dz)dy.\end{aligned}$$

The desired result follows. The corresponding result for $t\geq a$ follows similarly.

Our next result is on the weighted occupation time for X.

Proposition 6.3. *For any $f\in B_\rho^+(0,\infty)$ and $h\in B_\rho^+(0,\infty)$, we have*

$$\mathbf{E}e^{-\int_0^\infty h(t)\langle X_t,f\rangle dt} = \mathbf{E}e^{-\langle X_0,\omega_0\rangle}, \tag{6.5}$$

where ω is the unique nonnegative solution of the integral equation

$$\begin{aligned}\omega_t(x) - c^{-1}\int_t^\infty 1_{\{x>s-t\}}ds\int_0^\infty \Pi(dz)[1-e^{-\omega_s(z)}]\\ = \int_t^\infty h(s)f(x-s+t)1_{\{x>s-t\}}ds.\end{aligned} \tag{6.6}$$

Proof. By Theorem 5.3, similar to Section II.3 of Le Gall [17] we can show by induction together with the Markov property that for any $0\leq t_1<\ldots<t_p$ and any $f_1,\dots,f_p\in B_\rho^+(0,\infty)$,

$$\mathbf{E}e^{-\sum_{i=1}^p\langle X_{t_i},f_i\rangle} = e^{-\langle X_0,\omega_0\rangle}$$

where $(\omega_t(x), t\geq 0, x\in(0,\infty))$ is the unique nonnegative solution of the integral equation

$$\omega_t(x) - c^{-1}\int_t^\infty 1_{\{x>s-t\}}ds\int_0^\infty \Pi(dz)[1-e^{-\omega_s(z)}]$$

$$= \sum_{i=1}^{p} f_i(x - t_i + t) 1_{\{x > t_i - t\}}.$$

Further, by taking a limit on the Riemann sums we can show that (6.3) holds. Since the arguments for (6.6) is similar to (5.4), one could follow the proof of Corollary 9 in Section II.3 of [17] to get (6.5). We omit the details here. □

It is easy to recover Laplace transform for the total occupation time $\int_0^\infty \langle X_t, 1\rangle dt$. Observe that it is equal to the sum of a and sizes of all the jumps of S up to time τ_0^-, which is in turn equal to $c\tau_0^-$. We then have

$$\mathbf{E}_a e^{-q\int_0^\infty \langle X_t, 1\rangle dt} = \mathbf{E}_a e^{-qc\tau_0^-} = e^{-a\Phi(qc)}.$$

6.2 Properties of X^*

Properties of the measure-valued process X^* in Theorem 5.3 can also be investigated via the exit problems for process S^* with generator (5.1), the negative of a subordinator with positive drift.

Throughout this subsection, for $q \geq 0$, let $W^{(q)}$ be the scale function for the spectrally negative Lévy process S^*; i.e.; $W^{(q)}(x) = 0$ for $x < 0$ and on $[0, \infty)$, it is an increasing and continuous function taking values in $[0, \infty)$ with

$$\int_0^\infty e^{-\lambda x} W^{(q)}(x) dx = \frac{1}{\psi(\lambda) - q},$$

for $\lambda > \Phi(q) := \sup\{\lambda \geq 0 : \psi(\lambda) = q\}$, where $\psi(\lambda) = c\lambda - \int_0^\infty (1 - e^{-\lambda z})\Pi(dz)$. Write W for $W^{(0)}$. We also first present a result on the two-sided exit problems of S^*; see Chapter 8 of [13] and [2].

Lemma 6.4. *For any $y > 0 > z$,*

$$\mathbf{E} e^{-q\tau_0^-} = 1 - \frac{q}{\Phi(q)} W^{(q)}(0)$$

and

$$\mathbf{P}\{S^*_{\tau_0^- -} \in dy, S^*_{\tau_0^-} \in dz\} = W(0) e^{-\Phi(0)y} \Pi(dz - y) dy.$$

Proof. The first identity is just (8.6) of [13]. The second identity the equation right after (8.29) in [13]. □

For any $y > t$, let

$$g(y) := c^{-1} e^{-\Phi(0)(y-t)} \int_0^t \Pi(y - dz) \frac{W(z)}{W(t)}$$

and

$$h(y) := c^{-1}e^{-\Phi(0)(y-t)}\left\{\int_0^t \Pi(y-dz)\left(1-\frac{W(z)}{W(t)}\right)+\Pi((y,\infty))\right\}.$$

One will see from the proof of Proposition 6.5 that $\int_t^\infty g(y)dy+\int_t^\infty h(y)dy = 1$.

Fix $t > 0$ until the end of the following Proposition 6.5. We first proceed to recover distribution for the total mass for X_t^*. The proof of the following representation result is similar to Proposition 6.2 and is omitted.

