A Mathematical View on Quantum Mechanics

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Outline

- Background from quantum mechanics
- From Hermite to Hermitizable
- Differential operators.
  QM/QP & Mathematics

Nobel prize 1932. Werner Karl Heisenberg

"for the creation of quantum mechanics, the application of which has, inter alia, led to the discovery of the allotropic forms of hydrogen".

Received prize in 1933

Nobel prize 1933. E. Schrödinger & P. Dirac

"for the discovery of new productive forms of atomic theory"

Matrix mechanics. 1900, 1905, 1913

Founder of QM: Max Planck (N1918), $\hbar$

Albert Einstein (N1921), Niels Bohr(N1922).

Creator of MM: Werner Karl Heisenberg (N1932)$^{1925\text{-}7\text{-}29}$ (without matrix),

+Three fundamental papers:

Max Born (N1954)&Pascual Jordan$^{1925\text{-}9\text{-}27}$,

Paul Adrien Maurice Dirac (N1933)$^{1925\text{-}11\text{-}7}$,

Born, Heisenberg & Jordan$^{1925\text{-}11\text{-}16}$.

Complex+real spectrum. Hermite matrix

Extensions

1) Jordan algebra^{1934}. Efim Zel'manov^{1983}: The only 27 dim sys Albert algebra. Too small for quantum mechanics. 8 books

Field Medal^{1994} on Burnside Problem.

Book: "A Taste of Jordan Algebra", 2004

2) Non-Hermitian quantum mechanics:

$\mathscr{PT}$-Symmetric Quantum Mechanics:

"Non-Hermitian Quantum Mechanics"

by N. Moiseyev^{2011}

Wave mechanics

E. Schrödinger. "Physical Review"$^{1926\text{-}9\text{-}3}$

"An undulatory theory of the mechanics of atoms and molecules".

Schrödinger equation:

$$i\hbar\dot{\psi}(t) = (-\Delta + V)\psi(t) \quad \boxed{e^{2\pi i E t/\hbar}}$$

$\psi(t, x)$ wave function. $\hbar$: Planck const.

Stationary Schrödinger eq.: $(\Delta - V)\psi = 0$

$$(-\Delta + V)\psi = E_m\psi,\ E_m\text{: eigenvalue}$$

Centennial Debate

Born's Probability amplitude[1926]: $|\psi_m|^2$.

Schrödinger: never accepted the orthodox interpretation of 'almost psychical' advocated by the Copenhagen School.

Einstein: "God does not play dice!"

Bohr, said: "If anyone is not dizzy by quantum mechanics, then he must not understand quantum mechanics."

Einstein: "I think about quantum mechanics

Centennial Debate

a hundred times longer than I think about general relativity, but still not understand clearly"

Feynman: "It has been said that only three people in the world understand it, Relativity, I don't know who they are. However I think I can say with certainty that nobody understands quantum theory!"

"Do We Really Understand Quantum Mechanics?" by F. Laloë,

1st: 2012; 2nd: 2019, Cambridge U.

Equivalence of two mechanics (1926, 1929)

- Schrödinger; Clark Eckart 1926.
- Dirac 1926, book 1930, ... 4th ed.
- John von Neumann 1927–1929, book 1932 (German), 1955 (English), 2018 (New TeX edition).

Self-adjoint operator on Hilbert space. Given $\boldsymbol{A} = (\boldsymbol{a}_{ij})$, self-adjoint on $\boldsymbol{L}^2(\boldsymbol{\mu})$, $\boldsymbol{\mu}$?

J.C. Zambrini, Phys Review A (1986)

"In two forgotten publications, dating back to 1931 and 1932, Schrödinger tried to find a classical probabilistic derivation of an equation whose characteristics are as close as possible to those of his wave equation."

"Schrödinger Diffusion Processes" by R.Aebi^{1996}

Time-reversible Markov process

A.N. Kolmogorov (1936). Time-discrete finite Markov chains $\leftarrow$ Schrödinger[1931]
A.N.K. (1937): Diffusion processes.
Reversible.
Zen-Ting Hou & C. (1979): symmetrizable
C. (2018): Hermitizable $\longleftarrow$ Comp math

Definition

Complex matrix $\boldsymbol{A} = (\boldsymbol{a}_{ij})$ Hermitizable if $\exists \boldsymbol{\mu} = (\boldsymbol{\mu}_k) > \boldsymbol{0}$ such that $\boldsymbol{\mu}_i \boldsymbol{a}_{ij} = \boldsymbol{\mu}_j \bar{\boldsymbol{a}}_{ji}$ for every pair $(\boldsymbol{i}, \boldsymbol{j})$.

