# ***Proof for completion of $L^p(\Omega,\mu)$ with $p\in[1,\infty)$ (p. 27)***

Let $\{u_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $L^p(\Omega,\mu)$. To verify that $\{u_n\}_{n\in\mathbb{N}}$ converges in $L^p(\Omega,\mu)$, it suffices to show $\{u_n\}_{n\in\mathbb{N}}$ has a convergent subsequence (see Exercise 1.2.2, p. 13).

To this end, for $\frac{1}{2}$, there exists $N_1\in\mathbb{N}$ s. t. whenever $n,m>N_1$, it holds that $\|u_n-u_m\|<\frac{1}{2}$.

For $\frac{1}{2^2}$, there exists $N_2\in\mathbb{N}$ s. t. whenever $n,m>N_2$, it holds that $\|u_n-u_m\|<\frac{1}{2^2}$.

Choosing $n_1>N_1$ and $n_2>max(n_1,N_2,2)$, then $n_2>n_1$ and $\|u_{n_1}-u_{n_2}\|<\frac{1}{2}$.

Similarly, for $\frac{1}{2^3}$, there exists $N_3\in\mathbb{N}$ such that whenever $n,m>N_3$, we have that $\|u_n-u_m\|<\frac{1}{2^3}$.

Take $n_3>max(n_2,N_3,3)$, then $n_3>n_2>n_1$, and $\|u_{n_2}-u_{n_3}\|<\frac{1}{2^2}$.

Repeating the process, we obtain a subsequence $\{n_j\}_{j=1}^{\infty}$, s. t. $n_1<n_2<n_3<\cdots$, $n_j\to\infty$, and

$$\|u_{n_j}-u_{n_{j+1}}\|<\frac{1}{2^j}.$$

From this, it follows that

$$\left\|\sum_{j=1}^{\infty}|u_{n_j}-u_{n_{j+1}}|\right\|\le\sum_{j=1}^{\infty}\|u_{n_j}-u_{n_{j+1}}\|<\sum_{j=1}^{\infty}\frac{1}{2^j}=1.$$

Thus $\sum_{j=1}^{\infty}|u_{n_j}-u_{n_{j+1}}|\in L^p(\Omega,\mu)$, and then

$$\sum_{j=1}^{\infty}|u_{n_j}(x)-u_{n_{j+1}}(x)|<\infty\ a.e.\text{ in }\Omega.$$

For $x \in \Omega$ with $\sum_{j=1}^{\infty} |u_{n_j}(x) - u_{n_{j+1}}(x)| < \infty$ and $N \in \mathbb{N}$, if we set

$$S_N(x) = \sum_{j=1}^{N} \big( u_{n_j}(x) - u_{n_{j+1}}(x) \big) ,$$

we then have that for any $k \in \mathbb{N}$,

$$\begin{aligned} |S_{N+k}(x) - S_N(x)| &= \left| \sum_{j=N+1}^{N+k} (u_{n_j}(x) - u_{n_{j+1}}(x)) \right| \\ &\leq \sum_{j=N+1}^{N+k} \big| u_{n_j}(x) - u_{n_{j+1}}(x) \big| \to 0,\ N \to \infty. \end{aligned}$$

Thus $\{S_N(x)\}_{N \in \mathbb{N}}$ converges; and if we let

$$S_*(x) = \lim_{N \to \infty} S_N(x),$$

it then follows that $|S_*(x)| < \infty$ a. e. in $\Omega$. Moreover, $|S_*(x)|$ is measurable because of the measurability of $S_N(x)$. On the other hand, since $S_*(x) = \lim\limits_{N \to \infty} \big( u_{n_1}(x) - u_{n_{N+1}}(x) \big)$, we have

$$\lim_{N \to \infty} u_{n_{N+1}}(x) = u_{n_1}(x) - S_*(x) \triangleq u_*(x) \quad a.\, e. \text{ in } \Omega,$$

which together with the fact that

$$\left\| \sum_{j=1}^{N} \big( u_{n_j} - u_{n_{j+1}} \big) \right\| \leq \sum_{j=1}^{N} \big\| u_{n_j} - u_{n_{j+1}} \big\| < 1$$

and the Fatou Lemma gives that

$$\|S_*\| \leq 1.$$

Thus, the triangle inequality of norm yields

$$\|u_*\| \leq \|u_{n_1}\| + \|S_*\| \leq \|u_{n_1}\| + 1 < \infty,$$

which means $u_* \in L^p(\Omega, \mu)$.

Furthermore, noting that

$$\|S_{N+k} - S_N\| \leq \sum_{j=N+1}^{N+k} \left\|u_{n_j} - u_{n_{j+1}}\right\| < \sum_{j=N+1}^{N+k} \frac{1}{2^j},$$

and that

$$\begin{aligned} S_{N+k} - S_N &= \sum_{j=1}^{N+k} \big(u_{n_j}(x) - u_{n_{j+1}}(x)\big) - \sum_{j=1}^{N} \big(u_{n_j}(x) - u_{n_{j+1}}(x)\big) \\ &= \sum_{j=N+1}^{N+k} \big(u_{n_j}(x) - u_{n_{j+1}}(x)\big) \\ &= u_{n_{N+1}}(x) - u_{n_{N+k+1}}(x), \end{aligned}$$

we have

$$\|u_{n_{N+1}} - u_{n_{N+k+1}}\| < \sum_{j=N+1}^{N+k} \frac{1}{2^j}.$$

Taking $k \to \infty$ and applying the Fatou Lemma, we see that

$$\left\|u_{n_{N+1}} - u_*\right\| \leq \sum_{j=N+1}^{\infty} \frac{1}{2^j},$$

which implies that $\{u_{n_j}\}$ converges to $u_*$ in $L^p(\Omega, \mu)$, and therefore $\{u_n\}_{n\in\mathbb{N}}$ also converges to $u_*$ in $L^p(\Omega, \mu)$. Thus $L^p(\Omega, \mu)$ is complete.