Stochastic Models of Economic Optimization

Mu-Fa Chen*

(Beijing Normal University)

Abstract

This paper deals with some stochastic models of economic optimization. Due to the value in practice, the models are quite attractive. But our knowledge on them is still very limited, some fundamental problems remain open.

We begin with a short review of the study on some global economic models (or economy in large scale), the well-known input-output method and L. K. Hua's fundamental theorem for the stability of economy. Then, we show that it is necessary to study the stochastic models. A collapse theorem for a non-controlling stochastic economic system is introduced. In the analysis of the system, the products of random matrices play a critical role. Especially, the first eigenvalue, the corresponding eigenfunctions and an ergodic theorem of Markov chains play a nice role here. Partial proofs are included. Some challenge open problems are also mentioned.

AMS 2000 Subject classification: 60-02, 60F99, 62P20.

Keywords: Input–output method, L. K. Hua's theorem, stochastic model, collapse theorem, random matrix.

1 Input-output method

First, we fix the unit of the quantity of each product: kilogram, kilovolt and so on. Denote by $x = \left(x^{(1)}, x^{(2)}, \ldots, x^{(d)}\right)$ the quantity of the main products we are interested, it is called the *vector of products*. Throughout this paper, all vectors are row ones.

To understand the present economy, we need to examine three things: The input, the output and the structure matrix. Suppose that the starting vector of products last year was

$$x_0 = \left(x_0^{(1)}, x_0^{(2)}, \ldots, x_0^{(d)}\right).$$

For reproduction, assume that the j-th product distributed amount $x_{ij}^{(0)}$ to the i-th product, and the vector of the products this year becomes

$$x_1 = \left(x_1^{(1)}, x_1^{(2)}, \ldots, x_1^{(d)}\right).$$

*Research supported in part by NSFC (No. 10121101), RFDP and 973 Project.

Here, we suppose for a moment that all the products are used for the reproduction (*idealized model*). Next, set

$$a_{ij}^{(0)} = x_0^{(j)} / x_1^{(i)}, \qquad 1 \le i, \ j \le d .$$

The matrix $A_0 = \left(a_{ij}^{(0)}\right)$ is called a *structure matrix* (or matrix of *expending coefficients*). This matrix is essential since it describes the efficiency of the current economy: to produce one unit of i-th product, one needs $a_{ij}^{(0)}$ units of the j-th product. Clearly, $x_0 = x_1 A_0$. Similarly, we have $x_{n-1} = x_n A_{n-1}$ for all $n \ge 1$. Suppose that the structure matrices are time-homogeneous: $A_n = A$ for all $n \ge 0$ (This is reasonable if one consider a short time unit). Then we have a simple expression for the n-th output:

$$x_n = x_0 A^{-n}, \qquad n \ge 1. \tag{1.1}$$

Thus, once known the structure matrix and the input x_0, we may predict the future output, and so is called the *input-output method* or *Leontief's method* (cf., Leontief (1936, 1951, 1986)). It is a well known method. As far as I know, up to 1960's, more than 100 countries had used this method in their national economy.

2 L. K. Hua's fundamental theorem

Let us return to the original equation

$$x_1 = x_0 A^{-1}.$$

We now fix A, then x_1 is determined by x_0 only. The question is which choice of x_0 is the optimal one. Furthermore, in what sense of optimality are we talking about? The first choice would be "average". If one tells you that the average of the members' ages in a group is twenty, you may think that everyone in the group is strong, it may be a team of volleyball. However, the group may be a nursery, which consists of six babies and two older women, who are over seventies. The average of the ages in this group is still twenty. The misleading point is that the variance is too big in this situation and so the average is not a good tool in the present situation. To avoid this, we adopt the *minimax principle*: i.e., finding out the best solution among the worst cases. It is the most safe strategy and used widely in the optimalization theory and game theory. In other words, we want to find out x_0 such that $\min_{1\le j\le d} x_1^{(j)} / x_0^{(j)}$ attains the maximum below

$$\max_{x_1>0,\ x_0=x_1 A} \ \min_{1\le j\le d} x_1^{(j)} / x_0^{(j)}.$$

By using the classical Frobenius theorem, Hua (1984, Part III) proved the following result.

