New Skills of Matrix Computation

Mu-Fa Chen

(Jiangsu Normal U. & Beijing Normal U.)

"数值代数面临的机遇与挑战"研讨论坛

2022 年 4 月 9 日

Outline

- Isospectral transformation: economy $A \to P$.
- Symmetrizing and the first/second quasi-symmetrizing techniques.
- Mixing of three classical algorithms.
- A coupling technique.
- Hermitizable matrix. New framework, new spectral theory, and new algorithm of quantum mechanics (QM).

An isospectral transformation

Nonnegative, irreducible matrix $A = (a_{ij})$
$\to$ isospectral transition prob matrix $P = (p_{ij})$.
$P\mathbb{1} = \mathbb{1}$. Set $\sigma = A\mathbb{1}$, def $\mathrm{D}_\alpha = \mathrm{DiagMatrix}[\alpha]$,

$\mathbb{1} :=$ vector with components 1. Then

$$\mathrm{D}_{\sigma^{-1}} A\mathbb{1} = \mathbb{1}, \quad \sigma^\alpha := \text{vector } (\sigma_k^\alpha, \forall k).$$

Irreducibility: $\forall \{i, j\}, i \neq j, \exists$ path $i = i_0 \rightsquigarrow i_n = j$:

$$a_{i_0 i_1} a_{i_1 i_2} \cdots a_{i_{n-1} i_n} > 0.$$

A connected graph.

Isospectral transformation $\tilde{A} = \mathrm{D}_{w^{-1}} A\, \mathrm{D}_w$

Perron-Frobenius Thm $\exists!$ maximal $\rho(A)$,
left $u > 0$ (row), right $v > 0$ (column).

Three characteristics!

$$w = (w^{(k)} \neq 0,\ \forall k)$$

$$\frac{A}{\rho(A)} \mathrm{D}_v \mathbb{1} = \frac{Av}{\rho(A)} = v = \mathrm{D}_v \mathbb{1}$$

$$\Rightarrow \mathrm{D}_{v^{-1}} \frac{A}{\rho(A)} \mathrm{D}_v \mathbb{1} = \mathbb{1}.$$

$=$:P

— — — — — — — — — — — — — — — —

$$\mathrm{D}_{w^{-1}} \frac{A}{\rho(A)} \mathrm{D}_w \mathbb{1} = \mathbb{1} \quad \Rightarrow \quad w = v.$$

$v \to u$?

[C. 1992] Isospectral transformation: $A \to P$

$$P^n = \mathrm{D}_{v^{-1}} \frac{A^n}{\rho(A)^n} \mathrm{D}_v$$ (isospectral. unique).

$\forall \{i, j\}$: $p_{ij}^{(n)} > 0 \Leftrightarrow a_{ij}^{(n)} > 0$. Irreducibility!

Theorem (C. 2022)

For $x_n = x_0 A^{-n}$ & $\mu_n = \mu_0 P^{-n}$, $n \geqslant 0$, we have

$$\mu_n = \rho(A)^n x_n \odot v \longleftrightarrow x_n = \rho(A)^{-n} \mu_n \odot v^{-1}$$

for $n \geqslant 0$ where

$$x \odot y := (x^{(k)} y^{(k)}) \text{ if } x = (x^{(k)}) \text{ \& } y = (y^{(k)}).$$

L.K. Hua's model on economic opmitization

- Vector of products: $\boldsymbol{x} = \left(\boldsymbol{x}^{(1)}, \cdots, \boldsymbol{x}^{(d)}\right)$.
- Structure matrix (matrix of consumption coefficients): $\boldsymbol{A}=(\boldsymbol{a}_{ij})$. $\geqslant \mathbf{0}$, Irreducible.
- Input-output method: $\boldsymbol{x}_n = \boldsymbol{x}_0 \boldsymbol{A}^{-n}$, $\boldsymbol{n} \geqslant 1$.
- Optimal initial: $\boldsymbol{x}_0 = \boldsymbol{u}$, speed: $\boldsymbol{\rho}(\boldsymbol{A})^{-1}$.
- Collapse: If $\boldsymbol{x}_0 \neq \boldsymbol{u}$, then $\exists \boldsymbol{n}_0$ s.t. $\boldsymbol{x}_{n_0}$ contains $\pm$ components.

  unique equilibrium $\boldsymbol{u}$!
- $\boldsymbol{A} \to \boldsymbol{A}_\alpha := (1-\boldsymbol{\alpha})\boldsymbol{A} + \boldsymbol{\alpha} \boldsymbol{I}$, $\boldsymbol{\alpha} \in (\mathbf{0}, \mathbf{1})$ $\boldsymbol{P}_\alpha$

  $\boldsymbol{\rho}(\boldsymbol{A}_\alpha) = (1-\boldsymbol{\alpha})\boldsymbol{\rho}(\boldsymbol{A}) + \boldsymbol{\alpha}$, $\{\boldsymbol{A}_\alpha\}$ have same $\boldsymbol{u}$ & $\boldsymbol{v}$

  $\boldsymbol{\alpha}$: consumption parameter. H: 1984–87, C: 2021

Applications of transformation $A \to P$

- Alternative proof of Hua's collapse theorem, using P instead of A. 1992/8 [1989/9].
- Stability: computing collapse time.
- Algorithm for computing maximal eigenpair.
  Algorithm for computing $(\rho(A), v)$?
- ProductRank in economy.
- Optimization of economic structure matrix.

