Starting from Stochastic Model of Economic Optimization

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Outline

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Current economy

Fix the unit of the quantity of each product.

$$x = \left(x^{(1)}, x^{(2)}, \ldots, x^{(d)}\right)$$

vector of the quantities of the main products (*vector of products*).

Examining the current economy

Examine three things.

- The vector of products input last year:

$$x_0 = \left(x_0^{(1)}, x_0^{(2)}, \ldots, x_0^{(d)}\right).$$

- The output of the vector of products this year: $x_1 = \left(x_1^{(1)}, x_1^{(2)}, \ldots, x_1^{(d)}\right)).$
- The *structure matrix* (or *matrix of expend coefficient*): $A_0 = (a_{ij}^{(0)})$.
  Meaning: to produce one unit of i-th product, one needs $a_{ij}^{(0)}$ units of the j-th product.

$$x_0^{(j)} = \sum_{i=1}^{d} x_1^{(i)} a_{ij}^{(0)}, \qquad x_0 = x_1 A_0.$$

Suppose all the products are used for the reproduction (*idealized model*). Then

$$x_{n-1} = x_n A_{n-1}, \qquad n \geqslant 1.$$

Hence

$$x_0 = x_1 A_0 = x_2 A_1 A_0 = x_n A_{n-1} \cdots A_0.$$

Input–output method

Time-homogeneous: $A_n = A, \quad n \geqslant 0.$

Simple expression for the n-th output:

$$x_n = x_0 A^{-n}, \qquad n \geqslant 1.$$

Known the structure matrix and the input x_0, predict the future output.

Well known *input-output method*.

Up to the 1960s, more than 100 countries had used this method in their national economies.

Basic course in national economics.

Minimax principle

Return to the original equation $x_1 = x_0 A^{-1}$.

Fix A, then x_1 is determined by x_0 only.

Question: which choice of x_0 is the optimal one?

What sense?

Average of the members' ages in a group.

Minimax principle: Find out x_0 such that $\min_{1 \leqslant j \leqslant d} x_1^{(j)} / x_0^{(j)}$ attains the maximum below

$$\max_{x_0 :\, x_1 = x_0 A^{-1} > 0} \ \min_{1 \leqslant j \leqslant d} x_1^{(j)} / x_0^{(j)}.$$

Stability of an economy

By using the classical Frobenius theorem,

Theorem (L. K. Hua, 1983)

Given an irreducible non-negative matrix A, let u be the left eigenvector (positive) of A corresponding to the largest eigenvalue $\rho(A)$ of A. Then, up to a constant, the solution to the above problem is $x_0 = u$. In this case, we have

$$x_1^{(j)}/x_0^{(j)} = \rho(A)^{-1} \quad \textit{for all } j.$$

Called *the eigenvector method*.

Collapse Theorem

$$x_n = x_0 \rho(A)^{-n} \quad \text{whenever } x_0 = u.$$

What happen if $x_0 \neq u$ (up to a constant)?

Set $T^x = \inf\big\{n \geqslant 1 : x_0 = x \text{ and there is some } j \text{ such that } x_n^{(j)} \leqslant 0\big\}$

which is called the *collapse time* of the economic system.

Theorem [L.K. Hua, 1984–1985]

Under some mild conditions, if $x_0 \neq u$, then $T^{x_0} < \infty$.

Example [Hua]

Two products only: industry and agriculture. Let

$$A = \frac{1}{100}\begin{pmatrix} 25 & 14 \\ 40 & 12 \end{pmatrix}.$$

Then $u = (5(\sqrt{2400} + 13)/7, 20)$. 44.34397483.

We have

x_0	T^{x_0}
$(44, 20)$	?
$(44.344, 20)$	?
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x_0	T^{x_0}
$(44, 20)$	3
$(44.344, 20)$	8
$(44.34397483, 20)$	13

Interpretation

Particular case: $A = P$: P is a transition probability matrix. By ergodic theorem for MC,

$$P^n \to \mathbb{1}\pi \quad \text{as } n \to \infty,$$

where $\mathbb{1}$ is the row vector having elements 1, $\pi = \left(\pi^{(1)}, \pi^{(2)}, \ldots, \pi^{(d)}\right)$ is stationary distribution of the corresponding MC. Since the distribution is the only stable solution for the chain, it should have some meaning in economics even though the economic model goes in a converse way:

From the above facts, it is not difficult, as will be shown soon, to prove that if

$$x_0 \neq u = \left(\pi^{(1)}, \pi^{(2)}, \ldots, \pi^{(d)}\right)$$

up to a positive constant, then $T^{x_0} < \infty$.
Next, since the general case can be reduced to the above particular case, we think that this is a very natural way to understand the above Hua's theorem.

