概率论的进步

陈木法

(北京师范大学)

BNU第八届优秀大学生数学暑期夏令营

2016 年 7 月 19 日 [redacted]

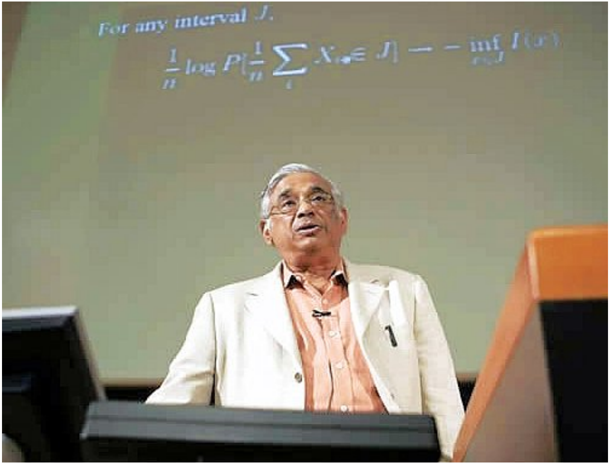
目录

- 概率论的成熟与成长
- 概率论与统计物理的交叉
- 概率论与数学其它分支的交叉

(1)2006 Gauss Prize   Kiyoshi Itô 伊藤清

[1915–2008] The prize honors his achievements in stochastic analysis, a field of mathematics based essentially on his groundbreaking work. 1942+64

(2) 2007 Abel Prize Srinivasa S.R.Varadhan



1940–

Fundamental contributions to probability theory and in particular for creating a unified theory of large deviations.

1975 with M.D. Donsker+42

(3) ICM2006 　 2.5/4

John Ball: "Probability swept most of the award", "2.5 of them, to be exact".

- Andrei Okounkov:概率论+表示论+代数几何.
- Wendelin Werner：二维布朗运动的几何和共形映照. 　 对数Sobole不等式
- Grigori Perelman：Ricci流的分析和几何结构.
- Terence Tao: PDE, 组合，调和分析及堆垒数论，随机矩阵.

(4) ICM2010 2/4

- Elon Lindenstrauss: 遍历理论.
- Ngô Báo Châu：Langlands 纲领(数论).
- Stanislav Smirnov：渗流的共形不变性与平面 Ising 模型.
- Cédric Villani：数学物理＋概率论.

(5) ICM2014 1/4

- Artur Avila: 动力系统理论.
- Manjul Bhargava: 数的几何(数论).
- Martin Hairer: 随机偏微分方程.
- Maryam Mirzakhani: 黎曼面的动力系统和几何及它们的模空间.

William Feller$^{1906-1970}$: An Introduction to Probability Theory and its Application

William Feller's vivid lecturing at IBM.

W.Feller: An Introduction to Probability Theory and its Application (Vol. 1 & 2)

Preface to the Third Edition (of Volume 1)

WHEN THIS BOOK WAS FIRST CONCEIVED (MORE THAN 25 YEARS AGO) few mathematicians outside of the Soviet Union recognized probability as legitimate [合法的, 正当的] branch of mathematics. ...... 1967 两书问世之前, 概率论除在前苏联而外, 尚未被数学界普遍认可.

W.Feller: An Introduction to Probability Theory and its Application (Vol. 1 & 2)

Preface to the First Edition (of Volume 2)

AT THE TIME THE FIRST VOLUME OF THIS BOOK WAS WRITTEN (BETWEEN 1941 AND 1948) the interest in probability was not yet widespread [流行; 普遍的]. 1965

Feller 边界理论

Andrei Nikolaevich Kolmogorov 1903-1987

A.N. Kolmogorov's book (in German, 1933);
英译(1950, 1956): Foundations of the Theory of Probability. 中译:概率论基本概念,1952.

Hilbert's period: 1933–1965. 公理化，基础

- 极限理论. B. Gnedenko and A.N. Kolmogorov (1954); 许宝騄
- 平稳过程. A.N. Kolmogorov (1941); 江泽培
- 马氏过程. J. Doob (1953); K.L. Chung (1960); E.B. Dynkin (1965); 王梓坤 (1965)

许: 中国概率统计的总司令.

江: 我国平稳过程之父.

王: 我国马氏过程之父.

