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STOCHASTIC MODEL OF ECONOMIC OPTIMIZATION —COLLAPSE THEOREM*

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ABSTRACT. This paper begins with a short survey on the study of some global economic models, including L. K. Hua's fundamental results. The main purpose of the paper is to proving a collapse theorem for a non–controlling stochastic economic system. In the analysis of the system, some recent progress on the products of random matrices plays a critical role.

0. INTRODUCTION

In this section, we first recall some necessary background on the subject, including the well–known input–output method and L. K. Hua's fundamental theorem for the stability of economy. Then, we show that it is necessary to study the stochastic models.

(i) Input–output method. Denote by $x = \left(x^{(1)}, x^{(2)}, \ldots, x^{(d)}\right)$ the quantity of the main products we are interested. Suppose that the starting vector of products last year is

$$x_0 = \left(x_0^{(1)}, x_0^{(2)}, \ldots, x_0^{(d)}\right).$$

For reproduction, assume that the j–th product distributed amount $x_{ij}^{(0)}$ to the i–th product, and the vector of the products this year becomes

$$x_1 = \left(x_1^{(1)}, x_1^{(2)}, \ldots, x_1^{(d)}\right).$$

Here, we suppose for a moment that all the products are used for the reproduction. Next, set

$$a_{ij}^{(0)} = x_0^{(j)}/x_1^{(i)}, \quad 1 \le i,\ j \le d.$$

The matrix $A_0 = \left(a_{ij}^{(0)}\right)$ is called a *structure matrix* (or matrix of *expend coefficient*). This matrix is essential since it describes the efficiency of the current economy. Clearly, $x_0 = x_1 A_0$. Similarly, we have $x_{n-1} = x_n A_{n-1}$, $n \ge 1$. Suppose that the structure matrices are time–homogeneous: $A_n = A$, $n \ge 0$. Then we have a simple expression for the n–th output:

$$x_n = x_0 A^{-n}, \quad n \ge 1. \tag{0.1}$$

Thus, once we know the structure matrix and the input x_0, we may predict the future output. This method is the well known *input-output method.*

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(ii) Hua's theorems. Let us return to the original equation

$$x_1 = x_0 A^{-1}.$$

We now fix A, then x_1 is determined by x_0 only. The question is which choice of x_0 is the optimal one. Furthermore, in what sense of optimality are we talking about?

We adopt the minimax principle: finding out a x_0 such that $\min_{1\le j\le d} x_1^{(j)}/x_0^{(j)}$ attains the maximum below

$$\max_{\substack{x_1>0\\ x_0=x_1A}} \min_{1\le j\le d} \frac{x_1^{(j)}}{x_0^{(j)}}.$$

By using the classical Frobenius theorem, Hua[3,(III)] proved the following result.

Theorem 0.1. Given an irreducible non–negative matrix A, let u be the left characteristic (positive) vector of A corresponding to the largest characteristic root $\rho(A)$ of A. Then, up to a constant, the solution to the above problem is $x_0 = u$. In this case, we have

$$\frac{x_1^{(j)}}{x_0^{(j)}} = \frac{1}{\rho(A)} \quad \text{for all } j.$$

In what follows, we call the above technique (i.e., setting $x_0 = u$) *the method of characteristic vector.*

Next, we are going further to study the stability of economy. From (0.1), we obtain the simple expression: $x_n = x_0\rho(A)^{-n}$ whenever $x_0 = u$. What happen if we take $x_0 \ne u$ (up to a constant)? Let us consider a particular case that $A = P$. That is, A is a transition probability matrix. Then, from the ergodic theorem, it follows that

$$P^n \to \Pi \quad \text{as } n \to \infty,$$

where Π is the matrix having the same row $\big(\pi^{(1)}, \pi^{(2)}, \ldots, \pi^{(d)}\big)$, which is just the stationary distribution of the corresponding Markov chain. Since the distribution is the only stable solution for the chain, it should have some meaning in economics even though the later one goes in a converse way:

$$x_n = x_0 P^{-n}, \quad n \ge 1.$$

Set

$$\tau^x = \inf\big\{n \ge 1 : x_0 = x \text{ and there is some } j \text{ such that } x_n^{(j)} \le 0\big\}$$

which is called the *collapse time* of the economic system. From the above facts, it is not difficult to prove that if $x_0 \ne u = \big(\pi^{(1)}, \pi^{(2)}, \ldots, \pi^{(d)}\big)$ up to a positive constant, then $\tau^{x_0} < \infty$. Next, since the general case can be reduced to the above particular case, we think that this is a very natural way to understand the following Hau's theorem (See [2,(I)] and [3,(Ⅲ),(Ⅸ)] for details).

Theorem 0.2. Under some mild conditions, if $x_0 \ne u$, then $\tau^{x_0} < \infty$.