Theorem 2.1 (Hua (1984, Part Ⅲ)). Given an irreducible non-negative matrix A, let u be the left eigenvector (positive) of A, corresponding to the largest eigenvalue $\rho(A)$ of A. Then, up to a constant, the solution to the above problem is $x_0 = u$. In this case, we have

$$x_1^{(j)}/x_0^{(j)} = \rho(A)^{-1} \qquad \text{for all } j.$$

In what follows, we call the above technique (i.e., setting $x_0 = u$) the *eigenvector's method*.

Next, we are going further to study the stability of economy. From (1.1), we obtain the simple expression:

$$x_n = x_0 \rho(A)^{-n}$$

whenever $x_0 = u$. What happens if we take $x_0 \neq u$ (up to a constant)?

Stability of economy

For convenience, set

$$T^x = \inf\big\{n \geq 1 : x_0 = x \text{ and there is some } j \text{ such that } x_n^{(j)} \leq 0\big\},$$

which is called the *collapse time* of the economic system.

We can now state Hua's important result as follows.

Theorem 2.2 (Hua (1984, Part Ⅲ; 1985, Part Ⅸ)). Under some mild conditions, if $x_0 \neq u$, then $T^{x_0} < \infty$.

In the case that the collapse time is bigger than 150 years, then we do not need to take care about the stability of the economy, since none of us will be still alive. However, the next example shows that we are not in this situation.

Example 2.3 (Hua (1984, Part I)). Consider two products only: industry and agriculture. Let

$$A = \frac{1}{100}\begin{pmatrix} 20 & 14 \\ 40 & 12 \end{pmatrix}.$$

Then $u = \big(5\big(\sqrt{2400} + 13\big)/7,\ 20\big) \approx (44.34397483,\ 20)$. We have

x_0	T^{x_0}
$(44, 20)$	3
$(44.344, 20)$	8
$(44.34397483, 20)$	13

This shows that the economy is very sensitive! We point out that this theorem is essential. Recall that the Frobenius theorem or Brouwer fixed point theorem, often used in the study on economics, do not provide any information about the collapse phenomena.

To understand Hua's theorem, for probabilists, it is very natural to consider a particular case that $A = P$. That is, A is a transition probability matrix. Then, from the ergodic theorem for Markov chains (irreducible and aperiodic), it follows that

$$P^n \to \Pi \quad \text{as } n \to \infty,$$

where Π is the matrix having the same row $\big(\pi^{(1)}, \pi^{(2)}, \ldots, \pi^{(d)}\big)$, which is just the stationary distribution of the corresponding Markov chain. Since the distribution is the only stable solution for the chain, it should have some meaning in economics even though the later one goes in a converse way:

$$x_n = x_0 P^{-n}, \qquad n \geq 1.$$

From the above facts, it is not difficult to prove, as shown in the next paragraph, that if

$$x_0 \neq u = \big(\pi^{(1)}, \pi^{(2)}, \ldots, \pi^{(d)}\big)$$

up to a positive constant, then $T^{x_0} < \infty$. Next, since the general case can be reduced to the above particular case, we think that this is a very natural way to understand the Hau's theorem.

Proof of Theorem 2.2. We need to show that if $x_n > 0$ for all n, then $x_0 = \pi$.

