MARKOV PROCESSES AND FIELD THEORY

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Let $P(t)=(p_{ij}(t))(t\geqslant 0)$ be a set of real-valued functions on a countable set E. $P(t)$ is called a Markov process if

$$P(t)\geqslant 0,\ P(t)1=1,\ P(s+t)=P(s)P(t),\quad \lim_{t\to 0}P(t)=I. \tag{1}$$

It is called weakly symmetrizable, if there is $(\pi_i>0:\ i\in E)$ such that

$$\pi_i p_{ij}(t)=\pi_j p_{ji}(t),\quad \forall i,\ j\in E,\ \forall t>0. \tag{2}$$

There are two questions: (i) When does Eq. (2) hold for a given $P(t)$? What is the corresponding criterion? (ii) How can we find all (π_i) which satisfies (2)?

It is known that for every Markov process $P(t)$ we have right-hand derivatives

$$\left.\frac{dP(t)}{dt}\right|_{t=0}=Q=(q_{ij}:\ i,j\in E) \tag{3}$$

and

$$0\leqslant q_{ij}<\infty(i\neq j),\quad 0\leqslant q_i\equiv -q_{ii}\leqslant\infty,\quad \sum_{j\neq i}q_{ij}\leqslant q_i,\quad (\forall i\in E). \tag{4}$$

Then $Q=(q_{ij})$ is called a Q-matrix. If $Q=(q_{ij})$ is finite, i. e. $q_i<+\infty$, $\forall i\in E$, then $P(t)$ satisfying (3) is called a Q-process. The two questions (i) and (ii) mentioned above also exist for the Q-matrix. Furthermore, there appears another question: (iii) When does there exist a weakly symmetrizable Q-process for a given finite Q-matrix? When does there exist the only one?

In this paper, the concept of an abstract field is established. We have given a powerful criterion of determining whether a field is a potential field. Having studied various properties of a potential field, we applied the results of the field theory to the Markov processes, hence questions (i) and (ii) for any $P(t)$ and for any Q-matrix and question (iii) for some Q-matrices have been solved.

Let E be any countable set, T be any index set. Let $a(t)=(a_{ij}(t):\ i,j\in E)$ be a set of functions on T to $[-\infty,\ +\infty]$. It satisfies the following fundamental hypotheses:

i) Non-diagonal elements of $a(t)$ are non-negative and finite;

ii) "Co-zero property":

$$a_{ij}(t)=0\Longleftrightarrow a_{ji}(t)=0.\quad \forall t\in T,\ \forall i,j\in E \tag{5}$$

Definition 1. If $a_{ij}(t) > 0$, then i is called "direct reaching" j at time t, and we write $i \to j$. If $i_1 \to i_2 \to \cdots \to i_{n+1}$ at time t, then j is "reachable" from i at time t, and we write $i \rightsquigarrow j$. Thus, $L(t) = (i_1, \cdots, i_{n+1})$ is called a path of $a(t)$ at time t.

The set of all paths of $a(t)$ at time t is denoted $\mathscr{L}(t)$.

If at time t, $i \to j$, then we set

$$\varphi_{ij}(t) = \log a_{ji}(t) - \log a_{ij}(t) \tag{6}$$

and we write $\varphi(t) = (\varphi_{ij}(t))$, where $\varphi_{ij}(t)$ is indefinite when $i \not\rightarrow j$ at time t.

Definition 2. $(E, a(t), \mathscr{L}(t), \varphi(t))$ (simply, $a(t)$) is called a field. If $\mathscr{L}(t) \ni L(t) = (i_1, i_2, \cdots, i_{n+1})$, then

$$\varphi(L(t)) \triangleq \sum_{k=1}^{n} \varphi_{i_k i_{k+1}}(t) \tag{7}$$

is called the work completed by the field $a(t)$ along $L(t)$.

Definition 3. $a(t)$ is called a potential field, if there is a set $U(t) = (u_i(t): i \in E)$ of real-valued functions such that

$$u_j(t) - u_i(t) = \varphi_{ij}(t), \quad \text{for } i \to j \text{ (at time } t\text{)}. \tag{8}$$

Then $U(t)$ is called a potential of the potential field $a(t)$.

