Potential Analysis, Vol. 23, No. 4, 2005, 303-322

CAPACITARY CRITERIA FOR POINCARÉ-TYPE INEQUALITIES

Mu-Fa Chen

(Beijing Normal University)
(Received: 30 September 2003; accepted: 10 February 2005)

Abstract. The Poincaré-type inequality is a unification of various inequalities including the F-Sobolev inequalities, Sobolev-type inequalities, logarithmic Sobolev inequalities, and so on. The aim of this paper is to deduce some unified upper and lower bounds of the optimal constants in Poincaré-type inequalities for a large class of normed linear (Banach, Orlicz) spaces in terms of capacity. The lower and upper bounds differ only by a multiplicative constant, and so the capacitary criteria for the inequalities are also established. Both the transient and the ergodic cases are treated. Besides, the explicit lower and upper estimates in dimension one are computed.

1. Introduction. In this section, we recall some necessary notation and state the main results of this paper.

Let E be a locally compact separable metric space with Borel σ-algebra $\mathscr{E}$, μ an everywhere dense Radon measure on E, and $(D, \mathscr{D}(D))$ a regular Dirichlet form on $L^2(\mu) = L^2(E;\mu)$. The starting point of our study is the following result, due to V. G. Maz'ya (1973) [cf. Maz'ya [17] for references] in the typical case and Z. Vondraček [22] in general. Its proof is simplified recently by M. Fukushima and T. Uemura [10].

Theorem 1.0. For a regular transient Dirichlet form $(D, \mathscr{D}(D))$, the optimal constant A in the Poincaré inequality

$$\|f\|^2 = \int_E f^2 \mathrm{d}\mu \leqslant A\, D(f), \qquad f \in \mathscr{D}(D) \cap C_0(E), \tag{1.1}$$

satisfies $B \leqslant A \leqslant 4B$, where $\|\cdot\|$ is the norm in $L^2(\mu)$ and

$$B = \sup_{\text{compact } K} \frac{\mu(K)}{\mathrm{Cap}(K)}. \tag{1.2}$$

2000 *Mathematics Subject Classification.* 60J55, 31C25, 60J35, 47D07.

Key words and phrases. Dirichlet form, isoperimetric constant, logarithmic Sobolev inequality, Poincaré-type inequality, Orlicz space.

Research supported in part NSFC (No. 10121101) and 973 Project.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Recall that

$$\mathrm{Cap}(K)=\inf\big\{D(f): f\in\mathscr{D}(D)\cap C_0(E), f|_K\geqslant 1\big\},$$

where $C_0(E)$ is the set of continuous functions with compact support. Certainly, in (1.1), one may replace "$\mathscr{D}(D)\cap C_0(E)$" by "$\mathscr{D}(D)$" or by the extended Dirichlet space "$\mathscr{D}_e(D)$", which is the set of $\mathscr{E}$-measurable functions f: $|f|<\infty$, μ-a.e., there exists a sequence $\{f_n\}\subset\mathscr{D}(D)$ such that $D(f_n-f_m)\to 0$ as $n,m\to\infty$ and $\lim_{n\to\infty}f_n=f$, μ-a.e. Refer to the standard books Fukushima, Oshima and Takeda [9], Ma and Röckner [14] for some preliminary facts about the Dirichlet forms theory.

Actually, inequality (1.1) in one-dimensional case was initiated by G. H. Hardy in 1920 and completed by B. Muckenhoupt in 1970 (see also Opic and Kufner [19]), in the context of diffusions (elliptic operators) with explicitly isoperimetric constant B.

The first goal of this paper is to extend (1.2) to the Poincaré-type inequality

$$\big\|f^2\big\|_{\mathbb{B}}\leqslant A_{\mathbb{B}}D(f),\qquad f\in\mathscr{D}(D)\cap C_0(E),\tag{1.3}$$

for a class of normed linear spaces $(\mathbb{B},\|\cdot\|_{\mathbb{B}},\mu)$ of real functions on E. To do so, we need the following assumptions on $(\mathbb{B},\|\cdot\|_{\mathbb{B}},\mu)$.

(H_1) $I_K\in\mathbb{B}$ for all compact K.

(H_2) If $h\in\mathbb{B}$ and $|f|\leqslant h$, then $f\in\mathbb{B}$.

(H_3) $\|f\|_{\mathbb{B}}=\sup_{g\in\mathscr{G}}\int_E|f|g\mathrm{d}\mu$,

where $\mathscr{G}$, to be specified case by case, is a class of nonnegative $\mathscr{E}$-measurable functions. By using Fatou's lemma and the completeness of $\mathscr{D}_e(D)$, one can also replace "$\mathscr{D}(D)\cap C_0(E)$" by "$\mathscr{D}_e(D)$" in (1.3).

We can now state our first result as follows.

Theorem 1.1. Assume (H_1)–(H_3). For a regular transient Dirichlet form $(D, \mathscr{D}(D))$, the optimal constant $A_{\mathbb{B}}$ in (1.3) satisfies

$$B_{\mathbb{B}}\leqslant A_{\mathbb{B}}\leqslant 4B_{\mathbb{B}},\tag{1.4}$$

where

$$B_{\mathbb{B}}:=\sup_{\text{compact }K}\frac{\|I_K\|_{\mathbb{B}}}{\mathrm{Cap}(K)}.\tag{1.5}$$

When $\mathbb{B}=L^p(\mu)\,(p\geqslant 1)$, Theorem 1.1 was proven by Fukushima and Uemura [10].

Next, we go to the ergodic case. Assume that $\mu(E)<\infty$ and set $\pi=\mu/\mu(E)$. Throughout this paper, we use the simplified notation: $\bar{f}=f-\pi(f)$, where $\pi(f)=\int f\mathrm{d}\pi$. We adopt a splitting technique. Let $E_1\subset E$ be open with $\pi(E_1)\in(0,1)$ and write $E_2=E_1^c\setminus\partial E_1$. Restricting the functions f in (1.1) to each E_i $\big($i.e., $f|_{E_i^c}=0$, μ-a.e.$\big)$, by Theorem 1.0, we obtain the corresponding constant B_i as follows.

$$B_i=\sup_{\text{compact }K\subset E_i}\frac{\mu(K)}{\mathrm{Cap}(K)},\qquad i=1,2.\tag{1.6}$$

This notation is meaningful because the restriction to an open set of a regular Dirichlet form is again regular (cf. Fukushima et al [9], Theorem 4.4.3). Moreover, since $(D,\mathscr{D}(D))$ is irreducible, its restrictions to E_1 and E_2 must be transient.

Theorem 1.2. Let $\mu(E) < \infty$. Then for a regular, irreducible, and conservative Dirichlet form, the optimal constant $\overline{A}$ in the Poincaré inequality

$$\|\bar{f}^2\| \leqslant \overline{A} D(f), \qquad f \in \mathscr{D}(D) \cap C_0(E), \tag{1.7}$$

satisfies

$$\begin{aligned} \overline{A} &\geqslant \sup_{\text{open } E_1 \text{ and } E_2} \max\left\{B_1 \pi(E_1^c),\, B_2 \pi(E_2^c)\right\} \geqslant \frac{1}{2} \sup_{\text{open } E_1:\, \pi(E_1)\in(0,1/2]} B_1, \\ \overline{A} &\leqslant 4 \sup_{\text{open } E_1:\, \pi(E_1)\in(0,1/2]} B_1. \end{aligned} \tag{1.8}$$

In particular, $\overline{A} < \infty$ iff $\sup_{\text{open } E_1:\, \pi(E_1)\in(0,1/2]} B_1 < \infty$.

The restriction of $\mathbb{B}$ to E_i gives us $(\mathbb{B}^i, \|\cdot\|_{\mathbb{B}^i}, \mu^i)$:

$$\begin{gathered} \mathbb{B}^i = \{f|_{E_i} : f \in \mathbb{B}\}, \quad \mu^i = \mu|_{E_i}, \quad \mathscr{G}^i = \{g|_{E_i} : g \in \mathscr{G}\}, \\ \|f\|_{\mathbb{B}^i} = \sup_{g\in\mathscr{G}^i} \int_{E_i} |f| g \mathrm{d}\mu^i = \sup_{g\in\mathscr{G}} \int_{E_i} |f| g \mathrm{d}\mu, \qquad i = 1, 2. \end{gathered}$$

Correspondingly, we have a restricted Dirichlet form $\big(D, \mathscr{D}_i\big)$ on $L^2(E_i, \mu^i)$, where $\mathscr{D}_i = \{f \in \mathscr{D}(D) :$ the quasi-version of f equals 0 on E_i^c, q.e.$\}$. The corresponding constants given by Theorem 1.1 are denoted by $A_{\mathbb{B}^i}$ and $B_{\mathbb{B}^i}$ $(i = 1, 2)$, respectively.

In the ergodic case, we also use the following assumptions.