Criterion for Hermitizability. $a_{ij} \geqslant 0,\ i \neq j$

Theorem (C. (2018))

Complex $A = (a_{ij})$ is Hermitizable iff two conditions hold simultaneously.

- For each pair i, j, either $a_{ij} \& a_{ji} = 0$ or $a_{ij} a_{ji} > 0$ ($\Leftrightarrow a_{ij}/\bar{a}_{ji} > 0$).
- The circle condition holds for each smallest closed path without round-trip.

$$i_0 \to i_1 \to \cdots \to i_n = i_0,\ a_{i_k i_{k+1}} \neq 0$$

$$\Rightarrow a_{i_0 i_1} \cdots a_{i_{n-1} i_n} = \bar{a}_{i_n i_{n-1}} \cdots \bar{a}_{i_1 i_0}.$$

One

Computation of Hermitizing measure μ

Fix reference point i_0 and set $\mu_{i_0} = 1$.
For each $j \neq i_0$, choose and fix a path

$$i_0 \to i_1 \to \cdots \to i_n = j,$$

then

$$\mu_j = \frac{a_{i_0 i_1}}{\bar{a}_{i_1 i_0}} \frac{a_{i_1 i_2}}{\bar{a}_{i_2 i_1}} \cdots \frac{a_{i_{n-1} i_n}}{\bar{a}_{i_n i_{n-1}}}.$$

Circle condition $\Rightarrow$ path-independence.
Irreducible $\Rightarrow$ unique μ up to $+$constant, determined by $A = (a_{ij})$ only.

Tridiagonal/Birth-death matrix

$T \sim (a_k, -c_k, b_k), \quad E = \{k \in \mathbb{Z}_+ : 0 \leqslant k < N+1\}$

$$\begin{matrix} T \\ Q \end{matrix} = \begin{pmatrix} -c_0 & b_0 & & & & 0 \\ a_1 & -c_1 & b_1 & & & \\ & a_2 & -c_2 & b_2 & & \\ & & \ddots & \ddots & b_{N-1} \\ 0 & & & a_N & -c_N \end{pmatrix},$$

$(a_k), (b_k), (c_k)$: complex sequences.

BD: $a_k > 0$, $b_k > 0$, $c_k = a_k + b_k$, $c_N \geqslant a_N$.

Tridiagonal matrix: $T \sim (a_k, -c_k, b_k)$

Theorem (C. 2018)

The tridiagonal T is Hermitizable iff the following two conditions hold simultaneously.

- The diagonals (c_k) are real.
- Either $a_{i+1} \& b_i = 0$ or $a_{i+1} b_i > 0$

分块 $(\Leftrightarrow b_i/\bar{a}_{i+1} > 0)$.

Then $$\mu_0 = 1, \ \mu_k = \mu_{k-1} \frac{b_{k-1}}{\bar{a}_k}.$$

Theorem/Algorithm

Given Hermitizable $T \sim (a_k, -c_k, b_k)$ with $c_k \geqslant |a_k| + |b_k|$ (or $\tilde{c}_k = c_k + m$) Then $\exists$ an explicit birth–death matrix $\widetilde{Q} \sim (\tilde{a}_k, -\tilde{c}_k, \tilde{b}_k)$ such that T is isospectral to $\widetilde{Q}$.

In general, we have $\tilde{c}_N \geqslant \tilde{a}_N$. We assume that $\tilde{c}_N > \tilde{a}_N$ in what follows. The case $\tilde{c}_N = \tilde{a}_N$ was also treated in the published paper (2018).