Let $x_0 > 0$ be normalized such that $x_0 \mathbb{1}^* = 1$, where $\mathbb{1}^*$ is the row vector having components 1 everywhere. Then

$$1 = x_0 \mathbb{1}^* = x_n P^n \mathbb{1}^* = x_n \mathbb{1}^*, \qquad n \geq 1.$$

Since the set $\{x : x \geq 0,\, x\mathbb{1}^* = 1\}$ is compact, exists a subsequence $\{x_{n_k}\}_{k \geq 1}$ and a vector $\bar{x}$ such that

$$\lim_{k\to\infty} x_{n_k} = \bar{x}, \qquad \bar{x} \geq 0, \qquad \bar{x}\mathbb{1}^* = 1.$$

Therefore,

$$x_0 = (x_0 P^{-n_k}) P^{n_k} = x_{n_k} P^{n_k} \to \bar{x}\Pi = \bar{x}\mathbb{1}^*\pi = \pi.$$

Thus, we must have $x_0 = \pi$. As mentioned before, the general case can be reduced to the above particular case and so we are done. $\square$

We have seen the critical role played by the largest or the first eigenvalue and its eigenvectors. For which, the computations are far non-trivial, especially for large scale of matrices. In the numerical computation of the largest eigenvalue, it is important to have a good initial data, which is just an application of the study on the estimation of the eigenvalue. Having the known eigenvalue at hand, the computation of eigenvectors is easier, for which, one needs only to solve a linear equation (in contract, the equation of eigenvalue is polynomial).

Economy in markets

In L. K. Hua's eleven reports (1984–1985), he also studied some more general models of economy. But the above two theorems are the key to his idea. The title of the reports (written in that specific period) may cost some misunderstanding since one may think that the theory works only for planned economy. Actually, the economy in markets was also treated in Hua (1984, Part VII). The only difference is that in the later case one needs to replace the structure matrix A with $V^{-1}AV$, where V is the diagonal matrix $\mathrm{diag}(v_i/p_i)$: (p_i) is the vector of prices in market and (v_i) is the right eigenvector of A. Note that the eigenvalue of $V^{-1}AV$ are the same as those of A. Corresponding to the eigenvalue $\rho(V^{-1}AV) = \rho(A)$, the left eigenvector of $V^{-1}AV$ becomes uV. Therefore, for the economy in markets, we have a new structure matrix $V^{-1}AV$ and a new left eigenvector uV, which are the all what we need in Hua's model. Thus, from mathematical point of view, the consideration of markets makes no essential difference in the Hua's model.

3 Stochastic model without consumption

In the case that the randomness does not play a critical role, one may simply ignore it and insist in the deterministic system. Thus, we started our study on examining the influence of a smaller random perturbation of Hua's example.

Consider the perturbation:

$$\begin{aligned} \widetilde{a}_{ij} &= a_{ij} \quad \text{with probability } 2/3, \\ &= a_{ij}(1 \pm 0.01) \quad \text{with probability } 1/6. \end{aligned}$$

Taking $(\widetilde{a}_{ij})$ instead of (a_{ij}), we get a random matrix. Next, let $\{A_n;\, n \geq 1\}$ be a sequence of independent random matrices with the same distribution as above, then $x_n = x_0 \prod_{k=1}^{n} A_k^{-1}$ gives us a stochastic model of an economy without consumption.

Again, starting from $x_0 = (44.344, 20)$ (remember the collapse time is 8 in the deterministic case), then the collapse probability in the above stochastic model is the following

$$\mathbb{P}[T^{x_0} = n] = \begin{cases} 0, & \text{for } n = 1, \\ 0.09, & \text{for } n = 2, \\ 0.65, & \text{for } n = 3. \end{cases}$$

Surprisingly, we have $\mathbb{P}[T \leq 3] \approx 0.74$. This observation tells us that the randomness plays a critical role in the economy. It also explains the reason why the traditional input-output is not very practicable, as people often think, because the randomness has been ignored and so the deterministic model is far away from the real practice.

Now, what is the analog of Hua's theorem for the stochastic case?

Theorem 3.1 (Chen (1992, Part II)). Under some mild conditions, we have

$$\mathbb{P}[T^{x_0} < \infty] = 1, \qquad \forall x_0 > 0.$$

Note that the limit theory of products of random matrices are quite different from the deterministic case (cf. Bougerol and Lacroix (1985)), the problem is non-trivial. We have to deal with the product of random matrices:

$$M_n = A_n A_{n-1} \cdots A_1.$$

The first result we learnt from the limit theory of products of random matrices is the *Liapynov exponent*, sometimes called "strong law of large numbers". Let $\|A\|$ denote the operator norm of A. Then, the main known result is as follows.

