Proof of Theorem 1.4.22 on p. 34

Assume that $\{e_1, \cdots, e_n\}$ is a basis of $\mathcal{X}$. Define

$$\begin{cases} T: \ \mathcal{X} \to \mathbb{K}^n, \\ x = \displaystyle\sum_{i=1}^{n} \xi_i e_i \mapsto Tx = \xi = (\xi_1, \cdots, \xi_n), \end{cases}$$

and $\|x\|_T = |Tx| = \left(\sum_{j=1}^{n} |\xi_j|^2\right)^{1/2}$. By (1.4.13) on p. 33, there exist constants $\widetilde{C_1} > 0$ and $\widetilde{C_2} > 0$ such that for all $x \in \mathcal{X}$,

$$\widetilde{C_1}\|x\|_T \le \|x\| \le \widetilde{C_2}\|x\|_T.$$

We next prove that P is bounded on $\{x \in \mathcal{X} : \|x\|_T = 1\}$. In fact, if we let $M = \max\{P(\pm e_i), i = 1, \cdots, n\}$, then $M < \infty$. Moreover, by the assumption that $P(x) = 0 \Longleftrightarrow x = \theta$, we easily see that $M > 0$. Now, if $x \in \mathcal{X}$ satisfying $\|x\|_T = 1$, then it follows by subadditivity and positive homogeneity of P together with the Hölder inequality that

$$\begin{aligned} P(x) = P\left(\sum_{i=1}^{n} \xi_i e_i\right) &\le \sum_{i=1}^{n} |\xi_i| P\left((\operatorname{sgn} \xi_i) e_i\right) \\ &\le M \sum_{i=1}^{n} |\xi_i| \\ &\le \sqrt{n} M \|x\|_T = \sqrt{n} M \triangleq M_0. \end{aligned}$$

Therefore, $P(x)$ is bounded on $\{x \in \mathcal{X} : \|x\|_T = 1\}$ with bound M_0. For $\xi \in \mathbb{K}^n$, setting $\widetilde{P}(\xi) = P(T^{-1}\xi)$, then if $\xi \neq \eta$, we have

$$\left|\widetilde{P}(\xi) - \widetilde{P}(\eta)\right|$$

$$
\begin{aligned}
&= \left|P\left(T^{-1}(\xi)\right) - P\left(T^{-1}(\eta)\right)\right| \\
&\overset{(1)}{\leq} \max\left\{P\left(T^{-1}(\xi-\eta)\right),\ P\left(T^{-1}(\eta-\xi)\right)\right\} \\
&\overset{(2)}{=} |\xi-\eta| \max\left\{P\left(T^{-1}\left(\frac{\xi-\eta}{|\xi-\eta|}\right)\right),\right. \\
&\qquad \left. P\left(T^{-1}\left(\frac{\eta-\xi}{|\eta-\xi|}\right)\right)\right\} \\
&\leq M_0|\xi-\eta|.
\end{aligned}
$$

It follows that $\widetilde{P}(\xi)$ is continuous and then has maximum and minimum on $\{\xi \in \mathbb{K}^n : |\xi| = 1\}$, which are both accessible, since $\{\xi \in \mathbb{K}^n : |\xi| = 1\}$ is compact. Let $\widetilde{P}(\xi_0) = \min_{\{\xi\in\mathbb{K}^n:\ |\xi|=1\}} \widetilde{P}(\xi)$. If $\widetilde{P}(\xi_0) = 0$, then

$$
P\left(T^{-1}(\xi_0)\right) = 0 \Leftrightarrow T^{-1}(\xi_0) = 0 \Leftrightarrow \xi_0 = 0.
$$

However, $|\xi_0| = 1$. This contradiction indicates that there exist constants C_1', $C_2' > 0$ such that for all ξ with $|\xi| = 1$,

$$
C_1' \leq \widetilde{P}(\xi) \leq C_2'.
$$

Thus, if $\xi \neq 0$,

$$
C_1' \leq \widetilde{P}\left(\frac{\xi}{|\xi|}\right) \leq C_2'
$$

$$
\Downarrow
$$

$$
C_1'|\xi| \leq \widetilde{P}(\xi) \leq C_2'|\xi|
$$

$$\Downarrow$$
$$C_1'|\xi| \leq P\left(T^{-1}(\xi)\right) \leq C_2'|\xi|.$$

Letting $T^{-1}(\xi) = x$, then $\xi = Tx$, and

$$\frac{C_1'}{\widetilde{C_2}}\|x\| \leq C_1'\|x\|_T \leq P(x) \leq C_2'|Tx| = C_2'\|x\|_T \leq \frac{C_2'}{\widetilde{C_1}}\|x\|.$$

Furthermore, if $\xi = 0$, then $x = 0$, and in this case the last inequalities still hold. Letting $C_1 = \frac{C_1'}{\widetilde{C_2}}$ and $C_2 = \frac{C_2'}{\widetilde{C_1}}$, then for $\forall x \in \mathcal{X}$,

$$C_1\|x\| \leq P(x) \leq C_2\|x\|,$$

which completes the proof of Theorem 1.4.22.