Starting From the First nontrivial Eigenvalue

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Outline

1. **The first (nontrivial) eigenvalue**
   - Motivation and the difficulty
   - Variational formulas and explicit bounds
2. **Different types of stability**
   - Basic inequalities
   - New picture of ergodicity
   - Explicit criteria
3. **Coupling method**
   - Lifting to the product space
   - Trilogy of couplings

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The first (nontrivial) eigenvalue

Consider the tridiagonal matrix Q:

$$Q = \begin{pmatrix} -b_0 & b_0 & 0 & 0 & \cdots \\ a_1 & -(a_1+b_1) & b_1 & 0 & \cdots \\ 0 & a_2 & -(a_2+b_2) & b_2 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

$$a_i > 0, \quad b_i > 0.$$

$Q\mathbb{1} = 0 \cdot \mathbb{1}$. Trivial eigenvalue: $\lambda_0 = 0$.
Question: Next eigenvalue of $-Q$: $\lambda_1 = ?$

L^2-exponential stability

Semigroup $P_t = e^{tQ}$. $L^2(\pi)$, $\|\cdot\|$, $(\cdot,\cdot)$.
Q self-adjoint: $(f, Qg) = (Qf, g)$.

$$\mathrm{Var}(P_t f) \leqslant \mathrm{Var}(f) e^{-2\lambda_1 t}$$

i.e. $\|P_t f - \pi(f)\| \leqslant \|f - \pi(f)\| e^{-\lambda_1 t}$

or $\|P_t - \pi\|_{2\to 2} \leqslant e^{-\lambda_1 t}, \quad \pi(f) := \int f \mathrm{d}\pi$

$$\lambda_1 = \inf\{(f, -Qf) : \pi(f) = 0, \|f\| = 1\}.$$

λ_1: infimum of $\mathrm{Spec}(-Q) \setminus \{0\}$, the first non-trivial eigenvalue of $-Q$.

Motivation: web search

Two steps in web search

- Collect the sites by using key words.
- Output according to the PageRank produced by using the eigenvector method.

Determine the PageRank

- Nonnegative matrix based on the connections of web sites.
- The first eigenvalue corresponds to a nonnegative eigenvector.

Need a fast algorithm.

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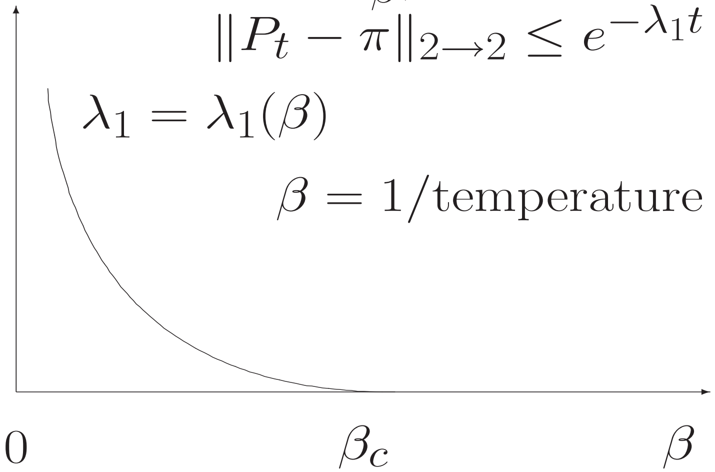
Need a fast algorithm.

