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REACTION–DIFFUSION PROCESSES

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Abstract. This paper surveys the main progress made in the past dozen years or more in the study of reaction-diffusion (abbrev. RD) processes. The processes are motivated from some typical models in the modern non-equilibrium statistical physics and consist an important class of interacting particle systems which is currently an active research field in probability and mathematical physics. The models are concrete but as a part of the infinite-dimensional mathematics, the topic is quite hard. It is explained how new problems arise, how some new ideas and new mathematical tools are introduced. Surprisingly, the mathematical tools produced from the study on the simple models then turn to have a lot of powerful applications not only in probability theory but also in other branches of mathematics. Nevertheless, the story is still far away to be finished, some important open problems are proposed for the further study. It is hoped that the paper would be readable for non-experts and new comers.

The paper consist of nine short sections. We begin with an introduction of the models (Section 1). Then we turn to the finite-dimensional case, in which the processes are indeed Markov chains (Section 2). The infinite-dimensional processes are constructed in Sections 3 and 5. The main tool of the construction is discussed in Section 4. The existence of the stationary distribution, the ergodicity and the phase transitions of the processes are discussed respectively in Sections 6–8. In the last section, the relation between the RD-processes and RD-equations are described.

1. The models.

Let $S = \mathbf{Z}^d$, the d-dimensional lattice. Consider a chemical reaction in a container. Divide the container into small vessels. Imagining each $u \in S$ as a small vessel in which there is a reaction. The reaction is described by some Markov chains (abbrev. MCs) with Q-matrices $Q_u = (q_u(i,j) : i,j \in \mathbf{Z}_+)$. That is, the rate of the MC jumping from i to $j \neq i$ is given by $q_u(i,j)$. Throughout the paper, we consider only totally stable and conservative Q-matrix: $-q_u(i,i) = \sum_{j\neq i} q_u(i,j) < \infty$ for all $i \in \mathbf{Z}_+$. Thus, the reaction part of the formal generator of the process is as follows:

$$\Omega_r f(x) = \sum_{u\in S} \sum_{k\in \mathbf{Z}\setminus\{0\}} q_u(x_u, x_u + k)\big[f(x + ke_u) - f(x)\big],$$

where e_u is the element in $E := \mathbf{Z}_+^S$ whose value at site u is equal to one, and at other sites are zero. Moreover, we have used the following convention: $q_u(i,j) = 0$ for $i \in \mathbf{Z}_+$, $j \notin \mathbf{Z}_+$ and $u \in S$. Mathematically, one may regard x_u as the u-th component of x in the product space ${\mathbf{Z}_+}^S$. The other part of the generator of the process consists of diffusions between the vessels, which are described by a transition probability matrix $(p(u,v) : u,v \in S)$ and a function $c_u\,(u \in S)$ on $\mathbf{Z}_+$. For instance, if

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there are k particles in u, then the rate function of the diffusion from u to v is $c_u(k)p(u,v)$, where c_u satisfies

$$c_u \ge 0, \qquad c_u(0) = 0, \qquad u \in S. \tag{1.1}$$

Thus, the diffusion part of the formal generator becomes

$$\Omega_d f(x) = \textstyle\sum_{u,v\in S} c_u(x_u)p(u,v)[f(x-e_u+e_v)-f(x)].$$

Finally, the whole formal generator of the process is $\Omega = \Omega_r + \Omega_d$.

Example 1.1 (Polynomial model). The diffusion rate is $c_u(k) = k$ and $p(u,v)$ is the simple random walk on $\mathbf{Z}^d$. The reaction rates are of birth-death type:

$$q_u(k,k+1) = b_k = \textstyle\sum_{j=0}^{m_0} \beta_j k^{(j)}, \qquad q_u(k,k-1) = a_k = \sum_{j=1}^{m_0+1} \delta_j k^{(j)},$$

where $k^{(j)} = k(k-1)\cdots(k-j+1)$, β_j, $\delta_j \ge 0$ and $\beta_0, \beta_{m_0}, \delta_1, \delta_{m_0+1} > 0$.

In particular, we have

Examples 1.2. (1). **Schlögl's first model**: $m_0 = 1$.

(2). **Schlögl's second model**: $m_0 = 2$ but $\beta_1 = \delta_2 = 0$.

All these examples have a single type of particles and so the number of the particles is valued in $\mathbf{Z}_+$. If we consider two types of particles, then the reaction part becomes a MC in $\mathbf{Z}_+^2$. Here is a typical example.

Example 1.3 (Brussel's model). For each type of the particles, the diffusion part of the formal generator is the same as in Example 1.1. As for the reaction part, the MC has the following transition behavior.

$$\begin{array}{lll} \mathbf{Z}_+^2 \ni (i,\,j) \to (i+1,\,j) & \text{at rate} & \lambda_1 \\ \qquad\quad \to (i-1,\,j) & \text{at rate} & \lambda_4 i \\ \qquad\quad \to (i-1,\,j+1) & \text{at rate} & \lambda_2 i \\ \qquad\quad \to (i+1,\,j-1) & \text{at rate} & \lambda_3 i(i-1)j/2 \end{array}$$

where λ_k's are positive constants.

These examples are typical models in non-equilibrium statistical physics. Refer to [1] or [2] for more information about the background and references. About 15 models are treated in these books. The author learnt the models from Prof. S. J. Yan in 1978[3].