Poincaré's period: 1964– 回归自然

Roland Lvovich Dobrushin 1929–1995

1988 年参观

长城和十三陵

时的留影

Roland Lvovich Dobrushin: Random Fields

- The existence conditions of the configuration integral of the Gibbs distribution, 1964.
- Methods of the theory of probability in statistical physics, 1964, Winter School.
- Existence of a phase transition in the two-dimensional and three-dimensional Ising models, 1965.

Roland Lvovich Dobrushin: Random Fields

- The existence conditions of the configuration integral of the Gibbs distribution, 1964.
- Methods of the theory of probability in statistical physics, 1964, Winter School.
- Existence of a phase transition in the two-dimensional and three-dimensional Ising models, 1965.
- Dobrushin: “目标是重新建立统计力学的数学基础”
- Robert A. Minlos:“开始时仅有一个结果已知,即自由能总存在”

Poincaré's period: 1964– 回归自然

- Random fields (1964)
- Interacting particle systems. R.L. Dobrushin, F. Spitzer (1970). D.A.Dawson
- Percolation. H. Kesten(1982): Percolation Theory for Mathematicians. Birkhäuser Verlag, Boston
- Large deviations, Malliavin calculus, stochastic differential geometry, quantum probability, Euclidean quantum field theory, free probability, Dirichlet forms, mathematical finance, stochastic PDE, etc

统计物理:

- 1977 之前, 平衡态(电影). Ising 模型.
- 1978 之后, 非平衡态(人体).
- 1977/12/8, I. Prigogine: Nobel prize lecture.
- 1977—, 北京师大数、理、化非平衡统计物理联合讨论班.

平衡态 $\longleftrightarrow$ 自共扼算子.

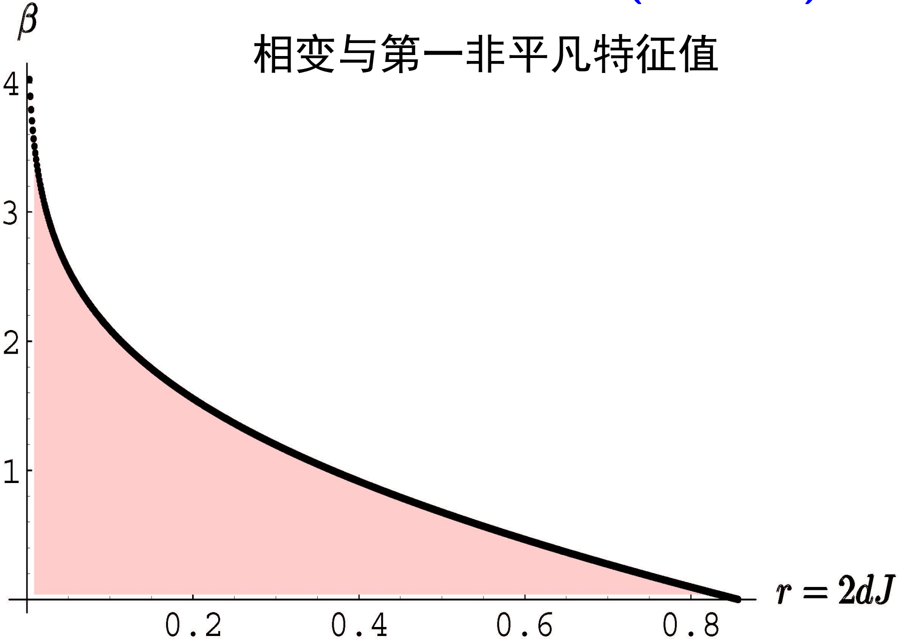
非平衡态 $\longleftrightarrow$ 非自共扼算子.
相变现象 $\longleftrightarrow$ 无穷维数学模型 $\longleftarrow$ 有限维极限.

概率论与统计物理的交叉 (15 年):

- 有限维模型解的存在唯一性? 1978—1983. 极限过渡.
- 耦合(coupling)三部曲: 马氏, 最优, 距离.
- 保序性(随机可比性).
- 无穷维反应扩散过程(Chen: 1985).

