Applied Probability Trust (9 October 2012)

LIMIT THEOREMS FOR CONTINUOUS TIME BRANCHING FLOWS

HUI HE,* *Beijing Normal University*

RUGANG MA,** *Central University of Finance and Economics*

Abstract

We construct a flow of continuous time and discrete state branching processes. Some scaling limit theorems for the flow are proved, which lead to the path-valued branching processes and nonlocal branching superprocesses over the positive half line studied in Li (2012).

Keywords: Stochastic flow; branching process; continuous-time; discrete-state; superprocess; nonlocal branching.

2010 Mathematics Subject Classification: Primary 60J68

Secondary 60J80; 60G57

1. Introduction

A genealogical tree is naturally associated with a Galton-Watson branching process. A continuous-state branching process (CB-process) can be obtained as the small particle limit of rescaled Galton-Watson processes; see, e.g., Lamperti (1967). The genealogical structures of binary branching CB-processes were investigated by introducing *continuum random trees* in the pioneer work of Aldous (1991, 1993). Continuum random trees corresponding to general branching mechanisms were constructed in Le Gall and Le Jan (1998a, 1998b) and were studied further in Duquesne and Le Gall (2002). By pruning a Galton-Watson tree, Aldous and Pitman (1998) and Abraham at

* Postal address: Laboratory of Mathematics and Complex Systems, School of Mathematical Sciences, Beijing Normal University, Beijing 100875, People's Republic of China.

E-mail:hehui@bnu.edu.cn

** Postal address: School of Applied Mathematics, Central University of Finance and Economics, Beijing 100081, People's Republic of China.

E-mail:marugang@mail.bnu.edu.cn

al. (2012) constructed a tree-valued Markov process. Tree-valued processes associated with general CB-processes were studied in Abraham and Delmas (2012) by pruning arguments.

Motivated by the study of genealogy trees for critical branching processes conditioned on non-extinction, Bakhtin (2011) studied a flow of binary branching continuous-state branching processes with immigration (CBI-processes) driven by a time-space Gaussian white noise. He also pointed out the connection of the model with a superprocess conditioned on non-extinction. In Li (2012), a class of path-valued branching processes were constructed and studied using the techniques of stochastic equations and superprocesses. The work is closely related to those of Bertoin and Le Gall (2006) and Dawson and Li (2012). In a special case, the path-valued branching processes in Li (2012) can be coded by the tree-valued processes of Abraham and Delmas (2012). In He and Ma (2012), two flows of discrete time and state Galton-Watson branching processes were introduced. There it was showed that suitable rescaled sequences of those flows converge to special forms of the flows of Dawson and Li (2012) and Li (2012), respectively. The limit theorems in He and Ma (2012) were given in the setting of the corresponding superprocesses. From those limit theorems the convergence of the finite-dimensional distributions of corresponding the path-valued processes was derived. The results give a better understanding of the connection between discrete and continuum tree-valued branching processes.

In this paper, we introduce a kind of flows of continuous time and discrete state branching processes. We shall prove the scaling limit theorems for those flows of the type of He and Ma (2012). In Section 2 a short review is given to the path-valued branching processes and nonlocal branching superprocesses studied in Li (2012). In Section 3 we construct a continuous time and discrete state branching processes as the strong solution of a stochastic integral equation. In Section 4 the construction is extended to branching flows by considering stochastic equation systems. In Section 5 we prove that suitable rescaled sequences of those flows converge to the nonlocal branching superprocess. From the limit theorem we also derive the convergence of the finite-dimensional distributions of corresponding the path-valued processes.

Let $\mathbb{N} = \{0, 1, 2, \cdots\}$ and $\mathbb{N}_+ = \{1, 2, \cdots\}$. Let $M[0, 1]$ be the set of finite Borel measures on $[0, 1]$ endowed with the topology of weak convergence. We identify $M[0, 1]$

with the set $F[0,1]$ of positive right continuous increasing functions on $[0,1]$. Let $B[0,1]$ be the Banach space of bounded Borel functions on $[0,1]$ endowed with the supremum norm $\|\cdot\|$. Let $C[0,1]$ denote its subspace of continuous functions. We use $B[0,1]^+$ and $C[0,1]^+$ to denote the subclasses of non-negative elements and $C[0,1]^{++}$ to denote the subset of $C[0,1]^+$ of functions bounded away from zero. For $\mu \in M[0,1]$ and $f \in B[0,1]$ write $\langle \mu, f\rangle = \int f d\mu$ if the integral exists. Let $D([0,\infty), M[0,1])$ denote the space of càdlàg paths from $[0,\infty)$ to $M[0,1]$ endowed with the Skorokhod topology. Throughout the paper, we only consider *continuous time* processes, so we shall often omit this phrase in the sequel.

2. Preliminaries

In this section, we recall some results established in Li (2012) on flows of CB-processes and nonlocal branching superprocesses over the positive half line. By a *branching mechanism* ϕ we mean a function ϕ on $[0,\infty)$ with the representation

$$\phi(z) = bz + \frac{1}{2}\sigma^2 z^2 + \int_0^\infty (e^{-zu} - 1 + zu) m(du), \tag{1}$$

where $\sigma \geq 0$ and b are constants and $(u \wedge u^2)m(du)$ is a finite measure on $(0,\infty)$. Consider a family of branching mechanisms $\{\phi_q : q \in [0,1]\}$ that is *admissible* in the sense that each ϕ_q is given by (1) with parameters $(b,m) = (b_q, m_q)$ depending on $q \in [0,1]$ and for each $z \geq 0$ the function $q \mapsto \phi_q(z)$ is decreasing and continuously differentiable with the derivative $\psi_\theta(z) = -(\partial/\partial\theta)\phi_\theta(z)$ of the form

$$\psi_\theta(z) = h_\theta z + \int_0^\infty (1 - e^{-zu}) n_\theta(du), \tag{2}$$

where $h_\theta \geq 0$ and $n_\theta(du)$ is a σ-finite kernel from $[0,1]$ to $(0,\infty)$ satisfying

$$\sup_{0\leq\theta\leq 1} \Big[h_\theta + \int_0^\infty u n_\theta(du)\Big] < \infty.$$

Let $m(dz, d\theta)$ be the measure on $(0,\infty)\times[0,1]$ defined by

$$m([c,d]\times[0,q]) = m_q[c,d], \quad q \in [0,1], d > c > 0.$$

Let $W(ds, du)$ be a white noise on $(0,\infty)^2$ based on the Lebesgue measure, $\tilde{N}(ds, dz, d\theta, du)$ be a compensated Poisson random measure on $(0,\infty)^2 \times [0,1] \times (0,\infty)$ with intensity

$dsm(dz, d\theta)du$. By the results in Li (2012), the following stochastic equation

$$\begin{aligned} Y_t(q) &= Y_0(q) - b_q \int_0^t Y_{s-}(q)ds + \sigma \int_0^t \int_0^{Y_{s-}(q)} W(ds, du) \\ &\quad + \int_0^t \int_0^\infty \int_{[0,q]} \int_0^{Y_{s-}(q)} z\tilde{N}(ds, dz, d\theta, du) \end{aligned} \tag{3}$$

has a unique solution flow $\{Y_t(q) : t \geq 0, q \in [0,1]\}$. For each $q \in [0,1]$, the one-dimensional process $\{Y_t(q) : t \geq 0\}$ is a CB-process with branching mechanism ϕ_q. The flow is increasing in $q \in [0,1]$. It was verified in Li (2012) that $\{(Y_t(q))_{t\geq 0} : q \in [0,1]\}$ can be identified as a path-valued branching process. Moreover, the flow induces a càdlàg $M[0,1]$-valued superprocess $\{Y_t : t \geq 0\}$ which is the unique solution of the following martingale problem: For every $G \in C^2(\mathbb{R})$ and $f \in C[0,1]$,

$$\begin{aligned} G(\langle Y_t, f\rangle) &= G(\langle Y_0, f\rangle) + \int_0^t G'(\langle Y_s, f\rangle)ds \int_{[0,1]} Y_s(dx) \int_{[0,1]} f(x \vee \theta)h_\theta d\theta \\ &\quad - b_0 \int_0^t G'(\langle Y_s, f\rangle)\langle Y_s, f\rangle ds + \frac{1}{2}\sigma^2 \int_0^t G''(\langle Y_s, f\rangle)\langle Y_s, f^2\rangle ds \\ &\quad + \int_0^t ds \int_{[0,1]} Y_s(dx) \int_0^\infty \Big[G(\langle Y_s, f\rangle + zf(x)) \\ &\quad - G(\langle Y_s, f\rangle) - zf(x)G'(\langle Y_s, f\rangle)\Big] m_0(dz) \\ &\quad + \int_0^t ds \int_{[0,1]} Y_s(dx) \int_{[0,1]} d\theta \int_0^\infty \Big[G(\langle Y_s, f\rangle + zf(x \vee \theta)) \\ &\quad - G(\langle Y_s, f\rangle)\Big] n_\theta(dz) + \text{local mart.} \end{aligned} \tag{4}$$