2. Finite-dimensional case.

Replacing $S = \mathbf{Z}^d$ with a finite set S (which is fixed in this section) in the above definitions of Ω_r and Ω_d, the corresponding processes are simply MCs since the state space $E = \mathbf{Z}_+^S$ (or $\left(\mathbf{Z}_+^2\right)^S$) is countable. At the beginning, one may think this step can be ignored because there have already been a well-developed theory on MCs. However, the subject is not so easy as it stands. Indeed, we did not know how to prove the uniqueness of the MCs for several years. The usual criterion for the uniqueness says that the equations

$$(\lambda - \Omega)u(x) = 0, \qquad 0 \le u(x) \le 1, \qquad x \in \mathbf{Z}_+^S$$

have only the solution zero for some (equivalently, for all) $\lambda > 0$. It should be clear that the equations are quite hard to handle especially in higher-dimensional case. The criterion does not take the geometry of the MC into account.

To overcome this difficulty, we regard the set $\{x : \sum_{u\in S} x_u = n\}$ as a single point n ($n \ge 0$). Construct a single birth process (i.e., when $k > 0$, $q_{i,i+k} > 0$ iff $k = 1$) on $\mathbf{Z}_+$ which dominates the original process. Since for single birth processes, we do have a computable criterion for the uniqueness and then we can prove the uniqueness of the original processes by a comparison argument. This and related results are presented in [4]. See also Chen[1] for some improvement.

By using an approximation of the processes with bounded rates (in this case, the process is always unique), a more general uniqueness result (even for Markov jump processes on general state space) was proved in [5]. The following result is also included in [1,2,6].

[1] Chen M F. Single birth processes, preprint, 1997.

Theorem 2.1. Let $Q = (q_{ij})$ be a Q-matrix on a countable set E. Suppose that there exist a sequence $\{E_n\}_1^\infty$ and a non-negative function φ such that $E_n \uparrow E$, $\sup_{i\in E_n}(-q_{ii}) < \infty$, $\lim_{n\to\infty} \inf_{i\notin E_n} \varphi_i = \infty$ and $\sum_j q_{ij}(\varphi_j - \varphi_i) \le c\,\varphi_i$ for all $i \in E$ and some $c \in \mathbf{R}$, then the process (MC) is unique.

To justify the power of the theorem, for the above examples, simply take $\varphi(x) = c[1 + \sum_u x_u]$ and $E_n = \{x : \sum_{u\in S} x_u \le n\}$ for some suitable constant c. Indeed, it can be proved that the conditions of the theorem are also necessary for the single birth processes (see [1] or [2]) and up to now we do not know any counterexample for which the process is unique but the conditions of Theorem 1.3 do not hold. The theorem now has a very wide range of applications. For instance, it is basic result used in study of the RD-processes (cf. [7]–[31]) and the mean field models (cf. [32]–[36]) which will be discussed later. It was used in Chen [37] to study an extended class of branching processes and it was actually a key in Song [38] (who knew the result from the author in 1984) to study the Markov decision programming in unbounded case. The theorem is also included in the book by Anderson [39; Corollary 2.2.16] and followed with some extension in [40]–[42]. The generalized result of Theorem 2.1 given in [5] is also meaningful in quantum mechanics, refer to [43] and references within.

We have seen that how the models lead us to resolve one of the classical problems for MCs and produce some effective results. Some new solutions to the recurrence and positive recurrence problems are also given in [4], [1] and [2]. However, the positive recurrence for the Brussel's model was proved only in 1990 by Han [22] in the case of $d = 1$ and by J. W. Chen (1991) for the general finite-dimensional situation ([2; Example 4.50]). From the papers listed above, one can see again a lot of applications of these results but we are not going to the details here. In conclusion, the finite-dimensional Schlögl's and Brussel's models are all ergodic and so have no phase transitions. Thus, in order to study the phase transition phenomena for the systems, we have to go to the infinite-dimensional situation.

Before moving further, let us compare the above models with the famous Ising model (refer to [1] or [2] for instance).

(1) The state space $E = \{-1, +1\}^{\mathbf{Z}^d}$ for Ising model is compact but for Schlögl's models, the state space $E = \mathbf{Z}_+^{\mathbf{Z}^d}$ is neither compact nor locally compact.
(2) The Ising model is reversible, its local Gibbs distributions are explicit. But the Schlögl's models has no such advantages except a very special case.
(3) The Ising model has at least one stationary distribution since every Feller process with compact state space does. But for non-compact case, the conclusion may not be true.
(4) The generator of the Ising model is locally bounded but it is not so for the Schlögl's models.

From these facts, it should be clear that the Ising and the Schlögl's models are very different.

3. Construction of the processes.

The diffusion part of the operator can not be ignored, otherwise, there is no interaction and then the RD-processes are reduced to the classical MCs. If we forget the reaction part, then the processes are reduced to the well-known zero range processes. For which, the construction was completed step by step by several authors. In a special case, the process was constructed by Holley (1970)[44], the general case was done by Liggett (1973)[45]. Then, Andjel (1982)[46], Liggett & Spitzer (1981)[47] simplified the construction. For all the models considered in the last paper, the coefficients of the operator are assumed to be locally bounded and linear. Thus, even in this simpler case, the construction is still not simple.