Last three enable us going to practice!

Commands on the study of economics

This year's report of government work has 81 times used the word "stability", which means "balance". Over the past few decades, it has often deviated from equilibrium. Hence, it is the time for further studying.

Now, mathematics has entered the front line of many fields of natural science and technology (weather forecast, medical treatment and surgery, etc.), it is hard to imagine why mathematics has not really entered the economy. We are looking forward to the last two items above. 谢颖超教授

Hua's model: two products

$$\boldsymbol{A} = \frac{1}{100}\begin{bmatrix} 25 & 14 \\ 40 & 12 \end{bmatrix},$$

$$\rho(\boldsymbol{A}) = \frac{1}{200}\left(37+\sqrt{2409}\right),$$

$$\approx \color{red}{0.43}0408, \quad 1/\rho(\boldsymbol{A}) \approx 2.3$$

$$\boldsymbol{u} = \left(\frac{5}{7}\left(13+\sqrt{2409}\right),\ 20\right),$$

$$\approx \left(\color{red}{44.343}97483,\ 20\right)$$

Computing collapse time by $\boldsymbol{A}$ and $\boldsymbol{P}$:

$x_0 = (44.344, 20)$. $T_{x_0} = 8$. $1/\rho(A) \approx 2.3$

n	1	2	3
x_n	(103.028, 46.4677)	(239.375, 107.96)	(556.111, 250.868)
n	4	5	6
x_n	(1292.85, 582.247)	(2990.66, 1362.96)	(7165.52, 2998.2)
n	7	8	
x_n	(13054.5, 9754.73)	(89821.2, -23501.9)	

$\mu_0 = (34.4118, 20)$. $T_{\mu_0} = 8$. $\rho(P) = 1$

n	1	2	3
μ_n	(34.4118, 20.0001)	(34.4122, 19.9996)	(34.4092, 20.0026)
n	4	5	6
μ_n	(34.4303, 19.9815)	(34.28, 20.1318)	(35.351, 19.0608)
n	7	8	
μ_n	(27.7201, 26.6917)	(82.0905, -27.6787)	

Differece $\frac{89821.2}{82.0905} = 1094$. Only 2×2 matrix.

Ex. 1 (Symmetrizable, symmetrizing)

BD-type matrix $Q \sim (1, -3, 2)$. flat nonzero elements

$$\lambda_k = 3 - 2\sqrt{2}\,\sin\frac{k\pi}{d+1}, \quad k = 1, \ldots, d,$$

$$g_k(j) = \left(\frac{1}{2}\right)^{j/2} \sin\frac{jk\pi}{d+1} \quad j, k = 1, \ldots, d.$$

Mathematica works up to $d = 81$ only. When $d = 2050$,(amplitude of g_1)$= 10^{307}$. Asymmetric!

Symm. tech. $Q^{\text{sym}} \sim (\sqrt{2}, -3, \sqrt{2})$, (amplitude of g_1) $= \sqrt{2}$. When $d = 10^4$, computed in 8's.

Ex. 2 (Symmetrizable, symmetrizing)

$$A = (a_{ij})_{i,j=1}^{d}, \quad a_{ij} = \frac{2^{\min\{i,j\}} - 1}{2^j}.$$ Not flat

Mathematica 'Eigensystem' works up to $d \leqslant 11$.

MatLab 'eig': $d \leqslant 50$. Output v contains $\pm$ terms

Lemma ($\exists \mu > 0$ s.t. $\mu_i a_{ij} = \mu_j a_{ji}, \forall i, j$)

