Ergodic Convergence Rates of Markov Processes

—Eigenvalues, Inequalities and Ergodic Theory

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Contents

- Introduction.
- Variational formulas of the first eigenvalue by couplings.
- Estimates of inequalities' constant by generalized Cheeger's method.
- A diagram of nine types of ergodicity and a table of explicit criteria in dimension one.

I. Introduction

1.1 Definition. The first (non-trivial) eigenvalue:

$$Q = \begin{pmatrix} -b_0 & b_0 & 0 & 0 & \cdots \\ a_1 & -(a_1+b_1) & b_1 & 0 & \cdots \\ 0 & a_2 & -(a_2+b_2) & b_2 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

$$a_i > 0, \ b_i > 0.$$

$Q\mathbb{1} = 0$. Trivial eigenvalue: $\lambda_0 = 0$.

Question: Next eigenvalue of $-Q$: $\lambda_1 =?$

Elliptic operator in $\mathbb{R}^d$; Laplacian on Riemannian manifolds. Importance: leading term.

1.2 Difficulties

Example 1: Trivial case(two points). Two parameters.

$$\begin{pmatrix} -b & b \\ a & -a \end{pmatrix}, \quad \lambda_1 = a + b.$$

λ_1 is increasing in each of the parameters!

Example 2: Three points. Four parameters.

$$\begin{pmatrix} -b_0 & b_0 & 0 \\ a_1 & -(a_1 + b_1) & b_1 \\ 0 & a_2 & -a_2 \end{pmatrix},$$

$$\lambda_1 = 2^{-1}\Big[a_1 + a_2 + b_0 + b_1 - \sqrt{(a_1 - a_2 + b_0 - b_1)^2 + 4a_1 b_1}\Big].$$

Example 3: Four points.

Six parameters: $b_0, b_1, b_2, a_1, a_2, a_3$.

$$\lambda_1 = \frac{D}{3} - \frac{C}{3 \cdot 2^{1/3}} + \frac{2^{1/3}\,\left(3\,B - D^2\right)}{3\,C},$$

where

$$D = a_1 + a_2 + a_3 + b_0 + b_1 + b_2,$$

$$B = a_3\,b_0 + a_2\,\left(a_3 + b_0\right) + a_3\,b_1 + b_0\,b_1 + b_0\,b_2 + b_1\,b_2 + a_1\,\left(a_2 + a_3 + b_2\right),$$

$$C = \left(A + \sqrt{4\left(3\,B - D^2\right)^3 + A^2}\right)^{1/3},$$

$$A = -2\,a_1^3 - 2\,a_2^3 - 2\,a_3^3 + 3\,a_3^2\,b_0 + 3\,a_3\,b_0^2 - 2\,b_0^3 + 3\,a_3^2\,b_1 - 12\,a_3\,b_0\,b_1 + 3\,b_0^2\,b_1 + 3\,a_3\,b_1^2 + 3\,b_0\,b_1^2 - 2\,b_1^3 - 6\,a_3^2\,b_2 + 6\,a_3\,b_0\,b_2 + 3\,b_0^2\,b_2 + 6\,a_3\,b_1\,b_2 - 12\,b_0\,b_1\,b_2 + 3\,b_1^2\,b_2 - 6\,a_3\,b_2^2 + 3\,b_0\,b_2^2 + 3\,b_1\,b_2^2 - 2\,b_2^3 + 3\,a_1^2\,(a_2 + a_3 - 2\,b_0 - 2\,b_1 + b_2) + 3\,a_2^2\,[a_3 + b_0 - 2\,(b_1 + b_2)] + 3\,a_2\left[a_3^2 + b_0^2 - 2\,b_1^2 - b_1\,b_2 - 2\,b_2^2 - a_3(4\,b_0 - 2\,b_1 + b_2) + 2\,b_0(b_1 + b_2)\right] + 3\,a_1\left[a_2^2 + a_3^2 - 2\,b_0^2 - b_0\,b_1 - 2\,b_1^2 - a_2(4\,a_3 - 2\,b_0 + b_1 - 2\,b_2) + 2\,b_0\,b_2 + 2\,b_1\,b_2 + b_2^2 + 2\,a_3(b_0 + b_1 + b_2)\right].$$

The role of each parameter is completely mazed!