[1] From Markov Chains to Nonequilibrium Particle Systems. World Sci. 1992/2004. Part IV
[2] Eigenvalues, Inequalities and Ergodic Theory. Springer 2005. Chapter 9
[3] Practical criterion'86 for uniq. of Q-processes, Chin. J. Appl. Prob. Stat. 2015, 31(2): 213–224

概率论与数学其它分支的交叉 (> 25 年)



The first (non-trivial) eigenvalue:

$$Q=\begin{pmatrix}-b_0 & b_0 & 0 & 0 & \cdots\\ a_1 & -(a_1+b_1) & b_1 & 0 & \cdots\\ 0 & a_2 & -(a_2+b_2) & b_2 & \cdots\\ \vdots & \vdots & \ddots & \ddots & \ddots\end{pmatrix}$$

$$a_i>0,\ b_i>0.$$

$Q\mathbb{1}=\mathbb{0}=0\cdot\mathbb{1}$. Trivial eigenvalue: $\lambda_0=0$.

Question: next eigenvalue of $-Q$, $\lambda_1=?$
Elliptic operator in $\mathbb{R}^d$; Laplacian on Riemannian manifolds. Importance: leading term.

Difficulties

Example 1: Trivial case (two points). Two parameters

$$\begin{pmatrix} -b & b \\ a & -a \end{pmatrix}, \quad \lambda_1 = a + b.$$

λ_1 is increasing in each of the parameters!

Example 2: Three points. Four parameters.

$$\begin{pmatrix} -b_0 & b_0 & 0 \\ a_1 & -(a_1 + b_1) & b_1 \\ 0 & a_2 & -a_2 \end{pmatrix},$$

$$\lambda_1 = 2^{-1}\Big[a_1 + a_2 + b_0 + b_1 - \sqrt{(a_1 - a_2 + b_0 - b_1)^2 + 4a_1 b_1}\Big].$$

Example 3: Four points.

Six parameters: $b_0, b_1, b_2, a_1, a_2, a_3$.

$$\lambda_1 = \frac{D}{3} - \frac{C}{3 \cdot 2^{1/3}} + \frac{2^{1/3}\ \left(3\,B - D^2\right)}{3\,C},$$

where

$$D = a_1 + a_2 + a_3 + b_0 + b_1 + b_2,$$

$$B = a_3\,b_0 + a_2\ (a_3 + b_0) + a_3\,b_1 + b_0\,b_1 + b_0\,b_2 + b_1\,b_2 + a_1\ (a_2 + a_3 + b_2)\,,$$

$$C = \left(A + \sqrt{4(3\,B - D^2)^3 + A^2}\right)^{1/3},$$

$$A = -2a_1^3 - 2a_2^3 - 2a_3^3 + 3a_3^2 b_0 + 3a_3 b_0^2 - 2b_0^3 + 3a_3^2 b_1 - 12a_3 b_0 b_1 + 3b_0^2 b_1 + 3a_3 b_1^2 + 3b_0 b_1^2 - 2b_1^3 - 6a_3^2 b_2 + 6a_3 b_0 b_2 + 3b_0^2 b_2 + 6a_3 b_1 b_2 - 12b_0 b_1 b_2 + 3b_1^2 b_2 - 6a_3 b_2^2 + 3b_0 b_2^2 + 3b_1 b_2^2 - 2b_2^3 + 3a_1^2\,(a_2 + a_3 - 2b_0 - 2b_1 + b_2) + 3a_2^2\,[a_3 + b_0 - 2\,(b_1 + b_2)] + 3a_2\left[a_3^2 + b_0^2 - 2b_1^2 - b_1 b_2 - 2b_2^2 - a_3(4b_0 - 2b_1 + b_2) + 2b_0(b_1 + b_2)\right] + 3a_1\left[a_2^2 + a_3^2 - 2b_0^2 - b_0 b_1 - 2b_1^2 - a_2(4a_3 - 2b_0 + b_1 - 2b_2) + 2b_0 b_2 + 2b_1 b_2 + b_2^2 + 2a_3(b_0 + b_1 + b_2)\right].$$

The role of each parameter is completely mazed!

Not solvable when space has more than five points!

Conclusion: Impossible to compute λ_1 explicitly!

How about the estimation of λ_1?

Perturbation of eigenvalues

Example 4: Infinite tri-diagonal matrix

(Birth-death processes).

$b_i\ (i \geqslant 0)$	$a_i\ (i \geqslant 1)$	λ_1	degree of eigenf. g
$i+\beta$ $(\beta > 0)$	$2i$	1	1
$i+1$	$2i+3$		
$i+1$	$2i+(4+\sqrt{2})$		

g: eigenfunction of λ_1.