A symmetrizable: $\exists\, \mu > 0$ s.t. $D_\mu A = A^* D_\mu$

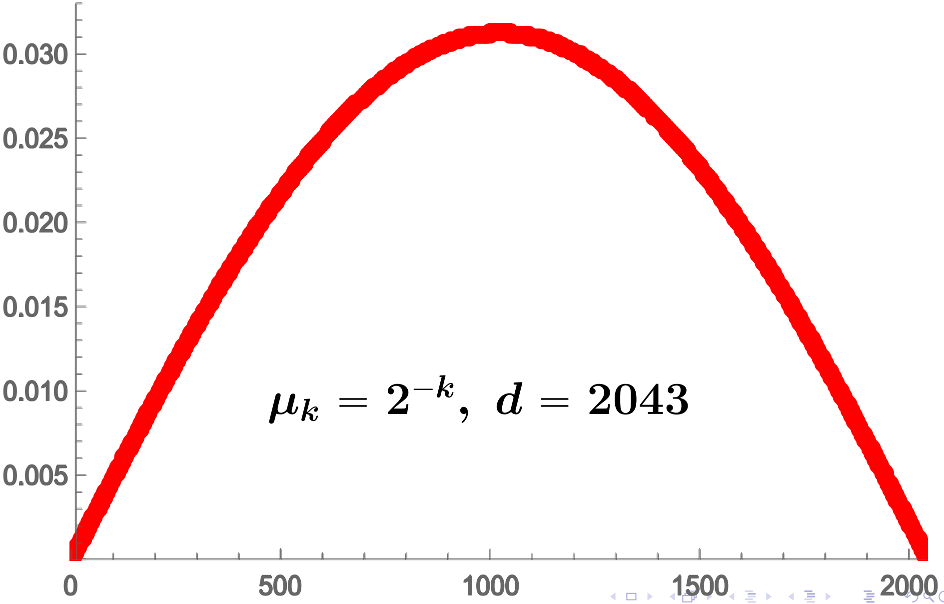
$\Leftrightarrow \hat{A} := D_{\mu^{1/2}} A\, D_{\mu^{-1/2}}$ symmetric $\hat{A}$: symmetrizing of A

where $\mu^\alpha = (\mu_k^\alpha, \forall k)$. A & $\hat{A}$ isospectral,

$g_A = D_{\mu^{-1/2}}\, g_{\hat{A}}$. $\mu_k = 2^{-k}$

Theory on this topic (Z.T. Hou & C. 1979).

Ex. 2 $\max(v)/\min(v) = 648.912$



Ex. 3 $A=(a_{ij})_{i,j=1}^d$, Not symmetrizable!

$$a_{ij} = \frac{2^{\min\{i,j\}}-1}{2^j} + \mathbb{1}_{\{j\leqslant i\}}.$$

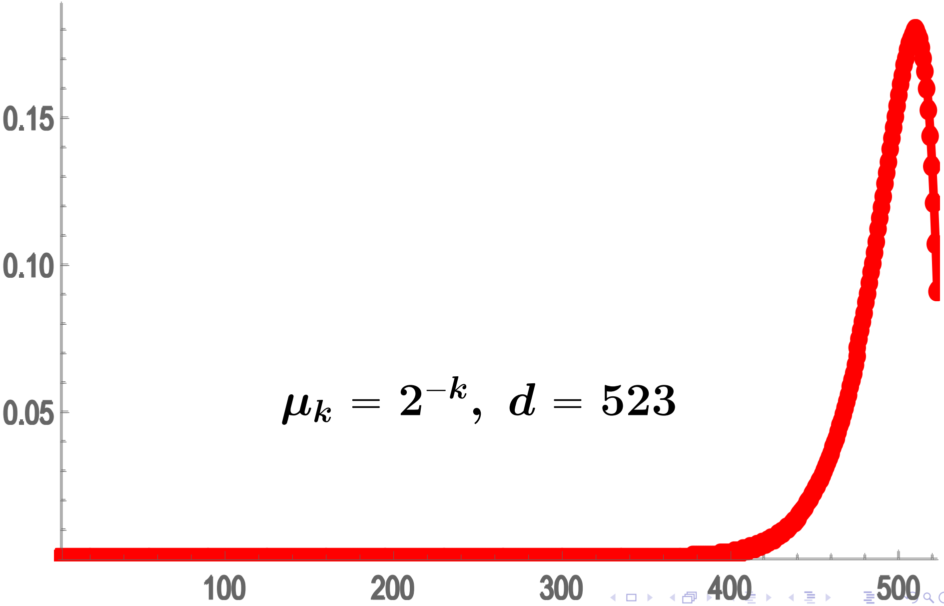
First quasi-symmetrizing technique: use $\hat{A}$ with suitable $\mu>0$ instead of A in the computation. The measure μ can be chosen arbitrarily, but it is reasonably to choose the one which is the symmetrizing measure of A, whenever A is symmetrizable. The solution of the last μ goes as follows. Let $Q=A-\mathrm{D}_{A\mathbb{1}}$. By the irreducibility, the maximal left eigenpair $(0,\mu)$ of Q

$$\mu Q=0,\ \ \mu=(\mu_1,\mu_2,\ldots,\mu_d),\ \ \ \mu_1=1$$

is what we required.

adopt $\mu_k=2^{-k}$ for simplicity

Ex. 3 $\max(v)/\min(v) = 1.79 \times 10^{14}$



Second quasi-symmetrizing technique

- First quasi-symmetrizing technique: leveling the matrix elements.
- Second quasi-symmetrizing technique: leveling the maximal eigenvector.