In L. K. Hua's eleven reports (1983–1985), he also studied some more general models of economy. But the above two theorems are the key to his idea. The title of the reports may cost some misunderstanding since one may think that the theory works only for planned economy. Actually, the economy of market was also treated in [3, (Ⅶ)]. The only difference is that in the later case one needs to replace the structure matrix A with $V^{-1}AV$, where V is the diagonal matrix $\mathrm{diag}(v_i/p_i)$: (p_i) is the vector of prices in market and (v_i) is the right characteristic vector of A. Note that the characteristic roots of $V^{-1}AV$ are the same as those of A. Corresponding to the largest characteristic root $\rho(V^{-1}AV) = \rho(A)$, the left characteristic vector of $V^{-1}AV$ becomes uV. Thus, from mathematical point of view, the consideration of market makes no essential difference in the Hua's model.

Before going further, let us look at a numerical example[3, (I)]. Take

$$A = \frac{1}{100}\begin{pmatrix} 25 & 14 \\ 40 & 12 \end{pmatrix}.$$

Its left characteristic vector is

$$u = \left(\frac{5}{7}\big(\sqrt{2409}+13\big),\, 20\right) \simeq (44.34397483,\, 20)\,.$$

Then we have:

$$\tau^x = \begin{cases} 3, & \text{for } x = (44,20)\,, \\ 8, & \text{for } x = (44.344,20)\,, \\ 13, & \text{for } x = (44.34397483,20)\,. \end{cases}$$

(iii) Stochastic model without consumption. Let us consider the above example again. We now allow a small random perturbation:

$$\begin{aligned} \widetilde{a}_{ij} &= a_{ij} \quad \text{with probability } 2/3, \\ &= a_{ij}(1 \pm 0.01) \quad \text{with probability } 1/6. \end{aligned}$$

Taking $(\widetilde{a}_{ij})$ instead of (a_{ij}), we get a random matrix. Next, let $\{A_n;\, n \geq 1\}$ be a sequence of independent random matrices with the same distribution as above, then

$$x_n = x_0 \prod_{k=1}^{n} A_k^{-1}$$

gives us a stochastic model of an economy without consumption.

Our starting point is to observe the influence of such a small random perturbation. But first, we use a smaller but more practical time S^x instead of τ^x:

$$S^x = \inf\{n \geq 1 :\, x_0 = x \text{ and } x_n \notin D\}\,,$$

where, for this model, we choose

$$D = \left\{(z^{(1)},\, z^{(2)}) \in \boldsymbol{R}^2 :\, 0 < z^{(2)} \leq z^{(1)} \leq 3.5 z^{(2)}\right\}\,.$$

The random time S^x is called the *dislocation time* of the economic system. For the above choice of D, in the deterministic case, this change makes a little difference comparing with the previous one:

$$S^x = \begin{cases} 3, & \text{for } x = (44,20)\,, \\ 8, & \text{for } x = (44.344,20)\,, \\ 12, & \text{for } x = (44.34397483,20)\,. \end{cases}$$

Now we would like to know what happen under such a small random perturbation. Starting from $x_0 = (44.344,\, 20)$, then the dislocation probability is the following

$$\boldsymbol{P}(S^{x_0} = m) = \begin{cases} 0, & \text{for } m = 1\,, \\ 0.09, & \text{for } m = 2\,, \\ 0.65, & \text{for } m = 3\,. \end{cases}$$

This observation tells us that randomness plays a critical role in the economy.

Now, a natural question arises: what is the analog of Hua's theorem for the stochastic case? Note that the limit theory of products of random matrices are quite different from the deterministic case (cf. [1], [4]–[6]), the problem is non-trivial.

Let $M_n = A_1 A_2 \cdots A_n$ and denote by $\|A\|$ the operator norm of A. Under some hypotheses, Kesten and Spitzer[5] proved that $M_n/\|M_n\|$ converges in distribution to a positive matrix $M = LR$ with rank one, where L and R are positive column vector and row vector respectively.

One of the authors has recently proved the following result[2,(II)]:

Theorem 0.4. Let R be the same as above. Under some mild conditions, we have

$$\boldsymbol{P}(\tau^x = \infty) \le \boldsymbol{P}(R^* = x/\|x\|) \quad \text{for any } x > 0\,,$$

where R^* is the transpose of R. In particular, if $\boldsymbol{P}(R^* = x/\|x\|) = 0$, then $\tau^x < \infty$, $a.s.$

(iv) Stochastic model with consumption. We now turn to the stochastic model with consumption. Let us allow a part of the productions turning into consumption, not used for reproduction. Suppose that every year we take the $\theta^{(i)}$-times amount of the increase of the i–th product as consumption. Then in the first year the vector of products which can be used for reproduction is

$$y_1 = x_0 + (x_1 - x_0)(I - \Theta)\,,$$

where I is the $d \times d$ unit matrix and $\Theta = \operatorname{diag}\left(\theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(d)}\right)$, which is called a *consumption matrix*. Therefore,

$$y_1 = y_0[A_0^{-1}(I - \Theta) + \Theta]\,,$$

where $y_0 = x_0$. Similarly, in the n–th year, the vector of the products which can be used for reproduction is