Not solvable when space has more than five points!

Conclusion: Impossible to compute λ_1 explicitly!

Perturbation of eigenvalues

Example: Infinite triangle matrix (Birth-death processes).

$b_i\ (i \geqslant 0)$	$a_i\ (i \geqslant 1)$	λ_1	degree of eigenfun.
$i+\beta$ $(\beta > 0)$	$2i$	1	1
$i+1$	$2i+3$		
$i+1$	$2i+\left(4+\sqrt{2}\right)$		

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$i+1$	$2i+3$	2	
$i+1$	$2i+\left(4+\sqrt{2}\right)$	3	

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$i+1$	$2i+3$	2	2
$i+1$	$2i+\left(4+\sqrt{2}\,\right)$	3	3

Sensitive. In general, it is too hard to estimate λ_1!

Ⅱ. New variational formulas of the first eigenvalue

2.1 Story of the study on λ_1 in geometry

(M, g): compact Riemannian manifold.

Discrete spectrum: $0 = \lambda_0 < \lambda_1 < \cdots$.

- g: Riemannian metric.
- d: dimension. $\mathbb{S}^d$: $D=\pi$, $\text{Ric}=d-1$.
- D: diameter.
- $\text{Ricci}_M \geqslant Kg$ for some $K \in \mathbb{R}$.

Idea: Use geometric quantities d, D and K to estimate λ_k's of Laplacian Δ.

Five books!

- Chavel, I. (1984): Eigenvalues in Riemannian Geometry, Academic Press
- Bérard, P. H. (1986): Spectral Geometry: Direct and Inverse Problem, LNM. vol. 1207, Springer-Verlag. Including 2000 references.
- Schoen, R. and Yau, S. T. (1988): Differential Geometry (In Chinese), Science Press, Beijing, China
- Li, P. (1993): Lecture Notes on Geometric Analysis, Seoul National U., Korea
- Ma, C. Y. (1993): The Spectrum of Riemannian Manifolds (In Chinese), Press of Nanjing U., Nanjing

Eight of the most beautiful lower bounds:

Case 1: $K \geqslant 0$

Author(s)	Lower bound
A. Lichnerowicz (1958)	$\frac{d}{d-1}K$
P. H. Bérard, G. Besson & S. Gallot (1985)	$d\left\{\frac{\int_0^{\pi/2}\cos^{d-1}t\,dt}{\int_0^{D/2}\cos^{d-1}t\,dt}\right\}^{2/d}$ $K=d-1$
P. Li & S. T. Yau (1980)	$\frac{\pi^2}{2\,D^2}$
J. Q. Zhong & H. C. Yang (1984)	$\frac{\pi^2}{D^2}$

Case 2: $K \leqslant 0$

Author(s)	Lower bound
P. Li & S. T. Yau (1980)	$\dfrac{1}{D^2(d-1)\exp\left[1+\sqrt{1+16\alpha^2}\right]}$
K. R. Cai (1991)	$\dfrac{\pi^2}{D^2}+K$
H. C. Yang (1989) & F. Jia (1991)	$\dfrac{\pi^2}{D^2}e^{-\alpha}, \quad \text{if } d \geqslant 5$
H. C. Yang (1989) & F. Jia (1991)	$\dfrac{\pi^2}{2\,D^2}e^{-\alpha'}, \quad \text{if } 2 \leqslant d \leqslant 4$

$$\alpha = D\sqrt{|K|(d-1)}/2,$$
$$\alpha' = D\sqrt{|K|((d-1)\vee 2)}/2.$$

2.2 New variational formulas

Theorem [C. & F. Y. Wang (1997)].