Motivation: phase transition

$$L = L_\beta,\ P_t = e^{tL}$$

$$\|P_t - \pi\|_{2\to 2} \le e^{-\lambda_1 t}$$



$$\lambda_1 = \lambda_1(\beta)$$

$$\beta = 1/\text{temperature}$$

Phase transition and λ_1

The difficulty of the problem

Example

Trivial case (two points). Two parameters.

$$\begin{pmatrix} -b & b \\ a & -a \end{pmatrix},$$

$$\lambda_1 = a + b.$$

Example

Four points. Six parameters: $b_0, b_1, b_2, a_1, a_2, a_3$.

$$\lambda_1 = \frac{D}{3} - \frac{C}{3 \cdot 2^{1/3}} + \frac{2^{1/3}\,\left(3\,B - D^2\right)}{3\,C},$$

where

$$D = a_1 + a_2 + a_3 + b_0 + b_1 + b_2,$$

$$B = a_3\,b_0 + a_2\,\left(a_3 + b_0\right) + a_3\,b_1 + b_0\,b_1 + b_0\,b_2$$
$$+ b_1\,b_2 + a_1\,\left(a_2 + a_3 + b_2\right),$$

$$C = \left(A + \sqrt{4(3\,B - D^2)^3 + A^2}\right)^{1/3},$$

The difficulty of the problem

$$\color{red}{A}\color{black} = -2a_1^3-2a_2^3-2a_3^3+3a_3^2b_0+3a_3b_0^2-2b_0^3+3a_3^2b_1-12a_3b_0b_1+3b_0^2b_1+3a_3b_1^2+3b_0b_1^2-2b_1^3-6a_3^2b_2+6a_3b_0b_2+3b_0^2b_2+6a_3b_1b_2-12b_0b_1b_2+3b_1^2b_2-6a_3b_2^2+3b_0b_2^2+3b_1b_2^2-2b_2^3+3a_1^2\left(a_2+a_3-2b_0-2b_1+b_2\right)+3a_2^2\left[a_3+b_0-2\left(b_1+b_2\right)\right]+3a_2\left[a_3^2+b_0^2-2b_1^2-b_1b_2-2b_2^2-a_3(4b_0-2b_1+b_2)+2b_0(b_1+b_2)\right]+3a_1\left[a_2^2+a_3^2-2b_0^2-b_0b_1-2b_1^2-a_2(4a_3-2b_0+b_1-2b_2)+2b_0b_2+2b_1b_2+b_2^2+2a_3(b_0+b_1+b_2)\right].$$

The difficulty of the problem

- The role of each parameter is completely mazed!
- Not solvable when space has more than five points!
- Conclusion: Impossible to compute λ_1 explicitly!

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Perturbation of eigenvalues and eigenfunctions

How about the estimation of λ_1?

$b_i\ (i \geqslant 0)$	$a_i\ (i \geqslant 1)$	λ_1	degree of g
$i+\beta$ $(\beta > 0)$	$2i$	1	1
$i+1$	$2i+3$		
$i+1$	$2i+\left(4+\sqrt{2}\right)$		

g: eigenfunction of λ_1.

Perturbation of eigenvalues and eigenfunctions

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$b_i\ (i \geqslant 0)$	$a_i\ (i \geqslant 1)$	λ_1	degree of g
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$i+1$	$2i+3$	2	
$i+1$	$2i+\left(4+\sqrt{2}\,\right)$	3	

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Perturbation of eigenvalues and eigenfunctions

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Perturbation of eigenvalues and eigenfunctions

Conclusion:

- Both of the eigenvalues and the eigenfunctions are very sensitive.
- In general, it is too hard to estimate λ_1!

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Results

Two notations:

$$\mu_0 = 1, \quad \mu_i = \frac{b_0 b_1 \cdots b_{i-1}}{a_1 a_2 \cdots a_i}, \quad i \geqslant 1.$$

$$Z = \sum_i \mu_i < \infty, \qquad \pi_i = \frac{\mu_i}{Z}.$$

$\mathscr{W} = \{w : w_0 = 0,\ w \text{ is strictly increasing}\}$,

$\widetilde{\mathscr{W}}$ = A slight modification of $\mathscr{W}$.

$$I_i(w) = \frac{1}{\mu_i b_i (w_{i+1} - w_i)} \sum\nolimits_{j=i+1}^{\infty} \mu_j w_j, \quad i \geqslant 0.$$

$\bar{w}_i = w_i - \sum_j \pi_j w_j$.