(H_4) $\mu(E) < \infty$.

(H_5) $1 \in \mathbb{B}$.

Denote by c_1 a constant such that

$$|\pi(f)| \leqslant c_1 \|f\|_{\mathbb{B}}, \qquad f \in \mathbb{B}. \tag{1.9}$$

For each $G \subset E$, denote by $c_2(G)$ a constant such that

$$|\pi(f I_G)| \leqslant c_2(G) \|f I_G\|_{\mathbb{B}}, \qquad f \in \mathbb{B}. \tag{1.10}$$

Theorem 1.3. Let $(D, \mathscr{D}(D))$ be a regular, irreducible, and conservative Dirichlet form. Assume that (H_2)–(H_5) hold and that

$$\sup_{\text{open } E_1:\, \pi(E_1)\in(0,1/2]} c_2(E_1)\pi(E_1)\|1\|_{\mathbb{B}} < 1.$$

Then the optimal constant $\overline{A}_{\mathbb{B}}$ in the Poincaré-type inequality

$$\|\bar{f}^2\|_{\mathbb{B}} \leqslant \overline{A}_{\mathbb{B}} D(f), \qquad f \in \mathscr{D}(D) \cap C_0(E), \tag{1.11}$$

satisfies

$$\overline{A}_{\mathbb{B}} \geqslant \underline{\kappa} \sup_{\text{open } E_1:\, \pi(E_1)\in(0,1/2]} A_{\mathbb{B}^1} \geqslant \underline{\kappa} \sup_{\text{open } E_1:\, \pi(E_1)\in(0,1/2]} B_{\mathbb{B}^1}, \tag{1.12}$$

$$\overline{A}_{\mathbb{B}} \leqslant \bar{\kappa} \sup_{\text{open } E_1:\, \pi(E_1)\in(0,1/2]} A_{\mathbb{B}^1} \leqslant 4\bar{\kappa} \sup_{\text{open } E_1:\, \pi(E_1)\in(0,1/2]} B_{\mathbb{B}^1}, \tag{1.13}$$

where

$$\underline{\kappa}=\big(1-\sup_{\text{open } E_1:\ \pi(E_1)\in(0,1/2]}\sqrt{c_2(E_1)\pi(E_1)\|1\|_{\mathbb{B}}}\,\big)^2,\quad \bar{\kappa}=\big(1+\sqrt{c_1\|1\|_{\mathbb{B}}}\,\big)^2.$$

A typical case for which one needs the Banach form of Poincaré-type inequality is the F-Sobolev inequality (cf. Wang [23], Gong and Wang [11]):

$$\int_E f^2F\big(f^2\big)\mathrm{d}\mu\leqslant A_F D(f),\qquad f\in\mathscr{D}(D)\cap C_0(E).\tag{1.14}$$

Recall that a function Φ: $\mathbb{R}\to\mathbb{R}$ is an N-function if it is nonnegative, continuous, convex, even (i.e., $\Phi(-x)=\Phi(x)$), and satisfies

$$\Phi(x)=0\ \text{ iff }\ x=0,\qquad \lim_{x\to 0}\Phi(x)/x=0,\qquad \lim_{x\to\infty}\Phi(x)/x=\infty.$$

We will often assume the following growth condition (or Δ_2-condition) for Φ:

$$\sup_{x\gg 1}\Phi(2x)/\Phi(x)<\infty\ \Big(\iff \sup_{x\gg 1}x\Phi'_-(x)/\Phi(x)<\infty\Big),$$

where Φ'_- is the left derivative of Φ. The conditions listed below for F guarantee that the function $\Phi(x):=|x|F(|x|)$, as an N-function, satisfies the above conditions.

Theorem 1.4. Let $F:\mathbb{R}_+\to\mathbb{R}_+$ satisfy the following conditions:

(1) $2F'+xF''\geqslant 0$ on $[0,\infty)$.
(2) $F\neq 0$ on $(0,\infty)$, $\lim_{x\to 0}F(x)=0$ and $\lim_{x\to\infty}F(x)=\infty$.
(3) $\sup_{x\gg 1}xF'(x)/F(x)<\infty$.

Then Theorem 1.1 is valid for the Orlicz space $\big(\mathbb{B}=L^{\Phi}(\mu),\|\cdot\|_{\mathbb{B}}=\|\cdot\|_{\Phi}\big)$ with N-function $\Phi(x)=|x|F(|x|)$:

$$L^{\Phi}(\mu)=\Big\{f:\int_E\Phi(f)\mathrm{d}\mu<\infty\Big\},\tag{1.15}$$

$$\|f\|_{\Phi}=\sup\Big\{\int_E|f|g\mathrm{d}\mu:\int\Phi_c(g)\mathrm{d}\mu\leqslant 1\Big\},\tag{1.16}$$

where Φ_c is the complementary function of Φ. [If we denote by φ_c the inverse function of the left-derivative of Φ, then Φ_c can be expressed as $\Phi_c(y)=\int_0^{|y|}\varphi_c$.] Furthermore the isoperimetric constant is given by

$$B_{\Phi}=\sup_{\text{compact } K}\frac{\alpha_*(K)^{-1}+\mu(K)F(\alpha_*(K))}{\mathrm{Cap}(K)},\tag{1.17}$$

where $\alpha_*(K)$ is the minimal positive root of the equation: $\alpha^2F'(\alpha)=\mu(K)$.

The corresponding ergodic case of Theorem 1.4 has been treated in [1; Theorems 11 and 12].

A more particular case is that $F=\log$. Then we have, in the ergodic case, the logarithmic Sobolev inequality

$$\int_E f^2\log\big[f^2/\pi(f^2)\big]\mathrm{d}\mu\leqslant A_{\mathrm{Log}}D(f),\qquad f\in\mathscr{D}(D)\cap C_0(E)\tag{1.18}$$

(due to L. Gross (1976), cf. Gross [12] and the references within). By examining the entropy carefully, using different Banach spaces (but not Orliczian) for the upper and lower bounds respectively, we obtain the following result.

Theorem 1.5. Let $(D, \mathscr{D}(D))$ be a regular, irreducible, and conservative Dirichlet form. Assume that (H_2)–(H_5) hold. Then we have

$$\frac{\log 2}{\log(1+2e^2)}\, B_{\mathrm{Log}}(e^2) \leqslant B_{\mathrm{Log}}(1/2) \leqslant A_{\mathrm{Log}} \leqslant 4\, B_{\mathrm{Log}}(e^2), \tag{1.19}$$

where

$$B_{\mathrm{Log}}(\gamma) = \sup_{\substack{\text{open } O:\ \pi(O)\in(0,1/2]\\ \text{compact } K\subset O}} \frac{\mu(K)}{\mathrm{Cap}(K)} \log\left(1+\frac{\gamma}{\pi(K)}\right). \tag{1.20}$$

One may regard Theorem 1.1–1.5 as extensions of the one-dimensional results obtained by S. G. Bobkov and F. Götze [3], Y. H. Mao [15, 16], F. Barthe and C. Roberto [2] and the author [5, 6]. However, the criteria and estimates given in the quoted papers are completely explicit, without using capacity. Even though the capacitary results in dimension one can also be made explicit, as shown in Section 4, the capacitary results here are much more involved; but this may be the price one has to pay for such a general setup (for the higher dimensions, in particular). Nevertheless, we have got the precise formula for the isoperimetric constant $B_{\mathbb{B}}$ (or B_Φ) in the general setup. Of course, it is valuable to work out more explicit expression for the constant in particular situations.

The proofs of Theorems 1.1–1.4 are presented in the next section. The proof of Theorem 1.5 and some related results are given in Section 3. In the last section, the isoperimetric constants in dimension one are computed explicitly.

2. Proofs of Theorems 1.1–1.4.

The key to prove Theorem 1.0 is the following result [cf. Fukushima and Uemura [10], Theorem 2.1]:

Theorem 2.1.

$$\int_0^\infty \mathrm{Cap}(\{x\in E: |f(x)|\geqslant t\})\mathrm{d}\big(t^2\big) \leqslant 4D(f), \qquad f\in\mathscr{D}(D)\cap C_0(E).$$

Having Theorem 2.1 in mind, the proof of Theorem 1.0 and more generally of Theorem 1.1 is quite standard. Here we follow the proof of Theorem 3.1 in Kaimanovich [13].