$$\lambda_1 \geqslant \sup_{f\in\mathscr{F}} \inf_{r\in(0,D)} \frac{4f(r)}{\int_0^r C(s)^{-1}\mathrm{d}s \int_s^D C(u)f(u)\mathrm{d}u}$$

Two notations:

$$C(r) = \cosh^{d-1}\left[\frac{r}{2}\sqrt{\frac{-K}{d-1}}\right], \quad r\in(0,D).$$

$$\mathscr{F} = \{f\in C[0,D] : f>0 \text{ on } (0,D)\}.$$

Classical variational formula:

$$\lambda_1 = \inf\{\textstyle\int_M \|\nabla f\|^2 : f\in C^1, \pi(f)=0, \pi(f^2)=1\}$$

Goes back to Lord S. J. W. Rayleigh(1877) or E. Fischer (1905). Gen. R. Courant (1924).

Elementary functions: $1, \sin(\alpha r), \cosh^{1-d}(\alpha r)\sin(\beta r)$,

$$\alpha = D\sqrt{|K|/(d-1)}/2, \qquad \beta = \frac{\pi}{2D}.$$

Corollary 1 [C. & F. Y. Wang (1997)].

$$\lambda_1 \geqslant \frac{dK}{d-1}\left\{1-\cos^d\left[\frac{D}{2}\sqrt{\frac{K}{d-1}}\right]\right\}^{-1}, \quad d>1, K\geqslant 0.$$

$$\lambda_1 \geqslant \frac{\pi^2}{D^2}\sqrt{1-\frac{2D^2K}{\pi^4}\cosh^{1-d}\left[\frac{D}{2}\sqrt{\frac{-K}{d-1}}\right]},$$

$$d>1, \qquad K\leqslant 0.$$

Corollary 2 [C., E. Scacciatelli and L. Yao (2002)].

$$\lambda_1 \geqslant \frac{\pi^2}{D^2}+\frac{K}{2}, \qquad K\in\mathbb{R}.$$

Representative test function:

$$f(r)=\left(\int_0^r C(s)^{-1}\mathrm{d}s\right)^{\gamma},\quad \gamma=\frac{1}{2},1.$$

$$C(s)=\cosh^{d-1}\left[\frac{s}{2}\sqrt{\frac{-K}{d-1}}\right].$$

$$\delta=\sup_{r\in(0,D)}\left(\int_0^r C(s)^{-1}\mathrm{d}s\right)\left(\int_r^D C(s)\mathrm{d}s\right).$$

Corollary 3 [C. (2000)]. $\lambda_1\geqslant\xi_1.$

$$4\delta^{-1}\geqslant(\delta_n')^{-1}\geqslant\xi_1\geqslant\delta_n^{-1}\geqslant\delta^{-1},$$

$$\text{Explicit }(\delta_n')^{-1}\downarrow,\quad \delta_n^{-1}\uparrow.$$

Compact M with convex ∂M. New, Lichnerowicz.

2.3 "Proof": Estimation of λ_1

Step 1. g: eigenfunction: $Lg = -\lambda_1 g$, $g \neq$const.
From semigroups theory,

$$\frac{d}{dt} T_t g(x) = T_t L g(x) = -\lambda_1 T_t g(x).$$

ODE. $T_t g(x) = g(x) e^{-\lambda_1 t}$. (1)

Step 2. Compact space. g Lipschitz w.r.t. ρ: c_g.