Theorem 3.2 (Oseledec (1968)). Let $\mathbb{E}\log^+ \|A_1\| < \infty$. Then

$$\frac{1}{n}\log\|M_n\| \xrightarrow{a.s.} \gamma \in \{-\infty\} \cup \mathbb{R},$$

where

$$\gamma = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}\log\|M_n\|.$$

However, this result is still not enough for our purpose. What we adopted is a much stronger result. To state the result, we need the following assumptions which are analogue of the irreducible and aperiodic conditions.

(H_1) $A_1 \geq 0$, a.s. and there exists an integer m such that

$$\mathbb{P}[M_m \text{ is positive}] > 0,$$

where $M_n = A_1 \cdots A_n$.

(H_2) $\mathbb{P}[A_1$ has zero row or column$] = 0$.

Theorem 3.3 (Kesten and Spitzer (1984)). Under (H_1) and (H_2), $M_n > 0$ for large n with probability one and $M_n/\|M_n\|$ converges in distribution to a positive matrix $M = L^* R$ with rank one, where L and R are independent, positive row vectors satisfying the normalizing condition:

$$\max_{1\leq i\leq d} R(i) = 1, \qquad \sum_{j=1}^{d} L(j) = 1. \tag{3.1}$$

By a change of the probabilistic frame, one may replace the "convergence in distribution" by "convergence almost surely" (Shkorohod Theorem). In this sense, the last result is really the strong law of large numbers. Having these remarks in mind, the proof of Theorem 3.1 is not difficult and is given in §10.5.

One may refer to Mukherjea (1991), Hennion (1997) and references within for more recent progress on the limit theory of products of random matrices.

4 Stochastic model with consumption

The model without consumption is idealized and so is not practice. More practical one should have consumption. That is, allow a part of the productions turning into consumption, not used for reproduction.

Suppose that every year we take the $\theta^{(i)}$-times amount of the increment of the i-th product to be consumed. Then in the first year, the vector of products which can be used for reproduction is

$$y_1 = x_0 + (x_1 - x_0)(I - \Theta),$$

where I is the $d \times d$ unit matrix and $\Theta = \operatorname{diag}\left(\theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(d)}\right)$, which is called a *consumption matrix*. Therefore,

$$y_1 = y_0[A_0^{-1}(I - \Theta) + \Theta], \qquad y_0 = x_0.$$

Similarly, in the n-th year, the vector of the products which can be used for reproduction is

$$y_n = y_0 \prod_{k=0}^{n-1} [A_{n-k-1}^{-1}(I - \Theta) + \Theta], \qquad n \geq 1.$$

Let

$$B_n = [A_{n-1}^{-1}(I - \Theta) + \Theta]^{-1}.$$

Then

$$y_n = y_0 \prod_{k=1}^{n} B_{n-k+1}^{-1}, \qquad n \geq 1.$$

We have thus obtained a *stochastic model with consumption.* In the deterministic case, a collapse theorem was obtained by Hua (1985, Part X), Hua and Hua (1985). The conclusion is that the system becomes more stable than the idealized model. More precisely, the dimension of (x_0) for which the economy will be not collapsed can be greater than one. This is consistence with our practice.

To state our result in this general case, we need some notation. Denote by $Gl(d, \mathbb{R})$ the general linear group of real invertible $d \times d$ matrices and by $O(d, \mathbb{R})$ the orthogonal matrices in $Gl(d, \mathbb{R})$. Next, denote by $\mathscr{G}_\mu$ the smallest closed semigroup of $Gl(d, \mathbb{R})$ containing the support of μ.

Definition 4.1.

- $\mathscr{G}$ is called strongly irreducible if exist no proper linear subspaces of $\mathbb{R}^d$, $\mathscr{V}_1, \cdots, \mathscr{V}_k$ such that

$$(\cup_{i=1}^k \mathscr{V}_i)B = \cup_{i=1}^k \mathscr{V}_i, \qquad \forall B \in \mathscr{G}.$$

- $\mathscr{G}$ is said to be contractive if exists $\{B_n\} \subset \mathscr{G}$ such that $\|B_n\|^{-1}B_n$ converges to a matrix with rank one.

