Spectra of A on p. 154

Example 2.6.2 Assume that $\mathcal{X} = L^2[0,1]$,

$$A: \ u(t) \mapsto -\frac{d^2}{dt^2}u(t),$$

where

$$D(A) = \{ u \in L^2[0,1], \ u \ \text{is second order differentiable, } u' \text{ is}$$

$$\text{absolutely continuous and } u'' \in L^2[0,1],$$

$$u(0) = u(1), \ u'(0) = u'(1)\}.$$

Then

$$\sigma(A) = \sigma_p(A) = \{(2n\pi)^2, \ n \in \mathbb{N}\}.$$

Remark 1 The domain $D(A)$ of Example 2.6.2 in page 154 of the book is not correct.

In order to obtain the spectra of A, we first establish the following lemma.

Lemma 1 *Let $\mathcal{X}$ be a B^*-space, $A: \ D(A) \subset \mathcal{X} \to \mathcal{X}$ be a closed linear operator, and $\lambda \in \mathbb{C}$. Then $\lambda I - A$ is a closed operator.*

Proof For any sequence $\{u_n\}_{n\in\mathbb{N}} \subset D(A)$ such that

$$\begin{cases} u_n \to u, \ n \to \infty; \\ (\lambda I - A)u_n \to v, \ n \to \infty, \end{cases}$$

we see that

$$\begin{cases} u_n \to u, \ n \to \infty \\ \\ Au_n \to \lambda u - v, \ n \to \infty. \end{cases}$$

By the fact that A is closed, we know that $u \in D(A)$ and $(\lambda I - A)u = v$. So $\lambda I - A$ is closed, which completes the proof of the lemma.

Next we compute the spectra set of A.

Let $\widetilde{\sigma}_p(A) = \{(2n\pi)^2 | n = 0,\ 1,\ 2,\ \cdots,\}$. Since

$$-\frac{d^2}{dt^2}\{\sin(2n\pi t)\} = (2n\pi)^2 \sin(2n\pi t)$$

and

$$-\frac{d^2}{dt^2}\{\cos(2n\pi t)\} = (2n\pi)^2 \cos(2n\pi t)$$

for $n = 0,\ 1,\ 2,\ \cdots$, we then see that $\widetilde{\sigma}_p(A) \subset \sigma_p(A)$. To show the desired conclusion, we only need prove that for any $\lambda \notin \widetilde{\sigma}_p(A)$, $\lambda \in \rho(A)$. Indeed, if this is true, since

$$\mathbb{C} = \widetilde{\sigma}_p(A) \cup (\mathbb{C} \setminus \widetilde{\sigma}_p(A) \subset \sigma_p(A) \cup \rho(A) \subset \mathbb{C}$$

and $\sigma_p(A) \cap \rho(A) = \emptyset$, we then have

$$\sigma(A) = \widetilde{\sigma}_p(A).$$

To show $\lambda \in \rho(A)$, we first prove that $(\lambda I - A)^{-1}$ exists. Indeed, assume that $u_1,\ u_2 \in D(A)$ and $u_1 \neq u_2$. If $-\frac{d^2}{dt^2}u_1 - \lambda u_1 =$

$-\frac{d^2}{dt^2}u_2 - \lambda u_2$, then

$$-\frac{d^2}{dt^2}(u_1 - u_2) = \lambda(u_1 - u_2).$$

Let $y := u_1 - u_2$. The equation above is equivalent to $-\frac{d^2}{dt^2}y = \lambda y$, that is, $y'' + \lambda y = 0$. Consider the equation $\gamma^2 + \lambda = 0$. From $\gamma^2 = -\lambda = |\lambda| e^{i(arg\lambda + \pi + 2k\pi)}$, $k \in \mathbb{Z}$, it follows immediately that

$$\gamma = \sqrt{|\lambda|} e^{i\left(\frac{arg\lambda}{2} + \frac{\pi}{2} + k\pi\right)},\ k \in \mathbb{Z}.$$