Let $f \mapsto \Psi(\cdot, f)$ be the operator on $C^+[0,1]$ defined by

$$\Psi(x, f) = \int_{[0,1]} f(x \vee \theta)h_\theta d\theta + \int_{[0,1]} d\theta \int_0^\infty (1 - e^{-zf(x\vee\theta)})n_\theta(dz). \tag{5}$$

Then the superprocess $\{Y_t : t \geq 0\}$ has local branching mechanism ϕ_0 and nonlocal branching mechanism Ψ. Its transition semigroup $(Q_t)_{t\geq 0}$ is given by

$$\int_{M[0,1]} e^{-\langle \nu, f\rangle} Q_t(\mu, d\nu) = \exp\Big\{ - \langle \mu, V_t f\rangle \Big\}, \qquad f \in C^+[0,1], \tag{6}$$

where $t \mapsto V_t f$ is the unique locally bounded positive solution of

$$V_t f(x) = f(x) - \int_0^t [\phi_0(V_s f(x)) - \Psi(x, V_s f)]ds, \qquad t \geq 0, x \in [0,1]. \tag{7}$$

The reader may refer Li (2012) for the derivations of the superprocess $\{Y_t : t \geq 0\}$.

Remark 1. Usually, one may only ask the Lévy measure m in ϕ in (1) to integrate $1 \wedge u^2$ and an indicator function is added in the integral. In fact, the assumption that $(u \wedge u^2)m(du)$ is a finite measure on $(0, \infty)$ is equivalent to that ϕ is locally Lipschitz; see Proposition 1.45 in Li (2011). This assumption is technically required to construct the scaling limits of Galton-Watson processes; see Proposition 3.40 in Li (2011) and Condition (5.A) below.

3. Stochastic equations for discrete state branching processes

In this section, we give a construction of the continuous time and discrete state branching process as the solution of a stochastic integral equation driven by Poisson random measure. Stochastic integral equations of this type were used in Li and Ma (2008) to construct catalytic branching processes. We here give all the details for completeness.

Let $g = g(z) = \sum_{i=0}^{\infty} p_i z^i$ be a probability generating function with $g'(1) < \infty$. Let $N(ds, dz, du)$ be a Poisson random measure on $(0, \infty) \times \mathbb{N} \times (0, \infty)$ with intensity $\sigma ds \pi(dz) du$, where $\sigma > 0$ is a constant and $\pi(dz) := \sum_{i=0}^{\infty} p_i \delta_i(dz)$. Suppose that X_0 is a non-negative integer-valued random variable satisfying $\mathbf{E}[X_0] < \infty$. We assume X_0 is independent of $N(ds, dz, du)$ and consider the stochastic integral equation

$$X_t = X_0 + \int_0^t \int_{\mathbb{N}} \int_0^{X_{s-}} (z-1) N(ds, dz, du). \tag{1}$$

By a *solution* of (1) we mean a non-negative càdlàg progressive process $\{X_t : t \geq 0\}$ satisfying the equation a.s. for each $t \geq 0$. We say *pathwise uniqueness* of solution holds for (1) if any two solutions of the equation with the same initial state are indistinguishable.

Theorem 1. *Suppose that $\{X_t^1\}$ and $\{X_t^2\}$ are two solutions of (1) satisfying $\mathbf{E}[|X_0^1 + X_0^2|] < \infty$. Then we have*

$$\mathbf{E}[|X_t^2 - X_t^1|] \leq \mathbf{E}[|X_0^2 - X_0^1|] \exp\{\sigma t(g'(1) + 1)\}. \tag{2}$$

Consequently, the pathwise uniqueness of solution holds for (1).

Proof. The pathwise uniqueness for (1) follows from Theorem 2.1 of Dawson and Li (2012). We present a proof of the result here for completeness. Let $\xi_t = X_t^2 - X_t^1$ for

$t \geq 0$. From (1) we have

$$\begin{aligned}\xi_t &= X_0^2 - X_0^1 + \int_0^t \int_{\mathbb{N}} \int_{X_{s-}^1}^{X_{s-}^2} (z-1) 1_{\{X_{s-}^1 \leq X_{s-}^2\}} N(ds, dz, du) \\ &\quad - \int_0^t \int_{\mathbb{N}} \int_{X_{s-}^2}^{X_{s-}^1} (z-1) 1_{\{X_{s-}^1 > X_{s-}^2\}} N(ds, dz, du).\end{aligned}$$

Let $\tau_m = \inf\{t \geq 0 : X_t^1 \geq m \text{ or } X_t^2 \geq m\}$. Then we have

$$\begin{aligned}\mathbf{E}[|\xi_{t\wedge\tau_m}|] &\leq \mathbf{E}[|\xi_0|] + \mathbf{E}\int_0^{t\wedge\tau_m} \int_{\mathbb{N}} \int_{X_{s-}^1}^{X_{s-}^2} (z+1) 1_{\{X_{s-}^1 \leq X_{s-}^2\}} N(ds, dz, du) \\ &\quad + \mathbf{E}\int_0^{t\wedge\tau_m} \int_{\mathbb{N}} \int_{X_{s-}^2}^{X_{s-}^1} (z+1) 1_{\{X_{s-}^1 > X_{s-}^2\}} N(ds, dz, du) \\ &= \mathbf{E}[|\xi_0|] + \mathbf{E}\int_0^{t\wedge\tau_m} ds \int_{\mathbb{N}} \xi_{s-} 1_{\{\xi_{s-} \geq 0\}} (z+1) \sigma\pi(dz) \\ &\quad + \mathbf{E}\int_0^{t\wedge\tau_m} ds \int_{\mathbb{N}} (-\xi_{s-}) 1_{\{\xi_{s-} < 0\}} (z+1) \sigma\pi(dz) \\ &\leq \mathbf{E}[|\xi_0|] + \int_0^t \mathbf{E}[|\xi_{s\wedge\tau_m}|] \sigma(g'(1)+1) ds.\end{aligned}$$

By Gronwall's inequality we get

$$\mathbf{E}[|\xi_{t\wedge\tau_m}|] \leq \mathbf{E}[|\xi_0|] \exp\{\sigma t(g'(1)+1)\}.$$

Then (2) follows by Fatou's lemma. □

By Theorem 2.5 in Dawson and Li (2012), there is a unique strong solution to (1). Here we give a simple direct proof of the existence of the solution. We first take an $n \in \mathbb{N}_+$ and consider the following stochastic equation

$$X_t = X_0 + \int_0^t \int_{\mathbb{N}} \int_0^{X_{s-}\wedge n} (z-1) N(ds, dz, du). \tag{3}$$

Proposition 3.1. *For each $n \geq 1$, there is a solution $\{X_t^n : t \geq 0\}$ of (3).*

Proof. Let $\{S_k : k = 1, 2, \cdots\}$ be the set of jump times of the Poisson process

$$t \mapsto \int_0^t \int_{\mathbb{N}} \int_0^n N(ds, dz, du).$$

We have clearly $S_k \to \infty$ as $k \to \infty$. For $0 \leq t < S_1$, set $X_t^n = X_0$. Suppose that X_t^n has been defined for $0 \leq t < S_k$ and let

$$X_t^n = X_{S_k-}^n + \int_{\{S_k\}} \int_{\mathbb{N}} \int_0^{X_{S_k-}^n \wedge n} (z-1) N(ds, dz, du), \quad S_k \leq t < S_{k+1}.$$

From the construction of $X^n_{S_k}$ we see $X^n_{S_k} - X^n_{S_{k-1}} \geq -1$. And since $X^n_{S_{k-1}} = 0$ implies $X^n_{S_k} = 0$, $X^n_{S_k} \in \mathbb{N}$. By induction that defines a non-negative process $\{X^n_t : t \geq 0\}$ which is clearly a solution to (3). $\square$