Theorem (C. 1996–2001)

- Dual variational formulas:

$$\inf_{w\in\widetilde{\mathscr{W}}}\sup_{i\geqslant 1} I_i(\bar{w})^{-1}=\lambda_1=\sup_{w\in\mathscr{W}}\inf_{i\geqslant 0} I_i(\bar{w})^{-1}.$$

- Explicit estimates: $Z\delta^{-1}\geqslant\lambda_1\geqslant(4\delta)^{-1}$,

$$\delta=\sup_{i\geqslant 1}\sum_{j\leqslant i-1}(\mu_j b_j)^{-1}\sum_{j\geqslant i}\mu_j.$$

- Approximation procedure: There are two explicit sequences $\tilde{\eta}_n$ and η_n such that $\tilde{\eta}_n^{-1}\geqslant\lambda_1\geqslant{\eta_n}^{-1}$.

Non-ergodic case

Ergodic condition: $\sum_i \mu_i < \infty$ and $\sum_{k=0}^{\infty} (b_k\mu_k)^{-1} \sum_{i=0}^{k} \mu_i = \infty$.

Non-ergodic case:

$$\lambda_0 = \inf\{D(f) : \|f\| = 1,\ f \text{ has finite support}\}$$

$$R_i(v) = a_i + b_i - a_i/v_{i-1} - b_i v_i, \qquad \mathscr{V},\ \widetilde{\mathscr{V}}$$

$$I_i(f) = \frac{1}{\mu_i b_i (f_i - f_{i+1})} \sum_{j \leqslant i} \mu_j f_j, \qquad \mathscr{F}_I,\ \widetilde{\mathscr{F}}_I$$

$$I\!I_i(f) = \frac{1}{f_i} \sum_{j \geqslant i} \frac{1}{\mu_j b_j} \sum_{k \leqslant j} \mu_k f_k, \qquad \mathscr{F}_{I\!I},\ \widetilde{\mathscr{F}}_{I\!I}$$

Theorem [C. 2008]

- Dual variational formulas:

$$\inf_{v\in\widetilde{\mathscr{V}}}\ \sup_{i\geqslant 0} R_i(v)^{-1} = \lambda_0 = \sup_{v\in\mathscr{V}}\ \inf_{i\geqslant 0} R_i(v)^{-1}.$$

$$\inf_{f\in\widetilde{\mathscr{F}}_I}\ \sup_{i\geqslant 0} I_i(f)^{-1} = \lambda_0 = \sup_{f\in\mathscr{F}_I}\ \inf_{i\geqslant 0} I_i(f)^{-1}.$$

$$\inf_{f\in\widetilde{\mathscr{F}}_{I\!I}}\ \sup_{i\in\mathrm{supp}(f)} I\!I_i(f)^{-1} = \lambda_0 = \sup_{f\in\mathscr{F}_{I\!I}}\ \inf_{i\geqslant 0} I\!I_i(f)^{-1}.$$

- Explicit estimates: $\delta^{-1} \geqslant \lambda_0 \geqslant (4\delta)^{-1}$,

$$\delta = \sup_{n\geqslant 0} \sum_{j\leqslant n} \mu_j \sum_{k\geqslant n} (\mu_k b_k)^{-1}.$$

- Approximation procedure.

Compact Riemannian mfd M, Laplacian $\triangle$

Aim: Estimating λ_1 in terms of dim d, diam D and the lower bound of Ricci curvature $K \in \mathbb{R}$: $\text{Ricci}_M \geqslant Kg$.