Proof. Need to show

$$x_n > 0 \text{ for all } n \Longrightarrow x_0 = \pi.$$

Let $x_0 > 0$ and $x_0\mathbb{1} = 1$. Then

$$1 = x_0\mathbb{1} = x_n P^n \mathbb{1} = x_n\mathbb{1}, \quad n \geqslant 1.$$

$\exists\{x_{n_k}\}_{k\geqslant 1}$ and $\bar{x}$: $\lim_{k\to\infty} x_{n_k} = \bar{x}$, $\bar{x} \geqslant 0$, $\bar{x}\mathbb{1} = 1$. Therefore,

$$x_0 = \left(x_0 P^{-n_k}\right) P^{n_k} = x_{n_k} P^{n_k} \to \bar{x}\mathbb{1}\pi = \pi.$$

We must have $x_0 = \pi$.

In L.K. Hua's eleven reports (1983–1985), he also studied some more general economic models. The above two theorems are the key to his idea. The market economies were also treated. The only difference is that in the latter case one needs to replace the structure matrix A with

$$V^{-1}AV,$$

where V is the diagonal matrix $\text{diag}(v_i/p_i)$:
(p_i) is the vector of prices in the market,
(v_i) is the right eigenvector of A.

Note that the eigenvalue of $V^{-1}AV$ is the same as those of A. Corresponding to the largest eigenvalue

$$\rho(V^{-1}AV) = \rho(A),$$

the left eigenvector of $V^{-1}AV$ becomes

$$uV.$$

Thus, from mathematical point of view, the consideration of market makes no essential difference in Hua's model.

Stochastic model without consumption

Small random perturbation:

$$\widetilde{a}_{ij} = a_{ij} \quad \text{with probability } 2/3,$$
$$= a_{ij}(1 \pm 0.01) \quad \text{with probability } 1/6.$$

Taking $(\widetilde{a}_{ij})$ instead of (a_{ij}), we get a random matrix. Next, let $\{A_n;\ n \geqslant 1\}$ be a sequence of independent random matrices with the same distribution as above, then

$$x_n = x_0 \prod_{k=1}^n A_k^{-1}$$

gives us a stochastic model of an economy without consumption.

Starting from $x_0 = (44.344,\ 20)$, then the collapse probability in stochastic model is the following

$$\mathbb{P}[T^{x_0} = n] = \begin{cases} 0, & \text{for } n = 1, \\ ?, & \text{for } n = 2, \\ ?, & \text{for } n = 3. \end{cases}$$

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$$\mathbb{P}[T^{x_0} = n] = \begin{cases} 0, & \text{for } n = 1, \\ 0.09, & \text{for } n = 2, \\ ?, & \text{for } n = 3. \end{cases}$$

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- This observation tells us that randomness plays a critical role in the economy.
- It also explains the reason why the input-output is not very practicable, as people often think, because the randomness has been ignored and so the deterministic model is far away from the real practice.

What is the analog of Hua's theorem for the stochastic case?

Theorem (C., 1992)

Under some mild conditions, we have

$$\mathbb{P}[T^{x_0} < \infty] = 1, \qquad \forall x_0 > 0.$$

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Proof is not easy! We have to deal with the product of random matrices:

$$M_n = A_n A_{n-1} \cdots A_1.$$

In deterministic case, leading term is $\prod_{j=1}^n \rho(A_j)$.
Hua (1984): $\prod_{j=1}^n A_j/\rho(A_j)$.

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By the Kolmogorov's SLLN:

$$\frac{1}{n}\log\prod_{j=1}^n \rho(A_j) \xrightarrow{\text{a.s.}} \mathbb{E}\log\rho(A_1), \qquad n\to\infty,$$

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Leading order $\rho(M_n)$ of M_n is not $\prod_{j=1}^n \rho(A_j)$.
Liapynov exponent ("strong law of large numbers"). $\|A\|$: the operator norm of A.
Theorem. Let $\mathbb{E}\log^+\|A_1\| < \infty$. Then

$$\frac{1}{n}\log\|M_n\| \xrightarrow{\text{a.s.}} \gamma \in \{-\infty\}\cup\mathbb{R},$$

where $\gamma = \lim_{n\to\infty}\frac{1}{n}\mathbb{E}\log\|M_n\|$.

Theory of random matrices

Theorem[H. Kesten & F. Spitzer: PTRF, 1984]

$M_n/\|M_n\|$ converges in distribution to a positive matrix $M = L^*R$ with rank one, where L and R are positive row vectors.