- We call $B = K\mathsf{diag}(a_i)U$ a polar decomposition if $K, U \in O(d, \mathbb{R})$ and $a_1 \geq a_2 \geq \cdots \geq a_d > 0$.

Theorem 4.2 (Chen and Li (1994)). Let $\{B_n\}$ be an i.i.d. sequence of random matrices with common distribution μ. Suppose that $\mathscr{G}_\mu$ is strongly irreducible, contractive and the sequence $\{K_n\}$ in the polar decomposition satisfies a "tightness condition". Then $\mathbb{P}[T^x < \infty] = 1$ for all $0 < x \in \mathbb{R}^d$.

Naturally, we have the following question.

Open Problem 4.3. How fast does the economy go to collapse?

As we have seen before, since the economy is very sensitive, one certainly expects the following large deviation result:

$$\mathbb{P}[T > n] \leq Ce^{-\alpha n}.$$

Clearly, Theorem 10.8 is still a distance from complete. Furthermore, in practice, collapse result is not expected and less useful. Now, another question arises.

Open Problem 4.4. How to control the economy and what is the optimal one?

Up to now, we have no idea how to handle this problem, we even do not understand what kind of optimality should be adopted here.

Finally, we mention that a probabilistic exploration of Hua's model, closed related to the ergodic theorem as used in the proof of Theorem 2.2, was investigated by Chung (1995). The topic of this article is now explored, with much more extension and recent references, in the book by Han and Hu (2003).

5 Proof of Theorem 3.1

Given i.i.d., nonnegative random matrices $\{A_n\}_{n=1}^{\infty}$, since we are working on the economic model

$$x_n = x_0 A_1^{-1} \cdots A_n^{-1},$$

it is natural to assume that

$$\mathbb{P}[\det A_1 = 0] = 0. \tag{5.1}$$

We study mainly on the collapse probability $\mathbb{P}[T < \infty]$, where T is the same as before,

$$T = \{n \geq 1 : \text{ there exists some } 1 \leq j \leq d \text{ such that } x_n^{(j)} \leq 0\}.$$

The following result is a more precise statement of Theorem 3.1.

Theorem 5.1 (Chen (1992, Part II)). Let (H_1), (H_2) and (5.1) hold. Given a deterministic $x_0 > 0$ with $\max_i x_0^{(i)} = 1$, we have

$$\mathbb{P}[T = \infty] \le \mathbb{P}[R = x_0].$$

In particular, if $\mathbb{P}[R = x_0] = 0$, then $\mathbb{P}[T = \infty] = 0$.

Proof. (a) Write $M_n = A_n \cdots A_1$ and set $\overline{M}_n = M_n / \|M_n^*\|$. Note that the product M_n is in different order of that in Theorem 3.3. From which, we know that $\overline{M}_n$ converges in distribution to $R^* L$, where R and L are independent, positive row vectors satisfying (3.1).

(b) By condition (5.1), we have $\|M_n^*\| > 0$, a.s. and so

$$x_n > 0 \Longleftrightarrow x_0 M_n^{-1} > 0 \Longleftrightarrow x_0 \overline{M}_n^{\,-1} > 0, \qquad n \ge 1.$$

Hence

$$\mathbb{P}[T = \infty] = \mathbb{P}[x_n > 0,\ \forall n \ge 1] = \mathbb{P}[x_0 \overline{M}_n^{\,-1} > 0,\ \forall n \ge 1].$$

Thus, we can use $\overline{M}_n$ instead of M_n.

