Proof of Example 1.1.14 on Page 7

We make a convention as that $v \in \mathbb{R}^m$, then

$$v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix}.$$

Let $r > 0$ and $C(\overline{B}(x_0, r), \mathbb{R}^m)$ be the set of continuous vector valued function defined on the closed ball $\overline{B}(x_0, r)$ and taking values in $\mathbb{R}^m$. The distance ρ in $C(\overline{B}(x_0, r), \mathbb{R}^m)$ is defined by

$$\rho(\varphi, \psi) \equiv \max_{\substack{z \in \overline{B}(x_0, r) \\ 1 \le i \le m}} |\varphi_i(x) - \psi_i(x)|$$

for any $\varphi = (\varphi_1, \cdots, \varphi_m)$, $\psi = (\psi_1, \cdots, \psi_m) \in C(\overline{B}(x_0, r), \mathbb{R}^m)$.

For $\varphi \in C(\overline{B}(x_0, r), \mathbb{R}^m)$, we consider the map $T : \varphi \mapsto T\varphi$, where

$$(T\varphi)(x) \equiv \varphi(x) - \left(\frac{\partial f}{\partial y}(x_0, y_0)\right)^{-1} f(x, \varphi(x)).$$

For any $x \in \mathbb{R}^n$ and $z_i \in \mathbb{R}^m$, $i = 1, \cdots, m$, define

$$D_y f(x, z_1, \cdots, z_m) \equiv \begin{pmatrix} \frac{\partial f}{\partial y_1}(x, z_1) & \cdots & \frac{\partial f}{\partial y_m}(x, z_1) \\ \cdots & \cdots & \cdots \\ \frac{\partial f}{\partial y_1}(x, z_m) & \cdots & \frac{\partial f}{\partial y_m}(x, z_m) \end{pmatrix}.$$

Since we assume that $\partial f / \partial y$ is continuous on $U \times V$, then

$$\lim_{\substack{x \to x_0 \\ z_1, \cdots, z_m \to y_0}} \left(\frac{\partial f}{\partial y}(x_0, y_0)\right)^{-1} D_y f(x, z_1, \cdots, z_m) = I,$$

where I is the Identity matrix (Unit matrix).

For $1 \leq i, j \leq m$, if letting $[A]_{ij}$ denote the ij-th element of the matrix A and

$$\delta_{ij} = \begin{cases} 1, & i = j; \\ 0, & i \neq j, \end{cases}$$

we then have

$$\lim_{\substack{x \to x_0 \\ z_1, \cdots, z_m \to y_0}} \left[\left(\frac{\partial f}{\partial y}(x_0,\, y_0) \right)^{-1} D_y f(x,\, z_1, \cdots, z_m) \right]_{ij} = \delta_{ij}.$$

This implies that there exists a constant $\delta > 0$ such that for any $x \in \overline{B}(x_0, \delta)$ and $z_1, \cdots, z_m \in \overline{B}(y_0, \delta)$,

$$(*) \qquad \left| \delta_{ij} - \left[\left(\frac{\partial f}{\partial y}(x_0,\, y_0) \right)^{-1} D_y f(x,\, z_1, \cdots, z_m) \right]_{ij} \right| < \frac{1}{2m}.$$

Let $d_i(x) = \varphi(x) - \psi_i(x)$ and v_i be the i-th element of the vector $v \in \mathbb{R}^m$ for $i = 1, \cdots, m$. By the differential median theorem, there exists $\theta_j \in (0, 1)$, $j = 1, \cdots, m$, such that

$$\begin{aligned}
&\Bigg| \left[\left(\frac{\partial f}{\partial y}(x_0,\, y_0) \right)^{-1} f(x,\, \psi(x)) \right]_i \\
&\qquad - \left[\left(\frac{\partial f}{\partial y}(x_0,\, y_0) \right)^{-1} f(x,\, \varphi(x)) \right]_i \Bigg| \\
&= \sum_{\ell=1}^{m} \left[\left(\frac{\partial f}{\partial y}(x_0,\, y_0) \right)^{-1} \right]_{i\ell} [f_\ell(x,\, \psi(x)) - f_\ell(x,\, \varphi(x))] \\
&= \sum_{\ell=1}^{m} \left[\left(\frac{\partial f}{\partial y}(x_0,\, y_0) \right)^{-1} \right]_{i\ell}
\end{aligned}$$