A standard tool in constructing Markov processes is the semigroup theory, as was used by Liggett (1985)[48] to construct a large class of interacting particle systems. However, the theory is not helpful in the present situation. Even one has a semigroup at hand, it is still quite a distance to construct the process since in our case we do not have the Riesz representation theorem for constructing the transition probability kernel. Moreover, from the author's knowledge, since the state space is so poor, the usual weak convergence (even on the path space) is not effective for the construction. What we adopt is a stronger convergence.

Recall for given two probability measures P_1 and P_2 on a measurable state space $(E, \mathcal{E})$, a **coupling** of P_1 and P_2 is a probability measure $\widetilde{P}$ on the product space $(E^2, \mathcal{E}^2)$ having the marginality: $\widetilde{P}(A \times E) = P_1(A)$ and $\widetilde{P}(E \times A) = P_2(A)$ for all $A \in \mathcal{E}$. Next, assume that $(E, \rho, \mathcal{E})$ is a metric

space with distance ρ. The **Wasserstein distance** $W(P_1, P_2)$ of P_1 and P_2 is defined by

$$W(P_1, P_2) = \inf_{\widetilde{P}} \int_{E^2} \rho(x_1, x_2)\widetilde{P}(dx_1, dx_2), \tag{3.1}$$

where $\widetilde{P}$ varies over all couplings of P_1 and P_2. Refer to [2] for further properties of the Wasserstein distance.

Now, our strategy goes as follows. Take a sequence of finite subsets $\{\Lambda_n\}$ of $S = \mathbf{Z}^d$, $\Lambda_n \uparrow S$. Using Λ_n instead of S, we obtain a MC $P_n(t, x, \cdot)$ as mentioned in the last section. For each $n < m$, one may regard $P_n(t, x, \cdot)$ as a MC on the space $E_m := \mathbf{Z}_+^{\Lambda_m}$ and hence for fixed $t \geq 0$ and $x \in E_m$, the distance $W(P_n(t, x, \cdot), P_m(t, x, \cdot)$ of $P_n(t, x, \cdot)$ and $P_m(t, x, \cdot)$ is well defined. Clearly, one key step in our construction is to prove that

$$W(P_n(t, x, \cdot),\ P_m(t, x, \cdot) \longrightarrow 0 \qquad \text{as } \ m,\ n \to \infty. \tag{3.2}$$

Certainly, it is no hope to compute exactly the W-distance since $P_n(t, x, \cdot)$ is not explicitly known. In virtue of (3.2), we need only a upper bound of the distance and moreover it follows from (3.1) that every coupling gives us such a bound. The problem is that a coupling measure of $P_n(t, x, \cdot)$ and $P_m(t, x, \cdot)$ for fixed t and x is still not easy to construct, again due to the fact that these marginal measures are not known explicitly. What we know is mainly the operators Ω_n obtained from Ω but replacing $\mathbf{Z}^d$ with Λ_n. Thus, in order to get some practical coupling, it is natural to restrict ourselves to the Markovian coupling, i.e., the coupling process itself is again a MC. This analysis leads us to explore a theory of couplings for time-continuous Markov processes, which dates back to 1983[49].

4. Markovian couplings.

Let us now start from the original point. Given MC $P_k(t)$ on E_k $(k = 1, 2)$, we want to construct a MC $\widetilde{P}(t)$ on the product space $E_1 \times E_2$ such that

$$\widetilde{P}(t)\tilde{f}_k(x_1, x_2) = P_k(t)f_k(x_k), \qquad x_k \in E_k,\ k = 1, 2 \tag{4.1}$$

for all bounded function f_k on E_k, where $\tilde{f}_k$ is a function on $E_1 \times E_2$ defined by $\tilde{f}_k(x_1, x_2) = f_k(x_k)$, $k = 1, 2$. Recall that there is a one-to-one correspondence between a Q-matrix $Q = (q_{ij})$ and its operator $\Omega f(i) = \sum_{j \neq i} q_{ij}(f_j - f_i)$. We now have the marginal operators Ω_1 and Ω_2 and furthermore an operator $\widetilde{\Omega}$ of $\widetilde{P}(t)$ on the product space $E_1 \times E_2$. In view of (4.1), it is easy to see that $\widetilde{\Omega}$ must satisfy the following marginality.

$$\widetilde{\Omega}\tilde{f}_k(x_1, x_2) = \Omega_k f_k(x_k), \qquad x_k \in E_k,\ k = 1, 2 \tag{4.2}$$

Any operator $\widetilde{\Omega}$ satisfying (4.2) is called a **coupling operator**.

The existence of a coupling operator is not a problem and indeed there exist infinitely many such operators. A real hard problem is the uniqueness of the process for the coupling operators. Fortunately, for which we do have a complete answer.

Theorem 4.1[5]**.** The marginal operators determine the processes uniquely iff so does a (equivalently, any) coupling operator. Moreover, under the uniqueness assumption, (4.1) and (4.2) are equivalent.