Proof of Theorem 1.1. Let $f\in\mathscr{D}(D)\cap C_0(E)$ and set $N_t=\{|f|\geqslant t\}$. Since N_t is compact, by (H_1), $I_{N_t}\in\mathbb{B}$. Next, since $|f|\leqslant \|f\|_\infty I_{\mathrm{supp}(f)}$, by (H_1) and (H_2), $f^2\in\mathbb{B}$. Note that

$$\int_0^\infty I_{N_t}\mathrm{d}\big(t^2\big) = 2\int_0^\infty t I_{\{|f|\geqslant t\}}\mathrm{d}t = 2\int_0^{|f|} t\,\mathrm{d}t = f^2 \quad \text{(co-area formula).}$$

By (H_3), the definition of $B_{\mathbb{B}}$, and Theorem 2.1, we obtain

$$\begin{aligned}\|f^2\|_{\mathbb{B}} &= \sup_{g\in\mathscr{G}}\int_E f^2 g\mathrm{d}\mu\\ &= \sup_{g\in\mathscr{G}}\int_E\bigg(\int_0^\infty I_{N_t}\mathrm{d}\big(t^2\big)\bigg)g\mathrm{d}\mu\\ &= \sup_{g\in\mathscr{G}}\int_0^\infty\bigg(\int_E I_{N_t}g\mathrm{d}\mu\bigg)\mathrm{d}\big(t^2\big)\\ &\leqslant \int_0^\infty \|I_{N_t}\|_{\mathbb{B}}\mathrm{d}\big(t^2\big)\\ &\leqslant B_{\mathbb{B}}\int_0^\infty \mathrm{Cap}(N_t)\mathrm{d}\big(t^2\big)\\ &\leqslant 4B_{\mathbb{B}}D(f).\end{aligned}$$

This implies that $A_{\mathbb{B}} \leqslant 4B_{\mathbb{B}}$.

Next, for every compact K and any function $f\in\mathscr{D}(D)\cap C_0(E)$ with $f|_K\geqslant 1$, by (H_2) and (H_3), we have

$$\|I_K\|_{\mathbb{B}}\leqslant\|f^2\|_{\mathbb{B}}\leqslant A_{\mathbb{B}}D(f).$$

Thus,

$$\|I_K\|_{\mathbb{B}}\leqslant A_{\mathbb{B}}\inf\{D(f): f\in\mathscr{D}(D)\cap C_0(E), f|_K\geqslant 1\}=A_{\mathbb{B}}\mathrm{Cap}(K).$$

Taking supremum over all compact K, it follows that $B_{\mathbb{B}}\leqslant A_{\mathbb{B}}$ and the proof is completed. □

To prove Theorem 1.2, we need the following result.

Lemma 2.2. Let $(D,\mathscr{D}(D))$ be a regular and conservative Dirichlet form, $\mu(E)<\infty$, $f\in\mathscr{D}(D)\cap C(E)$, and c a constant. Define $f^{\pm}=(f-c)^{\pm}$. Then we have $D(f)\geqslant D(f^+)+D(f^-)$.

Proof. Let $P_t(x,\mathrm{d}y)$ be the transition probability function determined by the Dirichlet form and set $\mu_t(\mathrm{d}x,\mathrm{d}y)=\mu(\mathrm{d}x)P_t(x,\mathrm{d}y)$. Then, by the spectral representation theorem, we have

$$\frac{1}{2t}\int\mu_t(\mathrm{d}x,\mathrm{d}y)[g(y)-g(x)]^2\uparrow D(g)\qquad\text{as } t\downarrow 0 \text{ for all } g\in L^2(\mu). \tag{2.1}$$

Next, $\{f^+>0\}$ and $\{f^->0\}$ are open sets on which the restricted Dirichlet forms are also regular. Moreover, since $1\in\mathscr{D}(D)$, we have $f^{\pm}\in\mathscr{D}(D)$; and hence $f^{\pm}$ belong to the corresponding restricted Dirichlet forms, respectively. Furthermore, it is easy to check the following crucial identity:

$$|f(y)-f(x)|=\big|f^+(y)-f^+(x)\big|+\big|f^-(y)-f^-(x)\big|. \tag{2.2}$$

Therefore

$$
\begin{aligned}
D(f) &= \lim_{t\downarrow 0} \frac{1}{2t} \int \mu_t(\mathrm{d}x, \mathrm{d}y)\big[f(y) - f(x)\big]^2 \\
&= \lim_{t\downarrow 0} \frac{1}{2t} \int \mu_t(\mathrm{d}x, \mathrm{d}y)\big[\big|f^+(y) - f^+(x)\big| + \big|f^-(y) - f^-(x)\big|\big]^2 \\
&\geqslant \lim_{t\downarrow 0} \frac{1}{2t} \int \mu_t(\mathrm{d}x, \mathrm{d}y)\big(f^+(y) - f^+(x)\big)^2 + \\
&\quad + \lim_{t\downarrow 0} \frac{1}{2t} \int \mu_t(\mathrm{d}x, \mathrm{d}y)\big(f^-(y) - f^-(x)\big)^2 \\
&= D\big(f^+\big) + D\big(f^-\big). \quad \square
\end{aligned} \tag{2.3}
$$

Proof of Theorem 1.2. The proof below is essentially the same as in Chen and Wang [7] and Chen [4, Theorem 3.1].

For each $\varepsilon > 0$, choose $f_\varepsilon \in \mathscr{D}(D) \cap C_0(E)$ with $\pi(f_\varepsilon) = 0$ and $\pi(f_\varepsilon^2) = 1$ such that $\overline{A}^{\,-1} + \varepsilon \geqslant D(f_\varepsilon)$. Next, choose c_ε such that $\pi(f_\varepsilon < c_\varepsilon) \leqslant 1/2$ and $\pi(f_\varepsilon > c_\varepsilon) \leqslant 1/2$. Set $f_\varepsilon^\pm = (f_\varepsilon - c_\varepsilon)^\pm$ and $G_\varepsilon^\pm = \{f_\varepsilon^\pm > 0\}$. Then $G_\varepsilon^\pm$ are open sets and Theorem 1.1 is meaningful for the restricted Dirichlet forms on $G_\varepsilon^\pm$. Denote by $A(G)$ the the optimal constant A in (1.1), when the functions are restricted on G. By Lemma 2.2, we obtain

$$
\begin{aligned}
1 &\leqslant 1 + c_\varepsilon^2 \\
&= \pi\big[\big(f_\varepsilon^+\big)^2 + \big(f_\varepsilon^-\big)^2\big] \\
&\leqslant A\big(G_\varepsilon^+\big)D\big(f_\varepsilon^+\big) + A\big(G_\varepsilon^-\big)D\big(f_\varepsilon^-\big) \\
&\leqslant \big[A\big(G_\varepsilon^+\big) \vee A\big(G_\varepsilon^-\big)\big]\,\big[D\big(f_\varepsilon^+\big) + D\big(f_\varepsilon^-\big)\big] \\
&\leqslant \big[A\big(G_\varepsilon^+\big) \vee A\big(G_\varepsilon^-\big)\big]\, D\big(f_\varepsilon\big) \\
&\leqslant \big[A\big(G_\varepsilon^+\big) \vee A\big(G_\varepsilon^-\big)\big]\big(\overline{A}^{\,-1} + \varepsilon\big) \\
&\leqslant \big(\overline{A}^{\,-1} + \varepsilon\big) \sup_{\text{open } O:\pi(O)\in(0,1/2]} A(O).
\end{aligned}
$$

Because ε is arbitrary, we obtain a upper bound of $\overline{A}$.

Next, for every $f \in \mathscr{D}(D)$ with $f|_{G^c} = 0$ and $\pi(f^2) = 1$, we have

$$
\pi\big(f^2\big) - \pi(f)^2 = 1 - \pi\big(fI_G\big)^2 \geqslant 1 - \pi\big(f^2\big)\pi(G) = 1 - \pi(G) = \pi(G^c).
$$

Hence

$$
\overline{A} \geqslant \frac{\pi\big(f^2\big) - \pi(f)^2}{D(f)} \geqslant \frac{\pi(G^c)}{D(f)}.
$$

This implies that $\overline{A} \geqslant A(G)\pi(G^c)$. By symmetry, we have

$$
\overline{A} \geqslant \max\big\{A(E_1)\pi(E_1^c),\ A(E_2)\pi(E_2^c)\big\}.
$$

Therefore

$$\begin{aligned}\overline{A} &\geqslant \sup_{\text{open } E_1 \text{ and } E_2} \max\big\{A(E_1)\pi(E_1^c),\ A(E_2)\pi(E_2^c)\big\}\\ &\geqslant \frac{1}{2}\sup_{\text{open } O:\ \pi(O)\in(0,1/2]} A(O).\end{aligned}$$

This gives us a lower bound of $\overline{A}$.

Finally, the assertion of Theorem 1.2 follows from Theorem 1.0. $\square$

To prove Theorem 1.3, we need the following proposition, taken from Chen [6, Proposition 2.4].

Proposition 2.3. Let $(E, \mathscr{E}, \pi)$ be a probability space and $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ a normed linear space, satisfying (H_5) and (H_2), of Borel measurable functions on $(E, \mathscr{E}, \pi)$.