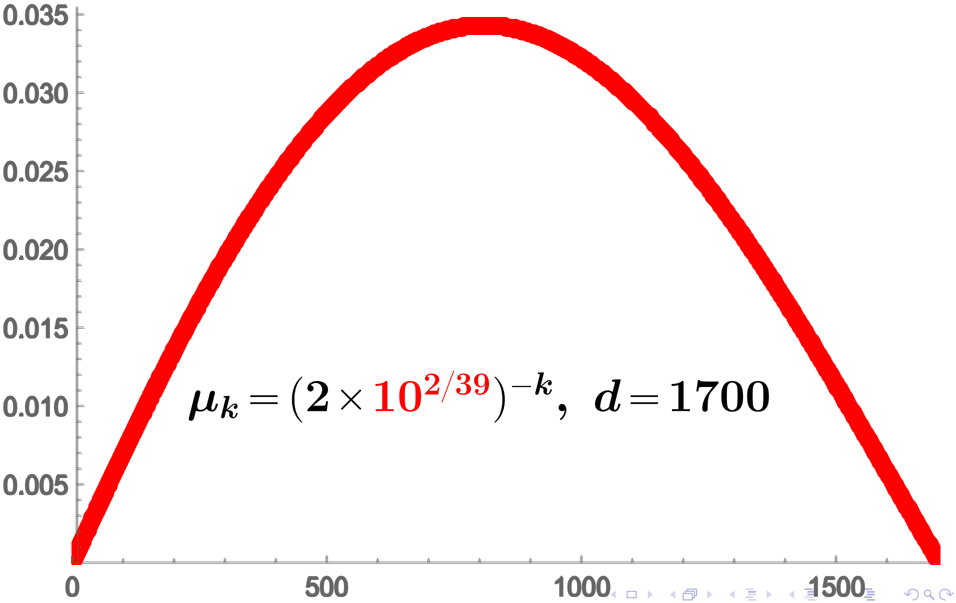
C.+Y.S. Li (2019): Make two endpoints at the same level.

$$\boldsymbol{\mu_k = \frac{1}{2^k} \quad \to \quad \mu_k = \frac{1}{2^k \cdot 10^{2k/39}}}.$$

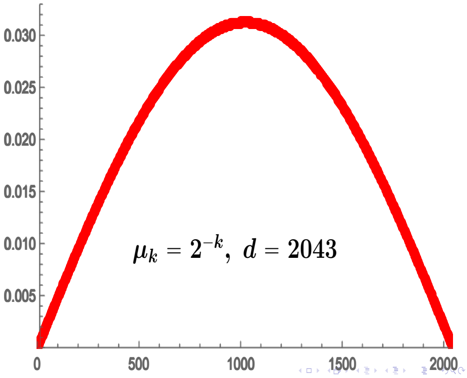
The last power: found by optimal search.

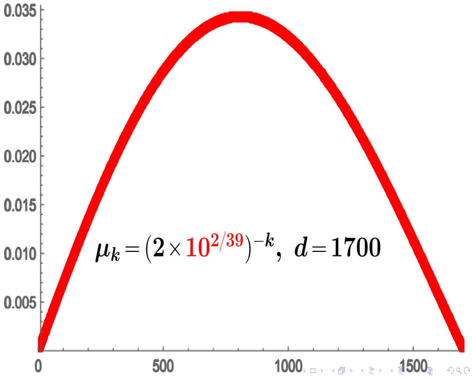
$$\boldsymbol{\frac{1.79\times10^{14}}{480} \approx 3.73 \times 10^{11}, \qquad \frac{1700}{523} \approx 3.25}$$

Second technique $\max(v)/\min(v) = 480$



Comparison of symmetrizable/not symm.





- Factor $10^{2/39}$, direct search (optimization).
- Method for leveling the maximal eigenvector v?

Algorithm for computing maximal eigenpair

Aim: Making the required eigenvector to be a nearly constant one.

$$D_{v^{-1}} A D_v \mathbb{1} = \rho(A)\mathbb{1} = \text{constant}\, \mathbb{1}$$

If $\|\tilde{v} - v\|_\infty \ll \varepsilon$, Then $\tilde{A} := D_{\tilde{v}^{-1}} A D_{\tilde{v}}$ satisfies

$$\tilde{A}\mathbb{1} \approx \text{constant}\, \mathbb{1}.$$

Hence we can use $\tilde{A}$ instead of A to compute the maximal eigenpair. Since the aimed eigenvector of $\tilde{A}$ is nearly a constant one, the convergence speed of the computation should be very fast (cf. C. 2022, in Chinese).

Mixing of three classical algorithms

Big data, AI, Machine Learning, tools are limited.