$$y_n = y_0 \prod_{k=0}^{n-1} [A_{n-k-1}^{-1}(I - \Theta) + \Theta]\,.$$

Let

$$B_n = [A_{n-1}^{-1}(I - \Theta) + \Theta]^{-1}\,.$$

Then

$$y_n = y_0 \prod_{k=1}^{n} B_{n-k+1}^{-1}\,.$$

We have thus obtained a *stochastic model with consumption.* In the deterministic case, a collapse theorem was obtained by Hua and Hua[4]. Now, the question in (iii) arises again: what is the analog of the theorem for the stochastic model with consumption? This is the main topic studied in this paper. Roughly speaking, we are looking for the conditions under which $\tau^x < \infty$, $a.s.$ for all positive x.

In Section 1, we introduce some necessary notations and definitions. Then we study some properties of the product of B_n and prove a collapse theorem (Theorem 2.9) in Section 2. Finally, we apply the collapse theorem to some stochastic models in Section 3.

1. Notations and definitions

We shall write $M(d, \boldsymbol{R})$ for the set of all $d \times d$ matrices with real entries, $Gl(d, \boldsymbol{R})$ the set of all invertible elements in $M(d, \boldsymbol{R})$, and $O(d, \boldsymbol{R})$ the set of all orthogonal matrices in $M(d, \boldsymbol{R})$. Denote by $\boldsymbol{R}^d$ the set of all d–dimensional real row vector. The transpose of a matrix M is denoted by M^*. Given a vector x and a subset $\mathcal{V}$ of $\boldsymbol{R}^d$, let x^* be the transpose of x and $\mathcal{V}^* = \{x^* : x \in \mathcal{V}\}$. Write

$$S(d) = \{x \in \boldsymbol{R}^d : \|x\| = 1\},$$

where $x = (x^{(1)}, x^{(2)}, \ldots, x^{(d)})$, $\|x\| = \left(\sum_{i=1}^d (x^{(i)})^2\right)^{1/2}$. For any $M \in M(d, \boldsymbol{R})$, set

$$\|M\| = \sup\{\|xM\| : x \in S(d)\},$$

which is just the operator norm of M.

A *topological semigroup* is a topological set $\mathcal{G}$ on which an associative product is defined, and the mapping $(M_1, M_2) \to M_1 M_2$ from $\mathcal{G}^2$ to $\mathcal{G}$ being continuous.

In what follows, we assume that $\{X_n; n \ge 1\}$ is a sequence of i.i.d. random matrices which are defined on a probability space $(\Omega, \mathcal{F}, \boldsymbol{P})$ with common distribution μ. Denote by $\mathcal{G}_\mu$ the *smallest closed semigroup* of $Gl(d, \boldsymbol{R})$ which contains the support of μ, $\operatorname{supp}(\mu)$.

Definition 1.1. Given a subset $\mathcal{G}$ of $Gl(d,\boldsymbol{R})$, we say that

(i) $\mathcal{G}$ is *irreducible*, if there is no proper linear subspace $\mathcal{V}$ of $\boldsymbol{R}^d$ such that $(\mathcal{V})M=\mathcal{V}$ for any M in $\mathcal{G}$.

(ii) $\mathcal{G}$ is *strongly irreducible*, if there does not exist a family of proper linear subspace of $\boldsymbol{R}^d$, $\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_k$ such that

$$\left(\bigcup_{i=1}^{k}\mathcal{V}_i\right)M=\bigcup_{i=1}^{k}\mathcal{V}_i$$

for any M in $\mathcal{G}$.

Definition 1.2. Given a subset $\mathcal{G}$ of $Gl(d,\boldsymbol{R})$, we define the *index* of $\mathcal{G}$ to be the least integer p such that there exists a sequence $\{M_n;\, n\geq 1\}$ in $\mathcal{G}$ for which $\|M_n\|^{-1}M_n$ converges to a matrix with rank p. We say that $\mathcal{G}$ is *contractive* if its index is equal to one.

Lemma 1.3. Let $\mathcal{G}\subset Gl(d,\boldsymbol{R})$. Then

(a) $\mathcal{G}$ is irreducible, if and only if so is $\mathcal{G}^*$.

(b) $\mathcal{G}$ is strongly irreducible, if and only if so is $\mathcal{G}^*$.

(c) The indices of $\mathcal{G}^*$ and $\mathcal{G}$ are the same.

Proof. (a) If $\mathcal{G}^*$ were not irreducible, then there would exist a proper linear subspace $\mathcal{V}$ of $\boldsymbol{R}^d$ such that

$$(\mathcal{V})M^*=\mathcal{V}$$

for any M in $\mathcal{G}$. Let $\mathcal{W}$ be the subspace orthogonal to $\mathcal{V}$, then

$$(\mathcal{W})M=\mathcal{W}$$

for any M in $\mathcal{G}$. This is in contradiction with the fact that $\mathcal{G}$ is irreducible.