Refer to [50] and [51] for additional information about this result. This is our first fundamental result on couplings. Since then, we have gone a long trip in the field: from MC to general jump processes[5], from discrete to continuous space[52], from Markovian coupling to optimal Markovian coupling[50], from the exponential convergence to the estimation of spectral gap[53],[50], from compact manifold to non-compact one[54]−[56] and from finite dimension to infinite one[8]−[13],[57]−[59]. No doubt, the coupling method is now a powerful tool and has many applications. The story of our study on couplings is out of the scope of the paper. The readers are urged to refer to the survey articles [60]–[62] for an account of the recent progress.

5. Construction of the processes (continued).

We now return to our main construction. We will restrict ourselves on single reactant for a while. Let (k_u) be a positive summable sequence and set $E_0 = \{x \in E : \|x\| := \sum_{u\in S} x_u k_u < \infty\}$, i.e., a L^1 subspace of E with respect to (k_u). Roughly speaking, the key of our construction (which is rather lengthy and technical) is to get the following estimates:

(1) $P_n(t)\|\cdot\|(x) \le (1 + \|x\|)e^{ct}$, $x \in E_0$ and
(2) $W_{\Lambda_n}(P_n(t, x, \cdot), P_m(t, x, \cdot)) \le c(t, \Lambda_n, x; n, m), \quad x \in E_0,$

where c is a constant independent of n, $c(t, \Lambda_n, x; n, m) \in \mathbf{R}_+$ satisfy $\lim_{m\ge n\to\infty} c(t, \Lambda_n, x; n, m) = 0$ and W_V is the Wasserstein distance restricted on $\mathbf{Z}_+^V$, with respect to the underlying distance $\sum_{u\in V} |x_u - y_u|k_u$. The second condition (2) shows that $\{P_n(t, x, \cdot) : n \ge 1\}$ is a Cauchy sequence in the W_V-distance (for fixed finite V). Note that our operators are not locally bounded and the particles from infinite sites may move to a single site, so the process may be explosive at some single site. This explains the reason why we use E_0 instead of E. Then, the first moment condition (1) ensures that E_0 is a closed set of the process. Finally, in order to prove that the limiting process satisfies the Chapman-Kolmogorov equation, some kind of uniform controlling in the second condition is also needed.

To state our main result, we need some assumptions.

$$\sup_v \textstyle\sum_u p(u, v) < \infty, \tag{5.1}$$

$$\textstyle\sum_{k\ne 0} q_u(i, i + k)|k| < \infty, \qquad u \in S, \tag{5.2}$$

$$\sup_{k,u} |c_u(k) - c_u(k + 1)| < \infty, \tag{5.3}$$

$$\sup\Big\{g_u(j_1, j_2) + h_u(j_1, j_2) : u \in S,\ j_2 > j_1 \ge 0\Big\} < \infty, \tag{5.4}$$

where

$$g_u(j_1, j_2) = \frac{1}{j_2 - j_1} \textstyle\sum_{k\ne 0} \big(q_u(j_2, j_2 + k) - q_u(j_1, j_1 + k)\big)k, \qquad j_2 > j_1 \ge 0,$$

$$h_u(j_1, j_2) = \frac{2}{j_2 - j_1} \textstyle\sum_{k=0}^{\infty} \Big[\big(q_u(j_2, j_1 - k) - q_u(j_1, 2j_1 - j_2 - k)\big)^+ + \big(q_u(j_1, j_2 + k) - q_u(j_2, 2j_2 - j_1 + k)\big)^+\Big]k, \qquad j_2 > j_1 \ge 0.$$

The next result is due to Chen (1985)[8], first reported at the Second International Conference on Random Fields, Hungary, 1984. See also [9], [1] and [2] for more general theorems.

Theorem 5.1. Denote by $\mathcal{E}_0$ the Borel σ-algebra generated by the distance $\|\cdot\|$ on E_0. Under (1.1) and (5.1)–(5.4), there exists a Markov process on $(E_0, \mathcal{E}_0)$, the corresponding semigroup (P_t) maps the set of Lipschitz functions on E_0 with respect to $\|\cdot\|$ into itself. Moreover, for every Lipschitz function f on E_0, the derivative of $P_t f$ at the origin coincides with Ωf in a dense set of E_0.

The conditions (1.1), (5.1) and (5.2) are technical but natural. For instance, when $p(u, v)$ is the simple random walk, (5.1) becomes trivial. However, the conditions (5.3) and (5.4) are essential in this construction, they are keys to the estimates (1) and (2) mentioned above and also to the study of mean field models discussed below. It is now a simple matter to justify the assumptions of Theorem 5.1 for Example 1.1 an Examples 1.2. However, up to now, we do not know how to choose a distance so that our general theorem[9,1,2] can be applied to obtain a Lipschitz semigroup for Example 1.3. In the case where the diffusion rates are bounded or growing at most as fast as $\log x_u$, a process corresponding to Example 1.3 was constructed by Tang[31] (see also [2; Example 13.38]) and Han[19]–[21] respectively. In the latter papers, the martingale approach was adopted but not the analytic one used here.

Open Problem 5.2. Construct a Markov process for the Brussel's model.

The next result is due to Li [24], which improves the author's [12].

Theorem 5.3. Under the same assumptions as in Theorem 5.1, if additionally,

$$\sup_u \sum_{k\neq 0} q_u(i,i+k)\big[(i+k)^m - i^m\big] \leq \text{const. } (1+i^m), \qquad i \in \mathbf{Z}_+ \tag{5.5}$$

for some $m > 1$, then the process constructed by Theorem 5.1 is also unique.