We then know that the equation $\gamma^2 + \lambda = 0$ has roots $\gamma_0 = \sqrt{|\lambda|} e^{i\left(\frac{arg\lambda}{2} + \frac{\pi}{2}\right)} = i\sqrt{|\lambda|} e^{i\frac{arg\lambda}{2}}$ and

$$\gamma_1 = \sqrt{|\lambda|} e^{i\left(\frac{arg\lambda}{2} + \frac{\pi}{2} + \pi\right)} = -i\sqrt{|\lambda|} e^{i\frac{arg\lambda}{2}} = -\gamma_0.$$

Since $\lambda \neq 0$, we then see that $\gamma_0 \neq 0$ and hence $e^{\pm\gamma_0} \neq 1$. Furthermore, $y'' + \lambda y = 0$ has a general solution

$$y(t) = C_1 e^{\gamma_0 t} + C_2 e^{-\gamma_0 t},$$

where C_1, $C_2 \in \mathbb{C}$. In other words, $u_1(t) - u_2(t) = C_1 e^{\gamma_0 t} + C_2 e^{-\gamma_0 t}$. Because $u_i(0) = u_i(1)$ and $u_i'(0) = u_i'(1)$, $i = 1, 2$, we have

$$
\begin{cases}
C_1 + C_2 = C_1 e^{\gamma_0} + C_2 e^{-\gamma_0} \\
C_1\gamma_0 - C_2\gamma_0 = C_1\gamma_0 e^{\gamma_0} - C_2\gamma_0 e^{-\gamma_0}
\end{cases}
$$

$$
\Longrightarrow \begin{cases}
C_1 + C_2 = C_1 e^{\gamma_0} + C_2 e^{-\gamma_0} \\
C_1 - C_2 = C_1 e^{\gamma_0} - C_2 e^{-\gamma_0}
\end{cases}
$$

$$
\Longrightarrow \begin{cases}
C_1 = C_1 e^{\gamma_0} \\
C_2 = C_2 e^{-\gamma_0}
\end{cases}
\Longrightarrow \begin{cases}
C_1 = 0 \\
C_2 = 0
\end{cases}
\Longrightarrow u_1(t) = u_2(t).
$$

This contradiction implies that $\lambda I - A$ is injective, that is, $(\lambda I - A)^{-1}$ exists.

We next show that $R(\lambda I - A) = L^2([0,\, 1])$. Choose any $f \in L^2([0,\, 1])$, and let

$$
c_n = \int_0^1 f(\tau) e^{-2\pi i n t}\, dt
$$

and $u_m(t) = \displaystyle\sum_{n=-m}^{m} \frac{c_n}{\lambda - (2n\pi)^2} e^{2\pi i n t}$, then it is clear that $u_m \in D(A)$. Since $\{e^{2\pi i n t}\}_{n\in\mathbb{Z}}$ is an orthogonal basis of $L^2([0,\, 1])$ (see

p. 62, Example 1.6.26), by Theorem 1.6.25 on p. 61 together with Corollary 1.6.24 on p. 60, we then see that

$$u(t) := \sum_{n\in\mathbb{Z}} \frac{c_n}{\lambda - (2n\pi)^2} e^{2\pi int} \in L^2[0,1],$$

$u_m \to u$ and

$$(\lambda I - A)u_m(t) = \left(\lambda + \frac{d^2}{dt^2}\right) u_m(t) = \sum_{n=-m}^{m} c_n e^{2\pi int}$$

$$\to f$$

in $L^2([0,\ 1])$. Moreover, since A is a closed operator, by Lemma 1, we know that $\lambda I - A$ is also a closed operator. Thus, we see that $u \in D(A)$ and $(\lambda I - A)u = f$, which shows $R(\lambda I - A) = L^2([0,\ 1])$. Thus, $\lambda \in \rho(A)$, which completes the proof.