- Power Iteration (PI), widely used. Less effective.
- Inverse Power Iteration (IPI). How to choose shift? Rayleigh Quotient: very dangerous. Except $A \geqslant \mathbf{0}$, nearly no practical estimates.
- Large scale, eigenvector contains many nearly zero components, often ignored. Reordering

IPI with varying shifts (IPI$_v$), and
IPI with fixed shift (IPI$_f$). Mixing of PI, IPI$_v$, IPI$_f$.

Mixing of three classical algorithms

Take the advantage and avoid the disadvantage of each of these three algorithms.

Allow matrix to be real/complex. Top six eigenpairs

Careful design for shifts of IPI, esp. for IPI_v.

Ex.'s, scale $\mathbf{6} \times \mathbf{10}^4$, $\mathbf{2} \times \mathbf{10}^6$, sparse, using PC.

C. & R.R. Chen (2022). Top Eigenpairs of Large Scale Matrices. CSIAM Trans. Appl. Math. 2022, 3 (1): 1–25.

Z.G. Jia, H.K. Pang. Two preprints (2022)

Trouble of non-symmetry

$$E = \{k \in \mathbb{Z}_+ : 0 \leqslant k < N+1\}.$$

$Q \sim (1, -3, 2)$ BD-type $\longrightarrow$ BD: $\tilde{Q} \sim (+, -?, +)$

$$\tilde{Q} = \begin{pmatrix} -3 & \frac{2^2-1}{2-1} & & & 0 \\ \frac{2^2-2}{2^2-1} & -3 & \frac{2^3-1}{2^2-1} & & \\ & \frac{2^3-2}{2^3-1} & -3 & \frac{2^4-1}{2^3-1} & \\ & & \ddots & \ddots & \ddots \\ 0 & & & \frac{2^{N+1}-2}{2^{N+1}-1} & -3 \end{pmatrix}$$

$$\tilde{Q}\mathbb{1} = \mathbb{0}', \quad \mathbb{0}' := (0, \ldots, 0, ?)^*, \ ? \leqslant 0.$$

$\widetilde{Q} \to Q^{\text{sym}}$. For $\widetilde{Q}/Q^{\text{sym}}$, $\frac{\text{Max}(v)}{\text{Min}(v)} \approx 10^{306}/1.4$

$$Q^{\text{sym}} = \begin{pmatrix} -3 & \sqrt{2} & & & 0 \\ \sqrt{2} & -3 & \sqrt{2} & & \\ & \sqrt{2} & -3 & \sqrt{2} & \\ & & \ddots & \ddots & \ddots \\ 0 & & & \sqrt{2} & -3 \end{pmatrix},$$

However, Q^{sym} becomes BD-type, but not BD.

Fall into an infinite loop.

Be alive in desperation! 2018/3/13 处于绝境. 奇思妙想, 绝地逢生

Coupling of $\tilde{Q}$ and Q^{sym}

Q [BD-type] $\xrightarrow{h}$ $\tilde{Q}$ [BD] $\xrightarrow{\text{sym}}$ Q^{sym} [BD-type]

Iteration equation:

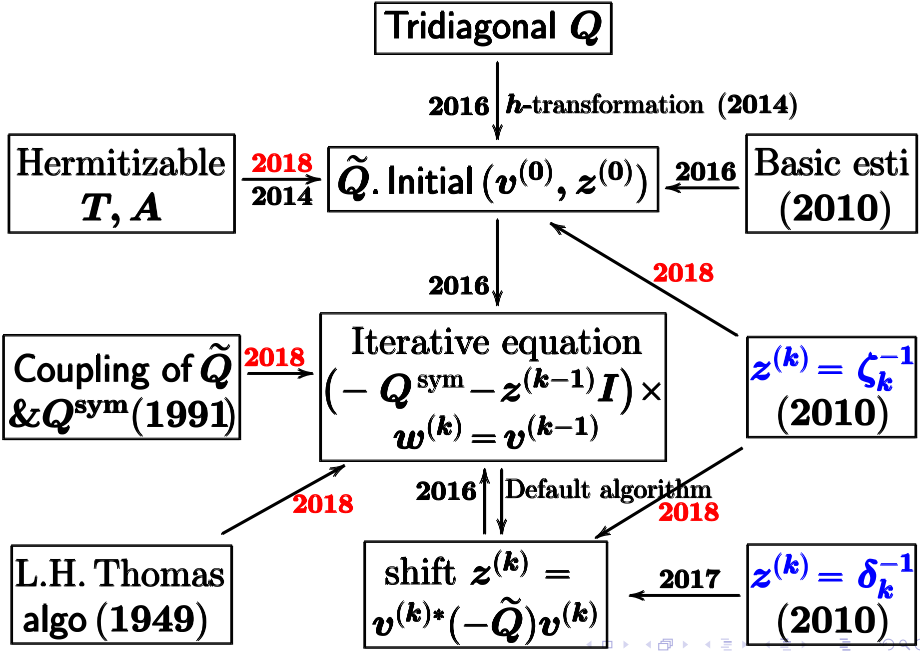
$$\left(-\tilde{Q} - z^{(k-1)}\right) w^{(k)} = v^{(k-1)} \qquad \text{— original}$$

$$\left(-Q^{\text{sym}} - z^{(k-1)}\right) w^{(k)} = v^{(k-1)} \qquad \text{— new}$$

$\tilde{Q} \to Q^{\text{sym}}$, at the same time, use $z^{(k-1)}$ deduced from $\tilde{Q}$ for which a complete theory.

