Proof of Proposition 1.3.13 on Page 18

The main ideas of the proof are similar to Example 1.1.7 on Page 2 and the difference is that the proof here needs to show that the limit function is continuous. Suppose that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $C(M)$, namely, for $\forall\epsilon > 0$, $\exists\ N(\epsilon) \in \mathbb{N}$, s. t. for any $m > N(\epsilon)$ and $n > N(\epsilon)$,

$$\max_{t\in M} |x_n(t) - x_m(t)| < \epsilon.$$

Therefore, for $\forall t \in M$, and for any $n, m > N(\epsilon)$,

$$(*) \qquad |x_n(t) - x_m(t)| < \epsilon.$$

This means that for any fixed $t \in M$, the sequence $\{x_n(t)\}_{n\in\mathbb{N}}$ of numbers is a Cauchy sequence in $\mathbb{R}$, and so the limit $\lim\limits_{n\to\infty} x_n(t)$ exists, which will be denoted by $x_0(t)$. By letting $m \to \infty$ in $(*)$, we see that for all $t \in M$, if $n > N(\epsilon)$, then

$$(**) \qquad |x_n(t) - x_0(t)| \le \epsilon.$$

Thus, for any fixed $n_0 > N(\epsilon)$, since x_{n_0} is continuous at any point $s \in M$, we have that by Definition 1.1.8 (p.2), for any sequence $\{t_m\}$ with $t_m \to s, m \to \infty$, $x_{n_0}(t_m) \to x_{n_0}(s), m \to \infty$, which implies that there exists $N_{n_0} \in \mathbb{N}$, s. t. whenever $m > N_{n_0}$, $|x_{n_0}(t_m) - x_{n_0}(s)| < \epsilon$. From this and $(**)$ with n_0, it then follows that for any $m > N_{n_0}$,

$$\begin{aligned} |x_0(t_m) - x_0(s)| &\le |x_0(t_m) - x_{n_0}(t_m)| + |x_{n_0}(t_m) - x_{n_0}(s)| \\ &\quad + |x_{n_0}(s) - x_0(s)| \\ &< 3\epsilon, \end{aligned}$$

which means that $\lim\limits_{m\to\infty} x_0(t_m) = x_0(s)$. This fact together with Definition 1.1.8 shows that $x_0 \in C(M)$. On the other hand, by $(**)$, for $\forall\epsilon > 0$, the inequality $d(x_0, x_n) < \epsilon$ holds for any $n > N(\epsilon)$. Therefore $x_n \to x_0$ in $C(M)$ and $(C(M), d)$ is complete.