(c) By Skorohod Theorem (cf., Ikeda and Watanabe (1981, page 9)), there exists a probability space $\big(\widetilde{\Omega}, \widetilde{\mathscr{F}}, \widetilde{\mathbb{P}}\big)$, on which, there are $\widetilde{M}_n$ and $\widetilde{M}$ such that

$$\begin{aligned} &\widetilde{M}_n = \overline{M}_n \quad \text{in distribution}, \quad \forall n \ge 1 \\ &\widetilde{M} = R^* L =: M \quad \text{in distribution}, \\ &\widetilde{M}_n \to \widetilde{M} \quad \text{as } n \to \infty, \quad \widetilde{\mathbb{P}}\text{-a.s.} \end{aligned} \tag{5.2}$$

In particular,

$$\widetilde{\mathbb{P}}\big[\widetilde{M} \text{ has rank } 1\big] = \mathbb{P}\big[M \text{ has rank } 1\big] = 1.$$

From these facts and the normalizing condition, it is easy to see that there exist positive $\widetilde{R}$ and $\widetilde{L}$, $\widetilde{\mathbb{P}}$-a.s. unique, such that $\widetilde{M} = \widetilde{R}^* \widetilde{L}$ and

$$\max_i \widetilde{R}(i) = 1, \qquad \sum_j \widetilde{L}(j) = 1, \qquad \widetilde{\mathbb{P}}\text{-a.s.}$$

Therefore, we must have

$$\mathbb{P}[x_0 \overline{M}_n^{\,-1} > 0,\ \forall n \ge 1] = \widetilde{\mathbb{P}}[x_0 \widetilde{M}_n^{-1} > 0,\ \forall n \ge 1].$$

Thus, we can ignore $\widetilde{\ }$ and use the original $(\Omega, \mathscr{F}, \mathbb{P})$ instead of $\big(\widetilde{\Omega}, \widetilde{\mathscr{F}}, \widetilde{\mathbb{P}}\big)$, and assume that $\overline{M}_n$ converges to $R^* L$ almost everywhere, rather than the convergence in distribution.

(d) By (5.2), there exists a $\mathbb{P}$-zero set Λ such that

$$\overline{M}_n^{\,*} \to R^* L, \qquad \text{as } n \to \infty \qquad \text{on } \Lambda^c.$$

Write $\bar{x}_n = x_0 \overline{M}_n^{\,-1}$. Fix $\omega \in \Lambda^c$. If

$$x_n(\omega) > 0, \qquad \forall n \ge 1,$$

because of the normalizing condition of x_0 and $\overline{M}_n$, there must exist a subsequence $\{n_k = n_k(\omega)\}$ such that

$$\lim_{k\to\infty} \bar{x}_{n_k}(\omega) =: \bar{x}(\omega) \in [0,\infty]^d.$$

But

$$\begin{aligned} x_0 &= \lim_{k\to\infty} \big[x_0 \overline{M}_{n_k}(\omega)^{-1} \overline{M}_{n_k}(\omega)\big] \\ &= \lim_{k\to\infty} \big[x_{n_k}(\omega) \overline{M}_{n_k}(\omega)\big] \\ &= \bar{x}(\omega) L^*(\omega) R(\omega). \end{aligned}$$

Combining this with the positivity of x_0, L and R, it follows that

$$c := \bar{x} L^* \in (0,\infty), \qquad \text{a.s.}$$

Furthermore, since $\max_i x_0(i) = \max_i R(i) = 1$, we know that $c = 1$, a.s. Therefore, we have

$$[T = \infty] \subset [R = x_0], \qquad \text{a.s.} \qquad \text{on } \Lambda^c,$$

as required. $\square$

Finally, we mention that the condition "$\mathbb{P}[R = x_0] = 0$" can be removed in some cases, as was proven in Chen and Li (1994).

Acknowledgment. This paper is based on the lecture presented at Dianjiang University and Zhongshan University, Taiwan (April 1998); Australian National University and Wollongong University, Australia (September 2000); and the Japanese National Conference on Stochastic Processes and Related Fields (Keio University, Tokyo, December 2002). The author is grateful to the invitations and financial supports from these universities and the hospitality of the following professors.