The proof of Theorem 5.3 is also non-trivial. It uses an infinite-dimensional version of the maximum principle, due to Tang [31] and Li [24]. This is the third mathematical tool developed from the study of RD-processes.

6. Existence of stationary distributions.

When the state space is compact, it is known that every Feller process has a stationary distribution. But for non-compact case, there is no such a general theorem and so one needs to work case by case. The next result is a particular case of [1] (in Chinese) and [10] (in English). See also [23].

Theorem 6.1. There always exists at least one stationary distribution for the polynomial model.

The intuition for the result is quite clear since the order of the death rate is higher than the birth one, the number of particles at each site is kept to be almost bounded and then we may return to the compact situation. However, the proof depends heavily on the construction of the process. We will not go to the details here.

7. Ergodicity.

There are two cases.

a) *The general case.* By using the coupling method again, some general sufficient conditions for the ergodicity of the processes were presented in [1] and in [10] . The result was then improved in Neuhauser [28] and further improved in [11]. In the case where the coefficients of the operator are translation invariant and with an absorbing state, some refined results are given in Li [25]. A particular result from [11] can be stated as follows.

Theorem 7.1. For the polynomial model, when $\beta_1, \cdots, \beta_{m_0}$ and $\delta_1, \cdots, \delta_{m_0+1}$ are fixed, the processes are ergodic for all large enough β_0.

We will come back to this topic at the end of the paper.

b) *The reversible case.* When the reaction part is a birth-death process with birth rates $b(k)$ and death rates $a(k)$, the RD-process is reversible iff $p(u,v) = p(v,u)$ and $(k+1)b(k)/a(k)$ =constant, independent of k[14].

The next result is due to Ding, Durrett and Liggett [17].

Theorem 7.2. For reversible polynomial model, the process is always ergodic.

The proof of the result is a nice illustration of the application of the free energy method. It also uses the power of the monotonicity of the processes. The result was then extended by Chen, Ding & Zhu [14] to the non-polynomial case.

If we replace $\beta_0 > 0$ with $\beta_0 = 0$, then we obtain two stationary distributions, one is trivial and the other one is non-trivial. The question is that starting from a non-trivial initial distribution, whether the process converges to the non-trivial stationary distribution (ergodic) or not. The affirmative assertion is called **Shiga's conjecture**, which was solved by Mountford [27].

Theorem 7.3. For the reversible polynomial model with $\beta_0 = 0$, under mild assumption, the Shiga's conjecture is correct.

8. Phase transitions.

a) *RD-processes with absorbing state.* The following result was first proved by Y. Li and X. G. Zheng (1988) by using color graph representation and then simplified by R. Durrett (1988) by using oriented percolation (see [2; Theorem 15.8]).

Theorem 8.1. Take $S = \mathbf{Z}$. Consider the RD-process with birth rates $b(k) = \lambda k$, arbitrary death rates $a(k) > 0\,(k \geq 1)$ and the diffusion coefficient $x_u p(u,v)$, where $p(u,v)$ is the simple random walk. Then for the process $X^0(t)$ starting from x^0: $x_0^0 = 1$ and $x_u^0 = 0$ for all $u \neq 0$, we have $\inf\{\lambda : \mathbb{P}[X^0(t) \not\equiv 0 \text{ for all } t > 0] > 0\} < \infty$. In other words, for some $\lambda > 0$, there exists a non-trivial stationary distribution except the trivial one.

b) *Mean field models.* In statistical physics, one often studies the mean field models as simplified approximation of the original ones. It is usually a common phenomena that the mean field models are easier to exhibit phase transitions. Roughly speaking, the mean field model of a RD-process is the time-inhomogeneous birth-death process on $\mathbf{Z}_+$ with death rates $a(k)$ as usual but with birth rates $b(k) + \mathbb{E}X(t)$, where $X(t)$ denotes the process. The term $\mathbb{E}X(t)$ represents the interaction of the particle at the present site with the particles at the other sites in the original models. The next result is due to Feng & Zheng [36].

Theorem 8.2. For the mean field of the second Schlögl model, there always exists at least one stationary distribution. There is precise one if $\delta_1, \delta_3 \gg 1$ and there are more than two if $\delta_1 < \delta_1^2 < 1/2 + (2\beta_2 + 1)/(3\delta_1 + 6\delta_3)$ and β_0 is small enough.

For more information about the study on the mean field models, refer to [32]–[36]. In [63], the models are treated as a measure-valued process. Here, we mention another model, the linear growth model which exhibits phase transitions, refer to Ding & Zheng [18]. However, we are still unable to answer the following problem.

Open Problem 8.3. Does there exist more than one stationary distribution for the polynomial model with no absorbing states?

The last phase means that $\beta_0 > 0$. In physics, this represents an exchange of the energy between inside and outside. From mathematical point of view, there is an essential difference between $\beta_0 = 0$ and $\beta_0 > 0$. For instance, when $\beta_0 = 0$, the process restricted on $\{x : \sum_u x_u < \infty\}$ is simply a MC but this is no longer true when $\beta_0 > 0$.