(1) Let c_1 be given by (1.9). Then

$$\|\bar{f}^2\|_{\mathbb{B}} \leqslant \big(1+\sqrt{c_1\|1\|_{\mathbb{B}}}\,\big)^2\|f^2\|_{\mathbb{B}}.$$

(2) Let $c_2(G)$ be given by (1.10). If $c_2(G)\pi(G)\|1\|_{\mathbb{B}} < 1$, then for every f with $f|_{G^c} = 0$, we have

$$\|f^2\|_{\mathbb{B}} \leqslant \|\bar{f}^2\|_{\mathbb{B}}\big/\big[1-\sqrt{c_2(G)\pi(G)\,\|1\|_{\mathbb{B}}}\,\big]^2.$$

Proof of Theorem 1.3. (a) Let $f \in \mathscr{D}(D)\cap C_0(E)$. Choose c_f such that $E_1 := \{f > c_f\}$ and $E_2 := \{f < c_f\}$ satisfy $\pi(E_1) \leqslant 1/2$ and $\pi(E_2) \leqslant 1/2$. Then E_1 and E_2 are open sets. Define $f_1 = (f - c_f)^+$ and $f_2 = (f-c_f)^-$. By part (1) of Proposition 2.3, it follows that

$$\|\bar{f}^2\|_{\mathbb{B}} = \|\overline{f-c_f}^2\|_{\mathbb{B}} \leqslant \Big(1+\sqrt{c_1\|1\|_{\mathbb{B}}}\Big)^2\|(f-c_f)^2\|_{\mathbb{B}}.$$

But

$$\|(f-c_f)^2\|_{\mathbb{B}} = \|f_1^2+f_2^2\|_{\mathbb{B}} \leqslant \|f_1^2\|_{\mathbb{B}} + \|f_2^2\|_{\mathbb{B}},$$

hence by Theorem 1.1 and Lemma 2.2, we get

$$\begin{aligned}\|\bar{f}^2\|_{\mathbb{B}} &\leqslant \Big(1+\sqrt{c_1\|1\|_{\mathbb{B}}}\Big)^2 A_{\mathbb{B}^1}D(f_1) + \Big(1+\sqrt{c_1\|1\|_{\mathbb{B}}}\Big)^2 A_{\mathbb{B}^2}D(f_2)\\ &\leqslant \Big(1+\sqrt{c_1\|1\|_{\mathbb{B}}}\Big)^2\big(A_{\mathbb{B}^1}\vee A_{\mathbb{B}^2}\big)\big(D(f_1)+D(f_2)\big)\\ &\leqslant \Big(1+\sqrt{c_1\|1\|_{\mathbb{B}}}\Big)^2\big(A_{\mathbb{B}^1}\vee A_{\mathbb{B}^2}\big)D(f).\end{aligned}$$

This means that

$$\overline{A}_{\mathbb{B}} \leqslant \Big(1+\sqrt{c_1\|1\|_{\mathbb{B}}}\Big)^2\big(A_{\mathbb{B}^1}\vee A_{\mathbb{B}^2}\big) \leqslant \Big(1+\sqrt{c_1\|1\|_{\mathbb{B}}}\Big)^2 \sup_{\text{open } E_1:\ \pi(E_1)\leqslant 1/2} A_{\mathbb{B}^1}.$$

(b) Conversely, assume that (1.11) holds. Let $f \in \mathscr{D}(D) \cap C_0(E)$, $f|_{E_1^c} = 0$ for some open E_1 with $\pi(E_1) \leqslant 1/2$. Then, from part (2) of Proposition 2.3 and (1.11), it follows that

$$\|f^2\|_{\mathbb{B}} \leqslant \frac{\|\bar{f}^2\|_{\mathbb{B}}}{\big(1-\sqrt{c_2(E_1)\pi(E_1)\|1\|_{\mathbb{B}}}\,\big)^2} \leqslant \frac{\overline{A}_{\mathbb{B}}}{\big(1-\sqrt{c_2(E_1)\pi(E_1)\|1\|_{\mathbb{B}}}\,\big)^2} D(f).$$

This means that

$$\overline{A}_{\mathbb{B}} \geqslant \big(1-\sqrt{c_2(E_1)\pi(E_1)\|1\|_{\mathbb{B}}}\,\big)^2 A_{\mathbb{B}^1}.$$

Noticing that $\sup_{\text{open } E_1:\, \pi(E_1)\leqslant 1/2} c_2(E_1)\pi(E_1)\|1\|_{\mathbb{B}} < 1$ by assumption, we obtain

$$\overline{A}_{\mathbb{B}} \geqslant \left(1 - \sup_{\text{open } E_1:\, \pi(E_1)\leqslant 1/2} \sqrt{c_2(E_1)\pi(E_1)\|1\|_{\mathbb{B}}}\right)^2 \sup_{\text{open } E_1:\, \pi(E_1)\leqslant 1/2} A_{\mathbb{B}^1}. \qquad \square$$

Proof of Theorem 1.4. By assumptions, $\Phi(x) = |x|F(|x|)$ is an N-function. From M. M. Rao and Z. D. Ren [20, §3.3, Theorem 13 and Proposition 14], it follows that

$$\|I_G\|_\Phi = \inf_{\alpha>0} \frac{1}{\alpha}\big(1+\mu(G)\Phi(\alpha)\big).$$

The infimum on the right-hand side is achieved at α^*, which is the minimal root of the equation: $\alpha^2 F'(\alpha) = \mu(G)$. Combining this with (1.5), we get (1.17). $\square$

3. Logarithmic Sobolev inequality.

This section is devoted to the logarithmic Sobolev inequality. First, we present a result as an illustration of the application of Theorem 1.3. Then we prove the refinement Theorem 1.5.

Theorem 3.1. Let $(D, \mathscr{D}(D))$ be a regular, irreducible, and conservative Dirichlet form. Assume that (H_2)–(H_5) hold. Next, let $\Phi(x) = |x|\log(1+|x|)$. Then the optimal A_{Log} in (1.18) satisfies

$$\frac{(\sqrt{2}-1)^2}{5} B_\Phi \leqslant A_{\mathrm{Log}} \leqslant \frac{51\times 16}{5} B_\Phi, \tag{3.1}$$

where

$$B_\Phi = \sup_{\substack{\text{open } O:\, \pi(O)\in(0,1/2]\\ \text{compact } K\subset O}} \frac{M(\mu(K))}{\mathrm{Cap}(K)}, \tag{3.2}$$

$$M(x) = \frac{1}{2}\big(\sqrt{1+4x}-1\big) + x\log\left(1+\frac{1+\sqrt{1+4x}}{2x}\right) \sim x\log x^{-1} \qquad \text{as } x\to 0.$$

Since the proof of Theorem 3.1 is essentially known, we sketch the main steps only for the reader's convenience.

From now on, we fix $\Phi(x) = |x|\log(1+|x|)$ and define $\Psi(x) = x^2\log(1+x^2)$. We need an equivalent norm $\|\cdot\|_{(\Phi)}$ of $\|\cdot\|_\Phi$ as follows

$$\|f\|_{(\Phi)} = \inf\left\{\alpha>0 : \int_E \Phi(f/\alpha)\mathrm{d}\mu \leqslant 1\right\},$$

which is usually easier to compute. The key observation is the following result:

Lemma 3.2. For any f with $f^2 \in L^{\Phi}(\mu)$, we have

$$\frac{4}{5}\big\|f-\pi(f)\big\|_{(\Psi)}^2 \leqslant \mathscr{L}(f) \leqslant \frac{51}{20}\big\|f-\pi(f)\big\|_{(\Psi)}^2,$$

where $\mathscr{L}(f) = \sup_{c\in\mathbb{R}} \operatorname{Ent}\big((f+c)^2\big)$ and $\operatorname{Ent}(f) = \int_{\mathbb{R}} f \log\big(f/\|f\|_{L^1(\pi)}\big)\mathrm{d}\mu$ for $f \geqslant 0$.

This result comes from Bobkov and Götze [3] and Deuschel and Stroock [8, p. 247], which go back to Rothaus [21]. An improvement of the coefficients is made in Chen [5]. Lemma 3.2 leads to the use of the Orlicz space $\mathbb{B} = L^{\Psi}(\mu)$ with norm $\|\cdot\|_{(\Psi)}$ and the following inequalities

$$\|f\|_{\mathbb{B}}^2 \leqslant A_{\mathbb{B}}' D(f), \qquad f \in \mathscr{D}(D) \cap C_0(E), \tag{3.3}$$

$$\big\|\bar{f}\big\|_{\mathbb{B}}^2 \leqslant \overline{A}_{\mathbb{B}}' D(f), \qquad f \in \mathscr{D}(D) \cap C_0(E), \tag{3.4}$$

as variants of (1.1) and (1.11). In parallel to Proposition 2.3, we have (cf. Chen [6, Proposition 3.4]) the following result.

Proposition 3.3. Everything in the premise is the same as in Proposition 2.3.