- Profs. T. S. Chiang, C. R. Hwang, Y. S. Chow and S. J. Sheu at Institute of Mathematics, Academy Sinica, Taipei.
- Prof. A. H. Xia at Melbourne University.
- Profs. C. Heyde, J. Gani and Dr. W. Dai at Australian National University.
- Prof. Y. X. Lin at Wollongong University.
- Profs. I. Shigekawa, Y. Takahashi, T. Kumagai, N. Yosida, S. Watanabe and Dr. Q. P. Liu at Kyoto University.
- Profs. M. Fukushima, S. Kotani, S. Aida and N. Ikeda at Osaka University.
- Profs. H. Osada, S. Liang and K. Sato at Nagoya University.
- Profs. T. Funaki and S. Kusuoka at Tokyo University.

The author acknowledges Profs. S. Albeverio, Z. M. Ma and M. Röckner for their invitation to the "First Sino-German Conference on Stochastic Analysis—A Satellite Conference of ICM 2002", The Sino-German Center, Beijing. The talk presented on this conference, entitled "Ten explicit criteria of one-dimensional processes" will be published in the "Proceedings of the Conference on Stochastic Analysis on Large Scale Interacting Systems", Advanced Studies in Pure Mathematics, by Mathematical Society of Japan.

References

P. Bougerol and J. Lacroix. *Products of Random Matrices with Applications to Schrödinger Operators.* Birkhäuser Boston, Inc., 1985.

M. F. Chen. Stochastic model of economic optimization. *Chin. J. Appl. Probab. and Statis.,* (I): 8(3), 289-294; (Ⅱ): 8(4), 374-377 (In Chinese), 1992.

M. F. Chen and Y. Li. Stochastic model of economic optimization. *J. Beijing Normal Univ.*, 30(2):185–194, 1994.

K. L. Chung. *A New Introduction to Stochastic Processes.* World Scientific, Singapore, 1995.

D. Han and X. J. Hu. *Mathematical Models of Economics and Finance — Theory and Practice.* Shanghai Jiaotong U. Press, Shanghai (In Chinese), 2003.

H. Hennion. Limit theorems for products of positive random matrices. *Ann. Probab.*, 25(4):1545–1587, 1997.

L. K. Hua. The mathematical theory of global optimization on planned economy. *Kexue Tongbao,* (I): 1984, No.12, 705–709. (Ⅱ): 1984, No.13, 769–772. (Ⅲ): 1984, No.16, 961–965. (V), (Ⅵ) and (Ⅶ): 1984, No.18, 1089–1092. (Ⅷ): 1984, No.21, 1281–1282. (Ⅳ): 1985, No.1, 1–2. (X): 1985, No.9, 641–646. (Ⅺ): 1985, No.24, 1841-1844 (In Chinese).

L. K. Hua and S. Hua. The study of the real square matrix with both left positive eigenvector and right positive eigenvector. *Shuxue Tongbao*, 8:30–32 (In Chinese), 1985.

N. Ikeda and S. Watanabe. *Stochastic Differential Equations and Diffusion Processes.* North-Holland, Kodansha, Tokyo, 1981.

H. Kesten and F. Spitzer. Convergence in distribution of products of random matrices. *Z. Wahrs.*, 67:363–386, 1984.

W. Leontief. Quantitive input–output relation in the economic system of the united states. *Review of Economics and Statistics*, XVⅢ(3):105–125, 1936.

W. Leontief. *The structure of the American Economy* 1919–1939. Oxford U. Press, New York, 1951.

W. Leontief. *Input-Output Economics.* 2nd Ed., Oxford U. Press, 1986.

A. Mukherjea. Tightness of products of i.i.d. random matrices. *Prob. Th. Rel. Fields*, 87:389–401, 1991.

V. I. Oseledec. A multiplicative ergodic theorem. Lyapunov characteristic number for dynamical systems. *Trans. Moscow Math. Soc.*, 19:197–231, 1968.

Department of mathematics, Beijing Normal University, Beijing 100875, China.
E-mail: mfchen bnu.edu.cn
Home page: http://www.bnu.edu.cn/˜chenmf/main_eng.htm