Because the RD-processes are quite involved, partially due to the non-compactness of the state space. Thus, one may construct some similar models with finite spin space to simplify the study. There are a lot of publications along this direction. Refer to Durrett et al [64]–[66] and references therein.

9. Hydrodynamic limits.

Consider again the polynomial model. However, we now study the process with the rescaled operator $\Omega^\varepsilon = \varepsilon^{-2}\Omega_d + \Omega_r$. Our main purpose is looking for the limiting behavior of the scaling processes as $\varepsilon \to 0$. To do so, let $\mu^\varepsilon(\varepsilon > 0)$ be the independent product of the Poisson measures for which $\mu^\varepsilon(x_u) = \rho(\varepsilon u),\ u \in \mathbf{Z}^d$, where ρ is a non-negative, bounded $C^2(\mathbf{R}^d)$-function with bounded first derivative.

Denote by $\mathbb{E}^\varepsilon_{\mu^\varepsilon}$ the expectation of the process with generator Ω^ε and initial distribution μ^ε. The next result is due to Boldrighini, DeMasi, Pellegrinotti & Presutti [7] (See [2], Theorem 16.1). Refer also to [67].

Theorem 9.1. For all $r = (r^1, \cdots, r^d) \in \mathbf{R}^d$ and $t \geq 0$, the limit $f(t,r) := \lim_{\varepsilon \to 0} \mathbb{E}^\varepsilon_{\mu^\varepsilon} X_{[r/\varepsilon]}(t)$, where $[r/\varepsilon] = ([r^1/\varepsilon], \cdots, [r^d/\varepsilon]) \in \mathbf{Z}^d$, exists and satisfies the RD-equation:

$$\begin{cases} \dfrac{\partial f}{\partial t} = \dfrac{1}{2}\displaystyle\sum_{i=1}^{d} \frac{\partial^2 f}{\partial (r^i)^2} + \sum_{j=0}^{m_0} \beta_j f^j - \sum_{j=1}^{m_0+1} \delta_j f^j \\ f(0,r) = \rho(r). \end{cases} \tag{9.1}$$

This result explains the relation between the RD-process and RD-equation and it is indeed the original reason why the processes was named RD-processes in [8]. Certainly, in that time, a result like Theorem 9.1 did not exist, we had only a rough impression that the RD-equations describe the macroscopic behavior of the physical systems and our aim was to introduce the processes as the microscopic description of the same systems.

To give some insight of the relation of these two subjects, we need some notation. Let $f_0(t)$ be a non-negative, spatially homogeneous solution to (9.1) satisfying

$$\sum_{j=0}^{m_0} \beta_j f^j - \sum_{j=1}^{m_0+1} \delta_j f^j = 0. \tag{9.2}$$

Then $f_0(t)$ is called **asymptotically stable** if there exists a $\delta > 0$ such that for any solution $f(t,r)$ to (9.1), whenever $|f(0,r) - f_0(0)| < \delta$, we have $\lim_{t\to\infty} |f(t,r) - f_0(t)| = 0$.

The following result is due to X. J. Xu (1991) (see [2; Theorem 16.2]).

Theorem 9.2. Denote by $\lambda_1 > \lambda_2 > \cdots > \lambda_k$ the non-negative roots of (9.2), where λ_j has multiplicity m_j. Then, $f(t,r) \equiv \lambda_i$ is asymptotically stable iff m_i is odd and $\sum_{j\le i-1} m_j$ is even.

All the known results is consistent with the assertion: a model has no phase transition iff every λ_j is asymptotically stable and it is the case of the Schlögl's first model. This leads to the following conjecture.

Conjecture 9.3. (1). The Schlögl's first model has no phase transition.

(2). The Schlögl's second model has phase transitions.

To conclude the paper, we want to show a use of the RD-equation. Note that for the Schlögl's second model, the role played by each of the parameters β_k and δ_k is not clear at all. It seems too hard and may not be necessary to consider the whole parameters. Based on the above observation and to keep the physical meaning, we fix $\beta_2 = 6\alpha\,(\alpha > 0)$, $\delta_1 = 9\alpha$ and $\delta_3 = \alpha$. Then, when $\beta_0 \in (0, 4\alpha)$, there are three roots $\lambda_1 > \lambda_2 > \lambda_3 \ge 0$, λ_1 and λ_3 are asymptotically stable but not λ_2. We have thus reduce the four parameters into one only. Now, we want to know for which region of α, the process can be ergodic. The following result is based on the new progress on couplings[50], it is a complemental to Theorem 7.1 and is also the most precise information we have known so far.

Theorem 9.4[13]**.** Consider the second Schlögl model with $\beta_0 = 2\alpha$, $\beta_2 = 6\alpha$, $\delta_1 = 9\alpha$ and $\delta_3 = \alpha$. Then, the processes are exponentially ergodic for all $\alpha \ge 0.7303$.