Theorem [Variational formula. C. & F.Y. Wang, 1997]

$$\lambda_1 \geqslant \sup_{f\in\mathscr{F}} \inf_{r\in(0,D)} \frac{4f(r)}{\int_0^r C(s)^{-1}\mathrm{d}s \int_s^D C(u)f(u)\mathrm{d}u},$$

$$\mathscr{F} = \{f \in C[0,D] : f \text{ is positive on } (0,D)\},$$

$$C(r) = \cosh^{d-1}\left[\frac{r}{2}\sqrt{\frac{-K}{d-1}}\right], \; r \in (0,D).$$

Classical variational formula:

$$\lambda_1 = \inf\left\{ \int_M \|\nabla f\|^2 : \int f = 0, \int f^2 = 1\right\}.$$

Rayleigh (1877), Fischer (1905).

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Stability corresponding to the inequalities

- $\text{Var}(P_t f) \leqslant \text{Var}(f) e^{-2\lambda_1 t} \Longleftrightarrow$ Poincaré ineq.
- Exponential convergence in entropy:

  $\text{Ent}(P_t f) \leqslant \text{Ent}(f) e^{-2\alpha_1 t}$

  $$\boxed{\text{Ent}(f) = \pi(f \log f) - \pi(f) \log \|f\|_1}$$

  $\Longleftrightarrow$ (Relative) entropy inequality

  $\Longleftarrow$ Logarithmic Sobolev inequality.
- $\text{Var}(P_t f) \leqslant C \|f\|_1^2 / t^{q-1} \Longleftrightarrow$ Nash ineq.

$$(E, \mathscr{E}, \pi), L^2(\pi), \|\cdot\|, D(f) = \frac{1}{2}\sum_{i,j\in E} q_{ij}(f_j - f_i)^2.$$

- Poincaré inequality$^{\text{[H. Poincaré, 1890]}}$:

$$\lambda_1 \mathrm{Var}(f) \leqslant D(f).$$

- Nash inequality$^{\text{[J. Nash, 1958]}}$: $\boxed{\|\cdot\|_p = \|\cdot\|_{L^p(\pi)}}$

$$\eta \mathrm{Var}(f) \leqslant D(f)^{1/p} \|f\|_1^{2/q}, \quad p^{-1} + q^{-1} = 1.$$

- Logarithmic Sobolev inequality$^{\text{[L.Gross,1976]}}$:

$$2^{-1}\sigma\, \mathrm{Ent}(f^2) \leqslant D(f). \quad \text{[G. Perelman, 2002]}$$

- (Relative) entropy inequality:

$$2\alpha_1\, \mathrm{Ent}(f) \leqslant D(f, \log f).$$

Fact: $\sigma \leqslant \alpha_1 \leqslant \lambda_1$.

Example. Two points $E = \{0, 1\}$,

$$Q = \begin{pmatrix} -\theta & \theta \\ 1-\theta & \theta-1 \end{pmatrix}, \quad \theta \in (0, 1/2].$$

- The first eigenvalue $\lambda_1 = 1$.
- The Nash constant $\eta = \left(\frac{\theta}{1-\theta}\right)^{1/q}$.
- The logarithmic constant $\sigma = \frac{2(1-2\theta)}{\log(1/\theta-1)}$.
- The entropy constant $\alpha_1 \approx (\lambda_1 + \sigma)/2$, open.

Fact: $\sigma \leqslant \alpha_1 \leqslant \lambda_1$.

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  open.

Three classical types of ergodicity

Semigroup $P_t = (p_{ij}(t)) = e^{tQ}$.

- Ergodicity[1953?]:

$$p_{ij}(t) \longrightarrow \pi_j, \quad t \longrightarrow \infty.$$

- Exponential ergodicity[D.G. Kendall, 1959]:

$$|p_{ij}(t) - \pi_j| \leqslant C_i e^{-\varepsilon_1 t}.$$

- Strong (uniform) ergodicity[1978]:

$$\sup_i |p_{ij}(t) - \pi_j| \leqslant C e^{-\varepsilon_2 t}.$$

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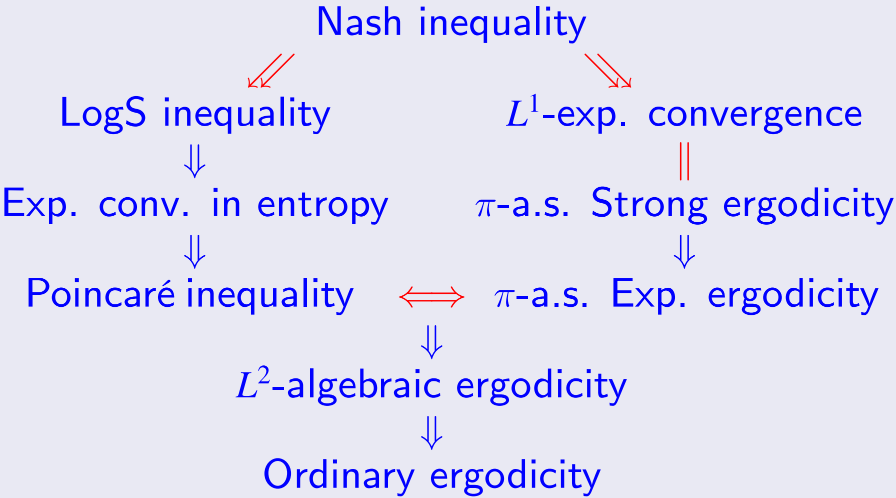
Theorem (New picture of ergodic theory)

For reversible Markov process with densities,



$$\begin{tikzcd}
 & \text{Nash inequality} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
\text{LogS inequality} \arrow[d, Rightarrow] & & L^1\text{-exp. convergence} \arrow[d, equal] \\
\text{Exp. conv. in entropy} \arrow[d, Rightarrow] & & \pi\text{-a.s. Strong ergodicity} \arrow[d, Rightarrow] \\
\text{Poincaré inequality} \arrow[rr, Leftrightarrow] & {} \arrow[d, Rightarrow] & \pi\text{-a.s. Exp. ergodicity} \\
 & L^2\text{-algebraic ergodicity} \arrow[d, Rightarrow] & \\
 & \text{Ordinary ergodicity} &
\end{tikzcd}$$

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Explicit criteria for the inequalities (or for different types of stability)

$$\mu_0 = 1, \quad \mu_n = \frac{b_0 \cdots b_{n-1}}{a_1 \cdots a_n},\ n \geqslant 1;$$

$$\mu[i,k] = \sum_{i \leqslant j \leqslant k} \mu_j.$$

Theorem [C. 2000–03, Y.H. Mao 2002, et al]

Ten criteria for different types of stability of the Q-semigroup are listed in the following table.

Property	**Criterion**
Uniq.	$\sum_{n\geqslant 0}\frac{1}{\mu_n b_n}\mu[0,n]=\infty \quad (*)$
Recur.	$\sum_{n\geqslant 0}\frac{1}{\mu_n b_n}=\infty$
Erg.	$(*)\ \&\ \mu[0,\infty)<\infty$
Exp. erg. Poincaré	$(*)\&\sup_{n\geqslant 1}\mu[n,\infty)\sum_{j\leqslant n-1}\frac{1}{\mu_j b_j}<\infty$
Dis. sp.	$(*)\ \lim_{n\to\infty}\sup_{k\geqslant n+1}\mu[k,\infty)\sum_{j=n}^{k-1}\frac{1}{\mu_j b_j}=0$
LogS	$(*)\&\sup_{n\geqslant 1}\mu[n,\infty)\log[\mu[n,\infty)^{-1}]\sum_{j\leqslant n-1}\frac{1}{\mu_j b_j}<\infty$
Str. erg. L^1-exp.	$(*)\&\sum_{n\geqslant 0}\frac{1}{\mu_n b_n}\mu[n+1,\infty)=\sum_{n\geqslant 1}\mu_n\sum_{j\leqslant n-1}\frac{1}{\mu_j b_j}<\infty$
Nash	$(*)\&\sup_{n\geqslant 1}\mu[n,\infty)^{(\nu-1)/\nu}\sum_{j\leqslant n-1}\frac{1}{\mu_j b_j}<\infty(\varepsilon)$

Criterion for exponential convergence in entropy?