Proposition 3.2. *Let $\{X^n_t\}$ be a solution of (3). Then we have*

$$\mathbf{E}\Big[\sup_{0\leq s\leq t} X^n_s\Big] \leq \mathbf{E}[X_0]\exp\{\sigma g'(1)t\}, \quad t \geq 0. \tag{4}$$

Proof. From (3) we have

$$\begin{aligned}\mathbf{E}\Big[\sup_{0\leq s\leq t} X^n_s\Big] &\leq \mathbf{E}[X_0] + \mathbf{E}\Big[\int_0^t \int_{\mathbb{N}} \int_0^{X^n_{s-}\wedge n} zN(ds,dz,du)\Big] \\ &= \mathbf{E}[X_0] + \mathbf{E}\Big[\int_0^t ds \int_{\mathbb{N}} (X^n_{s-}\wedge n)z\sigma\pi(dz)\Big].\end{aligned}$$

Thus $t \mapsto \mathbf{E}[\sup_{0\leq s\leq t} X^n_s]$ is a locally bounded function. Moreover,

$$\begin{aligned}\mathbf{E}\Big[\sup_{0\leq s\leq t} X^n_s\Big] &\leq \mathbf{E}[X_0] + \int_0^t ds \int_{\mathbb{N}} \mathbf{E}\Big[\sup_{0\leq r\leq s} X^n_r\Big] z\sigma\pi(dz) \\ &= \mathbf{E}[X_0] + \sigma g'(1)\int_0^t \mathbf{E}\Big[\sup_{0\leq r\leq s} X^n_r\Big] ds.\end{aligned}$$

By Gronwall's lemma we get the result. $\square$

By a modification of the proof of Theorem 1 we get the following Proposition.

Proposition 3.3. *Suppose that $\{X^{n,1}_t\}$ and $\{X^{n,2}_t\}$ are two solutions of (3). Then we have*

$$\mathbf{E}[|X^{n,2}_t - X^{n,1}_t|] \leq \mathbf{E}[|X^{n,2}_0 - X^{n,1}_0|]\exp\{\sigma t(g'(1)+1)\}. \tag{5}$$

Consequently, the pathwise uniqueness of solution holds for (3).

Proposition 3.4. *Let $\{X^n_t : t \geq 0\}$ be the solution of (3) with $n = 1, 2, \cdots$. Then the sequence $\{X^n_t : t \geq 0\}$ is tight in $D([0,\infty),\mathbb{N})$.*

Proof. By Proposition 3.2, it is easy to see that

$$t \mapsto C_t := \sup_{n\geq 1}\mathbf{E}\Big[\sup_{0\leq s\leq t} X^n_s\Big]$$

is locally bounded. Then for every fixed $t \geq 0$, the sequence of random variables X^n_t is tight. Moreover, in view of (3), if $\{\tau_n\}$ is a sequence of stopping times bounded above by $T \geq 0$, we have

$$\begin{aligned}\mathbf{E}[|X^n_{t+\tau_n} - X^n_{\tau_n}|] &= \mathbf{E}\Big[\Big|\int_{\tau_n}^{t+\tau_n}\int_{\mathbb{N}}\int_0^{X^n_{s-}\wedge n}(z-1)N(ds,dz,du)\Big|\Big] \\ &\leq \mathbf{E}\Big[\int_0^t ds\int_{\mathbb{N}}(X^n_{(s+\tau_n)-}\wedge n)(z+1)\sigma\pi(dz)\Big] \\ &\leq \sigma(g'(1)+1)\int_0^t \mathbf{E}[X^n_{(s+\tau_n)-}]ds \\ &\leq \mathbf{E}[X_0]\exp\{\sigma g'(1)(t+T)\}\sigma(g'(1)+1)t,\end{aligned}$$

where the last inequality follows by Proposition 3.2. Consequently, as $t \to 0$,

$$\sup_{n\geq 1}\mathbf{E}[|X^n_{t+\tau_n} - X^n_{\tau_n}|] \to 0.$$

Then $\{X^n_t : t \geq 0\}$ is tight in $D([0,\infty),\mathbb{N})$ by the criterion of Aldous (1978); see also Ethier and Kurtz (1986, pp.137-138). □

Theorem 2. *There is a solution* $\{X_t : t \geq 0\}$ *of (1).*

Proof. For each $n \geq 1$, let $\{X^n_t : t \geq 0\}$ be the solution of (3). Define $\tau_n = \inf\{t \geq 0 : X^n_t \geq n\}$. From Proposition 3.2 it follows that

$$\mathbf{E}[X^n_{t\wedge\tau_n}] \leq \mathbf{E}\Big[\sup_{0\leq s\leq t} X^n_s\Big] \leq \mathbf{E}[X_0]\exp\{\sigma g'(1)t\}, \quad t \geq 0.$$

Then we have

$$\mathbf{E}[X^n_{t\wedge\tau_n}1_{\{\tau_n\leq t\}}] \leq \mathbf{E}[X_0]\exp\{\sigma g'(1)t\}.$$

By the right continuity of $\{X^n_t\}$ we have $X^n_{\tau_n} \geq n$, so

$$n\mathbf{P}[\{\tau_n \leq t\}] \leq \mathbf{E}[X_0]\exp\{\sigma g'(1)t\}, \quad t \geq 0.$$

That implies $\tau_n \to \infty$ almost surely as $n \to \infty$. On the other hand, $\{X^n_t\}$ satisfies Equation (1) for $0 \leq t < \tau_n$. By the pathwise uniqueness of the solution of (1) we get, for any $i, j \in \mathbb{N}$,

$$X^i_t = X^j_t, \quad t < \tau_i \wedge \tau_j.$$

Let $\{X_t\}$ be the process such that $X_t = X^n_t$ for all $0 \leq t < \tau_n$ and $n \geq 1$. It is easily seen that $\{X_t\}$ is a solution of (1). □

Theorems 1 and 2 imply that (1) has unique *strong solution* and the solution $\{X_t : t \geq 0\}$ is a strong Markov process; see, e.g., Ikeda and Watanabe (1989, pp.163-166 and p.215). Let $B(\mathbb{N})$ denote the set of bounded functions on $\mathbb{N}$. By Itô's formula it is easy to see that $\{X_t : t \geq 0\}$ has generator A defined by

$$Af(x) = \sigma x \sum_{i=0}^{\infty} [f(x+i-1) - f(x)]p_i, \qquad x \in \mathbb{N},\ f \in B(\mathbb{N}).$$

Then $\{X_t : t \geq 0\}$ is a Galton-Watson branching process with σ-exponentially distributed life time and offspring distribution $\{p_i : i \geq 0\}$.

In fact, let $N^{(1)}(ds, dz, du)$ and $N^{(2)}(ds, dz, du)$ be two mutually independent Poisson random measures on $(0, \infty) \times \mathbb{N} \times (0, \infty)$ with the same intensity $\sigma ds\pi(dz)du$. Consider the following two stochastic equations

$$X_t^{(1)} = X_0^{(1)} + \int_0^t \int_{\mathbb{N}} \int_0^{X_{s-}^{(1)}} (z-1)N^{(1)}(ds, dz, du)$$

and

$$X_t^{(2)} = X_0^{(2)} + \int_0^t \int_{\mathbb{N}} \int_0^{X_{s-}^{(2)}} (z-1)N^{(2)}(ds, dz, du).$$

Clearly, $X_t^{(1)}$ and $X_t^{(2)}$ are mutually independent. Set $X_t = X_t^{(1)} + X_t^{(2)}$. Since the random measure

$$N'(ds, dz) := \int_{\{0<u\leq X_{s-}^{(1)}\}} N^{(1)}(ds, dz, du) + \int_{\{0<u\leq X_{s-}^{(2)}\}} N^{(2)}(ds, dz, du)$$

has predictable compensator $\sigma X_{s-} ds\pi(dz)$, by representation theorems for semimartingales, on an extension of the original probability space, there is a Poisson random measure on $(0, \infty) \times \mathbb{N} \times (0, \infty)$ with intensity $\sigma ds\pi(dz)du$ such that

$$X_t = X_0 + \int_0^t \int_{\mathbb{N}} \int_0^{X_{s-}} (z-1)N(ds, dz, du);$$

see, e.g., Ikeda and Watanabe (1989, p.93). Then the solution of (1) is a branching process (continuous time and discrete state). This gives another derivation of the branching property of $\{X_t : t \geq 0\}$.