References

1. Chen M F, *Jump Processes and Particle Systems* (In Chinese), Beijing Normal University Press, 1986.
2. Chen M F, *From Markov Chains to Non-Equilibrium Particle Systems*, Singapore, World Scientific, 1992.
3. Yan S J, Li Z B, *Probability models of non-equilibrium systems and the Master equations* (In Chinese), Acta Phys. Sin. 1980, 29:139–152.
4. Yan S J, Chen M F, *Multidimensional Q-processes*, Chin. Ann. Math. 1986, **7(B)**:1, 90–110.
5. Chen M F, *Coupling of jump processes*, Acta Math. Sin. New Ser. 1986, 2:123–136.
6. Chen M F, *On three classical problem for Markov chains with continuous time parameters*, J. Appl. Prob. 1991, 28:305–320.
7. Boldrighini C, DeMasi A, Pellegrinotti A, Presutti E, *Collective phenomena in interacting particle systems*, Stoch. Proc. Appl. 1987, 25:137–152.
8. Chen M F, *Infinite-dimensional reaction-diffusion processes*, Acta Math. Sin. New Ser. 1985, 1:3, 261–273.
9. Chen M F, *Existence theorems for interacting particle systems with non-compact state spaces*, Sci. Sin. 1987, 30:2, 148–156.
10. Chen M F, *Stationary distributions of infinite particle systems with non-compact state spaces*, Acta Math. Sci. 1989, 9:1, 7–19.
11. Chen M F, *Ergodic theorems for reaction-diffusion processes*, J. Statis. Phys. 1990, 58:5/6, 939–966.
12. Chen M F, *Uniqueness of reaction-diffusion processes*, Chin. Sci. Bulleten 1991, 36:12, 969–973.
13. Chen M F, *On ergodic region of Schlögl's model*, in Proc. Intern. Conf. on Dirichlet Forms & Stoch. Proc. Edited by Z. M. Ma, M. Röckner and J. A. Yan, Walter de Gruyter, 1994.
14. Chen M F, Ding W D, Zhu D J, *Ergodicity of reversible reaction-diffusion processes with general reaction rates*, Acta Math. Sin. New Ser. 1994, 10:1, 99–112.
15. Chen M F, Huang L P, Xu X J, *Hydrodynamic limit for reaction-diffusion processes with several species*, in "Probability and Statistics, Nankai Series of Pure and Applied Mathematics", edited by S. S. Chern and C. N. Yang, World Scientific, 1991.
16. DeMasi A, Presutti E, *Lectures on the Collective Behavior of Particle Systems*, LNM 1502, Springer, 1992.
17. Ding W D, Durrett R, Liggett T M, *Ergodicity of reversible theorems reaction-diffusion processes*, Prob. Th. Rel. Fields 1990, 85:1, 13–26.
18. Ding W D, Zheng X G, *Ergodic theorems for linear growth processes with diffusion*, Chin. Ann. Math. 1989, 10(B):3, 386–402.