(1) Assume that there is a constant c_1' such that $|\pi(f)| \leqslant c_1'\|f\|_{\mathbb{B}}$ for all $f \in \mathbb{B}$. Then

$$\big\|\bar{f}\big\|_{\mathbb{B}} \leqslant \big(1 + c_1'\|1\|_{\mathbb{B}}\big)\|f\|_{\mathbb{B}}.$$

(2) Next, for a given $G \in \mathscr{E}$, let $c_2'(G)$ be a constant such that $|\pi(fI_G)| \leqslant c_2'(G)\|fI_G\|_{\mathbb{B}}$ for all $f \in \mathbb{B}$. If $c_2'(G)\|1\|_{\mathbb{B}} < 1$, then for every f with $f|_{G^c} = 0$ we have

$$\|f\|_{\mathbb{B}} \leqslant \big\|\bar{f}\big\|_{\mathbb{B}}\big/\big[1 - c_2'(G)\|1\|_{\mathbb{B}}\big].$$

Denote by $A_{\mathbb{B}^i}'$ the optimal constant in (3.3) when the functions are restricted to E_i, $i = 1, 2$. By using Proposition 3.3 and following the proof of Theorem 1.3, we obtain the following result.

Theorem 3.4. Let $(D, \mathscr{D}(D))$ be a regular, irreducible, and conservative Dirichlet form. Assume that (H_2)–(H_5) hold and that

$$\sup_{\text{open } E_1\colon \pi(E_1)\in(0,1/2]} c_2'(E_1)\|1\|_{\mathbb{B}} < 1.$$

Then the optimal constants $A_{\mathbb{B}^i}'$ and $\overline{A}_{\mathbb{B}}'$ in (3.3) and (3.4), respectively, obey the following relation:

$$\Big(1 - \sup_{\text{open } E_1\colon \pi(E_1)\in(0,1/2]} \sqrt{c_2'(E_1)\|1\|_{\mathbb{B}}}\Big)^2 A_{\mathbb{B}^1}' \leqslant \overline{A}_{\mathbb{B}}' \leqslant 4\Big(1 + \sqrt{c_1'\|1\|_{\mathbb{B}}}\Big)^2 A_{\mathbb{B}^1}'.$$

Proof of Theorem 3.1.

(a) First, we compute the constants c_1' and $c_2'(E_1)$ used in Theorem 3.4. Actually, this can be done in the same way as in the proof of the last theorem in Chen [5] or [6]:

$$c_1' = \Psi^{-1}(Z^{-1}), \tag{3.5}$$

$$c_2'(E_1) = \Psi^{-1}(Z_1^{-1})Z_1/Z. \tag{3.6}$$

where Ψ^{-1} is the inverse of Ψ and $Z = \mu(E)$, $Z_1 = \mu(E_1)$. As an illustration, we now prove (3.6). Because of the convexity of Φ, we have for $f_1 := fI_{E_1}$ that

$$\begin{aligned}\|f_1\|_{(\Phi)} &= \inf\left\{\alpha > 0 : \frac{1}{Z_1}\int_{E_1}\Phi(|f|/\alpha)\mathrm{d}\pi \leqslant \frac{1}{Z_1}\right\}\\ &\geqslant \inf\left\{\alpha > 0 : \Phi\left(\frac{1}{Z_1}\int_{E_1}|f|\mathrm{d}\pi/\alpha\right) \leqslant \frac{1}{Z_1}\right\}\\ &= \frac{Z\pi(|f_1|)}{Z_1\Phi^{-1}(Z_1^{-1})}.\end{aligned}$$

Hence

$$\begin{aligned}\|f_1\|_{(\Psi)}^2 &= \|f_1^2\|_{(\Phi)}\\ &\geqslant \frac{Z}{Z_1\Phi^{-1}(Z_1^{-1})}\pi\big(f_1^2\big)\\ &\geqslant \frac{Z^2}{Z_1^2\Phi^{-1}(Z_1^{-1})}\big[\pi(f_1)\big]^2\\ &= \left[\frac{Z\pi(f_1)}{Z_1\Psi^{-1}(Z_1^{-1})}\right]^2.\end{aligned}$$

This means that one can choose $c_2'(E_1)$ as in (3.6).

(b) Next, since $\|1\|_{(\Psi)} = 1/\Psi^{-1}(Z^{-1})$, $Z_1 \leqslant Z/2$, and $\Psi^{-1}(x)/x$ is decreasing in x, it follows that

$$\begin{aligned}\sup_{\text{open } E_1:\, \pi(E_1)\in(0,1/2]} c_2'(E_1)\|1\|_{(\Psi)} &= \sup_{\text{open } E_1:\, \pi(E_1)\in(0,1/2]} \frac{Z_1\Psi^{-1}(Z_1^{-1})}{Z\Psi^{-1}(Z^{-1})}\\ &\leqslant \frac{\Psi^{-1}(2Z^{-1})}{2\Psi^{-1}(Z^{-1})}\\ &< 1,\end{aligned}$$

and so the assumption of Theorem 3.4 holds.

Note that

$$\left[1 - \frac{\Psi^{-1}(2Z^{-1})}{2\Psi^{-1}(Z^{-1})}\right]^2 \geqslant \frac{(\sqrt{2}-1)^2}{2}, \tag{3.7}$$

as proved at the end of Chen [5]. The estimates in (3.1) now follow from (3.5)–(3.7) and the following comparison of the norms: $\|f\|_{(\Phi)} \leqslant \|f\|_{\Phi} \leqslant 2\|f\|_{(\Phi)}$. $\square$

We now turn to prove Theorem 1.5. Since the N-function $\Phi(x) = |x| \log(1+|x|)$ used in Theorem 3.1 is a little different form the function $|x| \log |x|$ used in the entropy, it is natural to examine the entropy more carefully. The starting point is the classical variational formula for the entropy $\mathrm{Ent}(\varphi) = \int_E \varphi \log(\varphi/\pi(\varphi))\mathrm{d}\pi$:

$$\mathrm{Ent}(\varphi) = \sup\bigg\{\int_E \varphi g \mathrm{d}\pi : \int_E e^g \mathrm{d}\pi \leqslant 1\bigg\}, \qquad \varphi \geqslant 0. \tag{3.8}$$

The right-hand side is very much the same as the norm defined by (H_3). However, the only nonnegative function g in the constraint is zero. This leads us to consider the following upper and lower estimates, due to Barthe and Roberto [2].

Lemma 3.5. Let $(X, \mathscr{B}, \pi)$ be a probability space, $G \in \mathscr{B}$, and $\varphi \in \mathscr{B}_+$ with $\varphi|_{G^c} = 0$. Then we have

(1)

$$\begin{aligned}\mathrm{Ent}(\varphi) + 2\int_X \varphi \mathrm{d}\pi &\leqslant \sup\bigg\{\int_X \varphi g \mathrm{d}\pi : \int_X e^g \mathrm{d}\pi \leqslant e^2 + 1,\ g \geqslant 0\bigg\}\\ &= \sup\bigg\{\int_G \varphi g \mathrm{d}\pi : \int_G e^g \mathrm{d}\pi \leqslant e^2 + \pi(G),\ g \geqslant 0\bigg\}, \qquad \varphi \geqslant 0.\end{aligned}$$

(2) If moreover $\pi(G) < 1$, then

$$\mathrm{Ent}(\varphi) \geqslant \sup\bigg\{\int_G \varphi g \mathrm{d}\pi : \int_G e^g \mathrm{d}\pi \leqslant 1,\ g \geqslant 0\bigg\}, \qquad \varphi \geqslant 0.$$

To compute the bounds in Lemma 3.5, we also need the following result ([2; Lemma 6]).

Lemma 3.6. Let $(X, \mathscr{B}, \mu)$ be a finite measure space, $C \geqslant \mu(X)$, and $G \in \mathscr{B}$ with $\mu(G) > 0$. Then

$$\sup\bigg\{\int_X I_G h \mathrm{d}\mu : \int_X e^h \mathrm{d}\mu \leqslant C \text{ and } h \geqslant 0\bigg\} = \mu(G) \log\bigg(1 + \frac{C - \mu(X)}{\mu(G)}\bigg).$$

The two parts in Lemma 3.5 are used, respectively, for the upper and lower estimates given in Theorem 1.5. In view of (3.8), part (2) of the lemma is quite close to the entropy. As will be seen below, part (1) of the lemma leads us to define a norm by (H_3), using

$$\mathscr{G} = \bigg\{g \geqslant 0 : \int_E e^g \mathrm{d}\pi \leqslant e^2 + 1\bigg\}.$$

It corresponds to $\Phi_c(x) = e^{-2}(e^{|x|} - 1)$ and hence $\Phi(x) = |x| \log |x| + |x|$ which is not an N-function, since $\lim_{x\to 0} \Phi(x)/x = -\infty$; and is even not a Young function, since $\Phi \ngeqslant 0$. Thus, we are out of the Orlicz spaces. In contrast with Theorem 3.1, here two different norms are adopted rather than a single one.