$$\times \sum_{j=1}^{m} \frac{\partial f_\ell}{\partial y_j}(x,\, (1-\theta_\ell)\psi(x) + \theta_\ell \varphi(x))[\psi_j(x) - \varphi_j(x)]$$

$$= -\sum_{j=1}^{m} \sum_{\ell=1}^{m} \left[\left(\frac{\partial f}{\partial y}(x_0,\, y_0) \right)^{-1} \right]_{i\ell} \frac{\partial f_\ell}{\partial y_j}(x,\, \hat{y}_1, \cdots, \hat{y}_m) d_j(x)$$

$$= -\sum_{j=1}^{m} \left[\left(\frac{\partial f}{\partial y}(x_0,\, y_0) \right)^{-1} D_y f(x, \hat{y}_1, \cdots, \hat{y}_m) \right]_{ij} d_j(x),$$

where $\hat{y}_\ell = (1-\theta_j)\psi_\ell(x) + \theta_j \varphi_\ell(x)$.

Let $r < \delta$. If $\varphi(x), \psi(x) \in \overline{B}(y_0,\, \delta)$ for any $x \in \overline{B}(x_0,\, r)$, then noticing that $\hat{y}_1, \cdots, \hat{y}_m \in \overline{B}(y_0,\, \delta)$ in this case, by $(*)$, we have

$$\begin{aligned} &\rho(T\varphi,\, T\psi) \\ &\quad = \max_{\substack{x \in \overline{B}(x_0, r) \\ 1 \le i \le m}} \left| \varphi_i(x) - \left[\left(\frac{\partial f}{\partial y}(x_0,\, y_0) \right)^{-1} f(x,\, \varphi(x)) \right]_i \right. \\ &\qquad \left. -\varphi_j(x) + \left[\left(\frac{\partial f}{\partial y}(x_0,\, y_0) \right)^{-1} f(x,\, \psi(x)) \right]_i \right| \\ &\quad = \max_{\substack{x \in \overline{B}(x_0, r) \\ 1 \le i \le m}} \left| d_i(x) \right. \\ &\qquad \left. -\sum_{j=1}^{m} \left[\left(\frac{\partial f}{\partial y}(x_0,\, y_0) \right)^{-1} D_y f(x,\, \hat{y}_1, \cdots, \hat{y}_m) \right]_{ij} d_j(x) \right| \end{aligned}$$

$$< \frac{1}{2} \max_{\substack{x \in \overline{B}(x_0, r) \\ 1 \le i \le m}} |d_i(x)|$$

$$= \frac{1}{2} \rho(\varphi, \psi). \tag{1.1.11}$$

Let $0 < r < \delta$ and

$$\mathcal{X} = \Big\{ \varphi \in C(\overline{B}(x_0, r), \mathbb{R}^m) |\ \varphi(x_0) = y_0,$$

$$\varphi(x) \in \overline{B}(y_0, \delta), \forall\ x \in \overline{B}(x_0, r) \Big\} .$$

We then claim that $\mathcal{X}$ is a complete space.

In fact, let $\{\varphi^{(n)}\}_{n \in \mathbb{N}}$ be a Cauchy series in $\mathcal{X}$, where for $n \in \mathbb{N}$,

$$\varphi^{(n)}(x) = \begin{pmatrix} \varphi_1^{(n)}(x) \\ \vdots \\ \varphi_m^{(n)}(x) \end{pmatrix} .$$