Explicit $u_k := a_k b_{k-1} = |a_k b_{k-1}|$ & c_k

$$\tilde{b}_k = c_k - \cfrac{u_k}{c_{k-1} - \cfrac{u_{k-1}}{c_{k-2} - \cfrac{u_{k-2}}{\ddots\, c_2 - \cfrac{u_2}{c_1 - \cfrac{u_1}{c_0}}}}}$$

$$\tilde{b}_k = c_k - u_k / \tilde{b}_{k-1}, \quad \tilde{b}_0 = c_0$$

$$\tilde{a}_k = c_k - \tilde{b}_k, \quad k < N; \qquad \tilde{a}_N = u_N / \tilde{b}_{N-1}.$$

Discrete spectrum. $\mathbb{Z}_+ = \{0, 1, 2, \cdots\}$

Let $\widetilde{Q} \sim (\tilde{a}_k, -\tilde{c}_k, \tilde{b}_k)$. Define

$$\tilde{\mu}_0 = 1, \ \tilde{\mu}_k = \frac{\tilde{b}_0 \cdots \tilde{b}_{k-1}}{\tilde{a}_1 \cdots \tilde{a}_k}, \qquad k \geqslant 1.$$

Theorem (C. 2014. $\widetilde{Q} \to T$)

(1) Let $\sum_{k=0}^{\infty} \left(\tilde{\mu}_k \tilde{b}_k\right)^{-1} < \infty$. Then $\text{Spec}\big(\widetilde{Q}_{\min}\big)$ is discrete iff

$$\lim_{n\to\infty} \sum_{j=0}^{n} \tilde{\mu}_j \sum_{k=n}^{\infty} \left(\tilde{\mu}_k \tilde{b}_k\right)^{-1} = 0.$$

Discrete spectrum. $\mathbb{Z}_+ = \{0, 1, 2, \cdots\}$

Theorem (Continued)

(2) Let $\sum_{j=0}^{\infty} \tilde{\mu}_j < \infty$. Then Spec$\big(\tilde{Q}_{\max}\big)$ is discrete iff

$$\lim_{n\to\infty} \sum_{j=n+1}^{\infty} \tilde{\mu}_j \sum_{k=0}^{n} \big(\tilde{\mu}_k \tilde{b}_k\big)^{-1} = 0.$$

(3) Let $\sum_{k=0}^{\infty} \big(\tilde{\mu}_k \tilde{b}_k\big)^{-1} = \infty = \sum_{j=0}^{\infty} \tilde{\mu}_j$. Then Spec$\big(\tilde{Q}_{\min}\big)$ = Spec$\big(\tilde{Q}_{\max}\big)$ is not discrete.

$\sum_{i=0}^{\infty} \tilde{\mu}_i \sum_{j=i}^{\infty} \big(\tilde{\mu}_j \tilde{b}_j\big)^{-1} = \infty$

Remove condition "tridiagonal"

The results for Hermitizable tridiagonal matrices $\to$ general Hermitizable ones.

Lemma

$\boldsymbol{A} = (\boldsymbol{a_{ij}})$ is Hermitizable w.r.t. $\boldsymbol{\mu}$, i.e.,

$$\text{Diag}(\boldsymbol{\mu})\boldsymbol{A} = \boldsymbol{A}^{\boldsymbol{H}}\text{Diag}(\boldsymbol{\mu}) \quad \boxed{\boldsymbol{A}^{\boldsymbol{H}} := \bar{\boldsymbol{A}}^*}$$

iff $\boldsymbol{H} := \text{Diag}(\boldsymbol{\mu})^{1/2}\boldsymbol{A}\text{Diag}(\boldsymbol{\mu})^{-1/2}$ is Hermite.

Each theory/algorithm for Hermite $\to$ Hermitizable

Remove condition "tridiagonal"

Theorem (Householder transformation)

For each Hermite H, $\exists$ a sequence of extended reflection matrices $\{U_j\}$ such that for some $\ell \leqslant N$, $U := \prod_{j=0}^{\ell} U_j$ is unitary and $T := UHU^H$ becomes a real, symmetric tridiagonal matrix. ★

$\Rightarrow$Hermitizable

$$U_j = I + (\kappa - 1)uu^H$$

κ: constant with $|\kappa| = 1$, u: unit vector

Remove condition “tridiagonal”

Eigenproblem: started by C.G.J. Jacobi in 1846. 173 years

In 2000, two journals selected

“Top 10 algorithms in 20th century”

Three of them are on matrix eigenproblem,

Householder transformation

Remove condition "tridiagonal"

Theorem (C. 2018)

Each Hermitizable matrix is isospectral to a birth–death Q-matrix.