Key condition: $\widetilde{T}_t \rho(x,y) \leqslant \rho(x,y) e^{-\alpha t}$. (2)

$$\begin{aligned} e^{-\lambda_1 t}|g(x)-g(y)| &\leqslant \widetilde{T}_t \big|g(x)-g(y)\big| && \text{(by (1))} \\ &\leqslant c_g \widetilde{T}_t \rho(x,y) \\ &\leqslant c_g \rho(x,y) e^{-\alpha t} && \text{(by (2))} \end{aligned}$$

for all t. Hence $\lambda_1 \geqslant \alpha$. General!

$$(2) \iff \widetilde{L}\rho(x,y) \leqslant -\alpha\rho(x,y).$$

Two key points

- "Good" coupling: Classification of couplings. ρ-optimal couplings. "optimal mass transportation".
- "Good" distance: Ord.$\Rightarrow$ none. Reduce to dim. one.

$$L = a(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + b(x)\frac{\mathrm{d}}{\mathrm{d}x}, \qquad C(x) = \int_0^x \frac{b}{a}$$

$$f(x) \in C[0, D), \qquad f|_{(0,D)} > 0$$

$$g(x) = \int_0^x e^{-C(y)}\mathrm{d}y \int_y^D \frac{fe^C}{a} \uparrow$$

$$\rho_f(x, y) = |g(x) - g(y)|.$$

2.4 Triangle matrix (birth-death process)

Notation:

$$\mu_0 = 1, \quad \mu_i = \frac{b_0 b_1 \cdots b_{i-1}}{a_1 a_2 \cdots a_i}, \quad i \geqslant 1.$$

$$Z = \sum_i \mu_i < \infty, \qquad \pi_i = \frac{\mu_i}{Z}.$$

Two notations :

$\mathscr{F} = \{f : f_0 = 0,\ f \text{ is strictly increasing}\}$,

$\mathscr{F}' =$ A slight modification of $\mathscr{F}$.

Theorem [C. (1996, 2000,2001)].

- Dual variational formulas. Write $\bar{f} = f - \pi(f)$.

$$\lambda_1 = \sup_{f\in\mathscr{F}} \inf_{i\geqslant 0} \mu_i b_i (f_{i+1} - f_i) \Big/ \sum\nolimits_{j\geqslant i+1} \mu_j \bar{f}_j.$$

$$\lambda_1 = \inf_{f\in\mathscr{F}'} \sup_{i\geqslant 1} \mu_i b_i (f_{i+1} - f_i) \Big/ \sum\nolimits_{j\geqslant i+1} \mu_j \bar{f}_j.$$

- Explicit estimates. $Z\delta^{-1} \geqslant \lambda_1 \geqslant (4\delta)^{-1}$, where $\delta = \sup_{i\geqslant 1} \sum_{j\leqslant i-1} (\mu_j b_j)^{-1} \sum_{j\geqslant i} \mu_j$.

- Approximation procedure. $\exists$ explicit η_n', η_n'' such that $\eta_n'^{\,-1} \geqslant \lambda_1 \geqslant \eta_n''^{\,-1} \geqslant (4\delta)^{-1}$.

Ⅲ. Basic inequalities and new forms of Cheeger's constants

3.1 Basic inequalities

$(E, \mathscr{E}, \pi)$: prob. space, $L^p(\pi)$, $\|\cdot\|_p$, $\|\cdot\| = \|\cdot\|_2$.

Dirichlet form $(D, \mathscr{D}(D))$ on $L^2(\pi)$.

Poincaré inequality : $\text{Var}(f) \leqslant C D(f), \; C = \lambda_1^{-1}$.

Nash inequality : $\text{Var}(f) \leqslant C D(f)^{1/p} \|f\|_1^{2/q}$

J. Nash (1958) $\quad 1/p + 1/q = 1$

Logarithmic Sobolev inequality (L. Gross, 1976) :

$$\int_E f^2 \log \left(f^2/\|f\|^2\right) \text{d}\pi \leqslant C D(f).$$

3.2 Integral operator.

Symmetric form:

$$D(f) = \frac{1}{2}\int_{E\times E} J(\mathrm{d}x, \mathrm{d}y)[f(y) - f(x)]^2,$$

$$\mathscr{D}(D) = \{f \in L^2(\pi) : D(f) < \infty\},$$

$J \geqslant 0 :$ symmetric, no charge on $\{(x, x) : x \in E\}$.