4. The flow of discrete state branching processes

In this section, we give a formulation of the discrete state branching flow as the solution flow of a set of stochastic integral equations. Let $\{g_\theta : \theta \geq 0\}$ be a family of probability generating functions, that is, for each $\theta \geq 0$,

$$g_\theta(z) = \sum_{i=0}^{\infty} p_i(\theta) z^i, \quad |z| \leq 1,$$

where $p_i(\theta) \geq 0$ and $\sum_{k=0}^{\infty} p_i(\theta) = 1$. Moreover, we assume $\theta \mapsto g'_\theta(1)$ is continuous and $p_i(\theta_2) \geq p_i(\theta_1)$ holds for any $\theta_2 \geq \theta_1 \geq 0$ and $i \in \mathbb{N}_+$. Define a family of probability measures $\{\pi_\theta : \theta \geq 0\}$ on $\mathbb{N}$ by

$$\pi_\theta(dz) = \sum_{i=0}^{\infty} p_i(\theta)\delta_i(dz)$$

Then we have $\pi_{\theta_2}|_{\mathbb{N}_+} \geq \pi_{\theta_1}|_{\mathbb{N}_+}$ for any $\theta_2 \geq \theta_1 \geq 0$. Let $\bar{\pi}(dz, d\theta)$ be the measure on $\mathbb{N}_+ \times [0, \infty)$ defined by

$$\bar{\pi}(A \times [0, \theta]) = \pi_\theta(A), \quad A \subset \mathbb{N}_+, \theta \geq 0.$$

Notice that the positive function $\theta \mapsto b(\theta) := \pi_\theta(\{0\})$ is decreasing.

Let $q \mapsto X_0(q)$ be a deterministic non-negative right continuous non-decreasing function on $[0, \infty)$ and that takes values in $\mathbb{N}$. Let $N(ds, dz, d\theta, du)$ be a Poisson random measure on $(0, \infty) \times \mathbb{N} \times [0, \infty) \times (0, \infty)$ with intensity $\sigma ds \bar{\pi}(dz, d\theta)du$ and $N_0(ds, d\theta, du)$ a Poisson random measure on $(0, \infty)^3$ with intensity $\sigma ds d\theta du$. Suppose that $N(ds, dz, d\theta, du)$ and $N_0(ds, d\theta, du)$ are independent of each other. Consider the stochastic integral equation

$$\begin{aligned} X_t(q) \;&=\; X_0(q) + \int_0^t \int_{\mathbb{N}_+} \int_{[0,q]} \int_0^{X_{s-}(q)} (z-1) N(ds, dz, d\theta, du) \\ &\quad - \int_0^t \int_0^{b(q)} \int_0^{X_{s-}(q)} N_0(ds, d\theta, du). \end{aligned} \tag{1}$$

Note that for each $q \geq 0$,

$$\int_{\{0<\theta \leq b(q)\}} N_0(ds, d\theta, du)$$

is a Poisson random measure with intensity $\sigma b(q) ds du = \sigma \bar{\pi}_0(\mathbb{N} \times [0, q]) ds du$, where $\bar{\pi}_0(dz, d\theta)$ is a measure on $\mathbb{N} \times [0, \infty)$ defined by

$$\bar{\pi}_0(A \times [0, q]) = \pi_q(\{0\})\delta_0(A), \quad A \subset \mathbb{N},\ \theta \geq 0.$$

By representation theorems for semi-martingales, there is a Poisson random measure $N_1(ds, dz, d\theta, du)$ on $(0, \infty) \times \mathbb{N} \times [0, \infty) \times (0, \infty)$ with intensity $\sigma ds \bar{\pi}_0(dz, d\theta)du$ such that for every $E \in \mathscr{B}(0,\infty)$,

$$\int_0^t \int_0^{b(q)} \int_E N_0(ds, d\theta, du) = \int_0^t \int_{\mathbb{N}} \int_{[0,q]} \int_E N_1(ds, dz, d\theta, du);$$

see, e.g., Ikeda and Watanabe (1989, p.93). Define $N_2(ds, dz, du)$ by

$$N_2(ds, dz, du) = \int_{\{0 \le \theta \le q\}} N(ds, dz, d\theta, du) + \int_{\{0 \le \theta \le q\}} N_1(ds, dz, d\theta, du).$$

Then N_2 is a Poisson random measure on $(0,\infty)\times\mathbb{N}\times(0,\infty)$ with intensity $\sigma ds \pi_q(dz)du$ and Equation (1) can be rewrited as

$$X_t(q) = X_0(q) + \int_0^t \int_{\mathbb{N}} \int_0^{X_{s-}(q)} (z-1) N_2(ds, dz, du).$$

By Theorem 2 we see that for each $q \ge 0$, Equation (1) has a unique strong solution $\{X_t(q) : t \ge 0\}$.

Theorem 3. *Suppose that $q \ge p \ge 0$. Let $\{X_t(q)\}$ be the solution of (1) and $\{X_t(p)\}$ be the solution of the equation with q replaced by p. Then we have $\mathbf{P}\{X_t(q) \ge X_t(p)\ for\ all\ t \ge 0\} = 1$.*

Proof. Let $\zeta_t = X_t(p) - X_t(q)$ for $t \ge 0$. From (1) we have

$$\begin{aligned}\zeta_t \;=\; & \zeta_0 + \int_0^t \int_{\mathbb{N}_+} \int_{[0,p]} \int_{X_{s-}(q)}^{X_{s-}(p)} (z-1) N(ds, dz, d\theta, du) \\ & - \int_0^t \int_{\mathbb{N}_+} \int_{(p,q]} \int_0^{X_{s-}(q)} (z-1) N(ds, dz, d\theta, du) - \int_0^t \int_0^{b(q)} \int_{X_{s-}(q)}^{X_{s-}(p)} N_0(ds, d\theta, du) \\ & - \int_0^t \int_{b(q)}^{b(p)} \int_0^{X_{s-}(p)} N_0(ds, d\theta, du). \qquad (2)\end{aligned}$$

Let $\tau_m = \inf\{t \ge 0 : X_t(q) \ge m\ or\ X_t(p) \ge m\}$. It is easy to construct a sequence of functions $\{f_n\}$ on $\mathbb{R}$ such that $0 \le f_n'(z) \le 1$ for $z \ge 0$ and $f_n(z) = f_n'(z) = 0$ for $z \le 0$. Moreover, $f_n(z) \to z^+ := 0 \vee z$ increasingly as $n \to \infty$. By (2) and Itô's formula,

$$\begin{aligned} f_n(\zeta_{t\wedge\tau_m}) \;=\; & \int_0^{t\wedge\tau_m} \int_{\mathbb{N}_+} \int_{[0,p]} \int_{X_{s-}(q)}^{X_{s-}(p)} [f_n(\zeta_{s-} + z - 1) - f_n(\zeta_{s-})] 1_{\{\zeta_{s-}>0\}} N(ds, dz, d\theta, du) \\ & + \int_0^{t\wedge\tau_m} \int_{\mathbb{N}_+} \int_{(p,q]} \int_0^{X_{s-}(q)} [f_n(\zeta_{s-} - z + 1) - f_n(\zeta_{s-})] N(ds, dz, d\theta, du) \\ & + \int_0^{t\wedge\tau_m} \int_0^{b(q)} \int_{X_{s-}(p)}^{X_{s-}(q)} [f_n(\zeta_{s-} - 1) - f_n(\zeta_{s-})] 1_{\{\zeta_{s-}>0\}} N_0(ds, d\theta, du) \end{aligned}$$

$$
\begin{aligned}
&+\int_0^{t\wedge\tau_m}\int_{b(q)}^{b(p)}\int_0^{X_{s-}(p)}[f_n(\zeta_{s-}-1)-f_n(\zeta_{s-})]N_0(ds,d\theta,du)\\
\le\ &\sigma\int_0^{t\wedge\tau_m}\zeta_{s-}1_{\{\zeta_{s-}>0\}}ds\int_{\mathbb{N}_+}(z-1)\pi_p(dz)+\text{martingale}.
\end{aligned}
$$

Taking the expectation in both sides and letting $n\to\infty$ gives

$$
\mathbf{E}[\zeta^+_{t\wedge\tau_m}]\le\sigma(g'_p(1)-1+b(p))\int_0^t\mathbf{E}[\zeta^+_{s\wedge\tau_m}]ds.
$$

Then $\mathbf{E}[\zeta^+_{t\wedge\tau_m}]=0$ for all $t\ge 0$. Since $\tau_m\to\infty$ as $m\to\infty$, that proves the desired comparison result. □