(b) When $\mathcal{G}$ is strongly irreducible, there does not exist a finite family of proper linear subspaces of $\boldsymbol{R}^d$, $\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_k$ such that

$$M^*\left(\bigcup_{i=1}^{k}\mathcal{V}_i^*\right)=\bigcup_{i=1}^{k}\mathcal{V}_i^*$$

for any M in $\mathcal{G}$. From the proof of [1; Chapter III, Lemma 3.3], we see that there does not exist a finite family of proper linear subspaces $\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_k$ of $\boldsymbol{R}^d$ such that

$$(M^*)^*\left(\bigcup_{i=1}^{k}\mathcal{V}_i^*\right)=\bigcup_{i=1}^{k}\mathcal{V}_i^*$$

for any M in $\mathcal{G}$. Hence $\mathcal{G}^*$ is strongly irreducible.

The assertion (c) is obvious. □

Remark. By Lemma 1.3, the definition of irreducibility (resp., strongly irreducibility) coincides with the one given in [1; Chapter Ⅲ].

Definition 1.4. Given $M\in Gl(d,\boldsymbol{R})$, we say that

$$M=KAU$$

is a *polar decomposition* of M, if both K and U are in $O(d,\boldsymbol{R})$ and

$$A=\operatorname{diag}(a_1,a_2,\dots,a_d)$$

with $a_1\geq a_2\geq\cdots\geq a_d>0$.

Lemma 1.5. Any invertible matrix M has a polar decomposition. Moreover $a_1\geq a_2\geq\cdots\geq a_d>0$ are necessarily the square roots of the eigenvalues of M^*M.

The proof of this Lemma can be found from [1; Chapter Ⅲ, §1].

Lemma 1.6. Suppose that $M=KAU$ is a polar decomposition of M, then $\|M\|=\|M^*\|=a_1$.

Proof. The conclusion is clear since the matrices K and U are in $O(d,\mathbf{R})$. □

2. Proof of the Collapse theorem

Throughout the rest of the paper, we write $S_n = X_n X_{n-1} \cdots X_1$ and let the polar decomposition of S_n be as follows:

$$S_n = K_n A_n U_n \,,$$

where $A_n = \operatorname{diag}\big(a_n^{(1)}, a_n^{(2)}, \dots, a_n^{(d)}\big)$. Denote by $U_n^{(i)}$ (resp., $K_n^{(i)}$) the row (resp., column) vector which consists of the i–th row (resp., column) of U_n (resp., K_n).

For the reader's convenience, we copy a result from [1; Chapter Ⅲ, Proportion 3.2].

Proposition 2.1. Suppose that $\mathcal{G}_\mu$ is strongly irreducible with index p, then the following assertions hold.

(a) The subspace spanned by $\big\{U_n^{(1)}(\omega), U_n^{(2)}(\omega), \dots, U_n^{(p)}(\omega)\big\}$ converges a.s. to a p–dimensional subspace $\mathcal{V}(\omega)$.

(b) With probability one

$$\lim_{n\to\infty} \frac{a_n^{(p+1)}(\omega)}{\|S_n(\omega)\|} = 0 \quad \text{and} \quad \inf_{n\ge 1} \frac{a_n^{(p)}(\omega)}{\|S_n(\omega)\|} > 0\,.$$

(c) For any sequence $\{x_n;\, n\ge 1\}$ in $\boldsymbol{R}^d$ which converges to a non–zero vector, we have

$$\sup_{n\ge 1} \frac{\|S_n(\omega)\|}{\|S_n(\omega) x_n^*\|} < \infty \quad \text{a.s.}\,.$$

Lemma 2.2. Suppose that $\mathcal{G}_\mu$ is strongly irreducible and contractive. Then there exists a vector $R(\omega)$ in $S(d)$ such that with probability one $\{R(\omega), -R(\omega)\}$ is the set of all the cluster points of $\big\{U_n^{(1)}(\omega) :\, n\ge 1\big\}$ and $R(\omega)$ has at least one positive component.

Proof. By Proposition 2.1 and the assumption that $p=1$, with probability one, the subspace spanned by $\{U_n^{(1)}(\omega)\}$ converges to a one–dimensional subspace $\mathcal{V}(\omega)$. Take a unit vector $R(\omega)$ in $\mathcal{V}(\omega)$ such that it has at least one positive component. Since $U_n^{(1)}(\omega) \in S(d)$, it is obvious that $\{R(\omega), -R(\omega)\}$ is the set of all cluster points of $\big\{U_n^{(1)}(\omega)\big\}$. $\square$

For $\boldsymbol{R} \ni x > 0$, which means that x has positive components, write $\overline{x} = x/\|x\|$ and define

$$\tau^x(\omega) = \inf\{n\ge 1 :\, x S_n^{-1}(\omega) \text{ has at least one negative component}\}\,.$$

The next result may be regarded as a preliminary version of our collapse theorem.