$$\lambda_1 := \inf\{D(f) : \pi(f) = 0, \ \|f\| = 1\}.$$

Theorem [G. F. Lawler & A. D. Sokal (1988)] :

$$J(\mathrm{d}x, E)/\mathrm{d}\pi \leqslant M, \quad \lambda_1 \geqslant \frac{k^2}{2M}.$$

Six books:

- Chen, M. F. (1992), From Markov Chains to Non-Equilibrium Particle Systems, World Scientific, Singapore
- Sinclair, A. (1993), Algorithms for Random Generation and Counting: A Markov Chain Approach, Birkhäuser
- Colin de Verdière, Y. (1998), Spectres de Graphes, Publ. Soc. Math. France
- Chung, F. R. K. (1997), Spectral Graph Theory, CBMS, **92**, AMS, Providence, Rhode Island L. (1997),

- Saloff-Coste, L. (1997), Lectures on finite Markov chains, LNM **1665**, 301–413, Springer-Verlag
- Aldous, D. G. & Fill, J. A. (1994–), Reversible Markov Chains and Random Walks on Graphs

3.3 New results.

$$J^{(\alpha)}(\mathrm{d}x,\mathrm{d}y) = I_{\{r(x,y)^\alpha>0\}}\frac{J(\mathrm{d}x,\mathrm{d}y)}{r(x,y)^\alpha}$$

$r \geqslant 0$, symmetric. $\alpha \in [0,1]$

$J^{(1)}(\mathrm{d}x, E)/\pi(\mathrm{d}x) \leqslant 1, \quad \pi$-a.e.

New forms of Cheeger's constants

Inequalities	Constant $k^{(\alpha)}$
Poincaré (C.&W.)	$\inf_{\pi(A)\in(0,1)} \frac{J^{(\alpha)}(A\times A^c)}{\pi(A)\wedge\pi(A^c)}$
Nash (C.)	$\inf_{\pi(A)\in(0,1)} \frac{J^{(\alpha)}(A\times A^c)}{[\pi(A)\wedge\pi(A^c)]^{(2q-3)/(2q-2)}}$
LogS (Wang)	$\lim_{r\to 0}\inf_{\pi(A)\in(0,r]} \frac{J^{(\alpha)}(A\times A^c)}{\pi(A)\sqrt{\log[e+\pi(A)^{-1}]}}$
(C.)	$\lim_{\delta\to\infty}\inf_{\pi(A)>0} \frac{J^{(\alpha)}(A\times A^c)+\delta\pi(A)}{\pi(A)\sqrt{1-\log\pi(A)}}$

Theorem.
$k^{(1/2)} > 0 \Longrightarrow$ corresponding inequality holds.

Four papers:

C. and Wang (1998),
C. (1999, 2000),
Wang (2001).

Estimates $$\lambda_1 \geqslant \frac{{k^{(1/2)}}^2}{1+\sqrt{1-{k^{(1)}}^2}}.$$ Can be sharp!

IV. New picture of ergodic theory and explicit criteria

4.1 Importance of the inequalities

$(D, \mathscr{D}(\mathscr{D})) \longrightarrow$ semigroup (P_t): $P_t = e^{tL}$

Theorem [T.M.Liggett(89), L.Gross(76), C.(99)]

- Poincaré ineq. $\Longleftrightarrow \text{Var}(P_t f) \leqslant \text{Var}(f) e^{-2\lambda_1 t}$.
- LogS $\Longrightarrow$ exponential convergence in entropy: $\text{Ent}(P_t f) \leqslant \text{Ent}(f) e^{-2\sigma t}$, where $\text{Ent}(f) = \pi(f \log f) - \pi(f) \log \|f\|_1$.
- Nash ineq. $\Longleftrightarrow \text{Var}(P_t f) \leqslant C \|f\|_1^2 / t^{q-1}$.

Nash inequality weakest?