Proposition 4.1. *There is a locally bounded positive function $(t,u)\mapsto C(t,u)$ on $[0,\infty)^2$ so that, for any $t\ge 0$ and $p\le q\le u<\infty$,*

$$
\mathbf{E}\Big\{\sup_{0\le s\le t}[X_s(q)-X_s(p)]\Big\}\le C(t,u)\Big\{X_0(q)-X_0(p)+g'_q(1)-g'_p(1)\Big\}. \tag{3}
$$

Proof. Let $\xi_t=X_t(q)-X_t(p)$. From (1) we get

$$
\begin{aligned}
\sup_{0\le s\le t}\xi_s\ \le\ &\xi_0+\int_0^t\int_{\mathbb{N}_+}\int_{[0,q]}\int_{X_{s-}(p)}^{X_{s-}(q)}(z-1)N(ds,dz,d\theta,du)\\
&+\int_0^t\int_{\mathbb{N}_+}\int_{(p,q]}\int_0^{X_{s-}(p)}(z-1)N(ds,dz,d\theta,du)\\
&+\int_0^t\int_{b(q)}^{b(p)}\int_0^{X_{s-}(p)}N_0(ds,d\theta,du).
\end{aligned}
$$

Then

$$
\begin{aligned}
\mathbf{E}\Big[\sup_{0\le s\le t}\xi_s\Big]\ \le\ &\xi_0+\sigma[g'_p(1)-1+b(q)]\int_0^t\mathbf{E}[\xi_s]ds\\
&+\sigma[g'_q(1)-g'_p(1)]\int_0^t\mathbf{E}[X_s(p)]ds.
\end{aligned}
$$

Since $t\mapsto\mathbf{E}[X_t(p)]$ is locally bounded, by Gronwall's inequality we get the desired estimate. □

From the discussion above, given a constant $\sigma>0$ and a family of probability generating functions $\{g_\theta:\theta\ge 0\}$, we obtain a continuous time and discrete state branching process flow $\{X_t(q):t\ge 0,q\ge 0\}$ as the solution of equation (1). For any $t\ge 0$ define the random function $\tilde{X}_t\in F[0,1]$ by $\tilde{X}_t(1)=X_t(1)$ and

$$
\tilde{X}_t(q)=\inf\{X_t(u):\text{rational }u\in(q,1]\},\quad 0\le q<1. \tag{4}
$$

By Proposition 4.1, for each $q \in [0,1]$ we have

$$\mathbf{P}\{\tilde{X}_t(q) = X_t(q) \text{ for all } t \geq 0\} = 1.$$

Then $\{\tilde{X}_t(q) : t \geq 0\}$ is also càdlàg and solves (1) for every $q \in [0,1]$.

5. Scaling limits of the discrete branching flows

In this section, we prove some limit theorems for the discrete state branching flows, which will lead to the continuous state branching flows of Li (2012). We shall present the limit theorems in the settings of measure-valued processes and path-valued processes.

Suppose that for each $k \geq 1$, there is a positive constant σ_k and a family of generating functions $\{g_\theta^{(k)} : \theta \geq 0\}$ satisfying the assumptions specified at the beginning of the last section. Then we can define $\pi_\theta^{(k)}(dz)$ and $\bar{\pi}^{(k)}(dz, d\theta)$ in the same way as there. Let $\{X_t^{(k)}(q) : t \geq 0\}$ be the corresponding solution of (1) and $\{\tilde{X}_t^{(k)}(q) : t \geq 0, q \in [0,k]\}$ be defined in the same way as in (4). Define

$$Y_t^{(k)}(q) = \frac{1}{k}\tilde{X}_t^{(k)}(kq), \qquad q \in [0,1]. \tag{1}$$

From (1) we have

$$\begin{aligned} Y_t^{(k)}(q) \; &= \; Y_0^{(k)}(q) + \frac{1}{k}\int_0^t \int_{\mathbb{N}_+} \int_{[0,kq]} \int_0^{kY_{s-}^{(k)}(q)} (z-1) N(ds, dz, d\theta, du) \\ &\quad -\frac{1}{k}\int_0^t \int_0^{b_k(kq)} \int_0^{kY_{s-}^{(k)}(q)} N_0(ds, d\theta, du). \end{aligned} \tag{2}$$

One can use a standard stopping time argument to show that for any $q \in [0,1]$, the function $t \mapsto \mathbf{E}[Y_t^{(k)}(q)]$ is locally bounded. Then by an argument similar to the proof of Proposition 3.2 we have

Proposition 5.1. *For any $t \geq 0$ and $q \in [0,1]$, we have*

$$\mathbf{E}\Big[\sup_{0 \leq s \leq t} Y_s^{(k)}(q)\Big] \leq Y_0^{(k)}(q) \exp\Big\{t\sigma_k\Big((g_{kq}^{(k)})'(1) - 1 + b_k(kq)\Big)\Big\}. \tag{3}$$

The random function $Y_t^{(k)} \in F[0,1]$ induces a random measure $Y_t^{(k)} \in M[0,1]$ so that $Y_t^{(k)}([0,q]) = Y_t^{(k)}(q)$ for $q \in [0,1]$. We are interested in the asymptotic behavior of $\{Y_t^{(k)} : t \geq 0\}$ as $k \to \infty$. For any $f \in C^1[0,1]$ one can use Fubini's theorem to see

$$\langle Y_t^{(k)}, f\rangle = f(1)Y_t^{(k)}(1) - \int_0^1 f'(q)Y_t^{(k)}(q)dq. \tag{4}$$

Fix an integer $n \geq 1$ and let $q_i = i/2^n$ for $i = 0, 1, \cdots, 2^n$. By (2) we have

$$\begin{aligned}
\sum_{i=1}^{2^n} f'(q_i)Y_t^{(k)}(q_i) &= \sum_{i=1}^{2^n} f'(q_i)Y_0^{(k)}(q_i) \\
&\quad + \frac{1}{k}\sum_{i=1}^{2^n} f(q_i)\int_0^t\int_{\mathbb{N}_+}\int_{[0,kq_i]}\int_0^{kY_{s-}^{(k)}(q_i)} (z-1)N(ds,dz,d\theta,du) \\
&\quad - \frac{1}{k}\sum_{i=1}^{2^n} f'(q_i)\int_0^t\int_0^{b_k(kq_i)}\int_0^{kY_{s-}^{(k)}(q_i)} N_0(ds,d\theta,du) \\
&= \sum_{i=1}^{2^n} f'(q_i)Y_0^{(k)}(q_i) \\
&\quad + \frac{1}{k}\int_0^t\int_{\mathbb{N}_+}\int_{[0,k]}\int_0^{kY_{s-}^{(k)}(1)} F_n^{(k)}(s,\theta,u)(z-1)N(ds,dz,d\theta,du) \\
&\quad - \frac{1}{k}\int_0^t\int_0^{b_k(0)}\int_0^{kY_{s-}^{(k)}(1)} \tilde{F}_n^{(k)}(s,\theta,u)N_0(ds,d\theta,du),
\end{aligned} \tag{5}$$

where

$$F_n^{(k)}(s,\theta,u) = \sum_{i=1}^{2^n} f'(q_i)1_{\{\theta\leq kq_i\}}1_{\{u\leq kY_{s-}^{(k)}(q_i)\}}$$

and

$$\tilde{F}_n^{(k)}(s,\theta,u) = \sum_{i=1}^{2^n} f'(q_i)1_{\{\theta\leq b_k(kq_i)\}}1_{\{u\leq kY_{s-}^{(k)}(q_i)\}}.$$

By the right continuity of $q \mapsto Y_t^{(k)}(q)$ it is easy to see that, as $n \to \infty$,

$$2^{-n}F_n^{(k)}(s,\theta,u) \to F^{(k)}(s,\theta,u) := \int_0^1 f'(q)1_{\{\theta\leq kq\}}1_{\{u\leq kY_{s-}^{(k)}(q)\}}dq$$

and

$$2^{-n}\tilde{F}_n^{(k)}(s,\theta,u) \to \tilde{F}^{(k)}(s,\theta,u) := \int_0^1 f'(q)1_{\{\theta\leq b_k(kq)\}}1_{\{u\leq kY_{s-}^{(k)}(q)\}}dq.$$