Theorem 2.3. Under the conditions of Lemma 2.2, if there exist a subsequence $\big\{n_\omega(i);\, i\ge 1\big\}$ and a $K_\infty^{(1)}(\omega) \in \big(S(d)\big)^*$ such that

$$\lim_{i\to\infty} K_{n_\omega(i)}^{(1)}(\omega) = K_\infty^{(1)}(\omega) > 0\,, \quad a.s. \tag{H}$$

then for any $x>0$, we have

$$\{\tau^x(\omega) = \infty\} \subset \{\overline{x} = R(\omega)\} \quad \text{a.s.}\,,$$

$$\lim_{n\to\infty} U_n^{(1)}(\omega) I_{\{\tau^x(\omega)=\infty\}} = R(\omega) I_{\{\tau^x(\omega)=\infty\}} \quad \text{a.s.}\,.$$

Proof. Without loss of generality, assume that

$$\lim_{n\to\infty} K_n^{(1)}(\omega) = K_\infty^{(1)}(\omega) > 0\,, \quad \forall \omega\in\Omega. \tag{2.1}$$

Consider the polar decomposition $S_n = K_n A_n U_n$. Take a null set N such that for every $\omega \in \Omega \setminus N$, $\{R(\omega), -R(\omega)\}$ is the set of all cluster points of $\{U_n^{(1)}(\omega);\, n \geq 1\}$ and moreover

$$\lim_{n\to\infty} \frac{a_n^{(2)}(\omega)}{\|S_n(\omega)\|} = 0 \quad \text{and} \quad \inf_{n\geq 1} \frac{a_n^{(1)}(\omega)}{\|S_n(\omega)\|} > 0\,. \tag{2.2}$$

Fix an $\omega \in (\Omega \setminus N) \cap \{\tau^x = \infty\}$. For the remainder of the proof, we omit the variable ω in $S_n(\omega)$, $K_n(\omega)$, $U_n(\omega)$, $A_n(\omega)$ and so on. Take a subsequence $\{n(i);\, i \geq 1\}$ of $\{n;\, n \geq 1\}$ such that

$$\lim_{i\to\infty} U_{n(i)}^{(1)} =: B\,, \tag{2.3}$$

where $B \in \{R, -R\}$. Since $\tau^x = \infty$, we have

$$\frac{\overline{x} S_n^{-1}}{\|S_n\|^{-1}} > 0$$

for every $n \geq 1$. Hence, there is some $y \in [0,\infty]^d$ and a subsequence $\{m(i);\, i \geq 1\}$ of $\{n(i);\, i \geq 1\}$ such that

$$\lim_{i\to\infty} \overline{x} S_{m(i)}^{-1} \|S_{m(i)}\| = y\,. \tag{2.4}$$

Since K_n and $U_n \in O(d, \boldsymbol{R})$, there exist K_∞, $U_\infty \in O(d, \boldsymbol{R})$, and a subsequence $\{\ell(i); i \geq 1\}$ of $\{m(i); i \geq 1\}$ such that

$$\lim_{i\to\infty} K_{\ell(i)} = K_\infty \quad \text{and} \quad \lim_{i\to\infty} U_{\ell(i)} = U_\infty\,.$$

Noticing that $\|S_{\ell(i)}\| = a_{\ell(i)}^{(1)}$, by (2.2), we have

$$\lim_{i\to\infty} \frac{S_{\ell(i)}}{\|S_{\ell(i)}\|} = \lim_{i\to\infty} K_{\ell(i)} \left\{ A_{\ell(i)} / a_{\ell(i)}^{(1)} \right\} U_{\ell(i)} = K_\infty^{(1)} B\,. \tag{2.5}$$

Put

$$J_1 = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}, \qquad J_2 = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

By (2.4), (2.1) and (2.3), we obtain

$$\begin{aligned}
\overline{x} &= \overline{x}\big(U_{\ell(i)}\big)^* J_1 U_{\ell(i)} + \overline{x}\big(U_{\ell(i)}\big)^* J_2 U_{\ell(i)} \\
&= \overline{x}\big(U_{\ell(i)}\big)^* \big(A_{\ell(i)}\big)^{-1} \big(K_{\ell(i)}\big)^* K_{\ell(i)} A_{\ell(i)} J_1 U_{\ell(i)} + \overline{x}\big(U_{\ell(i)}\big)^* J_2 U_{\ell(i)} \\
&= \Big(\overline{x} S_{\ell(i)}^{-1} \|S_{\ell(i)}\|\Big) \Big(K_{\ell(i)}^{(1)} U_{\ell(i)}^{(1)}\Big) + \overline{x}\big(U_{\ell(i)}\big)^* J_2 U_{\ell(i)} \\
&\to y K_\infty^{(1)} B + \overline{x} - \overline{x} B^* B\,, \quad \text{as } i \to \infty\,.
\end{aligned}$$

This shows that $y \in [0,\infty)^d$. On the other hand, since

$$\overline{x} = \Big(\overline{x} S_{\ell(i)}^{-1} \|S_{\ell(i)}\|\Big) \left(\frac{S_{\ell(i)}}{\|S_{\ell(i)}\|}\right),$$

by (2.4) and (2.5), we have

$$\overline{x} = \lim_{i\to\infty} \Big[\overline{x} S_{\ell(i)}^{-1} \|S_{\ell(i)}\|\Big] \left[\frac{S_{\ell(i)}}{\|S_{\ell(i)}\|}\right] = y K_\infty^{(1)} B\,.$$

Hence $y K_\infty^{(1)} = 1$ and $\overline{x} = B$. Because of $x > 0$, we have $B = R$ and $\overline{x} = R$.

