A complex version of Theorem 2.5.18 on p. 142

We first prove the following **Proposition**: **Let $\mathcal{X}$ be a normed linear space, and M be any subset of $\mathcal{X}$, $x_0 \in \mathcal{X}$, then $x_0 \in \overline{co(M)}$ if and only if for any $f \in \mathcal{X}^*$ and any $\alpha \in \mathbb{R}$, if for any $x \in M$, $Ref(x) \leq \alpha$, then $Ref(x_0) \leq \alpha$.**

Proof: " $\Rightarrow$ ". Suppose $x_0 \in \overline{co(M)}$, then there exists a sequence $\{y_n\} \subset co(M)$ such that $y_n \to x_0$. Let $y_n = \sum_{l=1}^{k_n} \lambda_l^{(n)} \xi_l^{(n)}$, where $\xi_l^{(n)} \in M$, $\sum_{l=1}^{k_n} \lambda_l^{(n)} = 1$ and $0 \leq \lambda_l^{(n)} \leq 1$. Then

$$\begin{aligned} f(y_n) &= f\left(\sum_{l=1}^{k_n} \lambda_l^{(n)} \xi_l^{(n)}\right) \\ &= \sum_{l=1}^{k_n} \lambda_l^{(n)} f\left(\xi_l^{(n)}\right) \\ &= \sum_{l=1}^{k_n} \lambda_l^{(n)} Ref\left(\xi_l^{(n)}\right) \\ &\quad + i \sum_{l=1}^{k_n} \lambda_l^{(n)} Imf\left(\xi_l^{(n)}\right). \end{aligned}$$

For any $f \in \mathcal{X}^*$ and any $\alpha \in \mathbb{R}$, if $Ref(x) \leq \alpha$ for any $x \in M$, then

$$Ref(x_0) = \lim_{n\to\infty} Ref(y_n) = \lim_{n\to\infty} \sum_{l=1}^{k_n} \lambda_l^{(n)} Ref\left(\xi_l^{(n)}\right) \leq \alpha.$$

" $\Leftarrow$ ". For any $f \in \mathcal{X}^*$ and any $\alpha \in \mathbb{R}$, if for any $x \in M$, $Ref(x) \leq \alpha$, then $Ref(x_0) \leq \alpha$. We now show that $x_0 \in \overline{co(M)}$. If $x_0 \notin \overline{co(M)}$, we first regard $\mathcal{X}$ as a real linear normed space and by Corollary 2.4.16 on p. 117, we then have that there exists $g \in \mathcal{X}^*$, $\alpha \in \mathbb{R}$, such that

$$g(x) < \alpha < g(x_0), \forall x \in \overline{co(M)}.$$

Let $f(x) = g(x) - ig(ix)$, then $f(ix) = g(ix) - ig(-x) = i(g(x) - ig(ix)) = if(x)$. In other words, f is complex linear, and for all $x \in \mathcal{X}$,

$$\begin{aligned} |f(x)| = \sqrt{g(x)^2 + g(ix)^2} &\leq \sqrt{\|g\|^2\|x\|^2 + \|g\|^2\|ix\|^2} \\ &= \sqrt{2}\|g\|\|x\|. \end{aligned}$$

Thus $f \in \mathcal{X}^*$, that is, f is a bounded linear functional on the complex linear space $\mathcal{X}$, and for any $x \in M$, $Ref(x) = g(x) < \alpha$, but, $Ref(x_0) = g(x_0) > \alpha$ which contracts to $Ref(x_0) \leq \alpha$. Thus $x_0 \in \overline{co(M)}$, and we complete the proof of the proposition.

We now verify that Theorem 2.5.18 (Mazur) also holds for complex B^* space:

Let $M \triangleq \overline{co(\{x_n\}_{n\in\mathbb{N}})}$. Then M is a closed convex set in $\mathcal{X}$. To show $x_0 \in M$, by the proposition above, it suffices to show that for any $f \in \mathcal{X}^*$, and any $\alpha \in \mathbb{R}$, if for any $n \in \mathbb{N}$, $Ref(x_n) \leq \alpha$, then $Ref(x_0) \leq \alpha$. In fact, since $x_n \rightharpoonup x_0$, so $f(x_n) \to f(x_0)$ as $n \to \infty$. Therefore, $Ref(x_n) \to Ref(x_0)$, which implies that $Ref(x_0) \leq \alpha$. Thus we have $x_0 \in M$ and complete the proof.