Then by (5) we have, almost surely,

$$\begin{aligned}
\int_0^1 f'(q)Y_t^{(k)}(q)dq &= \int_0^1 f'(q)Y_0^{(k)}(q)dq \\
&\quad + \frac{1}{k}\int_0^t\int_{\mathbb{N}_+}\int_{[0,k]}\int_0^{kY_{s-}^{(k)}(1)} F^{(k)}(s,\theta,u)(z-1)N(ds,dz,d\theta,du)
\end{aligned}$$

$$-\frac{1}{k}\int_0^t \int_0^{b_k(0)} \int_0^{kY_{s-}^{(k)}(1)} \tilde{F}^{(k)}(s,\theta,u)N_0(ds,d\theta,du). \tag{6}$$

From (2), (4) and (6) it follows that, almost surely,

$$\begin{aligned}
\langle Y_t^{(k)}, f\rangle \;=\; & \langle Y_0^{(k)}, f\rangle \\
& +\frac{1}{k}\int_0^t \int_{\mathbb{N}_+}\int_{[0,k]}\int_0^{kY_{s-}^{(k)}(1)} [f(1)-F^{(k)}(s,\theta,u)](z-1)N(ds,dz,d\theta,du) \\
& -\frac{1}{k}\int_0^t \int_0^{b_k(k)}\int_0^{kY_{s-}^{(k)}(1)} [f(1)-\tilde{F}^{(k)}(s,\theta,u)]N_0(ds,d\theta,du) \\
& +\frac{1}{k}\int_0^t \int_{b_k(k)}^{b_k(0)}\int_0^{kY_{s-}^{(k)}(1)} \tilde{F}^{(k)}(s,\theta,u)N_0(ds,d\theta,du).
\end{aligned} \tag{7}$$

Proposition 5.2. *Suppose that $Y_0^{(k)}(1)$ converges to some $Y_0(1)$ as $k\to\infty$ and*

$$\sup_{k\geq 1}\sigma_k\Big[(g_k^{(k)})'(1)-1+b_k(0)\Big]<\infty.$$

Then $\{Y_t^{(k)}: t\geq 0\}$, $k=1,2,\cdots$ is a tight sequence in $D([0,\infty),M[0,1])$.

Proof. For any $t\geq 0$ and $f\in C[0,1]$, by Proposition 5.1 it is easy to see that

$$t\mapsto C_t := \sup_{k\geq 1}\mathbf{E}\Big[\sup_{0\leq s\leq t}\langle Y_s^{(k)}, f\rangle\Big]$$

is locally bounded. Then for every fixed $t\geq 0$, the sequence $\langle Y_t^{(k)}, f\rangle$ is tight. Let τ_k be a bounded stopping time for $\{Y_t^{(k)}: t\geq 0\}$ and assume the sequence $\{\tau_k: k=1,2,\cdots\}$ is bounded above by $T\geq 0$. Let $f\in C^1[0,1]$. By (7) we see

$$\begin{aligned}
&\mathbf{E}\Big[\Big|\langle Y_{\tau_k+t}^{(k)}, f\rangle-\langle Y_{\tau_k}^{(k)}, f\rangle\Big|\Big] \\
&\quad\leq \frac{\sigma_k}{k}\mathbf{E}\Big[\int_0^t ds\int_{\mathbb{N}_+}\int_{[0,k]}\int_0^{kY_{s+\tau_k}^{(k)}(1)}(z-1)|f(1)-F^{(k)}(s+\tau_k,\theta,u)|\bar{\pi}^{(k)}(dz,d\theta)du\Big] \\
&\qquad+\frac{\sigma_k}{k}\mathbf{E}\Big[\int_0^t ds\int_0^{b_k(k)}d\theta\int_0^{kY_{s+\tau_k}^{(k)}(1)}|f(1)-\tilde{F}^{(k)}(s+\tau_k,\theta,u)|du\Big] \\
&\qquad+\frac{\sigma_k}{k}\mathbf{E}\Big[\int_0^t ds\int_{b_k(k)}^{b_k(0)}d\theta\int_0^{kY_{s+\tau_k}^{(k)}(1)}|\tilde{F}^{(k)}(s+\tau_k,\theta,u)|du\Big].
\end{aligned} \tag{8}$$

For $s,\theta,u>0$ let $Y_{s,k}^{-1}(u)=\inf\{q\geq 0: Y_s^{(k)}(q)>u\}$ and $b_k^{-1}(u)=\inf\{q\geq 0: b_k(q)>u\}$. It is easy to see that $\{q\geq 0: u\leq kY_s^{(k)}(q)\}=[Y_{s,k}^{-1}(u/k),\infty)$ and

$\{q \geq 0 : \theta \leq b_k(kq)\} = [0, b_k^{-1}(\theta)/k]$ except for at most countably many $u > 0$ and $\theta > 0$, respectively. Then in the above we can replace $f(1) - F^{(k)}(s, \theta, u)$ by

$$f(1) - \int_{\theta/k}^{1} f'(q) 1_{\{Y_{s,k}^{-1}(u/k) \leq q\}} dq = f\Big(Y_{s,k}^{-1}(\frac{u}{k}) \vee \frac{\theta}{k}\Big)$$

and $\tilde{F}^{(k)}(s, \theta, u)$ can be replaced by

$$\begin{aligned}&\int_0^1 f'(q) 1_{\{q \leq b_k^{-1}(\theta)/k\}} 1_{\{Y_{s,k}^{-1}(u/k) \leq q\}} dq \\ &\qquad = \Big[f(1 \wedge (b_k^{-1}(\theta)/k)) - f(Y_{s,k}^{-1}(u/k))\Big] 1_{\{Y_{s,k}^{-1}(u/k) \leq b_k^{-1}(\theta)/k\}}.\end{aligned}$$

Then from (8) we have

$$\begin{aligned}&\mathbf{E}\Big[\Big|\langle Y_{\tau_k+t}^{(k)}, f\rangle - \langle Y_{\tau_k}^{(k)}, f\rangle\Big|\Big] \\ &\quad \leq \sigma_k \mathbf{E}\Big[\int_0^t ds \int_0^1 Y_{s+\tau_k}^{(k)}(dx) \int_{\mathbb{N}_+} \int_{[0,k]} (z-1)|f(x \vee \theta)| \bar{\pi}^{(k)}(dz, d\theta)\Big] \\ &\quad\quad + \sigma_k \mathbf{E}\Big[\int_0^t ds \int_0^{b_k(k)} d\theta \int_0^1 |f(x)| Y_{s+\tau_k}^{(k)}(dx)\Big] \\ &\quad\quad + \sigma_k \mathbf{E}\Big[\int_0^t ds \int_{b_k(k)}^{b_k(0)} d\theta \int_0^1 |f(b_k^{-1}(\theta)/k) - f(x)| Y_{s+\tau_k}^{(k)}(dx)\Big] \\ &\quad \leq \|f\| \sigma_k \int_0^t \mathbf{E}\Big[Y_{s+\tau_k}^{(k)}(1)\Big] ds \int_{\mathbb{N}_+} (z-1) \pi_k^{(k)}(dz) \\ &\quad\quad + \|f\| \sigma_k b_k(k) \mathbf{E} \int_0^t \Big[Y_{s+\tau_k}^{(k)}(1)\Big] ds \\ &\quad\quad + 2\|f\| \sigma_k [b_k(0) - b_k(k)] \int_0^t \mathbf{E}\Big[Y_{s+\tau_k}^{(k)}(1)\Big] ds \\ &\quad \leq \|f\| \sigma_k \Big((g_k^{(k)})'(1) - 1 + 2b_k(0)\Big) \int_0^t \mathbf{E}\Big[Y_{s+\tau_k}^{(k)}(1)\Big] ds \\ &\quad \leq 2\|f\| Y_0^{(k)}(1) t \sigma_k A_k \exp\Big\{\sigma_k A_k (t+T)\Big\},\end{aligned} \tag{9}$$

where $A_k = (g_k^{(k)})'(1) - 1 + b_k(0)$ and the last inequality follows by Proposition 5.1. For $f \in C[0, 1]$ the above inequality follows by an approximation argument. Then we have

$$\lim_{t \to 0} \sup_{k \geq 1} \mathbf{E}\Big[\Big|\langle Y_{\tau_k+t}^{(k)}, f\rangle - \langle Y_{\tau_k}^{(k)}, f\rangle\Big|\Big] = 0.$$

By a criterion of Aldous (1978), the sequence $\{\langle Y_t^{(k)}, f\rangle : t \geq 0\}$ is tight in $D([0, \infty), \mathbb{R})$; see also Ethier and Kurtz (1986, pp.137-138). Then the tightness criterion of Roelly (1986) implies $\{Y_t^{(k)} : t \geq 0\}$ is tight in $D([0, \infty), M[0, 1])$. $\square$

For any $z \geq 0$ define

$$\phi_\theta^{(k)}(z) = k\sigma_k \Big[g_{k\theta}^{(k)}(e^{-z/k}) - e^{-z/k} \Big]. \tag{10}$$

Let us consider the following condition:

Condition (5.A) *For each $l \geq 0$ the sequence $\{\phi_\theta^{(k)}(z)\}$ is Lipschitz with respect to z uniformly on $[0,1] \times [0,l]$ and there is an admissible family of branching mechanisms $\{\phi_\theta(z) : \theta \geq 0\}$ with $(\partial/\partial\theta)\phi_\theta(z) = -\psi_\theta(z)$ such that $\phi_\theta^{(k)}(z) \to \phi_\theta(z)$ uniformly on $[0,1] \times [0,l]$ as $k \to \infty$.*

Let $\{Y_t : t \geq 0\}$ be the càdlàg superprocess with transition semigroup defined by (6) and (7).