Furthermore, the discreteness of spectrum of an Hermitizable matrix may be justified by the birth–death Q-matrix, in terms of an approximating procedure. wave $e^{i\theta}$

A: complex $L^2(\mu) \to \widetilde{Q}$: real $L^2(|h|^2 \mathrm{d}x)$

Criteria for discrete spec, common frame

Remove condition "tridiagonal"

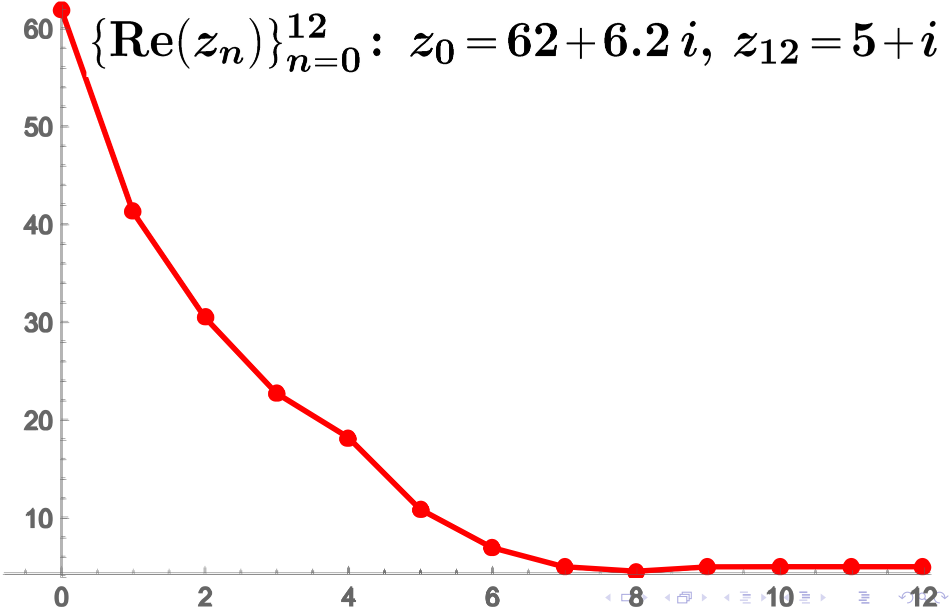
Schrödinger's cat[1935].

Bohr[1927]: "the principle of complementarity". "Electrons are both waves and particles, when you observe them, it exists in the form of particles; without observation, it exists in the form of waves. The so-called wave-particle duality, just depends on the way we observe it. "

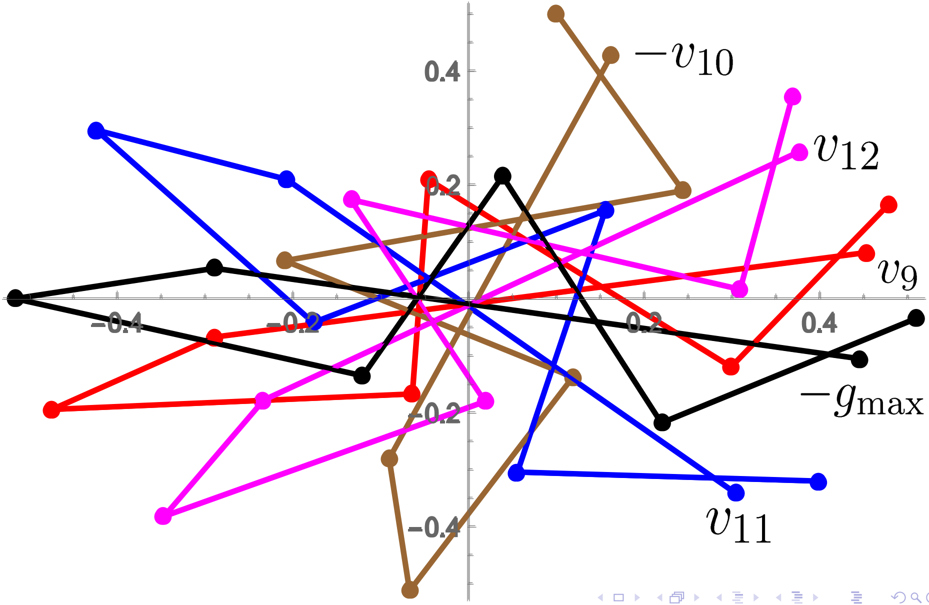
$|h|^2$ Observable, but not h.