Definition 4. $a(t)$ is called weakly symmetrizable, if there is a set $V(t) = (v_i(t): i \in E)$ of real-valued functions such that

i) $v_i(t) > 0, \ \forall i \in E, \ \forall t \in T,$ (9)

ii) $v_i(t) a_{ij}(t) = v_j(t) a_{ji}(t), \ \forall t \in T, \forall i, \ j \in E.$ (10)

Then $V(t)$ is called the symmetrizing sequence of $a(t)$.

Definition 5. We say that $a(t)$ is independent of the path if for every closed path $L(t) = (i, i_1, \cdots, i_n, i)$

$$\varphi(L(t)) = 0. \tag{11}$$

Theorem 1. *The following three properties of a field $a(t)$ are equivalent.* (i) *$a(t)$ is a potential field*; (ii) *$a(t)$ is weakly symmetrizable*; *and* (iii) *$a(t)$ is path-independent*.

Furthermore, if $a(t)$ has a potential $U(t) = (u_i(t))$, then $(\exp[-u_i(t)]: i \in E)$ is a symmetrizing sequence of $a(t)$. Conversely, if $a(t)$ has a symmetrizing sequence $V(t) = (v_i(t))$, then $(-\log v_i(t): i \in E)$ is a potential of $a(t)$.

For fixed $t \in T$, define an equivalence relation $\sim$ as follows:

$$i \sim j \text{ if and only if } i \leftrightsquigarrow j \text{ or } i = j, \tag{12}$$

where "$i \leftrightsquigarrow j$" means both "$i \rightsquigarrow j$ and $j \rightsquigarrow i$". Thus, we may divide E into equivalent classes $(E_l(t): l \in D(t))$ by the equivalence relation. For each $l \in D(t)$, we choose $\Delta_l \in E_l(t)$ at will; for each $i \in E_l(t)$, $i \neq \Delta_l$; and we also choose arbitrarily a path

$$L_i(t) = (\Delta_l, i_1, \cdots, i_k, i), \tag{13}$$

and put

$$\hat{a}_{\Delta_l i}(t)=a_{\Delta_l i_1}(t)a_{i_1 i_2}(t)\cdots a_{i_k i}(t) \tag{14}$$

and

$$\hat{a}_{i\Delta_l}(t)=a_{ii_k}(t)a_{i_k i_{k-1}}(t)\cdots a_{i_1\Delta_l}(t). \tag{15}$$

Let

$$\hat{a}_{\Delta_l\Delta_l}(t)\equiv 1 \tag{16}$$

and

$$v_i(t)=\hat{a}_{i\Delta_l}(t)/\hat{a}_{i\Delta_l}(t). \tag{17}$$

The criterion of determining whether a field is a potential field is as follows.

Theorem 2. *A field $a(t)$ is a potential field if and only if*

$$\hat{a}_{\Delta_l i}(t)a_{ij}(t)\hat{a}_{j\Delta_l}(t)=\hat{a}_{\Delta_l j}(t)a_{ji}(t)\hat{a}_{i\Delta_l}(t)$$
$$\forall t\in T,\ \ \forall i,\ j\in E_l(t),\ \ \forall l\in D(t). \tag{18}$$

Some common fields (in particular, n-dimensional lattice fields) are to be discussed in detail and several more delicate criteria are to be given. Now, take 2-dimensional lattice field for an example to illustrate them.

Take $E_1=(0,\ \pm1,\ \pm2,\cdots)$, $E_2=E_1\times E_1$, and denote points of E_2 by (i,j), (k,l) or e, e'.

Definition 6. A field $a=(a(e,e'):e,e'\in E_2)$ is called a 2-dimensional lattice field (in the narrow sense) if

$$a(e,e')\begin{cases}>0, & \text{for } d(e,e')=1,\\ =0, & \text{for } d(e,e')>1,\end{cases}\quad \forall e,\ e'\in E_2 \tag{19}$$

where $d(\cdot,\ \cdot)$ is the ordinary Euclidean distance.

Theorem 3. *A 2-dimensional lattice field $a=(a(i,\ j;\ k,\ l))$ is a potential field if and only if*

$$a(i,j;i+1,j)a(i+1,j;i+1,j+1)a(i+1,j+1;i,j+1)a(i,j+1;i,j)$$
$$=a(i,j;i,j+1)a(i,j+1;i+1,j+1)a(i+1,j+1;i+1,j)a(i+1,j;i,j)$$
$$\forall(i,j)\in E_2. \tag{20}$$

The condition (20) cannot be improved.