Theorem 4. *Suppose that Condition (5.A) holds and $\sup_{k\geq 1} \sigma_k b_k(0) < \infty$. If $Y_0^{(k)}$ converges weakly to $Y_0 \in M[0,1]$, then $\{Y_t^{(k)} : t \geq 0\}$ converges to the superprocess $\{Y_t : t \geq 0\}$ in distribution on $D([0,\infty), M[0,1])$.*

Proof. Under the assumption, we have

$$\sup_{k\geq 1} \sigma_k \Big[(g_k^{(k)})'(1) - 1 + b_k(0) \Big] < \infty.$$

By Proposition 5.2 and Skorokhod's representation theorem, to simplify the notation we pass to a subsequence and simply assume $\{Y_t^{(k)} : t \geq 0\}$ converges a.s. to a process $\{Z_t : t \geq 0\}$ in the topology of $D([0,\infty), M[0,1])$. Since the solution of the martingale problem (4) is unique, it suffices to prove the weak limit point $\{Z_t : t \geq 0\}$ of the sequence $\{Y_t^{(k)} : t \geq 0\}$ is the solution of the martingale problem. Let $Y_{s,k}^{-1}(u)$ and $b_k^{-1}(u)$ be defined as in Proposition 5.2. For every $G \in C^2(\mathbb{R})$ and $f \in C^1[0,1]$ we use (7) and Itô's formula to get

$$\begin{aligned} G(\langle Y_t^{(k)}, f\rangle) \;=\; & G(\langle Y_0^{(k)}, f\rangle) + \sigma_k \int_0^t ds \int_{\mathbb{N}_+} \int_{[0,k]} \int_0^{kY_{s-}^{(k)}(1)} \Big\{ G\Big(\langle Y_{s-}^{(k)}, f\rangle \\ & + k^{-1}(z-1)[f(1) - F^{(k)}(s,\theta,u)] \Big) - G(\langle Y_{s-}^{(k)}, f\rangle) \Big\} \bar{\pi}^{(k)}(dz, d\theta) du \\ & + \sigma_k \int_0^t ds \int_0^{b_k(k)} d\theta \int_0^{kY_{s-}^{(k)}(1)} \Big\{ G\Big(\langle Y_{s-}^{(k)}, f\rangle - k^{-1}[f(1) - \tilde{F}^{(k)}(s,\theta,u)] \Big) \\ & - G(\langle Y_{s-}^{(k)}, f\rangle) \Big\} du + \sigma_k \int_0^t ds \int_{b_k(k)}^{b_k(0)} d\theta \int_0^{kY_{s-}^{(k)}(1)} \Big\{ G\Big(\langle Y_{s-}^{(k)}, f\rangle \\ & + k^{-1}\tilde{F}^{(k)}(s,\theta,u) \Big) - \; G(\langle Y_{s-}^{(k)}, f\rangle) \Big\} du + \text{local mart.} \end{aligned}$$

$$
\begin{aligned}
&= G(\langle Y_0^{(k)}, f\rangle) + \sigma_k \int_0^t ds \int_{\mathbb{N}_+} \int_{[0,k]} \int_0^{kY_{s-}^{(k)}(1)} \Big\{ G\Big(\langle Y_{s-}^{(k)}, f\rangle \\
&\quad + k^{-1}(z-1) f(Y_{s,k}^{-1}(u/k) \vee (\theta/k))\Big) - G(\langle Y_{s-}^{(k)}, f\rangle)\Big\} \bar{\pi}^{(k)}(dz, d\theta) du \\
&\quad + \sigma_k \int_0^t ds \int_0^{b_k(k)} d\theta \int_0^{kY_{s-}^{(k)}(1)} \Big\{ G\Big(\langle Y_{s-}^{(k)}, f\rangle - k^{-1} f(Y_{s,k}^{-1}(u/k))\Big) \\
&\quad - G(\langle Y_{s-}^{(k)}, f\rangle)\Big\} du + \sigma_k \int_0^t ds \int_{b_k(k)}^{b_k(0)} d\theta \int_0^{kY_{s-}^{(k)}(1)} \Big\{ G\Big(\langle Y_{s-}^{(k)}, f\rangle \\
&\quad + k^{-1}[f(b_k^{-1}(\theta)/k) - f(Y_{s,k}^{-1}(u/k))] 1_{\{Y_{s,k}^{-1}(u/k) \le b_k^{-1}(\theta)/k\}}\Big) \\
&\quad - G(\langle Y_{s-}^{(k)}, f\rangle)\Big\} du + \text{local mart.} \\
&= G(\langle Y_0^{(k)}, f\rangle) + k\sigma_k \int_0^t ds \int_{[0,1]} Y_{s-}^{(k)}(dx) \int_{\mathbb{N}_+} \int_{[0,1]} \Big\{ G\Big(\langle Y_{s-}^{(k)}, f\rangle \\
&\quad + k^{-1}(z-1) f(x \vee \theta)\Big) - G(\langle Y_{s-}^{(k)}, f\rangle)\Big\} \bar{\pi}^{(k)}(dz, kd\theta) \\
&\quad + k\sigma_k b_k(k) \int_0^t ds \int_{[0,1]} \Big\{ G\Big(\langle Y_{s-}^{(k)}, f\rangle - k^{-1} f(x)\Big) - G(\langle Y_{s-}^{(k)}, f\rangle)\Big\} Y_{s-}^{(k)}(dx) \\
&\quad + k\sigma_k \int_0^t ds \int_{[0,1]} Y_{s-}^{(k)}(dx) \int_1^x \Big\{ G\Big(\langle Y_{s-}^{(k)}, f\rangle + k^{-1}[f(\theta) - f(x)]\Big) \\
&\quad - G(\langle Y_{s-}^{(k)}, f\rangle)\Big\} b_k(kd\theta) + \text{local mart.} \\
&= G(\langle Y_0^{(k)}, f\rangle) + k\sigma_k \int_0^t ds \int_{[0,1]} Y_{s-}^{(k)}(dx) \int_{\mathbb{N}} \int_{[0,1]} \Big\{ G\Big(\langle Y_{s-}^{(k)}, f\rangle \\
&\quad + k^{-1}(z-1) f(x \vee \theta)\Big) - G(\langle Y_{s-}^{(k)}, f\rangle)\Big\} \bar{\pi}^{(k)}(dz, kd\theta) \\
&\quad + k\sigma_k \int_0^t ds \int_{[0,1]} Y_{s-}^{(k)}(dx) \int_{\{0\}} \int_x^1 \epsilon_k(s, x, \theta) \bar{\pi}^{(k)}(dz, kd\theta) \\
&\quad + \text{local mart.},
\end{aligned} \tag{11}
$$

where

$$
\begin{aligned}
\epsilon_k(s, \theta, x) &= \Big\{ G\Big(\langle Y_{s-}^{(k)}, f\rangle - k^{-1} f(x)\Big) - G\Big(\langle Y_{s-}^{(k)}, f\rangle - k^{-1} f(\theta)\Big)\Big\} \\
&\quad - \Big\{ G\Big(\langle Y_{s-}^{(k)}, f\rangle + k^{-1}[f(\theta) - f(x)]\Big) - G(\langle Y_{s-}^{(k)}, f\rangle)\Big\}.
\end{aligned}
$$