想法来之不易, 成功弥足珍贵

$O(d)$ algorithm for tridiagonal matrix



Symmetrizable→Hermitizable $[a_{ii} \text{ real}]$

$A = (a_{ij})$ is Hermitizable [可厄米] if

$\exists \mu = (\mu_k > 0)$ s.t. $\mu_i a_{ij} = \mu_j \bar{a}_{ji}$ $\forall i, j$. co-zero

A: symmetric/selfadjoint on complex $L^2(\mu)$

$$\mathrm{D}_\mu A = A^H \mathrm{D}_\mu \quad \boxed{A^H := (\bar{A})^*}$$

$$\Leftrightarrow \mathrm{D}_\mu A\, \mathrm{D}_\mu^{-1} = A^H$$

$$\Leftrightarrow \mathrm{D}_\mu^{1/2} A \mathrm{D}_\mu^{-1/2} = \mathrm{D}_\mu^{-1/2} A^H \mathrm{D}_\mu^{1/2}$$

$$\boxed{=: \hat{A} \text{ Hermitizing of } A}$$

Theory/algorithm of Hermitian matrix → Hermitizable one

Criterion for Hermitizability

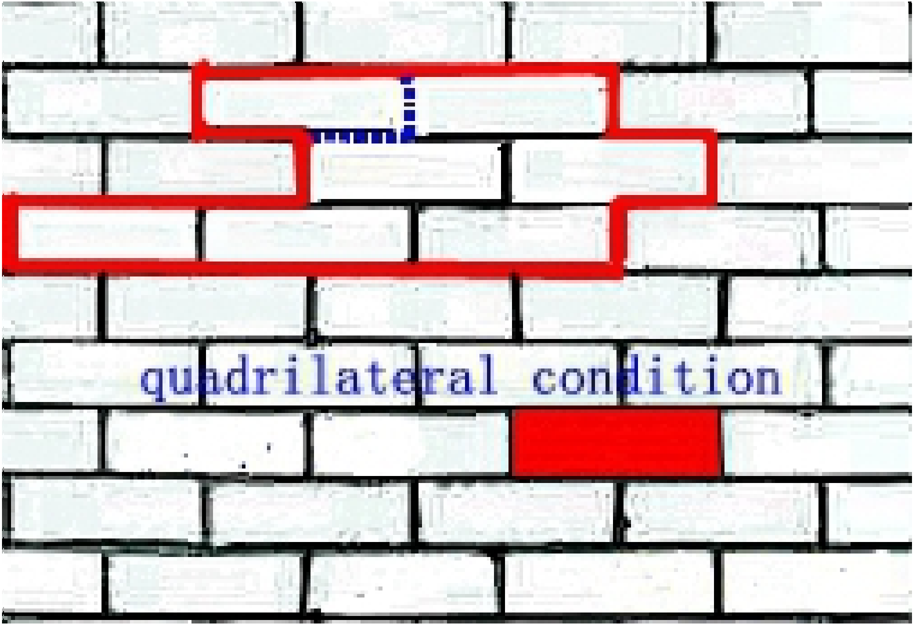
Theorem (C. 2018)

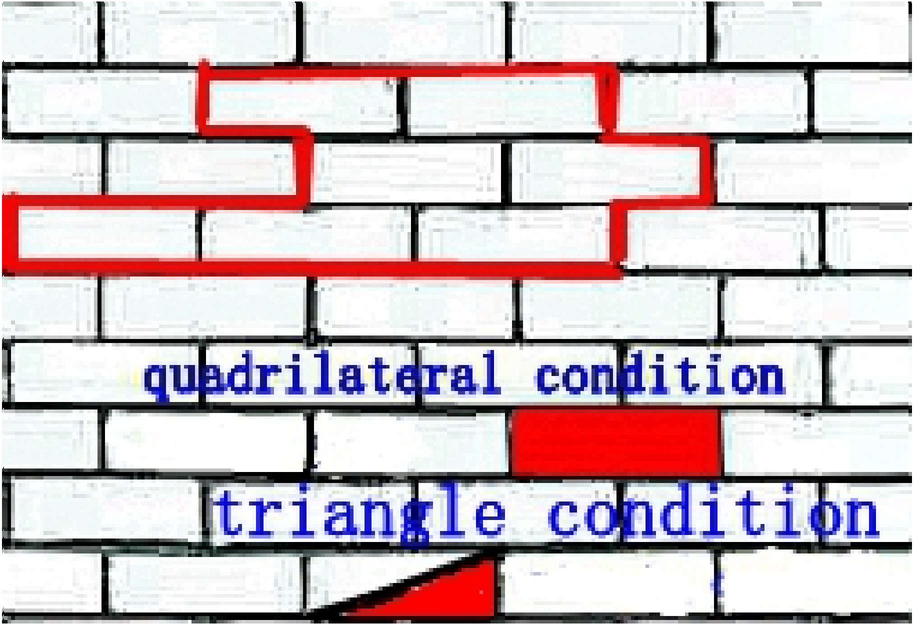
Complex $A = (a_{ij})$ is Hermitizable iff two conditions hold simultaneously.