4.2 Three traditional types of ergodicity.

$$\|\mu - \nu\|_{\text{Var}} = 2\sup_A |\mu(A) - \nu(A)|$$

Ordinary erg. : $\lim_{t\to\infty} \|P_t(x,\cdot) - \pi\|_{\text{Var}} = 0$

Exp. erg. : $\lim_{t\to\infty} e^{\alpha t}\|P_t(x,\cdot) - \pi\|_{\text{Var}} = 0$

Strong erg. : $\lim_{t\to\infty} \sup_x \|P_t(x,\cdot) - \pi\|_{\text{Var}} = 0$

$$\iff \lim_{t\to\infty} e^{\beta t} \sup_x \|P_t(x,\cdot) - \pi\|_{\text{Var}} = 0$$

Classical Theorem :

Strong ergodicity $\Longrightarrow$ Exp. erg. $\Longrightarrow$ Ordinary erg.

LogS $\Longrightarrow$ Exp. conv. in entropy $\Longrightarrow$ Poincaré

4.3 New picture of ergodic theory

Theorem. For rever. Markov proc. with densities,

Nash inequality

$\swarrow\!\!\swarrow$ $\qquad\qquad$ $\searrow\!\!\searrow$

LogS inequality $\qquad\qquad$ L^1-exp. convergence

$\Downarrow$ $\qquad\qquad$ $\|$

Exp. conv. in entropy $\qquad\qquad$ Strong ergodicity

$\Downarrow$ $\qquad\qquad$ $\Downarrow$

Poincaré inequality $\Longleftrightarrow$ Exponential ergodicity

$\Downarrow$

L^2-algebraic ergodicity

$\Downarrow$

Ordinary ergodicity

4.4 Explicit criteria for several types of ergodicity

Birth-death processes

$$\mu_0 = 1, \quad \mu_n = \frac{b_0 \cdots b_{n-1}}{a_1 \cdots a_n}, \ n \geqslant 1;$$
$$\mu[i,k] = \textstyle\sum_{i \leqslant j \leqslant k} \mu_j.$$

Theorem. Ten criteria for birth-death processes are listed in the following table [C. (2001)].

Property	Criterion
Uniq.	$\sum_{n\geqslant 0}\frac{1}{\mu_n b_n}\mu[0,n]=\infty \quad (*)$
Recur.	$\sum_{n\geqslant 0}\frac{1}{\mu_n b_n}=\infty$
Erg.	$(*)$ & $\mu[0,\infty)<\infty$
Exp. erg. Poincaré	$(*)\&\sup_{n\geqslant 1}\mu[n,\infty)\sum_{j\leqslant n-1}\frac{1}{\mu_j b_j}<\infty$
Dis. sp.	$(*)\&\lim_{n\to\infty}\sup_{k\geqslant n+1}\mu[k,\infty)\sum_{j=n}^{k-1}\frac{1}{\mu_j b_j}=0$
LogS	$(*)\&\sup_{n\geqslant 1}\mu[n,\infty)\log[\mu[n,\infty)^{-1}]\sum_{j\leqslant n-1}\frac{1}{\mu_j b_j}<\infty$
Str. erg. L^1-exp.	$(*)\&\sum_{n\geqslant 0}\frac{1}{\mu_n b_n}\mu[n+1,\infty)=\sum_{n\geqslant 1}\mu_n\sum_{j\leqslant n-1}\frac{1}{\mu_j b_j}<\infty$
Nash	$(*)\&\sup_{n\geqslant 1}\mu[n,\infty)^{(q-2)/(q-1)}\sum_{j\leqslant n-1}\frac{1}{\mu_j b_j}<\infty(\varepsilon)$

Contributors:

- Bobkov, S. G. and Götze, F. (1999a, b)
- C. (1991, 1996, 2000, 2001)
- Miclo, L. (1999, 2000)
- Mao, Y. H. (2001, 2002a, b)
- Wang, F. Y. (2001)
- Zhang, H. Z., Lin, X. and Hou, Z. T. (2000)
- Zhang, Y. H. (2001)
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The end!

Thank you, everybody!

32