Since $\mathcal{X} \subset C(\overline{B}(x_0, r), \mathbb{R}^m)$, $\{\varphi^{(n)}\}_{n \in \mathbb{N}}$ is also a Cauchy series in $C(\overline{B}(x_0, r), \mathbb{R}^m)$. From the completeness of $C(\overline{B}(x_0, r), \mathbb{R}^m)$, which can be proved similarly to Example 1.1.7 in Page 2, it follows that there exists a $\varphi \in C(\overline{B}(x_0, r), \mathbb{R}^m)$ such that $\varphi^{(n)} \to \varphi$ in $C(\overline{B}(x_0, r), \mathbb{R}^m)$ as $n \to \infty$. To verify $\varphi \in \mathcal{X}$, for $\forall\ x \in \overline{B}(x_0, r)$, we have

$$
\begin{aligned}
|\varphi(x)-\varphi^{(n)}(x)| &\le \sqrt{m}\max_{x\in\overline{B}(x_0,r),\ 1\le i\le m}|\varphi_i(x)-\varphi_i^{(n)}(x)| \\
&= \sqrt{m}\rho(\varphi,\varphi^{(n)})\to 0,\ n\to\infty,
\end{aligned}
$$

which yields that

$$
\begin{aligned}
|\varphi(x)-y_0| &\le |\varphi(x)-\varphi^{(n)}(x)|+|\varphi^{(n)}(x)-y_0| \\
&\le \delta+|\varphi(x)-\varphi^{(n)}(x)|.
\end{aligned}
$$

Letting $n\to\infty$, we obtain $|\varphi(x)-y_0|\le\delta$ and $\varphi(x_0)=y_0$, namely, for all $x\in\overline{B}(x_0,r)$, $\varphi(x)\in\overline{B}(x_0,\delta)$ and $\varphi(x_0)=y_0$. Thus, $\varphi\in\mathcal{X}$, and by Definition 1.1.5, $\mathcal{X}$ is a complete metric space which is desired.

We now verify $T:\ \mathcal{X}\to\mathcal{X}$. Note that

$$
\begin{aligned}
\rho(T\varphi,y_0) &\le \rho(T\varphi,Ty_0)|+\rho(Ty_0,y_0) \\
&\overset{(1.1.11)}{<} \frac{1}{2}\rho(\varphi,y_0) \\
&\quad+\max_{\substack{x\in\overline{B}(x_0,r)\\ 1\le i\le m}}\left|\left[\left(\frac{\partial f}{\partial y}(x_0,y_0)\right)^{-1}f(x,y_0)\right]_i\right|,
\end{aligned}
$$

and by the continuity of f, there exists $\eta>0$ such that for $\gamma<\eta$ and $x\in\overline{B}(x_0,r)$,

$$
\max_{\substack{x\in\overline{B}(x_0,r)\\ 1\le i\le m}}\left|\left[\left(\frac{\partial f}{\partial y}(x_0,y_0)\right)^{-1}f(x,y_0)\right]_i\right|
$$

$$\overset{f(x_0,y_0)=0}{=} \max_{\substack{x\in\overline{B}(x_0,r)\\ 1\le i\le m}} \left| \left[\left(\frac{\partial f}{\partial y}(x_0,y_0) \right)^{-1} (f(x,y_0) - f(x_0,y_0)) \right]_i \right|$$
$$< \frac{\delta}{2}.$$

Thus, for $0 < r < \min(\eta, \delta)$, we have $\rho(T\varphi, y_0) < \delta$; and moreover,

$$T\varphi(x_0) = y_0 + \left(\frac{\partial f}{\partial y}(x_0,y_0) \right)^{-1} f(x_0, \varphi(x_0)) = y_0,$$

which indicates that $T: \mathcal{X} \to \mathcal{X}$.

Since (1.1.11) implies that T is contraction map, applying Theorem 1.1.11, we obtain a unique $\varphi \in \mathcal{X}$ such that $T\varphi = \varphi$, i. e., there exists a unique $\varphi \in C(\overline{B}(x_0,\, r), \mathbb{R}^m)$ such that $\varphi(x_0) = y$, $\varphi(x) \in \overline{B}(y_0,\, \delta)$ and $f(x,\, \varphi(x)) = 0$ for any $x \in \overline{B}(x_0, r)$. Let $U_0 = B(x_0,\, r)$ and $V_0 = B(y_0,\, (1 + 1/m)\delta)$. We then obtain the desired results. This completes the proof of Example 1.1.14.