Proposition 6.5. *X_t^* has the same distribution as $\sum_{i=0}^{N-1}\delta_{Y_i}$, where N and (Y_i) are independent random variables.*

- $$\mathbf{P}\{N=0\} = 1-\frac{1}{cW(t)} \tag{6.7}$$

  and for any $n \geq 1$

  $$\mathbf{P}\{N=n\} = \frac{1}{cW(t)}\left(\int_t^\infty g(y)dy\right)^{n-1}\int_t^\infty h(y)dy. \tag{6.8}$$

- *Y_0 has the density function $h(t+y)/\int_t^\infty h(r)dr, y>0$ and $Y_i, i=1,2,\ldots,$ share the common density function $g(t+y)/\int_t^\infty g(r)dr, y>0$.*

Proof. Since $N=0$ if and only if the whole excursion of S^* stays below level t up to time τ_0^-, the probability (6.7) just follows from Lemma 6.1. Observe that the total mass $X_t^*(0,\infty)$ is exactly the number of up-crossings (the same as the number of down-crossings) of level t by process S^* until the time τ_0^-. Each up-and-down-crossing of level t corresponds to an excursion starting at level t. All of such excursions end at level t except that the last one ends below 0 at time τ_0^-, where the last excursion determines the residual life time of a particle that can be either the ancestor or an offspring. Using solutions to the two-sided exit problem in Lemmas 6.1 and 6.4 together with the strong Markov property repeatedly at those up-crossing times of level t we have

$$\begin{aligned}
&\mathbf{P}\{X_t^*(0,\infty)=n\}\\
&=\mathbf{P}\{\tau_t^+<\tau_0^-\}\left(\int_0^t \mathbf{P}_t\{S^*_{\tau_t^-}\in dz\}\mathbf{P}_z\{\tau_t^+<\tau_0^-\}\right)^{n-1}\\
&\qquad\times\left(\int_0^t \mathbf{P}_t\{S^*_{\tau_t^-}\in dz\}\mathbf{P}_z\{\tau_t^+>\tau_0^-\}+\mathbf{P}_t\{S^*_{\tau_0^-}\leq 0\}\right)\\
&=\frac{W(0)}{W(t)}\left(\int_t^\infty e^{-\Phi(0)(y-t)}W(0)dy\int_0^t \Pi(y-dz)\frac{W(z)}{W(t)}\right)^{n-1}\\
&\qquad\times\Bigg(\int_t^\infty e^{-\Phi(0)(y-t)}W(0)dy\\
&\qquad\qquad\times\left\{\int_0^t \Pi(y-dz)\left(1-\frac{W(z)}{W(t)}\right)+\Pi((y,\infty))\right\}\Bigg).
\end{aligned}\tag{6.9}$$

Therefore, the probability (6.8) follows.

Given $X_t^*(0,\infty)=n$, the support of $X_t^*(0,\infty)$ consists of those distances between the pre-down-crossing (of level t) values of S^* and t for the n excursions from t. By the strong Markov property all these distances are independent. By Lemma 6.4 the distances for the first $n-1$ excursions following the same distribution of

$$\begin{aligned}
&\int_0^t \mathbf{P}_t\{S^*_{\tau_t^- -}\in t+dy, S^*_{\tau_t^-}\in dz\}\\
&\qquad\times\mathbf{P}_z\{\tau_t^+<\tau_0^-\}\left(\int_0^t \mathbf{P}_t\{S^*_{\tau_t^-}\in dz\}\mathbf{P}_z\{\tau_t^+<\tau_0^-\}\right)^{-1}\\
&=e^{-\Phi(0)y}W(0)dy\int_0^t \Pi(t+y-dz)\frac{W(z)}{W(t)}\left(\int_t^\infty g(r)dr\right)^{-1}\\
&=g(t+y)dy\left(\int_t^\infty g(r)dr\right)^{-1}.
\end{aligned}$$

The distance for the last excursion follows the distribution of

$$\begin{aligned}
&\left(\int_0^t \mathbf{P}_t\{S^*_{\tau_t^- -}\in t+dy, S^*_{\tau_t^-}\in dz\}\mathbf{P}_z\{\tau_t^+>\tau_0^-\}\right.\\
&\qquad\qquad\left.+\mathbf{P}_t\{S^*_{\tau_t^- -}\in t+dy, S^*_{\tau_t^-}\le 0\}\right)\\
&\times\left(\int_t^\infty\int_0^t \mathbf{P}_t\{S^*_{\tau_t^- -}\in t+dy, S^*_{\tau_t^-}\in dz\}\mathbf{P}_z\{\tau_t^+>\tau_0^-\}\right.\\
&\qquad\qquad\left.+\mathbf{P}_t\{S^*_{\tau_t^- -}\in t+dy, S^*_{\tau_t^-}\le 0\}\right)^{-1}\\
&=h(t+y)dy\left(\int_t^\infty h(r)dr\right)^{-1}.
\end{aligned}$$

□

Our next result is on the weighted occupation time for X^*. The proof is similar to Proposition 6.3 and is omitted.