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Erg.	$(*)\ \&\ \mu[0,\infty)<\infty$
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LogS	$(*)\&\sup_{n\geqslant 1}\mu[n,\infty)\log[\mu[n,\infty)^{-1}]\sum_{j\leqslant n-1}\frac{1}{\mu_j b_j}<\infty$
Str. erg. L^1-exp.	$(*)\&\sum_{n\geqslant 0}\frac{1}{\mu_n b_n}\mu[n+1,\infty)=\sum_{n\geqslant 1}\mu_n\sum_{j\leqslant n-1}\frac{1}{\mu_j b_j}<\infty$
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Criterion for exponential convergence in entropy?

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Lifting to the product space

Example (Normal distribution)

$$\int_{\mathbb{R}} e^{-x^2/2}\mathrm{d}x = \sqrt{2\pi}.$$

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)/2}\mathrm{d}x\mathrm{d}y,$$

use the polar coordinate system,

due to Simón-Denis Poisson (1781–1840).

$$\mathrm{d}x \longrightarrow \mathrm{d}x\,\mathrm{d}y.$$

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Lifting to the product space

Example [FKG-inequality, 1971]

$$f, g \uparrow \ +? \ \Longrightarrow \int_{\mathbb{R}} fg\,\mathrm{d}\nu \geqslant \int_{\mathbb{R}} f\,\mathrm{d}\nu \int_{\mathbb{R}} g\,\mathrm{d}\nu.$$

$$\|fg\|_1 \leqslant \|f\|_p \|g\|_q, \quad p > 1,\ p^{-1} + q^{-1} \geqslant 1,$$
$$\|fg\|_1 \geqslant \|f\|_p \|g\|_q, \quad p < 1,\ p^{-1} + q^{-1} \leqslant 1.$$

$$\int_{\mathbb{R}} \int_{\mathbb{R}} [f(x) - f(y)][g(x) - g(y)]\nu(\mathrm{d}x)\nu(\mathrm{d}y) \geqslant 0.$$

$\nu \longrightarrow \nu \times \nu.$ (Independent coupling)

Lifting to the product space

Example [FKG-inequality, 1971]

$$f, g \uparrow \ +? \implies \int_{\mathbb{R}} fg\mathrm{d}\nu \geqslant \int_{\mathbb{R}} f\mathrm{d}\nu \int_{\mathbb{R}} g\mathrm{d}\nu.$$

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Coupling and Markovian coupling

Coupling measure $\tilde{\nu}$. Given ν_i on $(E_i, \mathscr{E}_i)$, $i=1, 2$, $\tilde{\nu}$ on product space satisfies marginality:

$$\tilde{\nu}(A_1 \times E_2) = \nu_1(A_1), \quad \tilde{\nu}(E_1 \times A_2) = \nu_2(A_2)$$

for all $A_i \in \mathscr{E}_i$, $i = 1, 2$.

Coupling = copulas, transport.

Coupling operator $\widetilde{L}$. Given operators L_i, $i=1, 2$, $\widetilde{L}$ on product space satisfies marginality:

$$\widetilde{L}\tilde{f}_1 = L_1 f_1, \quad \widetilde{L}\tilde{f}_2 = L_2 f_2$$

for every lifted $\tilde{f}_i \in \mathscr{E}_1 \times \mathscr{E}_2$.