Perturbation of eigenvalues

Example 4: Infinite tri-diagonal matrix

(Birth-death processes).

$b_i\ (i \geqslant 0)$	$a_i\ (i \geqslant 1)$	λ_1	degree of eigenf. g
$i+\beta$ $(\beta > 0)$	$2i$	1	1
$i+1$	$2i+3$	2	
$i+1$	$2i+(4+\sqrt{2})$	3	

Perturbation of eigenvalues

Example 4: Infinite tri-diagonal matrix

(Birth-death processes).

$b_i \ (i \geqslant 0)$	$a_i \ (i \geqslant 1)$	λ_1	degree of eigenf. g
$i+\beta$ $(\beta > 0)$	$2i$	1	1
$i+1$	$2i+3$	2	2
$i+1$	$2i+\left(4+\sqrt{2}\,\right)$	3	3

Sensitive. In general, it is too hard to estimate λ_1!

Bare-handed. Visited 黎曼几何 3 年, 调和分析 13 年 等.

Numerical computation

Two algorithms for computing the maximal eigenpair.

- Power iteration. Given $v_0|_g \neq 0$, define
  $v_k = \frac{Av_{k-1}}{\|Av_{k-1}\|}$ and $z_k = v_k^* A v_k$.
- Rayleigh quotient iteration Given $(v_0, z_0) \sim (g, \lambda_{\max}(A))$, define
  $v_k = \frac{(A - z_{k-1}I)^{-1} v_{k-1}}{\|(A - z_{k-1}I)^{-1} v_{k-1}\|}$ and $z_k = v_k^* A v_k$.

Then $v_k \to g$ and $z_k \to \lambda_{\max}(A)$.

Example: $a_{k+1} = b_k = (k+1)^2$, $N = 7$. 10^3

Large N. $\lambda_0 = 1/4$ if $N = \infty$. $\leqslant 30$ Sec

Use $\tilde{v}_0$ and δ_1. Let $z_0 = 7/(8\delta_1) + v_0^*(-Q)v_0/8$.

$N+1$	z_0	z_1	$z_2 = \lambda_0$
10^4	0.31437	0.302586	0.302561

Large N. $\lambda_0 = 1/4$ if $N = \infty$. $\leqslant 30$ Sec

Use $\tilde{v}_0$ and δ_1. Let $z_0 = 7/(8\delta_1) + v_0^*(-Q)v_0/8$.

$N+1$	z_0	z_1	$z_2 = \lambda_0$	upper/lower
8	0.523309	0.525268	0.525268	$1+10^{-11}$
100	0.387333	0.376393	0.376383	$1+10^{-8}$
500	0.349147	0.338342	0.338329	$1+10^{-7}$
1000	0.338027	0.327254	0.32724	$1+10^{-7}$
5000	0.319895	0.30855	0.308529	$1+10^{-7}$
7500	0.316529	0.304942	0.304918	$1+10^{-7}$
10^4	0.31437	0.302586	0.302561	$1+10^{-7}$

Unified speed estimation of various stabilities

Theorem (Informal! 1988 ⟶ 2010–2014)

For tridiagonal matrix Q or one-dim elliptic operator (order 2) with/without killing on a finite/infinite interval, in each of 20 cases, there exist explicit δ, δ_1, δ_1' (and then δ_n, δ_n', recursively) such that $\delta_n' \uparrow$, $\delta_n \downarrow$ and

$$(4\delta)^{-1} \leqslant \delta_n^{-1} \leqslant \lambda_0 \leqslant {\delta_n'}^{-1} \leqslant \delta^{-1}, \qquad n \geqslant 1.$$

Besides, $1 \leqslant {\delta_1'}^{-1}/\delta_1^{-1} \leqslant 2$.

State space $E = (-M, N)$, $M, N \leqslant \infty$.

Eigenequation: $Lg = -\lambda g, \quad g \neq 0.$

Four boundaries. Use codes 'D' and 'N'.

D: (Abs.) Dirichlet boundary $g(-M)=0$ $\boxed{\lim_M}$

N: (Ref.) Neumann boundary $g'(-M) = 0$.

- λ^{NN}: Neumann boundaries at $-M$ and N.
- λ^{DD}: Dirichlet boundaries at $-M$ and N.
- λ^{DN}: Dirichlet at $-M$ and Neumann at N.
- λ^{ND}: Neumann at $-M$ and Dirichlet at N.

$$L = a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx}, \qquad C(x) = \int_\theta^x \frac{b}{a}$$

Speed $\frac{d\mu}{dx} = \frac{e^C}{a}$, scale $\frac{d\hat{\nu}}{dx} = e^{-C}$.