Theorem 4. *If the 2-dimensional lattice field a has a potential, then its potential is $(-\log\pi(i,j):(i,j)\in E_2)$:*

$$\pi(i,j)=\pi(0,0)\prod_{l=s(i)}^{i}\frac{a(l-s(i),0;l,0)}{a(l,0;l-s(i),0)}\prod_{k=s(j)}^{j}\frac{a(i,k-s(j);i,k)}{a(i,k;i,k-s(j))}, \tag{21}$$

where $\pi(0,0)$ is an arbitrary positive constant, and

$$s(i)=\begin{cases}1, & \text{for } i>0,\\ 0, & \text{for } i=0,\\ -1, & \text{for } i<0.\end{cases} \tag{22}$$

We apply the results of the field theory to Markov chains, N-tuple random walks, Markov processes and Q-matrices, and obtain a series of results. Now, we take Markov processes for an example to illustrate them.

Theorem 5. *For co-zero Markov process $P(t)$, the following assertions are equiva-*

lent. (i) $P(t)$ *is a potential field*; (ii) $P(t)$ *is weakly symmetrizable*; (iii) $P(t)$ *is path independent*; *and* (iv) *for every* $l \in D$, $\forall i, j \in E$, *we have*

$$\hat{p}_{\Delta_l i}(t) p_{ij}(t) \hat{p}_{j \Delta_l}(t) = \hat{p}_{\Delta_l j}(t) p_{ji}(t) \hat{p}_{i \Delta_l}(t). \quad \forall t \in T \tag{23}$$

Definition 7. The potential $U(t)$ is called a conservative one if it does not depend on t.

Theorem 6. *The potential of the Markov process $P(t)$ which has a potential is determined completely by its conservative potential.*

From this theorem, we have only to discuss conservative potential and symmetrizing sequence which does not depend on t. "Weakly symmetrizable" will be taken in this sense later on. Thus, let $Q = (q_{ij})$ be finite, $Q = (q_{ij})$ is conservative if

$$\sum_j q_{ij} = 0.$$

The Markov process (Q-process) which has a potential will be called a potential Markov process (potential Q-process).

Theorem 7. *The minimal Q-process $(p_{ij}^{\min}(t))$ is a potential Q-process if and only if its Q-matrix is too.*

Theorem 8. *Let $P(t)$ is a potential Q-process, then*

$$p'_{ij}(t) = \sum_k p_{ik}(t) q_{kj}$$

holds for a pair $i, j \in E$ if and only if

$$p'_{ji}(t) = \sum_k q_{jk} \, p_{ki}(t)$$

holds.

We have discussed whether birth-and-death processes, conservative both-side birth-and-death processes and single exit Q-processes are all potential processes or not. Then, all of potential Q-processes have been found out.

Definition 8. A Q-matrix $Q = (q_{ij})$ is called a single exit if it is conservative and if the equation

$$\left.\begin{array}{l} \lambda u_i - \sum_j q_{ij} u_j = 0 \\ 0 \leqslant u_i \leqslant 1 \end{array} \quad (i \in E, \ \lambda > 0) \right\} \tag{24}$$

has only one non-zero linear independent solution. A Q-process with such a Q-matrix is called a single exit Q-process.

Theorem 9. *Let Q be a single exit Q-matrix, then there exists a single exit potential Q-process if and only if Q is weakly symmetrizable and*

$$\sum_i \pi_i x_i(\lambda) < \infty \quad (\lambda > 0). \tag{25}$$

There exists at most one single exit potential Q-process, i. e.,

$$p_{ij}(\lambda) = p_{ij}^{\min}(\lambda) + \frac{x_i(\lambda) x_j(\lambda) \pi_j}{\lambda \sum_k \pi_k x_k(\lambda)}, \tag{26}$$

where $(p_{ij}(\lambda))$ *is the Laplace transformation of* $(p_{ij}(t))$, (π_i) *is the symmetrizing sequence of* Q *and*

$$x_i(\lambda) = 1 - \lambda \sum_j p_{ij}^{\min}(\lambda). \tag{27}$$

Definition 9. A potential process $P(t)$ is symmetrizable, if it is weakly symmetrizable and its symmetrizing sequence is a probability measure on E.

Definition 10. A Q-process $P(t)$ is reversible, if it has a symmetrizing probability measure (π_i) and

$$\lim_{t\to\infty} p_{ij}(t) = \pi_j, \quad \forall i, j \in E. \tag{28}$$

We have also discussed symmetrizable Markov processes and reversible Q-processes in detail.