Examples of Markovian coupling

Independent coupling $\widetilde{Q}_0$:

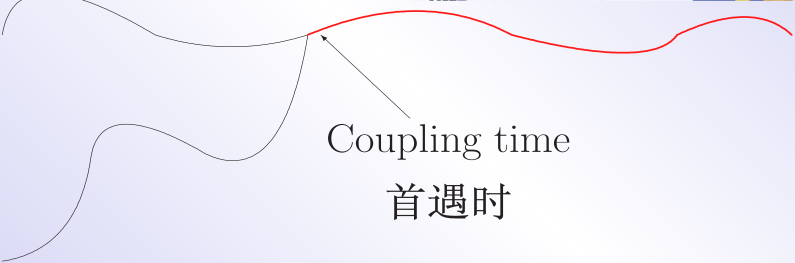
$$\widetilde{Q}_0 f(i_1, i_2) = [Q_1 f(\cdot, i_2)](i_1) + [Q_2 f(i_1, \cdot)](i_2).$$

Classical coupling $\widetilde{Q}_c$: $Q_1 = Q_2 =: Q$,

$$\widetilde{Q}_c f(i_1, i_2) = 1_{\{i_1 \neq i_2\}} \widetilde{Q}_0 f(i_1, i_2) + 1_{\{i_1 = i_2\}} Qf(\cdot, \cdot)(i_1).$$

Chinese love story

梁山伯与祝英台



Boy

梁山伯

Coupling time

首遇时

Girl

祝英台

“Fall in love at first sight” “一见钟情”耦合

Trilogy of Couplings

- Markovian couplings.
- Optimal Markovian couplings.
- Good distance with respect to optimal Markovian couplings.

$$\widetilde{Q}\rho(i_1, i_2) \leqslant -\alpha\rho(i_1, i_2)$$

$$\Longrightarrow \lambda_1 \geqslant \alpha.$$

Reduce higher dim to dim one.

"Proof": estimation of λ_1:

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“Proof”: estimation of λ_1:

Step 1. g: eigenfunction: $Lg = -\lambda_1 g$, $g \neq$const.

$T_t = e^{tL}$. From semigroups theory,

$$\frac{d}{dt} T_t g(x) = T_t L g(x) = -\lambda_1 T_t g(x).$$

ODE. $\quad T_t g(x) = g(x) e^{-\lambda_1 t}. \qquad (1)$

Step 2. Compact space. g Lipschitz w.r.t. ρ: c_g.

Key condition: $\widetilde{T}_t \rho(x, y) \leqslant \rho(x, y) e^{-\alpha t}. \qquad (2)$

$$\begin{aligned} e^{-\lambda_1 t} |g(x) - g(y)| &\leqslant \widetilde{T}_t \big| g(x) - g(y) \big| && \text{[by (1)]} \\ &\leqslant c_g \widetilde{T}_t \rho(x, y) \\ &\leqslant c_g \rho(x, y) e^{-\alpha t} && \text{[by (2)]} \end{aligned}$$

for all t. Hence $\lambda_1 \geqslant \alpha$. General method!

$$(2) \Longleftrightarrow \widetilde{L} \rho(x, y) \leqslant -\alpha \rho(x, y).$$

Design of the distance

Let $g \sim \lambda_1$. Then $g_i \uparrow\uparrow$.

$$\rho(i,j) = |g_i - g_j|.$$

Mimic of g:

$$\tilde{g}_i = \sum_{j \leqslant i-1} \frac{1}{\mu_j b_j} \sum_{k \geqslant j+1} \mu_k f_k$$

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Summary

- Variational formulas.
- Coupling method.

For Further Reading I

- M.F. Chen.
  Eigenvalues, Inequalities, and Ergodic Theory.
  Springer, 2005.

- M.F. Chen.
  Ergodic Convergence Rates of Markov Processes — Eigenvalues, Inequalities and Ergodic Theory. Book[4]
  http://math.bnu.edu.cn/~chenmf, 2001–.

For Further Reading II

- Wang, F.Y. (2005),
  Functional inequalities, Markov semigroups and spectral theory.
  Science Press, Beijing.

The end!

Thank you, everybody!

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