Speed of L^2-exponential stability $P_t = e^{tL}$

Theorem (C. 2010). For each # of 4 cases, the

unified estimates $(4\kappa^{\#})^{-1} \leqslant \lambda^{\#} \leqslant (\kappa^{\#})^{-1}$,

where $\mu(\alpha, \beta) = \int_\alpha^\beta d\mu.$

$$(\kappa^{\text{NN}})^{-1} = \inf_{x<y} \{\mu(-M, x)^{-1} + \mu(y, N)^{-1}\} \hat{\nu}(x, y)^{-1}$$

$$(\kappa^{\text{DD}})^{-1} = \inf_{x\leqslant y} \{\hat{\nu}(-M, x)^{-1} + \hat{\nu}(y, N)^{-1}\} \mu(x, y)^{-1}$$

$$\kappa^{\text{DN}} = \sup_{x\in(-M,N)} \hat{\nu}(-M, x)\, \mu(x, N)$$

1920–1972

$$\kappa^{\text{ND}} = \sup_{x\in(-M,N)} \mu(-M, x)\, \hat{\nu}(x, N)$$

μ and $\hat{\nu}$, factor 4

In particular, $\lambda^{\#} > 0$ iff $\kappa^{\#} < \infty$.

1988–2010. Three probabilistic tools + 5 steps.

Exp stability in L^2 or entropy

Semigroup $P_t = e^{tL}$. $L^2(\pi)$, $\|\cdot\|$, $(\cdot,\cdot)$.

L self-adjoint: $(f, Lg) = (Lf, g)$.

$\|P_t f - \pi(f)\| \leqslant \|f\| e^{-\varepsilon t}$, $\varepsilon_{\max} = \lambda_1 := \lambda^{\text{NN}}$.

Exponential stability in entropy

$$Ent(P_t f) \leqslant Ent(f) e^{-2\sigma t}, \qquad t \geqslant 0,$$

$$Ent(f) := H(\mu \| \pi) = \int_E f \log f \, d\pi \quad \text{if} \quad \frac{d\mu}{d\pi} = f$$

The φ^4 Euclidean quantum field on the lattice

$$\mathbb{R}^{\mathbb{Z}^d} \ni x : \mathbb{Z}^d \to \mathbb{R}. \quad H(x) = -2J \sum_{\langle ij \rangle} x_i x_j, \quad J, \beta \geqslant 0.$$

$$L = \sum_{i \in \mathbb{Z}^d} \left[\partial_{ii} - (u'(x_i) + \partial_i H) \partial_i \right], \quad u(x_i) = x_i^4 - \beta x_i^2$$

Theorem (C. 2008)

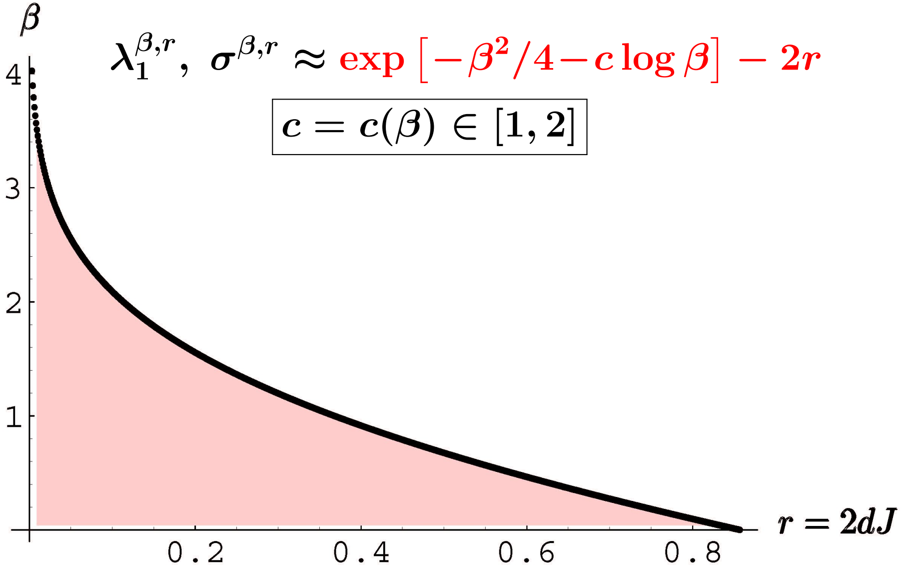
$$\inf_{\Lambda \Subset \mathbb{Z}^d} \inf_{\omega \in \mathbb{R}^{\mathbb{Z}^d}} \lambda_1^{\beta, J}(\Lambda, \omega) \approx \inf_{\Lambda \Subset \mathbb{Z}^d} \inf_{\omega \in \mathbb{R}^{\mathbb{Z}^d}} \sigma^{\beta, J}(\Lambda, \omega)$$