Proof of Theorem 1.5. For convenience, we replace the finite measure μ with the probability measure $\pi = \mu/\mu(E)$ in this proof. This makes no change of A_{Log} in (1.18).

(a) We now consider the normed linear space $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$, where the norm $\|\cdot\|_{\mathbb{B}}$ is defined by (H_3) in terms of

$$\mathscr{G} = \left\{ g \geqslant 0 : \int_E e^g \mathrm{d}\pi \leqslant e^2 + 1 \right\}.$$

Following the proof (a) of Theorem 1.3, for a given $f \in \mathscr{D}(D) \cap C_0(E)$, let c_f be a median of f and set $f_1 = (f - c_f)^+$ and $f_2 = (f - c_f)^-$. By Lemma 9 in Rothaus [21], we have

$$\mathrm{Ent}\big(f^2\big) \leqslant \inf_{c \in \mathbb{R}} \big\{ \mathrm{Ent}\big((f-c)^2\big) + 2\|f - c\|^2 \big\}. \tag{3.9}$$

Applying part (1) of Lemma 3.5 with $G = E$, Theorem 1.1, and Lemma 2.2, we obtain

$$\begin{aligned}
\mathrm{Ent}\big(f^2\big) &\leqslant \big\|(f - c_f)^2\big\|_{\mathbb{B}} \quad \text{(by (3.9) and Lemma 3.5)}\\
&= \big\|f_1^2 + f_2^2\big\|_{\mathbb{B}}\\
&\leqslant \big\|f_1^2\big\|_{\mathbb{B}} + \big\|f_2^2\big\|_{\mathbb{B}}\\
&\leqslant 4B_{\mathbb{B}^1} D(f_1) + 4B_{\mathbb{B}^2} D(f_2) \quad \text{(by Theorem 1.1)}\\
&\leqslant 4\big(B_{\mathbb{B}^1} \vee B_{\mathbb{B}^2}\big)\big(D(f_1) + D(f_2)\big)\\
&\leqslant 4\big(B_{\mathbb{B}^1} \vee B_{\mathbb{B}^2}\big) D(f) \quad \text{(by Lemma 2.2)},
\end{aligned}$$

where $B_{\mathbb{B}^i}$ is given by Theorem 1.1. More precisely, by part (1) of Lemma 3.5 with $G = E_i$, we have $\|f_i\|_{\mathbb{B}} = \|f_i\|_{\mathbb{B}^i}$ with respect to the class

$$\mathscr{G}^i = \left\{ g \geqslant 0 : \int_{E_i} e^g \mathrm{d}\pi \leqslant e^2 + \frac{1}{2} \right\}$$

of functions on $E_i := \{f_i > 0\}$, $i = 1, 2$. We have thus proved that

$$A_{\text{Log}} \leqslant 4\big(B_{\mathbb{B}^1} \vee B_{\mathbb{B}^2}\big). \tag{3.10}$$

By Lemma 3.6, we have

$$\|I_K\|_{\mathbb{B}^i} = \pi(K) \log\left(1 + \frac{e^2 + 1/2 - 1/2}{\pi(K)}\right) = \pi(K) \log\left(1 + \frac{e^2}{\pi(K)}\right).$$

Combining this with (3.10), (1.5), and (1.20), we obtain $A_{\text{Log}} \leqslant 4\, B_{\text{Log}}(e^2)$.

(b) To prove the lower bound, assume (1.18). Let E_1 be open with $\pi(E_1) \leqslant 1/2$ and let $f \in \mathscr{D}(D) \cap C_0(E)$ with $f|_{E_1^c} = 0$. Then by part (2) of Lemma 3.5,

$$\mathrm{Ent}(f^2) \geqslant \sup\left\{ \int_{E_1} f^2 g \mathrm{d}\pi : \int_{E_1} e^g \mathrm{d}\pi \leqslant 1,\ g \geqslant 0 \right\}.$$

The right-hand side is the norm of f, denoted by $\|f\|_{\widetilde{\mathbb{B}}^1}$, with respect to a new class $\widetilde{\mathscr{G}}^1 = \{g \geqslant 0 : \int_{E_1} e^g \mathrm{d}\pi \leqslant 1\}$ of functions on E_1. To compute this norm, we use Lemma 3.6 again,

$$\begin{aligned}\|I_K\|_{\widetilde{\mathbb{B}}^1} &= \sup_{g\in\widetilde{\mathscr{G}}^1} \int_{E_1} I_K g \mathrm{d}\pi \\ &= \pi(K)\log\left(1+\frac{1-\pi(E_1)}{\pi(K)}\right) \\ &\geqslant \pi(K)\log\left(1+\frac{1}{2\pi(K)}\right), \qquad K\subset E_1.\end{aligned}$$

Combining this estimate with (1.18) and applying Theorem 1.1 to $(\widetilde{\mathbb{B}}^1, \|\cdot\|_{\widetilde{\mathbb{B}}^1}, \mu^1)$, we obtain $A_{\text{Log}} \geqslant B_{\text{Log}}(1/2)$ as required.

The factor $\log 2/\log(1+e^2)$ in the lower estimate of the theorem is due to the fact that

$$\frac{\log(1+e^2/x)}{\log(1+1/2x)} \uparrow \frac{\log(1+e^2)}{\log 2} < 4 \qquad \text{as } x\uparrow 1/2. \quad \square$$

4. Computation of isoperimetric constant in dimension one.

It is known that in general, the optimal constant A in (1.1) is not explicitly computable even in dimension one. However, the next two results show that the isoperimetric constant B in (1.2) in dimension one is computable and coincides with the Muckenhoupt-type bound (cf., [18], [4]).

Corollary 4.1. Consider an ergodic birth–death process with birth rates b_i $(i \geqslant 0)$ and death rates a_i $(i \geqslant 1)$. Define

$$\mu_0 = 1, \quad \mu_n = \frac{b_0 b_1 \cdots b_{n-1}}{a_1 a_2 \cdots a_n}, \quad n \geqslant 1.$$

Then the isoperimetric constant $B_{\mathbb{B}}$ in (1.5) with Dirichlet boundary at 0 can be expressed as follows:

$$B_{\mathbb{B}} = \sup_{n\geqslant 1} \|I_{[n,\infty)}\|_{\mathbb{B}} \sum_{i=0}^{n-1} \frac{1}{\mu_i b_i}.$$

Proof. (a) We show that in the definition of $\mathrm{Cap}(K)$, one can replace "$f|_K \geqslant 1$" by "$f|_K = 1$".

Because $1 \in \mathscr{D}(D)$, we have $f\wedge 1 \in \mathscr{D}(D)\cap C_0(E)$ if so is f. Then the assertion follows from $D(f) \geqslant D(f\wedge 1)$.

(b) Next, let K_i $(i = 1, 2, \ldots, k)$ be disjoint intervals with natural order. Set $K = [\min K_1, \max K_k]$, where $\min K = \min\{i : i \in K\}$ and $\max K = \max\{i : i \in K\}$. We show that

$$\frac{\|I_K\|_{\mathbb{B}}}{\mathrm{Cap}(K)} \geqslant \frac{\|I_{K_1+\cdots+K_k}\|_{\mathbb{B}}}{\mathrm{Cap}(K_1+\cdots+K_k)}.$$

In other words, the ratio for a disconnected compact set is less than or equal to that of the corresponding connected one. For f with $f|_{K_1+\cdots+K_k} = 1$, the restriction of f to the intervals $[\max K_i, \min K_{i+1}]$ may not be a constant. Thus, if we define $\tilde{f} = f$ on K^c and $\tilde{f}|_K = 1$, then $D(\tilde{f}) \leqslant D(f)$, due to the character of birth–death processes. This means that $\mathrm{Cap}(K) \leqslant \mathrm{Cap}(K_1 + \cdots + K_k)$. In fact, equality holds, because for f with $f|_K = 1$, we must have $f|_{K_1+\cdots+K_k} = 1$ and so the inverse inequality is trivial. Since $K \supset K_1 + \cdots + K_k$ and (H_3), we have $\|I_K\|_{\mathbb{B}} \geqslant \|I_{K_1+\cdots+K_k}\|_{\mathbb{B}}$. This proves the required assertion.

(c) Because of (b), to compute the isoperimetric constant, it suffices to consider the compact sets having the form $K = \{n, n+1, \dots, m\}$ for $m \geqslant n \geqslant 1$. We now fix such a compact set K and compute $\mathrm{Cap}(K)$.