It is elementary to see that

$$
k\sigma_k \int_{\{0\}} \int_x^1 \epsilon_k(s, x, \theta) \bar{\pi}^{(k)}(dz, kd\theta)
$$

tends to zero uniformly as $k \to \infty$. Let $G(x) = e^{-x}$, by letting $k \to \infty$ in (11) we get (4) for $f \in C^1[0,1]$. A simple approximation shows the martingale problem (4) actually holds for any $f \in C[0,1]$. By the proof of Theorem 7.13 in Li (2011) we get the result. $\square$

Let $\{0 \leq a_1 < a_2 < \cdots < a_n = 1\}$ be an ordered set of constants. Denote by $\{Y_{t,a_i} : t \geq 0\}$ and $\{Y^{(k)}_{t,a_i} : t \geq 0\}$ the restriction of $\{Y_t : t \geq 0\}$ and $\{Y^{(k)}_t : t \geq 0\}$ to $[0, a_i]$, respectively. Let $Y_t(a_i) := Y_t[0, a_i]$ and $Y^{(k)}_t(a_i) := Y^{(k)}_t[0, a_i]$ for every $t \geq 0$, $i = 1, 2, \cdots, n$. By arguments similar to those in He and Ma (2012) we get following results.

Theorem 5. *Suppose that Condition (5.A) is satisfied and* $\sup_{k\geq 1} \sigma_k b_k(0) < \infty$. *If* $Y^{(k)}_0$ *converges weakly to* $Y_0 \in M[0, 1]$, *then* $\{(Y^{(k)}_{t,a_1}, \cdots, Y^{(k)}_{t,a_n}) : t \geq 0\}$ *converges to* $\{(Y_{t,a_1}, \cdots, Y_{t,a_n}) : t \geq 0\}$ *in distribution on* $D([0, \infty), M[0, a_1] \times \cdots \times M[0, a_n])$.

Corollary 1. *Suppose that Condition (5.A) is satisfied and* $\sup_{k\geq 1} \sigma_k b_k(0) < \infty$. *If* $(Y^{(k)}_0(a_1), \cdots, Y^{(k)}_0(a_n))$ *converges to* $(Y_0(a_1), \cdots, Y_0(a_n))$, *then* $\{(Y^{(k)}_t(a_1), \cdots, Y^{(k)}_t(a_n)) : t \geq 0\}$ *converges to* $\{(Y_t(a_1), \cdots, Y_t(a_n)) : t \geq 0\}$ *in distribution on* $D([0, \infty), \mathbb{R}^n_+)$.

Example 1. Suppose that ϕ is defined in (1). Let Θ_ϕ be the set of $\theta \geq 0$ such that

$$\int_1^\infty u e^{\theta u} m(du) < \infty.$$

Then a particular choice of ϕ_θ is

$$\phi_\theta(\cdot) = \phi(\cdot - \theta) - \phi(-\theta), \quad \theta \in \Theta_\phi.$$

Suppose $[0, 1] \subset \Theta_\phi$. Let $\{g^{(k)}_k : k \geq 1\}$ be a sequence of generating functions such that

$$k\sigma_k[g^{(k)}_k(e^{-z/k}) - e^{-z/k}] \to \phi(z - 1) - \phi(-1), \qquad k \to \infty. \tag{12}$$

Since $\phi_1(\cdot)$ is a branching mechanism, then by Li (2011, p.93) (12) holds for some σ_k and $g^{(k)}_k$. Then for $\theta \in [0, 1]$, define

$$g^{(k)}_{k\theta}(z) = 1 - e^{\frac{1-\theta}{k}} g^{(k)}_k(e^{-\frac{1-\theta}{k}}) + e^{\frac{1-\theta}{k}} g^{(k)}_k(e^{-\frac{1-\theta}{k}} z).$$

Then one could check that Condition (5.A) holds with $\phi_\theta(\cdot) = \phi(\cdot - \theta) - \phi(-\theta)$. In fact, if $g^{(k)}_k$ corresponds to a probability measure $\{p^{(k)}_i : i \geq 0\}$, then for each $\theta \in [0, 1]$, $g^{(k)}_{k\theta}$ is the generating function of probability measure

$$p^{(k)}_i(\theta) = p^{(k)}_i e^{-\frac{(1-\theta)(i-1)}{k}}, \quad i \geq 1,$$

and $p^{(k)}_0(\theta) = 1 - \sum_{i\geq 1} p^{(k)}_i(\theta)$.

Acknowledgements

We would like to give our sincere thanks to Professor Zenghu Li for his encouragement and helpful discussions. The authors also wish to thank an anonymous referee for his/her remarks that improved significantly the presentation of the paper. The work is supported by NSFC (No.11126037, 11201030, 11071021) and Ministry of Educatioin (985 Project).

References

[1] Abraham, R. and Delmas, J.-F. (2012). A continuum tree-valued Markov process. *Ann. Probab.* **40**, 1167-1211.

[2] Abraham, R., Delmas, J.-F. and He, H.(2012). Pruning Galton-Watson Trees and Tree-valued Markov Processes. *Annales de l'Institut Henri Poincare: Probabilites et Statistiques* **48**, 688-705.

[3] Aldous, D. (1978). Stopping times and tightness. *Ann. Probab.* **6**, 335–340.

[4] Aldous, D. (1991). The continuum random tree I. *Ann. Probab.* **19**, 1–28.

[5] Aldous, D. (1993). The continuum random tree III. *Ann. Probab.* **21**, 248–289.

[6] Aldous, D. and Pitman, J. (1998). Tree-valued Markov chains derived from Galton-Watson processes. *Ann. Inst. H. Poincaré Probab. Statist.* **34**, 637–686.

[7] Bakhtin, Y. (2011). SPDE approximation for random trees. *Markov Process. Related Fields* **17**, 1–36.

[8] Bertoin, J. and Le Gall, J.-F. (2006). Stochastic flows associated to coalescent processes III. Limit theorems. *Illinois J. Math.* **50**, 147–181.

[9] Dawson, D.A. and Li, Z. (2006). Skew convolution semigroups and affine Markov processes. *Ann. Probab.* **34**, 1103–1142.

[10] Dawson, D.A. and Li, Z. (2012). Stochastic equations, flows and measure-valued processes. *Ann. Probab.* **40**, 813-857.

[11] Duquesne, T. and Le Gall, J.-F. (2002). *Random Trees, Lévy Processes and Spatial Branching Processes.* Astérisque **281**.

[12] Ethier, S.N. and Kurtz, T.G. (1986). *Markov Processes: Characterization and Convergence.* Wiley, New York.

[13] Fu, Z. and Li, Z. (2010). Stochastic equations of non-negative processes with jumps. *Stochastic Process. Appl.* **120**, 306-330.

[14] He, H. and Ma, R. (2012). Some limit theorems for flows of branching processes. *http://arxiv.org/abs/1204.1248*.

[15] Ikeda, N. and Watanabe, S. (1989). *Stochastic Differential Equations and Diffusion Processes*. Second Edition. North-Holland/Kodasha, Amsterdam/Tokyo.

[16] Kawazu, K. and Watanabe, S. (1971). Branching processes with immigration and related limit theorems. *Theory Probab. Appl.* **16**, 36-54.

[17] Lamperti, J. (1967). The limit of a sequence of branching processes. *Z. Wahrsch. verw. Geb.* **7**, 271-288.

[18] Le Gall, J.-F. and Le Jan, Y. (1998a). Branching processes in Lévy processes. The exploration process. *Ann. Probab.* **26**, 213-252.

[19] Le Gall, J.-F. and Le Jan, Y. (1998b). Branching processes in Lévy processes. Laplace functionals of snakes and superprocesses. *Ann. Probab.* **26**, 1407-1432.

[20] Li, Z. (2006). A limit theorem for discrete Galton-Watson branching processes with immigration. *J. Appl. Probab.* **43**, 289–295.

[21] Li, Z. (2011). *Measure-Valued Branching Markov Processes*. Springer, Berlin.

[22] Li, Z. (2012). Path-valued branching processes and nonlocal branching superprocesses. *Ann. Probab.* To appear.

[23] Li, Z. and Ma, C. (2008). Catalytic discrete state branching models and related limit theorems. *J. Theoret. Probab.* **21**, 936–965.

[24] Roelly, S. (1986). A criterion of convergence of measure-valued processes. Application to measure branching processes. *Stochastics* **17**, 43–65.