Now we need only to show that

$$\lim_{n\to\infty} U_n^{(1)} = R\,.$$

Equivalently, R is the unique cluster point of $\{U_n^{(1)};\, n \geq 1\}$ since $U_n^{(1)}$ is contained in the compact set $S(d)$. In fact, if R were not the unique cluster point of $\{U_n^{(1)};\, n \geq 1\}$, then by Lemma 2.2, there would exist a subsequence $\{\tilde{n}(i);\, i \geq 1\}$ of $\{n;\, n \geq 1\}$ such that

$$\lim_{i\to\infty} U_{\tilde{n}(i)}^{(1)} = -R\,.$$

From this, the above proof would imply that $\overline{x} = -R$, which is a contradiction. □

The following result is a consequence of Theorem 2.3.

Corollary 2.4. Under the conditions of Theorem 2.3, we have

$$\boldsymbol{P}(\tau^x = \infty) \leq \boldsymbol{P}(\overline{x} = R)$$

for any positive $x \in \boldsymbol{R}^d$.

For a subset $\mathcal{G}$ of $Gl(d, \boldsymbol{R})$, we write

$$\mathcal{G}^{-1} = \{M :\ M^{-1} \in \mathcal{G}\}\,.$$

Lemma 2.5. Given a subset $\mathcal{G}$ of $Gl(d, \boldsymbol{R})$, if $\mathcal{G}$ is irreducible (resp., strongly irreducible), then so is $\mathcal{G}^{-1}$.

Proof. Noticing that

$$\left(\bigcup_{i=1}^{k} \mathcal{V}_i\right) M = \bigcup_{i=1}^{k} \mathcal{V}_i \quad \text{for any } M \in \mathcal{G}\,,$$

if and only if

$$\left(\bigcup_{i=1}^{k} \mathcal{V}_i\right) M^{-1} = \bigcup_{i=1}^{k} \mathcal{V}_i \quad \text{for any } M \in \mathcal{G},$$

where $\mathcal{V}_i \subset \boldsymbol{R}^d$ $(1 \leq i \leq k)$. The conclusion follows easily. □

Proposition 2.6. Let $T_n(\omega) = S_n(\omega)/\|S_n(\omega)\|$. Under the conditions of Lemma 2.2, for every sequence $\{x_n;\, n \geq 1\}$ in $\boldsymbol{R}^d$ which converges to a non–zero vector, we have

$$\sup_{n\geq 1} \frac{\|T_n^{-1}(\omega)\|}{\|x_n T_n^{-1}(\omega)\|} < \infty \quad \text{a.s.}\,.$$

Proof. It is obvious that

$$\frac{\|T_n^{-1}(\omega)\|}{\|x_n T_n^{-1}(\omega)\|} = \frac{\|S_n^{-1}(\omega)\|}{\|x_n S_n^{-1}(\omega)\|}\,.$$

Let $\widetilde{X}_i = \left(X_i^{-1}\right)^*$, $\widetilde{S}_n = \widetilde{X}_n \widetilde{X}_{n-1} \cdots \widetilde{X}_1$. Then $\{\widetilde{X}_n;\, n \geq 1\}$ is again a sequence of i.i.d. random matrices. Suppose that their common distribution is $\widetilde{\mu}$. By Lemma 1.3 and Lemma 2.5, $\mathcal{G}_{\widetilde{\mu}}$ is also strongly irreducible. Therefore, by Proposition 2.1 (c), we obtain

$$\sup_{n\geq 1} \frac{\|\widetilde{S}_n(\omega)\|}{\|\widetilde{S}_n(\omega) x_n^*\|} < \infty \quad \text{a.s.}\,.$$

Hence

$$\sup_{n\geq 1} \frac{\|T_n^{-1}(\omega)\|}{\|x_n T_n^{-1}(\omega)\|} < \infty \quad \text{a.s.}\,.$$

Lemma 2.7. Suppose that $\{M_n;\, n \geq 1\}$ is a sequence in $Gl(d, \boldsymbol{R})$ which converges to a matrix M with rank one. Then

$$\lim_{n\to\infty} \|M_n^{-1}\| = \infty\,.$$

Proof. If the conclusion were not true, there would exist a sequence $\{n(i);\, i \geq 1\}$ of $\{n;\, n \geq 1\}$ and $\widetilde{M}$ in $M(d, \boldsymbol{R})$ such that

$$\widetilde{M} = \lim_{i\to\infty} M_{n(i)}^{-1}\,.$$

On the other hand,

$$I = M_{n(i)} M_{n(i)}^{-1} \to M\widetilde{M}\,, \quad i \to \infty\,,$$

where I is the $d \times d$ unit matrix. This is in contradiction with the fact that the rank of M is one. □

By Lemma 1.6, Proposition 2.1, Proposition 2.6 and Lemma 2.7, we have the following result.