"QM–A Simplified Approach", 2019, p.71

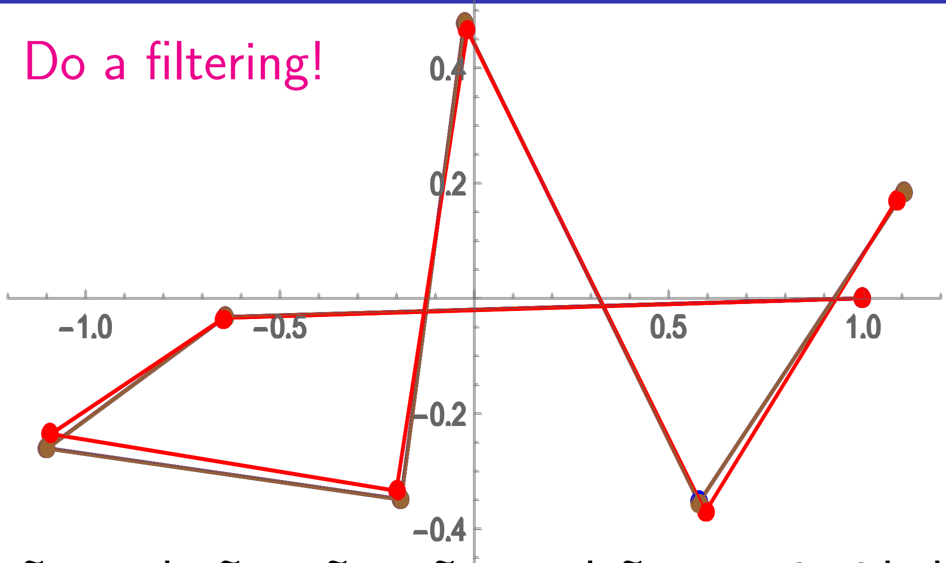
$v_n \to g_{\max}$, $\mathrm{Re}(z_n) \to \max_j \mathrm{Re}(\lambda_j)$.



Vectors $\{v_n\}_{n=9}^{12}$ converge? 7-dim



$\|e^{i\theta_n}x\| = \|x\|$. $\tilde{v} := v/v(0)$. Conformal



$\tilde{v}_9$: red. $\tilde{v}_{10}$, $\tilde{v}_{11}$, $\tilde{v}_{12}$ and $\tilde{g}$ are coincided

News. From Hong-Yu Liu & Hai-Gang Li

Z.K. Minev et al, Nature v. 570, 200–204 (2019/6/3)

Research Letter. "The experimental results demonstrate that the evolution of each completed jump is continuous, coherent and deterministic".

M. Pitkänen (2019/6/20). arXiv

Copenhagen interpretation dead:... YouTube

https://www.youtube.com/watch?v=MNNm1uurr9Y

•Feynman-Kac semigroup:

$$T_t f(x) = \mathbb{E}_x \left\{ \exp\left[\int_0^t V(w_s) \mathrm{d}s \right] f(w_t) \right\}.$$

•h-transform [C. & Xu Zhang, 2014]:

$$L = \Delta + V \to \widetilde{L} = \Delta + \tilde{b}^h \nabla, \quad Lh = 0,$$

L on $L^2(\mathrm{d}x)$ is isospectral to $\widetilde{L}$ on $L^2(\tilde{\mu}) := L^2(|h|^2 \mathrm{d}x)$.

Criteria for discrete spectrum (one dim)

Differential operators. b: magnetic potential

$$L = D^*(aD) - c. \quad D = \partial + b.$$

Theorem (C. 2018)

Dirichlet boundary. Operator L is selfadjoint (formally) on complex $L^2(\mathrm{d}x)$ iff a is Hermitian: $a^H (:= \bar{a}^*) = a$, $\bar{b} = -b$ and $\bar{c} = c$. If so, then

$$(-Lf, f) = (aDf, Df) + (cf, f)$$
$$= \int a|Df|^2 \mathrm{d}x + \int c|f|^2 \mathrm{d}x.$$

Isospectral differential operators

Theorem (C. 2018)