$$\approx \exp\left[-\beta^2/4 - c \log \beta \right] - 4dJ \quad \boxed{c = c(\beta) \in [1, 2]}$$

Phase transition: the φ^4 model

$$\lambda_1^{\beta,r},\ \sigma^{\beta,r} \approx \exp\left[-\beta^2/4 - c\log\beta\right] - 2r$$

$$\boxed{c = c(\beta) \in [1, 2]}$$

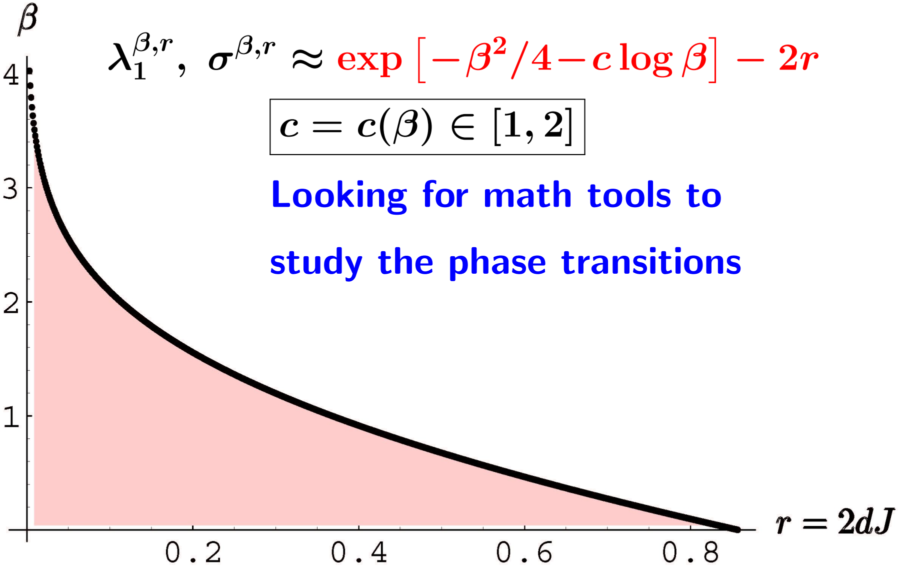


Phase transition: the φ^4 model

$$\lambda_1^{\beta,r},\ \sigma^{\beta,r} \approx \exp\left[-\beta^2/4 - c\log\beta\right] - 2r$$

$$\boxed{c = c(\beta) \in [1, 2]}$$

Looking for math tools to study the phase transitions



For Further Reading I

C. (2008)
Spectral gap and logarithmic Sobolev constant for continuous spin systems
Acta Math. Sin. New Ser. 24(5): 705–736

C. (2010)
Speed of stability for birth–death processes
Front. Math. China. 5:3, 379–515

For Further Reading II

- C. (2014)

  Criteria for discrete spectrum of 1D operators

  Commun. Math. Stat. 2014, 2(3): 279 – 309

- C. (2016) Survey article, free downloadable!

  Unified speed estimation of various stabilities

  Chin. J. Appl. Probab. Statis. 32(1), 1–22

For Further Reading III

- C. (2016): *Efficient initials for computing the maximal eigenpair*, preprint
- C. and Zhang, X. (2014).
  Isospectral operators.
  Commu. Math. Stat. 2, 17–32

For Further Reading IV

- C. and Zhang, Y.H. (2014).
  Unified representation of formulas for single birth processes.
  Front. Math. China 9(4): 761–796
- 陈 (2005). 谈谈概率论与其它学科的若干交叉

http://math0.bnu.edu.cn/~chenmf 4 volumes

18 popularizing papers

The end!

Thank you, everybody!

谢谢大家！

42