19. Han D, *Existence of solution to the martingale problem for multispecies infinite dimensional reaction-diffusion particle systems*, Chin. J. Appl. Prob. Statis. 1990, 6:3, 265–278.
20. Han D, *Uniqueness of solution to the martingale problem for infinite-dimensional reaction-diffusion particle systems with multispecies* (In Chinese), Chin. Ann. Math. 1992, 13A:2, 271–277.
21. Han D, *Uniqueness for reaction-diffusion particle systems with multispecies* (In Chinese), Chin. Ann. Math. 1995, 16A:5, 572–578.
22. Han D, *Ergodicity for one-dimensional Brussel's model* (In Chinese), J. Xingjiang Univ. 1991, 8:3, 37.
23. Huang L P, *The existence theorem of the stationary distributions of infinite particle systems* (In Chinese), Chin. J. Appl. Prob. Statis. 1987, 3:2, 152–158.
24. Li Y, *Uniqueness for infinite-dimensional reaction-diffusion processes*, Chin. Sci. Bulleten (Chinese version) 1991, 22:1681–1684.
25. Li Y, *Ergodicity of a class of reaction-diffusion processes with translation invariant coefficients* (In Chinese), Chin. Ann. Math. 1995, 16A:2, 223–229.
26. Lü J S, *The optimal coupling for single-birth reaction-diffusion processes and its applications* (In Chinese), J. Beijing Normal Univ. 1997, 33:1, 10–17.
27. Mountford T S, *The ergodicity of a class of reversible reaction-diffusion processes*, Prob. Th. Rel. Fields 1992, 92:2, 259-274.
28. Neuhauser C, *An ergodic theorem for Schlögl models with small migration*, Prob. Th. Rel. Fields 1990, 85:1, 27–32.
29. Shiga T, *Stepping stone models in population, genetics and population dynamics*, in "Stochastic Processes in Physics and Engineering", edited by S. Albeverio et al. 1988, 345–355.
30. Zheng X G, Ding W D, *Existence theorems for linear growth processes with diffusion*, Acta Math. Sin. New Ser. 1987, 7:1, 25–42.
31. Tang S Z, *The existence and uniqueness of reaction-diffusion processes with multi-species and bounded diffusion rates* (In Chinese), J. Chin. Appl. Prob. Stat. 1985, 1:11-22.
32. Dawson D A, Zheng X, *Law of large numbers and central limit theorem for unbounded jump mean field models*, Adv. Appl. Math. 1991, 12:293–326.
33. Feng S, *Large deviations for empirical process of interacting particle system with unbounded jumps*, Ann. Prob. 1994, 22:4, 2122–2151.
34. Feng S, *Large deviations for Markov processes with interaction and unbounded jumps*, Prob. Th. Rel. Fields 1994, 100:227–252.
35. Feng S, *Nonlinear master equation of multitype particle systems*, Stoch. Proc. Appl. 1995, 57:247–271.
36. Feng S, Zheng X, *Solutions of a class of nonlinear master equations*, Stoch. Proc. Appl. 1992, 43:65–84.
37. Chen R R, *An extended class of time-continuous branching processes*, J. Appl. Prob. 1997, 34:1, 14–23.
38. Song J S, *Time-continuous MDP with unbounded rate*, Sci. Sin. 1987, 12:1258–1267.
39. Anderson W J, *Continuous-Time Markov Chains*, Springer Series in Statistics, 1991.
40. Hamza K, Klebaner F C, *Conditions for integrability of Markov chains*, J. Appl. Prob. 1995, 32:541–547.
41. Kersting G, Klebaner F C, *Sharp conditions for nonexplosions in Markov jump processes*, Ann. Prob. 1995, 23:1, 268–272.
42. Meyn S P, Tweedie R L, *Stability of Markovian processes* Ⅲ*: Foster-Lyapunov criteria for continuous-time processes*, Adv. Appl. Prob. 1993, 25:518-548.
43. Konstantinov A A, Maslov V P, Chebotarev A M, *Probability representations of solutions of the Cauchy problem for quantum mechanical solutions*, Russian Math. Surveys 1990, 45:6, 1-26.
44. Holley R, *A class of interaction in an infinite particle systems*, Adv. Math. 1970, 5:291–309.
45. Liggett T M, *An infinite particle system with zero range interactions*, Ann. of Prob. 1973, 1:240–253.
46. Andjel E D, *Invariant measures for zero range process*, Ann. of Prob. 1982, 10:3, 527–547.
47. Liggett T M, Spitzer F, *Ergodic theorems for coupled random walks and other systems with locally interacting components*, Z. Wahrs. 1981, 56:443–468.
48. Liggett T M, *Interacting Particle Systems*, Springer-Verlag, 1985.
49. Chen M F, *Basic couplings of Markov chains* (In Chinese), J. Beijing Normal Univ. 1984, 4:3–10.
50. Chen M F, *Optimal Markovian couplings and applications*, Acta Math. Sin. New Ser. 1994, 10:3, 260–275.
51. Zhang Y H, *Conservativity of couplings for jump processes* (In Chinese), J. Beijing Normal Univ. 1994, 30:3, 305–307.
52. Chen M F, Li S F, *Coupling methods for multi-dimensional diffusion processes*, Ann. of Probab. 1989, 17:1, 151–177.
53. Chen M F, Wang F Y, *Application of coupling method to the first eigenvalue on manifold*, Sci. Sin.(A), 1993, 23:11 (Chinese Edition), 1130–1140; 1994, 37:1 (English Edition), 1–14.
54. Chen M F, Wang F Y, *Estimation of the first eigenvalue of second order elliptic operators*, J. Funct. Anal. 1995, 131:2, 345–363.
55. Chen M F, Wang F Y, *Estimation of spectral gap for elliptic operators*, Trans. Amer. Math. Soc. 1997, 349:3, 1239–1267.
56. Chen M F, Wang F Y, *General formula for lower bound of the first eigenvalue on Riemannian manifolds*, Sci. Sin. 1997, 27:1, 34–42 (Chinese Edition); 1997, 40:4, 384–394 (English Edition).
57. Wang F Y, *Ergodicity for infinite-dimensional diffusion processes on manifolds*, Sci. Sin. Ser A, 1994, 37(2), 137–146.

58. Wang F Y, *Uniqueness of Gibbs states and the L^2-convergence of infinite-dimensional reflecting diffusion processes*, Sci. Sin. Ser A, 1995, 32:8, 908–917.
59. Wang F Y, *Estimation of the first eigenvalue and the lattice Yang-Mills fields*, Chin. J. Math. 1996, 17A:2 (In Chinese), 147–154; Chin. J. Contem. Math. 1996, 17:2 (In English), 119–126.
60. Chen M F, *Optimal Markovian couplings and application to Riemannian geometry*, in Prob. Theory and Math. Stat., Eds. Grigelionis B et al, 1994, 121–142 VPS/TEV.
61. Chen M F, *Trilogy of couplings and general formulas for lower bound of spectral gap*, in "Proceedings of the Symposium on Probability Towards the Year 2000", Edited by Accardi L, Heyde C. 1997.
62. Chen M F, *Coupling, spectral gap and related topics* (I), (Ⅱ), (Ⅲ), Chin. Sci. Bull. 1997, 42:16, 1321; 42:17, 1409; 42:18, 1497.
63. Djehiche B, Kaj I, *The rate function for some measure-valued jump processes*, Ann. Prob. 1995, 23:3, 1414–1438.
64. Durrett R, Levin S, *The importance of being discrete* (*and spatial*), Theoret. Pop. Biol. 1994, 46:363–394.
65. Durrett R, *Ten lectures on particle systems, St. Flour Lecture Notes*, LNM 1608, 1995.
66. Durrett R, Neuhauser C, *Particle systems and reaction-diffusion equations*, Ann. Prob. 1994, 22:1, 289–333.
67. Feng J F, *The hydrodynamic limit for the reaction diffusion equation—an approach in terms of the GRP method*, J. Theor. Prob., 1996, 9:2, 285–299.

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