Let L selfadjoint, $L^0 = L - D^*(ab) + c$ and h: $Lh = 0$, $h \neq 0$ (a.e.). Then L is isospectral to

$$\widetilde{L} = L^0 + \left[\mathbb{1}_{[h\neq 0]}\frac{2}{h}(\partial h)^* \mathrm{Re}[a]\right]\partial.$$

$V = D^*(ab) - c, \qquad \tilde{b}^h = [\cdots]$

$$L = \Delta + V, \quad \widetilde{L} = \Delta + \left[\mathbb{1}_{[h\neq 0]}\frac{2}{h}\nabla h\right]\bullet\nabla.$$

Discrete criteria for spectrum in dim one

$$L = a(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + b(x)\frac{\mathrm{d}}{\mathrm{d}x} - c(x)$$

$$a(x) > 0,\ \ c(x) \geqslant 0 \qquad\qquad \text{on } \mathbb{R}$$

$$\mu(\mathrm{d}x) = \frac{e^{C(x)}}{a(x)}\mathrm{d}x, \qquad \nu(\mathrm{d}x) = e^{C(x)}\mathrm{d}x,$$

$$C(x) = \int_\theta^x \frac{b}{a}(y)\mathrm{d}y \qquad \hat{\nu}(\mathrm{d}x) = e^{-C(x)}\mathrm{d}x,$$

$\theta \in \mathbb{R}$: reference point. Symmetric case!

Discrete criteria for spectrum in dim one

Theorem (C. 2014)

Let $Lh = 0$, $h \neq 0$-a.e. on $\mathbb{R}$.

(1) If $\hat{\nu}(h^{-2}) < \infty$, then $\text{Spec}(L_{\min})$ is discrete iff

$$\lim_{x\to\infty}\Big[\mu\big(h^2 \mathbb{1}_{(0,x)}\big)\hat{\nu}\big(h^{-2}\mathbb{1}_{(x,\infty)}\big) + \mu\big(h^2\mathbb{1}_{(-x,0)}\big)\hat{\nu}\big(h^{-2}\mathbb{1}_{(-\infty,-x)}\big)\Big] = 0.$$

Discrete criteria for spectrum in dim one

Theorem (C. 2014)

(2) If $\mu(h^2) < \infty$, then Spec$(L_{\max})$ is discrete iff

$$\lim_{x\to\infty}\Big[\mu\big(h^2\mathbb{1}_{(x,\infty)}\big)\hat{\nu}\big(h^{-2}\mathbb{1}_{(0,x)}\big) + \mu\big(h^2\mathbb{1}_{(-\infty,-x)}\big)\hat{\nu}\big(h^{-2}\mathbb{1}_{(-x,0)}\big)\Big] = 0.$$

(3) If $\hat{\nu}\big(h^{-2}\mathbb{1}_{(-\infty,0)}\big) = \hat{\nu}\big(h^{-2}\mathbb{1}_{(0,\infty)}\big) = \infty = \mu\big(h^2\mathbb{1}_{(-\infty,0)}\big) = \mu\big(h^2\mathbb{1}_{(0,\infty)}\big)$, then Spec$(L)$ is not discrete.

Complex stochastic processes

- SLE theory. One-dim complex BM by P. Lévy. "Conformally Invariant Processes in the Plane". G.F. Lawler^{2005}
- Dirichlet form theory. M. Fukushima and M. Okada^{1987}. "On Dirichlet forms for plurisubharmonic functions". 5 papers
- Probability amplitude theory.
  "Complex Markov chains". V.P. Maslov^{1970}
- "Schrödinger Diffusion Process". R. Aebi^{1996}
- "Stochastic Processes in QP" [M.Nagasawa]^{2000}

Quantum & Modern Mathematics

- Quantum probability. Quantum logic.
- PDE, Hilbert space, spectral theory.
- Operator algebra: C^*-, $\boldsymbol{W}$-, Weyl-, Jordan-, von Neumann-algebra.
- Heisenberg-, Quantum-group.
- Non-commutative geometry. MatLab, book.
- Quantum+Geometry, supersymmetric, string.

"Math Structure of QM", by F.Strocchi, 2$^{\text{nd}}$, 2008
M Atiyah: "Quantum" is still a big word for him.
His dream: establish "union of quantum & math"

http://math0.bnu.edu.cn/~chenmf

The end!

Thank you, everybody!

谢谢大家!

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