Proof of Proposition 1.6.13 on p. 57

Necessity:

$$\begin{aligned}
&\|x+y\|^2 + \|x-y\|^2 \\
&\quad = (x+y, x+y) + (x-y, x-y) \\
&\quad = (x,x) + (y,y) + (x,y) + (y,x) \\
&\qquad + (x,x) + (y,y) - (y,x) - (x,y) \\
&\quad = 2\{(x,x) + (y,y)\} = 2(\|x\|^2 + \|y\|^2).
\end{aligned}$$

Sufficiency: If $\mathbb{K} = \mathbb{R}$, define

$$(*) \qquad (x,y)_1 = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right), \ x, \ y \in \mathcal{X}.$$

We next verify $(\cdot,\cdot)_1$ is surely an inner product. In fact, it is clear that $(x,y)_1 = (y,x)_1$. By $(*)$, $(0,z)_1 = 0$. Furthermore, it follows by (1.6.8) and $(*)$ that

$(**)$

$$\begin{aligned}
&(x,z)_1 + (y,z)_1 \\
&\quad = \frac{1}{4}\left(\|x+z\|^2 - \|x-z\|^2 + \|y+z\|^2 - \|y-z\|^2\right) \\
&\quad = \frac{1}{4}\left\{\left(\left\|\frac{x+y}{2} + z + \frac{x-y}{2}\right\|^2 + \left\|\frac{x+y}{2} + z - \frac{x-y}{2}\right\|^2\right)\right. \\
&\qquad \left. - \left(\left\|\frac{x+y}{2} - z + \frac{x-y}{2}\right\|^2 + \left\|\frac{x+y}{2} - z - \frac{x-y}{2}\right\|^2\right)\right\} \\
&\quad = \frac{1}{4}\left\{2\left(\left\|\frac{x+y}{2} + z\right\|^2 + \left\|\frac{x-y}{2}\right\|^2\right)\right. \\
&\qquad \left. -2\left(\left\|\frac{x+y}{2} - z\right\|^2 + \left\|\frac{x-y}{2}\right\|^2\right)\right\}
\end{aligned}$$

$$= \frac{1}{2}\left(\left\|\frac{x+y}{2}+z\right\|^2 - \left\|\frac{x+y}{2}-z\right\|^2\right)$$
$$= 2\left(\frac{x+y}{2}, z\right)_1 .$$

In $(**)$, taking $y = 0$, we have

$$(x, z)_1 = 2\left(\frac{x}{2}, z\right)_1 .$$

It then follows by substituting $x+y$ for x in the last equation and making use of $(**)$ that

$$(***) \qquad (x+y, z)_1 = 2\left(\frac{x+y}{2}, z\right)_1 = (x, z)_1 + (y, z)_1.$$

For given x, $z \in \mathcal{X}$, define

$$f(t) = (tx, z)_1$$

for $t \in \mathbb{R}$. It is easy to see by $(***)$ that the function $f(t)$ satisfies the equation

$$(\star_1) \qquad f(t_1 + t_2) = f(t_1) + f(t_2), \quad t_1,\ t_2 \in \mathbb{R}.$$

On the other hand, when $t_n \to t$,

$$\Big|\|t_n x \pm z\| - \|tx \pm z\|\Big| \le \|t_n x - tx\| = |t_n - t|\|x\| \to 0,$$

which together with $(*)$ implies that $f(t)$ is continuous. However, a continuous function $f(t)$ satisfying $(\star_1)$ must have the following form (see the appendix for the proof of this fact):

$$f(t) = f(1)t.$$

Therefore for x, $z \in \mathcal{X}$ and $t \in \mathbb{R}$,

$$(tx, z)_1 = t(x, z)_1. \tag{$\star_2$}$$

Thus $(\cdot, \cdot)_1$ is an inner product. If we let $x = y$ in $(*)$, we then obtain that

$$(x, x)_1 = \|x\|^2.$$

So when $\mathcal{X}$ is a real space, $(\cdot, \cdot)_1$ is an inner product, and the norm induced by the inner product $(\cdot, \cdot)_1$ is just the originally given $\|\cdot\|$, and satisfies (1.6.7).

If $\mathcal{X}$ is a complex normed linear space, define

$$\begin{aligned}(x, y) &= \frac{1}{4}\Big(\|x + y\|^2 - \|x - y\|^2 \\ &\quad + i\,\|x + iy\|^2 - i\,\|x - iy\|^2\Big) \\ &= (x, y)_1 + i(x, iy)_1,\end{aligned} \tag{$\star_3$}$$

where $(x, y)_1$ is determined by $(*)$. Now we verify that $(\cdot, \cdot)$ is an inner product of $\mathcal{X}$. In fact, by $(***)$,

$$(x, z) + (y, z) = (x + y, z).$$

By $(\star_2)$, for any $\alpha \in \mathbb{R}$,

$$(\alpha x, y) = \alpha(x, y). \tag{$\star_4$}$$

And $(\star_3)$ shows via a simple computation that

$$(ix, y) = i(x, y) \quad (i^2 = -1),$$

from which it follows that $(\star_4)$ also holds for $\forall \alpha \in \mathbb{C}$. It is easy to see by $(\star_3)$ that $(y, x) = \overline{(x, y)}$. Consequently, $(\cdot, \cdot)$ is an inner product. If taking $y = x$ in $(\star_3)$, one also has

$$(x, x) = \|x\|^2.$$

So $(\cdot, \cdot)$ also satisfies (1.6.7). Thus $(\cdot, \cdot)$ is the required inner product.

Appendix: Assume that the function $f(t)$ $(t \in \mathbb{R})$ is continuous and satisfies the equation:

$$f(t_1 + t_2) = f(t_1) + f(t_2), \ t_1, \ t_2 \in \mathbb{R}, \tag{1}$$

then for $\forall t \in \mathbb{R}$,

$$f(t) = tf(1).$$

Proof: First, we show that for any natural number n,

$$f(nt) = nf(t), \ t \in \mathbb{R}. \tag{2}$$

It is clearly that (2) holds when $n = 1$. Supposing it also holds for n, then by (1),

$$f((n+1)t) = f(nt) + f(t) = (n+1)f(t).$$

Therefore (2) also holds for $n + 1$. By induction, (2) holds for all natural numbers n. For any positive rational number $\frac{n}{m}$, it follows by making use of (2) twice that

$$f\left(\frac{n}{m}\right) = nf\left(\frac{1}{m}\right) = \frac{n}{m}f(1).$$

In (1), taking $t_1 = t_2 = 0$, we obtain $f(0) = 0$; if we take $t_1 = -t_2 = t$, we see that

$$f(-t) = -f(t).$$

Thus, for all rational numbers t, $f(t) = tf(1)$. By the continuity of both $f(t)$ and $tf(1)$, we finally obtain that for all real numbers t,

$$f(t) = tf(1),$$

which completes the proof.