- For each pair i, j, either $a_{ij} \& a_{ji} = 0$ or $a_{ij}a_{ji} > 0$ ($\Leftrightarrow a_{ij}/\bar{a}_{ji} > 0$).
- The circle condition holds for each smallest closed path without round-trip.

Path: $i_0 \to i_1 \to \cdots \to i_n = i_0$, $a_{i_k i_{k+1}} \neq 0$

$\Rightarrow a_{i_0 i_1} \cdots a_{i_{n-1} i_n} = \bar{a}_{i_n i_{n-1}} \cdots \bar{a}_{i_1 i_0}$. Circle cond.


quadrilateral condition


quadrilateral condition
triangle condition

Tridiagonal/Birth-death (BD) matrix
$T\sim(a_k, -c_k, b_k),\ \ E=\{k\in\mathbb{Z}_+ : 0\leqslant k<N+1\}$

$$\begin{matrix} T \\ Q \end{matrix} = \begin{pmatrix} -c_0 & b_0 & & & 0 \\ a_1 & -c_1 & b_1 & & \\ & a_2 & -c_2 & b_2 & \\ & & \ddots & \ddots & b_{N-1} \\ 0 & & & a_N & -c_N \end{pmatrix},$$

[T] $(a_k), (b_k), (c_k)$: complex sequences.

[Q] BD: $a_k>0,\ b_k>0,\ c_k=a_k+b_k,\ c_N\geqslant a_N$.

Tridiagonal matrix: $T \sim (a_k, -c_k, b_k)$

Theorem

The tridiagonal T is Hermitizable iff the following two conditions hold simultaneously.

- The diagonals (c_k) are real.
- Either $a_{i+1} \& b_i = 0$ or $a_{i+1} b_i > 0$

  分块 $(\Leftrightarrow b_i / \bar{a}_{i+1} > 0)$.

Then

$$\mu_0 = 1, \quad \mu_k = \mu_{k-1} \frac{b_{k-1}}{\bar{a}_k}.$$

Theorem/Algorithm

Given Hermitizable $T\sim(a_k, -c_k, b_k)$ with $c_k \geqslant |a_k| + |b_k|$ (or $\tilde{c}_k = c_k + m$) Then $\exists$ an explicit birth–death matrix $\tilde{Q} \sim (\tilde{a}_k, -\tilde{c}_k, \tilde{b}_k)$ such that T is isospectral to $\tilde{Q}$.

In general, we have $\tilde{c}_N \geqslant \tilde{a}_N$. We assume that $\tilde{c}_N > \tilde{a}_N$ in what follows. The case $\tilde{c}_N = \tilde{a}_N$ was also treated in the published papers (2018, 2019).

Explicit $u_k := a_k b_{k-1} > 0$ & $c_k =: \tilde{c}_k$

$$\tilde{b}_k = c_k - \cfrac{u_k}{c_{k-1} - \cfrac{u_{k-1}}{c_{k-2} - \cfrac{u_{k-2}}{\ddots \; \cfrac{}{c_2 - \cfrac{u_2}{c_1 - \cfrac{u_1}{c_0}}}}}}$$

$$\tilde{b}_k = c_k - u_k / \tilde{b}_{k-1}, \quad \tilde{b}_0 = c_0$$

$$\tilde{a}_k = c_k - \tilde{b}_k, \quad k < N; \qquad \tilde{a}_N = u_N / \tilde{b}_{N-1}.$$

Ex: $Q \sim (1, -3, 2) \to \tilde{Q}.$

Theorem

Every irreducible$^+$ Hermitizable tridiagonal matrix T on complex $L^2(\mu)$ is isospectral to a birth-death (BD) Q-matrix $\tilde{Q}$ on real $L^2(\tilde{\mu})$: $\tilde{\mu} = |h|^2\mu$, where $Th = 0$ on $[0, N)$. Explicit h

$$h_0 = 1, \quad h_n = h_{n-1}\frac{\tilde{b}_{n-1}}{b_{n-1}}, \quad 1 \leqslant n \leqslant N.$$

$$\tilde{Q} = D_h^{-1}\, T\, D_h.$$

Remove "tridiagonal" condition

$\boldsymbol{A}$ is Hermitizable $\Longleftrightarrow$ $\mathrm{D}_{\mu}^{1/2}\boldsymbol{A}\,\mathrm{D}_{\mu}^{-1/2}$ is Hermitian

Hermite $\xrightarrow{\boldsymbol{U}}$ tridiagonal, real, symmetric

$\rightarrow$ BD (by our result) [Blocks]

Eigenproblem: C.G.J. Jacobi in 1846, 173 years.