Proposition 6.6. *For any $f\in B_\rho^+(0,\infty)$ and $h\in B_\rho^+(0,\infty)$, we have*

$$\mathbf{E}e^{-\int_0^\infty h(t)\langle X_t^*,f\rangle dt}=\mathbf{E}e^{-\langle X_0^*,\omega_0\rangle}, \tag{6.10}$$

where ω is the unique nonnegative solution of the integral equation

$$\begin{aligned}
\omega_t(x)-c^{-1}\int_t^\infty 1_{\{x>s-t\}}ds\int_0^\infty \Pi^+(dz)[1-e^{-\omega_s(z)}]\\
=\int_t^\infty h(s)f(x-s+t)1_{\{x>s-t\}}ds.
\end{aligned}$$

Observe that the total occupation time $\int_0^\infty \langle X_t^*, 1\rangle dt$ is just the sum of the sizes of all the jumps of process S^* before time τ_0^- together with $S^*_{\tau_0^- -}$. Further, this sum is equal to $c\tau_0^-$ since $S_0^* = 0$. By Lemma 6.4 we then have

$$\mathbf{E}e^{-q\int_0^\infty \langle X_t^*,1\rangle dt} = \mathbf{E}e^{-qc\tau_0^-} = 1 - \frac{qc}{\Phi(qc)}W^{(qc)}(0) = 1 - \frac{q}{\Phi(qc)}. \tag{6.11}$$

7 Connections with the CMJ model

Informally, the Crump-Mode-Jagers branching processes or the CMJ process counts the size of a branching population system with random characteristics. Informally, a particle, say x, of this process is characterized by there random process

$$(\lambda_x, \zeta_x(\cdot), \omega_x)$$

which is an i.i.d. copy of $(\lambda, \zeta(\cdot), \omega)$ and the reproduction scheme is given in the following sense: if x was born at time σ_x, then

1. λ_x is the life length of x;
2. $\zeta_x(\cdot) = \{0 < \zeta_x^1 < \zeta_x^2 < \cdots < \lambda_x\}$ is a point process defined on $(0, \lambda_x)$. $\{\zeta_x^i + \sigma_x : i = 1, \cdot\}$ is the collection of splitting times of x at which it produces offspring.
3. ω_x^i is the number of children produced by x at time $\sigma_x + \zeta_x^i$.

Let $Z(t)$ denote the total number of individuals in the system at time t with $Z(0)$ ancestors. In general, the process $\{Z(t) : t \geq 0\}$ is not Markovian unless λ_x is exponentially distributed. Now assume that

1. The distribution of λ is determined by a probability measure $\eta(dx)$ on $(0, \infty)$;
2. $\zeta(\cdot)$ is a Poisson point process with parameter α;
3. The distribution of ω^i is determined by a generating function $g(\cdot)$.

According to the argument in Section 2 and [3], we may define a measure-valued Markov process $Y = \{Y(t) : t \geq 0\}$ with transition probabilities given by

$$\int_{N(0,\infty)} e^{-\langle \nu, f\rangle} Q_t(\mu, d\nu) = e^{-\langle \mu, U_t f\rangle}, \qquad f \in B^+(0, \infty), \tag{7.1}$$

where $(t, x) \mapsto U_t f(x)$ is the unique locally bounded positive solution of

$$U_t f(x) = f(x-t)1_{\{x>t\}} + \alpha \int_0^t 1_{\{x>t-s\}}\big[1 - g(\langle \eta, e^{-U_s f}\rangle)\big] ds. \tag{7.2}$$

Then the CMJ process $\{Z(t) : t \geq 0\}$ is just the total mass process of Y; i.e. $Z(t) = \langle Y(t), 1\rangle$.

The connection between Lévy processes and CMJ processes was first investigated by Lambert in [14] which showed that the contour process of a splitting tree defined from a suitable CMJ process is a spectrally positive Lévy process with negative drift killed when it hits 0. The starting position of the Lévy process is just the life time of the ancestor. Equivalently, given such a Lévy process, one could construct a CMJ process; see also [15]. In those works, the Lévy measure, say γ, is assumed to be a σ-finite measure on $(0, \infty]$ with $\int_{(0,\infty]} 1 \wedge z\gamma(dz) < \infty$. Our main result, Theorem 3.2, also gives similar relationships between one-sided Lévy processes of bounded variation and CMJ processes.

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