Given f with $f|_K = 1$ and $\mathrm{supp}(f) = \{1, \dots, N\}$, $N \geqslant m$, we have

$$D(f) = \sum_{i=0}^{n-1} \mu_i b_i (f_{i+1} - f_i)^2 + \sum_{i=m}^{N} \mu_i b_i (f_{i+1} - f_i)^2, \tag{4.1}$$

where $f_0 = 0$ and $f_{N+1} = 0$. Then

$$\begin{aligned}\frac{\partial D}{\partial f_j} &= -2\mu_j b_j (f_{j+1} - f_j) + 2\mu_{j-1} b_{j-1} (f_j - f_{j-1}) \\ &= -2\mu_j b_j v_j + 2\mu_{j-1} b_{j-1} v_{j-1}, \qquad 1 \leqslant j \leqslant n-1 \ \text{ or } \ m+1 \leqslant j \leqslant N,\end{aligned}$$

where $v_i = f_{i+1} - f_i$. The condition $\partial D/\partial f_j = 0$ gives us

$$v_j = \frac{\mu_{j-1} b_{j-1}}{\mu_j b_j} v_{j-1}, \qquad 1 \leqslant j \leqslant n-1 \ \text{ or } \ m+1 \leqslant j \leqslant N.$$

Hence

$$v_j = \frac{\mu_0 b_0 v_0}{\mu_j b_j}, \quad 0 \leqslant j \leqslant n-1, \ \text{ and } \ v_j = \frac{\mu_m b_m v_m}{\mu_j b_j}, \quad m \leqslant j \leqslant N. \tag{4.2}$$

Therefore

$$\begin{aligned} f_j &= \sum_{i=0}^{j-1} v_i = \mu_0 b_0 v_0 \sum_{i=0}^{j-1} \frac{1}{\mu_i b_i}, \qquad 0 \leqslant j \leqslant n, \\ f_j &= \sum_{i=m}^{j-1} v_i + 1 = \mu_m b_m v_m \sum_{i=m}^{j-1} \frac{1}{\mu_i b_i} + 1, \qquad m \leqslant j \leqslant N. \end{aligned}$$

On the other hand, since $f_n = 1$ and $v_N = f_{N+1} - f_N = -f_N$, we get

$$1 = \mu_0 b_0 v_0 \sum_{i=0}^{n-1} \frac{1}{\mu_i b_i}, \qquad \frac{\mu_m b_m v_m}{\mu_N b_N} = -\mu_m b_m v_m \sum_{i=m}^{N-1} \frac{1}{\mu_i b_i} - 1.$$

Then

$$\mu_0 b_0 v_0 = \left(\sum_{i=0}^{n-1} \frac{1}{\mu_i b_i} \right)^{-1}, \qquad \mu_m b_m v_m = -\left(\sum_{i=m}^{N} \frac{1}{\mu_i b_i} \right)^{-1}. \tag{4.3}$$

Inserting (4.2) and (4.3) into (4.1), we obtain

$$\begin{aligned} D(f) &= \sum_{i=0}^{n-1} \mu_i b_i v_i^2 + \sum_{i=m}^{N} \mu_i b_i v_i^2 \\ &= (\mu_0 b_0 v_0)^2 \sum_{i=0}^{n-1} \frac{1}{\mu_i b_i} + (\mu_m b_m v_m)^2 \sum_{i=m}^{N} \frac{1}{\mu_i b_i} \\ &= \bigg(\sum_{i=0}^{n-1} \frac{1}{\mu_i b_i}\bigg)^{-1} + \bigg(\sum_{i=m}^{N} \frac{1}{\mu_i b_i}\bigg)^{-1}. \end{aligned}$$

Since the process is recurrent, $\sum_{i=m}^{\infty} 1/\mu_i b_i = \infty$, we have

$$\mathrm{Cap}(K) = \inf\{D(f), f_0 = 0, f \text{ has finite support}, f|_K \geqslant 1\} = \bigg(\sum_{i=0}^{n-1} \frac{1}{\mu_i b_i}\bigg)^{-1},$$

which is independent of m. Therefore

$$B_{\mathbb{B}} = \sup_K \frac{\|I_K\|_{\mathbb{B}}}{\mathrm{Cap}(K)} = \sup_{1 \leqslant n \leqslant m} \frac{\|I_{[n,m]}\|_{\mathbb{B}}}{\mathrm{Cap}([n,m])} = \sup_{n \geqslant 1} \|I_{[n,\infty)}\|_{\mathbb{B}} \sum_{i=0}^{n-1} \frac{1}{\mu_i b_i}$$

as required. □

We remark that once we know the solution f that minimizes $D(f)$, the proof (c) above can be done in a different way as illustrated in the next proof.

Corollary 4.2. Consider an ergodic diffusion on $(0, \infty)$ with operator

$$L = a(x)\mathrm{d}^2/\mathrm{d}x^2 + b(x)\mathrm{d}/\mathrm{d}x$$

and reflecting boundary. Suppose that the corresponding Dirichlet form $(D, \mathscr{D}(E))$ is regular, having the core $\mathbb{C}_d[0,\infty)$: the set of all continuous functions with piecewise continuous derivatives and having compact support. Define $C(x) = \int_0^x b/a$ for $x \geqslant 0$. Then for Dirichlet boundary at 0, we have

$$B_{\mathbb{B}} = \sup_{x \geqslant 0} \|I_{[x,\infty)}\|_{\mathbb{B}} \int_0^x e^{-C}.$$

Proof. In view of (b) in the above proof, to compute the isoperimetric constant, we need only consider the compact $K = [n, m]$, $m > n$, $m, n \in \mathbb{R}_+$. Define

$$g(x) = \begin{cases} \int_0^x e^{-C} \big/ \int_0^n e^{-C}, & \text{if } 0 \leqslant x \leqslant n, \\ 1, & \text{if } n \leqslant x \leqslant m, \\ 1 - \int_m^{x \wedge N} e^{-C} \big/ \int_m^N e^{-C}, & \text{if } x \geqslant m. \end{cases}$$

We now show that $\mathrm{Cap}(K)$ can be computed in terms of $g \in \mathbb{C}_d[0,\infty)$. Note that

$$\mathrm{Cap}(K) = \inf\{D(f) : f \in \mathbb{C}_d[0,\infty) : f|_K = 1\}.$$

Next, let $f_1 \in \mathbb{C}_d[0,n]$ with $f_1(0) = f_1(n) = 0$, $f_2 \in \mathbb{C}_d[m,N]$ with $f_2(m) = f_2(N) = 0$, and study the following variational problem with respect to ε_1 and ε_2:

$$H(\varepsilon_1, \varepsilon_2) = \int_0^n (g' + \varepsilon_1 f_1')^2 e^C + \int_m^N (g' + \varepsilon_2 f_2')^2 e^C.$$

If necessary, one may regard $\int_0^n$ as $\int_{0+}^{n-}$ and similarly for $\int_m^N$. Without loss of generality, assume that $f_k' \neq 0$. Otherwise, we can set $\varepsilon_k = 0$. Clearly, H should have a minimum in a bounded region. From $\partial H/\partial \varepsilon_k = 0$, it follows that

$$\begin{aligned}
\varepsilon_1 &= -\frac{\int_0^n g' f_1' e^C}{\int_0^n {f_1'}^2 e^C} = -\frac{\int_0^n f_1'}{\big(\int_0^n {f_1'}^2 e^C\big)\big(\int_0^n e^{-C}\big)} = -\frac{f_1(n) - f_1(0)}{\big(\int_0^n {f_1'}^2 e^C\big)\big(\int_0^n e^{-C}\big)} = 0,\\
\varepsilon_2 &= -\frac{\int_m^N g' f_2' e^C}{\int_m^N {f_2'}^2 e^C} = -\frac{\int_m^N f_2'}{\big(\int_m^N {f_2'}^2 e^C\big)\big(\int_m^N e^{-C}\big)} = -\frac{f_2(N) - f_2(m)}{\big(\int_m^N {f_2'}^2 e^C\big)\big(\int_m^N e^{-C}\big)} = 0.
\end{aligned}$$

More precisely, if f' is discontinuous at $n_1, \ldots, n_k$, then

$$\int_0^n f' = \int_0^{n_1} f' + \cdots + \int_{n_k}^n f' = \big(f(n_1) - f(0)\big) + \cdots + \big(f(n) - f(n_k)\big) = f(n) - f(0) = 0,$$

since f is continuous. Thus, $H(\varepsilon_1, \varepsilon_2)$ attains its minimum

$$D(g) = \bigg(\int_0^n e^{-C}\bigg)^{-1} + \bigg(\int_m^N e^{-C}\bigg)^{-1}$$

at $\varepsilon_1 = \varepsilon_2 = 0$. Moreover, due to the recurrence, we have $\int_m^\infty e^{-C} = \infty$. Collecting these facts, we obtain $\mathrm{Cap}(K) = \big(\int_0^n e^{-C}\big)^{-1}$. The assertion now follows immediately. $\square$

Because of the linear order in the real line, it is easy to write down the explicit estimates of the logarithmic Sobolev constant A_{Log}, in terms of Theorem 1.5 and Corollaries 4.1 and 4.2.