In 2000, two journals selected

"Top 10 algorithms in 20th century". $\mathbf{1/3}$ matrix: Householder transformation

Hermite$\rightarrow$Hermitizable: Homogeneous medium $\rightarrow$ Inhomogeneous medium $\rightarrow$ BD.

Differential operators

Let $\boldsymbol{a} = (\boldsymbol{a}_{ij})_{i,j=1}^{d}$, $\boldsymbol{b} = (\boldsymbol{b}_i)_{i=1}^{d}$ and $\boldsymbol{c}$ are complex but $\boldsymbol{V}$ is real on $\mathbb{R}^d$. Define $\mathrm{d}\boldsymbol{\mu} = \boldsymbol{e}^{\boldsymbol{V}}\mathrm{d}\boldsymbol{x}$ and $\boldsymbol{L} = \nabla(\boldsymbol{a}\nabla) + \boldsymbol{b}\cdot\nabla - \boldsymbol{c}$.

Theorem (C. & J.Y. Li 2020)

Dirichlet boundary. Operator $\boldsymbol{L}$ is Hermitizable w.r.t. $\boldsymbol{\mu}$ iff $\boldsymbol{a}^{\boldsymbol{H}} = \boldsymbol{a}$ and

$$\mathrm{Re}\,\boldsymbol{b} = (\mathrm{Re}\,\boldsymbol{a})(\nabla \boldsymbol{V}),$$

$$2\,\mathrm{Im}\,\boldsymbol{c} = -\big((\nabla \boldsymbol{V})^{*} + \nabla^{*}\big)\big((\mathrm{Im}\,\boldsymbol{a})(\nabla \boldsymbol{V}) + \mathrm{Im}\,\boldsymbol{b}\big).$$

Isospectral differential operators

Theorem (C. & J.Y. Li 2020)

Denote $\mathscr{D}(L)$ be the domain of the Hermitizable L as above in $L^2(\mu)$ and let h: $Lh = 0$, $h \neq 0$ (a.e.). Then L is isospectral to $(\tilde{L},\ \mathscr{D}(\tilde{L}))$:

$$\begin{cases} \tilde{L} = \nabla(a\nabla) + \tilde{b}\cdot\nabla - 0, \\ \mathscr{D}(\tilde{L}) = \{\tilde{f} \in L^2(\tilde{\mu}) : \tilde{f}h \in \mathscr{D}(L)\}; \end{cases}$$

$$\tilde{b} := b + 2\,\mathrm{Re}(a)\mathbb{1}_{[h\neq 0]}\frac{\nabla h}{h}, \qquad \tilde{\mu} := |h|^2\mu.$$

New framework, new spectral theory, and new algorithm of QM. 1979→Statis. Phys., 2018→QM

References

[1] Economics. New Progress on L.K. Hua’ s Optimization Theory of Economics (in Chinese). Chin. J. Appl. Probab. Statis. 2022, 38(2): 159—178.

[2] Quasi-symmetrizing [1]+. With Y.S. Li, Improved global algorithms for maximal eigenpair. Front. Math. China 2019, 14(6): 1077–1116.

[3] Hermitizable operators. Hermitizable, isospectral complex matrices or differential operators.Front. Math. China 2018, 13(6): 1267 – 1311.

References

[4] QM[3]+. A New Mathematical Perspective of Quantum Mechanics (in Chinese). Advances in Math. (China) 2021, 50(3): 321–334.

[5] Coupling technique [3]+. With Y.S. Li, Development of powerful algorithm for maximal eigenpair. Front. Math. China 2019, 14(3): 493–519.

[6] Mixing of three algorithms. With Z.G. Jia and H.K. Pang. Computing top eigenpairs of Hermitizable matrix. Front. Math. China 2021, 16(2): 345–379.

[7] Mixing[6]+. With R.R. Chen. Top Eigenpairs of Large Scale Matrices. CSIAM Trans. Appl. Math. 2022, 3(1): 1–25.

数学研究院(2019.11), 办公大楼(建设中).
富有前景的迷人新方向: 数理经济等应用数学, 计算, 量子力学, 随机数学, 各种交叉研究.
急需一批人才, 有志向和能力的年青学者, 有基础和经验的访问学者, 高层次人才...

http://math0.bnu.edu.cn/~chenmf

The end!

Thank you, everybody!

谢谢大家!

40