Proposition 2.8. Under the conditions of Lemma 2.2,

$$\lim_{n\to\infty} \|x_n T_n^{-1}(\omega)\| = \infty\,, \quad \text{a.s.}$$

for any sequence $\{x_n;\, n \geq 1\}$ in $\boldsymbol{R}^d$ which converges to a non–zero vector.

Now we are ready to prove our main theorem.

Theorem 2.9 (Collapse theorem). Suppose that $\{X_n;\, n \geq 1\}$ is a sequence of i.i.d. random matrices with common distribution μ. If $\mathcal{G}_\mu$ is strongly irreducible, contractive and (H) holds, then for any positive $x \in \boldsymbol{R}^d$, we have

$$\boldsymbol{P}(\tau^x = \infty) = 0\,.$$

Proof. By Theorem 2.3 and Proposition 2.8, there exists a null set N such that for any $\omega \in \Omega \setminus N$,

$$\lim_{n\to\infty} U_n^{(1)}(\omega) I_{\{\tau^x(\omega)=\infty\}} = R(\omega) I_{\{\tau^x(\omega)=\infty\}}$$

and

$$\lim_{n\to\infty} \|\overline{x} T_n^{-1}(\omega)\| = \infty\,. \tag{2.6}$$

We now fix $\omega \in \Omega \setminus N$ and prove that $\tau^x(\omega) < \infty$. Again, we drop ω for a moment. If the conclusion were not true, then there would exist a subsequence $\{n(i);\, i \geq 1\}$ of $\{n;\, n \geq 1\}$, some $y \in [0, \infty]^d$ and $\big(K_\infty^{(1)}\big)^* \in S(d)$ such that

$$\lim_{i\to\infty} \overline{x} T_{n(i)}^{-1} = y \quad \text{and} \quad \lim_{i\to\infty} \big(K_{n(i)}^{(1)}\big)^* = \big(K_\infty^{(1)}\big)^* \in S(d)\,.$$

Now

$$\overline{x} = \Big(\overline{x} T_{n(i)}^{-1}\Big) T_{n(i)} \quad \text{and} \quad \lim_{i\to\infty} T_{n(i)} = K_\infty^{(1)} R\,.$$

So

$$\overline{x} = \lim_{i\to\infty} \Big(\overline{x} T_{n(i)}^{-1}\Big) T_{n(i)} = y[K_\infty^{(1)} R]\,.$$

This is means that $y \in [0, \infty)^d$ and hence

$$\lim_{i\to\infty} \|\overline{x} T_{n(I)}^{-1}\| = \|y\| < \infty\,,$$

which is in contradiction with (2.6). □

3. Application

To apply the Collapse Theorem to the stochastic models mentioned in the Section 0, we still need some preparations.

Write

$$Gl^+(d,\boldsymbol{R}) = \left\{A \in Gl(d,\boldsymbol{R}) : A \text{ is a non-negative matrix}\right\}.$$

For any $x \in \boldsymbol{R}^d \setminus \{0\}$, identify $-x/\|x\|$ with $x/\|x\|$ and let $\tilde{x}$ denote this equivalent class. Next, set

$$P(d) = \left\{\tilde{x} : x \in \boldsymbol{R}^d \setminus \{0\}\right\},$$

and endow $P(d)$ with the induced topology by the equivalent relation. A probability measure ν on $P(d)$ is said to be *proper* if for any hyperplane H in $\boldsymbol{R}^d$,

$$\nu\left(\tilde{x} : x \in \boldsymbol{R}^d \bigcap \{H \setminus \{0\}\}\right) = 0.$$

Given a probability measure μ on $Gl(d,\boldsymbol{R})$ and a probability measure ν on $P(d)$, we say that ν is *μ-invariant* if for any bounded measurable f on $P(d)$,

$$\int f\, d\nu = \iint f(\tilde{x}\cdot g)\nu(d\tilde{x})\mu(dg),$$

where $g \in Gl(d,\boldsymbol{R})$ and $\tilde{x}\cdot g = \widetilde{(xg)}$.

Lemma 3.1. Suppose that $\mathcal{G}_\mu \subset Gl^+(d,\boldsymbol{R})$ is strongly irreducible and contractive, then $R(\omega) > 0$ a.s.