Corollary 4.3. For ergodic birth–death processes, let m satisfy

$$\pi[0,m) := \sum_{j=0}^{m-1} \mu_j/Z \leqslant 1/2 \quad \text{and} \quad \pi(m,\infty) := \sum_{j=m+1}^{\infty} \mu_j/Z \leqslant 1/2,$$

where $Z = \sum_{k=0}^\infty \mu_k$. Then we have

$$\frac{\log 2}{\log(1+2e^2)}\, B_{\mathrm{Log}}(e^2) \leqslant B_{\mathrm{Log}}(1/2) \leqslant A_{\mathrm{Log}} \leqslant 4\, B_{\mathrm{Log}}(e^2), \tag{4.4}$$

where $B_{\mathrm{Log}}(\gamma) = B_+(\gamma) \vee B_-(\gamma)$ and

$$\begin{aligned}
B_+(\gamma) &= \sup_{n>m} \mu[n,\infty) \log\bigg(1 + \frac{\gamma}{\pi[n,\infty)}\bigg) \sum_{j=m}^n \frac{1}{\mu_j b_j},\\
B_-(\gamma) &= \sup_{0 \leqslant n < m} \mu[0,n] \log\bigg(1 + \frac{\gamma}{\pi[0,n]}\bigg) \sum_{j=n}^{m-1} \frac{1}{\mu_j b_j}.
\end{aligned} \tag{4.5}$$

Proof. Here we prove the upper estimate only since the proof for the lower estimate is similar. Set $E_1 = \{m+1, m+2, \dots\}$ and $E_2 = \{0, \dots, m-1\}$. Following the proof (a) of Theorem 1.5, we obtain (3.10) with respect to E_1 and E_2. Applying Corollary 4.1 to each E_i, we get $B_\pm(e^2)$. We remark that in the application of Corollary 4.1 to E_1, the Dirichlet boundary is setting at m rather than at 0. In other words, we need to consider the inverse order on E_1.

For one-dimensional diffusion, a similar result of Corollary 4.3 was obtained by Barthe and Roberto [2].

Corollary 4.4. Let μ and ν be Borel measures on $\mathbb{R}$ with $\mu(\mathbb{R}) < \infty$ and denote by h the derivative of the the absolutely continuous part of ν with respect to the Lebesgue measure. Next, set $\pi = \mu/\mu(\mathbb{R})$ and let m be the median of π. Then the optimal constant A_{Log} in the inequality

$$\int_{\mathbb{R}} f^2 \log\big(f^2/\pi(f^2)\big)\mathrm{d}\mu \leqslant A_{\mathrm{Log}} \int_{\mathbb{R}} {f'}^2 \mathrm{d}\nu, \qquad f \in \mathbb{C}_d(\mathbb{R}), \tag{4.6}$$

(cf. Corollary 4.2 for definition of $\mathbb{C}_d(\mathbb{R})$) satisfies

$$\frac{\log 2}{\log(1+2e^2)}\, B_{\mathrm{Log}}(e^2) \leqslant B_{\mathrm{Log}}(1/2) \leqslant A_{\mathrm{Log}} \leqslant 4\, B_{\mathrm{Log}}(e^2), \tag{4.7}$$

with $B_{\mathrm{Log}}(\gamma) = B_+(\gamma) \vee B_-(\gamma)$, where

$$\begin{aligned} B_+(\gamma) &= \sup_{x>m}\ \mu[x,\infty) \log\left(1 + \frac{\gamma}{\pi[x,\infty)}\right) \int_m^x \frac{1}{h}, \\ B_-(\gamma) &= \sup_{x<m}\ \mu(-\infty,x] \log\left(1 + \frac{\gamma}{\pi(-\infty,x]}\right) \int_x^m \frac{1}{h}. \end{aligned} \tag{4.8}$$

Actually, Corollaries 4.3 and 4.4 can be further improved by using the variational formulas presented in Chen [5, 6].

Acknowledgements. A large part of this paper is completed during the author's visit to Japan (Oct. 16, 2002–Jan. 17, 2003). The author is grateful for the kind invitations, financial supports and the warm hospitality made by many Japanese probabilists and their universities: I. Shigekawa, Y. Takahashi, T. Kumagai and N. Yosida at Kyoto Univ.; M. Fukushima, S. Kotani and S. Aida at Osaka Univ.; H. Osada, S. Liang at Nagoya Univ.; T. Funaki and S. Kusuoka at Tokyo Univ. A special appreciation is given to the senior N. Ikeda, S. Watanabe, K. Sato and my Chinese friend Mr. Q. P. Liu at Kyoto Univ. Moreover, the author would like to thank M. Fukushima for providing the preprint by him and Uemura [10] before publication.

Finally, the author deeply acknowledges the referees for their kind and crucial comments on an earlier version of the paper.

References

[1]. Barthe, F., Cattiaux, P. and Roberto, C. (2004), *Interpolated inequalities between exponential and Guassian, Orlicz hypercontractivity and isoperimetry*, arXiv:math.PR /0407219.

[2]. Barthe, F. and Roberto, C. (2003), *Sobolev inequalities for probability measures on the real line*, Studia Math. **159**(3), 481–497.

[3]. Bobkov, S. G. and Götze, F. (1999), *Exponential integrability and transportation cost related to logarithmic Sobolev inequalities*, J. Funct. Anal., **163**, 1–28.

[4]. Chen, M. F. (2000), *Explicit bounds of the first eigenvalue*, Sci. in China, Ser A, **43**(10), 1051–1059.

[5]. Chen, M. F. (2002), *Variational formulas of Poincaré-type inequalities in Banach spaces of functions on the line*, Acta Math. Sin. Eng. Ser. **18**(3), 417–436.

[6]. Chen, M. F. (2003), *Variational formulas of Poincaré-type inequalities for birth–death processes*, Acta Math. Sin. Eng. Ser **19**(4), 625–644.

[7]. Chen, M. F. and Wang, F. Y. (1998), *Cheeger's inequalities for general symmetric forms and existence criteria for spectral gap*, Abstract. Chin. Sci. Bulletin **43**(18), 1516–1519. Ann. Prob. 2000, **28**(1), 235–257.

[8]. Deuschel, J. D. and Stroock, D. W. (1989), *Large Deviations*, Academic Press, New York.

[9]. Fukushima, M., Oshima, Y. and Takeda, M. (1994), *Dirichlet forms and symmetric Markov processes*, Walter de Gruyter.

[10]. Fukushima, M. and Uemura, T. (2003), *Capacitary bounds of measures and ultracontractivity of time changed processes*, J. Math. Pure et Appliquees **82**(5), 553-572.

[11]. Gong, F. Z. and Wang, F. Y. (2002), *Functional inequalities for uniformly integrable semigroups and application to essential spectrums*, Forum. Math. **14**, 293–313.

[12]. Gross, L. (1993), *Logarithmic Sobolev inequalities and contractivity of semigroups*, LNM. **1563**.

[13]. Kaimanovich, V. A. (1992), *Dirichlet norms, capacities and generalized isoperimetric inequalities for Markov operators*, Potential Analysis **1**, 61-82.

[14]. Ma, Z. M. and Röckner, M. (1992), *Introduction to the Theory of* (*non-symmetric*) *Dirichlet Forms*, Springer.

[15]. Mao, Y. H. (2002), *The logarithmic Sobolev inequalities for birth–death process and diffusion process on the line*, Chin. J. Appl. Prob. Statis. **18**(1), 94–100.

[16]. Mao, Y. H. (2002), *Nash inequalities for Markov processes in dimension one*, Acta Math. Sin. Eng. Ser. **18**(1), 147–156.

[17]. Maz'ya, V. G. (1985), *Sobolev Spaces*, Springer.

[18]. Muckenhoupt, B. (1972), *Hardy's inequality with weights*, Studia Math. **XLIV**, 31–38.

[19]. Opic, B. and Kufner, A. (1990), *Hardy-type Inequalities*, Longman, New York.

[20]. Rao, M. M. and Ren, Z. D. (1991), *Theory of Orlicz Spaces*, Marcel Dekker, Inc New York.

[21]. Rothaus, O. S. (1985), *Analytic inequalities, isoperimetric inequalities and logarithmic Sobolev inequalities*, J. Funct. Anal. **64**, 296–313.

[22]. Vondraček, Z. (1996), *An estimate for the L^2-norm of a quasi continuous function with respect to a smooth measure*, Arch. Math. **67**, 408-414.

[23]. Wang, F. Y. (2000), *Functional inequalities for empty essential spectrum*, J. Funct. Anal. **170**, 219–245.

School of Mathematical Sciences, Beijing Normal University, Beijing 100875, The People's Republic of China. E-mail: mfchen@bnu.edu.cn
Home page: http://math.bnu.edu.cn/~chenmf/main_eng.htm