— Strong law of large numbers

Recent paper: Ann. of Probab. (1997), No. 4.
"Limit theorems for products of random matrices"

Theory of random matrices

[1] Mehta, M. L., Random Matrices, 2^{nd}, Academic Press, 1991 [562 pages]

[2] Girko, V. L., Theory of Random Determinants, Kluwer Acad. Publ., 1990 [702 pages]

- Statistics. Hsu, P. L., 1939
- Physics.
- Number theory, Riemannian hypothesis.
- Free probability theory.

Wigner's semicircle law (1955)

$a_{ij}(1 \leqslant i < j \leqslant N)$. i.i.d., $\mathbb{E}a_{ij} = 0$, $\mathbb{E}a_{ij}^2 = v$
$a_{ii}(1 \leqslant i \leqslant N)$. i.i.d., $\mathbb{E}a_{ii} = 0$, $\mathbb{E}a_{ii}^2 = 2v$

$$A_N = \left[\frac{1}{\sqrt{N}} a_{ij}\right]_{i,j=1}^{N}, \qquad F_N(x) = \frac{1}{N} \#\{\lambda_N \leqslant x\}$$

Semicircle law: $F_N \to F$ weakly as $N \to \infty$, density f:

$$f(x) = \begin{cases} \sqrt{4v^2 - x^2}, & \text{if } |x| \leqslant 2v \\ 0, & \text{if } |x| > 2v \end{cases}$$

- Algebra. Linear, multiplication.
- Banach algebra. Norm.
- C^*-algebra. Conjugate transpose.
- Free probability. Independence. ICM 2002

(1) Haagerup, U., Random matrices, free probability and the invariant subspace problem relative to a von Neumann algebra

(2) Biane, P., Free probability and combinarorics

(3) Ge, L. M., Free probability, free entropy and applications to von Neumann algebra

Stochastic model with consumption

The model without consumption is idealized!
More practical one should have consumption.
Allow a part of the production to turn into consumption and not be used for reproduction.

Suppose that every year we take the $\theta^{(i)}$ times amount of the increment of the i-th product to be consumed. Then in the first year, the vector of products which can be used for reproduction is

$$y_1 = x_0 + (x_1 - x_0)(I - \Theta),$$

where I is the $d \times d$ unit matrix and $\Theta = \operatorname{diag}\left(\theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(d)}\right)$, which is called a *consumption matrix*. Therefore,

$$y_1 = y_0[A_0^{-1}(I - \Theta) + \Theta], \qquad y_0 = x_0.$$

Similarly, in the n-th year, the vector of the products that can be used for reproduction is

$$y_n = y_0 \prod_{k=0}^{n-1} [A_{n-k-1}^{-1}(I - \Theta) + \Theta].$$

Let
$$B_n = [A_{n-1}^{-1}(I - \Theta) + \Theta]^{-1}.$$

Then
$$y_n = y_0 \prod_{k=1}^{n} B_{n-k+1}^{-1}.$$

We have thus obtained a *stochastic model with consumption*. In the deterministic case, a collapse theorem was obtained by L. K. Hua and S. Hua (1985). More stable! Dim $(x_0) > 1$.

- $Gl(d,\mathbb{R})$: General linear group of real invertible $d \times d$ matrices.
- $O(d,\mathbb{R})$: Orthogonal matrices in $Gl(d,\mathbb{R})$.
- $\mathscr{G}_\mu$: Smallest closed semigroup of $Gl(d,\mathbb{R})$ containing $\operatorname{supp}(\mu)$.
- $\mathscr{G}$ strongly irreducible: $\nexists$ proper linear subspaces of $\mathbb{R}^d$, $\mathscr{V}_1, \cdots, \mathscr{V}_k$ such that $(\cup_{i=1}^k \mathscr{V}_i)B = \cup_{i=1}^k \mathscr{V}_i, \forall B \in \mathscr{G}$.
- $\mathscr{G}$ contractive: $\exists\{B_n\} \subset \mathscr{G}$ such that $\|B_n\|^{-1}B_n$ converges to a matrix with rank 1.
- Polar decomposition: $B = K\text{diag}(a_i)U$, $K, U \in O(d,\mathbb{R})$, $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_d > 0$.

Theorem [C., Y. Li (1994)]

Let $\{B_n\}$ i.i.d. $\sim \mu$. Suppose that $\mathscr{G}_\mu$ strongly irreducible, contractive and "K" satisfies a tightness condition. Then

$$\mathbb{P}[T^x < \infty] = 1 \text{ for all } 0 < x \in \mathbb{R}^d.$$

Open problems

- How fast does the economy go to collapse?

$$\mathbb{P}[T > n] \leqslant Ce^{-\alpha n}.$$

- How to control the economy and what is the optimal one?

http://math.bnu.edu.cn/~chenmf

The end!

Thank you, everybody!

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