Proof. By Lemma 1.6 and Proposition 2.1, with probability one,

$$\lim_{n\to\infty} \frac{a_n^{(i)}(\omega)}{\|S_n(\omega)\|} = 0, \quad 1 < i \le d.$$

Noticing that the cluster points of $S_n(\omega) = K_n(\omega)A_n(\omega)U_n(\omega)$ are non-negative, we have $R(\omega) \ge 0$ by Lemma 2.2. On the other hand, by [1; Chapter Ⅲ, Theorem 3.1], there exists a unique μ-invariant measure ν such that

$$\nu(\tilde{x} : x > 0) = \boldsymbol{P}(R > 0) \quad \text{and} \quad \nu(\tilde{x} : x \ge 0) = \boldsymbol{P}(R \ge 0).$$

Furthermore, by [1; Chapter III, Proposition 2.3], ν is proper . Hence

$$\nu(\tilde{x} : x > 0) = \nu(\tilde{x} : x \ge 0).$$

Combining these facts gives us

$$\boldsymbol{P}(R > 0) = 1. \quad \square$$

Proposition 3.2. Suppose that $\mathcal{G}_\mu \subset Gl^+(d,\boldsymbol{R})$ is strongly irreducible and contractive, then (H) holds.

Proof. Consider the polar decomposition

$$S_n(\omega) = K_n(\omega)A_n(\omega)U_n(\omega).$$

Let

$$k(n,\omega) = \min\left\{|k_i| : k_i \text{ is the } i\text{-th component of } K_n^{(1)}(\omega)\right\}.$$

Obviously, $k(n,\omega) \ge 0$ for all $n \ge 1$ and $\omega \in \Omega$. Moreover, there exist a subsequence $\{k(m(i);\omega),\ i \ge 1\}$ and a $k(\omega) \in [0,1]$ such that

$$k(\omega) = \limsup_{n\to\infty} k(n,\omega) = \lim_{i\to\infty} k(m(i),\omega).$$

On the other hand, there exist a subsequence $\{n(i);\, i \geq 1\}$ of $\{m(i);\, i \geq 1\}$ and a $K_\infty^{(1)}(\omega) \in \big(S(d)\big)^*$ such that

$$K_\infty^{(1)}(\omega) = \lim_{i\to\infty} K_{n(i)}^{(1)}(\omega).$$

Hence

$$k(\omega) = \min\big\{|k_i| :\, k_i \text{ is the } i\text{-th component of } K_\infty^{(1)}(\omega)\big\}.$$

Thus, for any fixed $\varepsilon > 0$,

$$\begin{aligned}&\big\{K_\infty^{(1)}(\omega) \text{ has a zero-component}\big\}\\ &\quad = \big\{k(\omega) = 0\big\}\\ &\quad \subset \big\{\text{there exists a } N(\omega) > 0 \text{ such that } k(n,\omega) < \varepsilon \text{ for all } n > N(\omega)\big\}.\end{aligned}$$

Next, for $\alpha \in [0,1]$, let

$$\begin{aligned}A(\alpha) = \big\{\tilde{x} :\, & x = (x_1, \ldots, x_d) \in \boldsymbol{R}^d \text{ and there exists}\\ & \text{an } i \text{ such that } 1 \leq i \leq d \text{ and } |x_i| \leq 2\alpha\big\}.\end{aligned}$$

Then

$$I_{A(0)}\big(\tilde{y}\cdot\big[R^*(\omega)(K_\infty^{(1)})^*\big]\big) \leq \liminf_{i\to\infty} I_{A(\alpha)}\big(\tilde{y}\cdot S_{n(i)}^*(\omega)\big), \quad y > 0.$$

Denote by μ^* the distribution of X_1^*. From the proof of Lemma 3.1, it follows that

$$\nu^*(\tilde{x} :\, x > 0) = 1,$$

where ν^* is a μ^*-invariant measure. So we have

$$\begin{aligned}\boldsymbol{P}\big(K_\infty^{(1)} \text{ has a zero-component}\big) &\leq \iint \liminf_{i\to\infty} I_{A(\alpha)}\big(\tilde{x}\cdot S_{n(i)}^*(\omega)\big)\nu^*(d\tilde{x})\boldsymbol{P}(d\omega)\\ &\leq \liminf_{i\to\infty} \iint I_{A(\alpha)}\big(\tilde{x}\cdot S_{n(i)}^*(\omega)\big)\nu^*(d\tilde{x})\boldsymbol{P}(d\omega)\\ &= \nu^*(\tilde{x} \in A(\alpha)).\end{aligned}$$

Letting $\alpha \downarrow 0$, we obtain

$$\boldsymbol{P}\big(K_\infty^{(1)} \text{ has a zero-component}\big) \leq \nu^*\big(\tilde{x} :\, x \text{ has s zero-component}\big) = 0. \quad \square$$

To conclude this paper, let us return to the example given in (iii) of Section 0:

$$A = \frac{1}{100}\begin{pmatrix} 25 & 14\\ 40 & 12\end{pmatrix} = (a_{ij}), \qquad X_1 = (a_{ij} \pm 0.01).$$

It is easy to see that $\mathcal{G}_\mu \subset Gl^+(2, \boldsymbol{R})$ is strongly irreducible and contractive, therefore

$$\boldsymbol{P}(\tau^x = \infty) = 0 \quad \text{for all } x > 0.$$

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