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Ergodicity of Reversible Reaction Diffusion Processes with General Reaction Rates

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Abstract. In this paper, a class of reaction diffusion processes with general reaction rates is studied. A necessary and sufficient condition for the reversibility of this calss of reaction diffusion processes is given, and then the ergodicity of these processes is proved.

§1. Introduction

Reaction diffusion processes come from non-equilibrium statistical physics. Existence theorems for most models have been established [1, 2, 4, 11]. In the case when the reaction rate is linear, the ergodicity was studied in [5], where the structure of the set of invariant measures was partially described. Some sufficient conditions for the ergodicity of general reaction diffusion processes were presented in [1]. However, a complete answer for the ergodicity is known only in the reversible case with a polynomial reaction rate. In this paper, we consider the general reaction rates. We give a necessary and sufficient condition for the reversibility of the reaction diffusion processes, and then extend the results in [6] to the general case.

First of all, we introduce some notations. Let Z be a set of integers and Z_+ be a set of nonnegative integers endowed with discrete topology. Let $X = Z_+^Z$ with product topology, and let $\mathcal{F}$ denote the Borel σ-algebra on X. Throughout this paper, we always assume (H): (i) β, δ are two nonnegative functions on Z_+ with $\delta(0) = 0$ and $\beta(n) > 0, \delta(n+1) > 0$ for $n \geq 0$, and (ii) $p(x, y)$ is the transition probability of an irreducible symmetric random walk on Z with $p(x, x) = 0, x \in Z$.

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The reaction diffusion processes studied in this paper are continuous time Markov processes on X with a formal generator as follows:

$$\begin{aligned}\Omega f(\eta) = & \sum_x \Big\{\beta(\eta(x))[f(\eta+e_x)-f(\eta)]+\delta(\eta(x))[f(\eta-e_x)-f(\eta)] \\ & +\sum_y \eta(y)p(y,x)[f(\eta+e_x-e_y)-f(\eta)]\Big\},\end{aligned} \tag{1.1}$$

where the sums are taken over all x and y in Z and $e_x \in X$ has values $e_x(x) = 1$ and $e_x(y) = 0$ for all $y \neq x$. The processes considered here are constructed on a subspace of X. Since the transition probability $(P(x,y))_{x,y\in Z}$ is symmetric, it follows that there exists a positive function $\rho(\cdot)$ on Z and a positive constant M such that

$$\sum_y p(x,y)\rho(y) \leq M\rho(x), \quad x \in Z, \tag{1.2}$$

$$\sum_x \rho(x) < \infty. \tag{1.3}$$

In what follows we fix a ρ with Properties (1.2) and (1.3) and set

$$X_m = \left\{\eta \in X : \|\eta\|_m = \sum_x \eta(x)^m \rho(x) < \infty\right\}.$$

When $m = 1$, simply write $\|\eta\|_1$ as $\|\eta\|$. Let $\mathcal{L}$ be a set of Lipschitz functions with respect to the distance $\|\eta-\zeta\| = \sum_x |\eta(x)-\zeta(x)|\rho(x)$ and denote by $c(f)$ the Lipschitz constant of $f \in \mathcal{L}$.

Theorem 0 (Existence and Uniqueness Theorem) [2,4]. *Let β, δ satisfy the following conditions:*

$$K := \sup_n\{\beta(n+1)-\beta(n)+\delta(n)-\delta(n+1)\} < \infty, \tag{1.4}$$

and there are constants $C_i, (i = 1,2)$ and a positive integer k such that for all $n \geq 1$

$$\beta(n)+\delta(n) < C_1(1+n^k), \tag{1.5}$$

and

$$\beta(n)[(n+1)^k-n^k]+\delta(n)[(n-1)^k-n^k] \leq C_2(1+n^k). \tag{1.6}$$

Then there exists a unique positive operator semigroup $P(t), t \geq 0$ such that $P(0) = I; P(t)$ is strong contraction on the uniform closure $\overline{\mathcal{L}}$ of $\mathcal{L}$ and

(i) $P(t)f \in \mathcal{L}, c(P(t)f) \leq c(f)\exp[(K+M+2)t]$,

(ii) $\frac{d}{dt}P(t)f(\eta) = \Omega P(t)f(\eta) = P(t)\Omega f(\eta), t \geq 0, f \in \mathcal{L}, \eta \in X_k$.

Moreover, there exists a Markov process $(\{\eta_t\}_{t\geq 0}, P^\eta, \eta \in X_1)$ on X_1 so that

$$P(t)f(\eta) = E_\eta f(\eta_t) = \int f(\xi)P(t,\eta,d\xi),$$

where $P(t,\eta,A)$ denotes the transition function of the process.

In what follows, when we are talking about (1.5) or (1.6), the positive integer k will be fixed.

Our first result is concerned with the necessity of the reversibility. Denote by $\mathcal{R}(\Omega)$ the set of all reversible measures of the process.

Theorem 1. *Let* (H)(i), (1.4), (1.5) *and* (1.6) *hold. Suppose that* $\mathcal{R}(\Omega) \neq \emptyset$ *and there is a* $\mu \in \mathcal{R}(\Omega)$ *with* $\int \|\eta\|_k \mu(d\eta) < \infty$. *Then there is a constant* $\lambda > 0$ *such that* $\delta(n+1) = (n+1)\beta(n)/\lambda$ *and* $p(x,y) = p(y,x)$ *for all* $x, y \in Z$. *Moreover, the reversible measure is unique, and it is the product measure* ν *in which each marginal distribution is Poissonian with mean* λ.

Theorem 2. *Let* (H) *and the following conditions be satisfied:*

(i) *There is a positive constant* $\lambda > 0$ *such that* $\delta(n) = n\beta(n-1)/\lambda$ *for all* $n \geq 1$,

(ii) $\sup_{n\geq 0}\{\beta(n+1) - (1 + \frac{n+1}{\lambda})\beta(n) + \frac{n}{\lambda}\beta(n-1)\} < \infty$ *and* $\overline{\lim}_{n\to\infty} \dfrac{\beta(n)}{n\beta(n-1)} < \dfrac{1}{\lambda}$,

(iii) *There are constants* $\varepsilon > 0, c > 0, d > 0$ *and a positive integer* k *such that*

$$cn^{1+\varepsilon} \leq \delta(n) \leq dn^k, \quad \textit{for all} \quad n \geq 0.$$

Then there exists a unique reaction diffusion process $\{\eta_t\}_{t\geq 0}$, *which is ergodic with a unique invariant measure* ν *mentioned above.*

In view of Theorem 1, Condition (i) means that Theorem 2 deals with the reversible processes. The first condition in (ii) is for the Lipschitz property of the semigroup. Actually, it implies (1.4). The second condition in (ii) is quite natural. For instance, considering a birth-death process with birth rate β and death rate $\delta(n) = n\beta(n-1)/\lambda$, the uniqueness criterion of the process leads naturally to this condition. It is interesting that this condition also implies (1.6). To see this, take $\varepsilon < 1$, sufficiently near 1, and take an integer $N_1 > 0$ such that for all $n \geq N_1$

$$\frac{\beta(n)}{n\beta(n-1)} \leq \frac{\varepsilon}{\lambda} \leq \frac{1}{\lambda}.$$

On the other hand, for a fixed $\varepsilon < 1$ there is an integer $N_2 > 0$ such that for all $n \geq N_2$,

$$\varepsilon^{\frac{1}{(k-1-m)}}(n+1) < \varepsilon^{\frac{1}{(k-1)}}(n+1) < (n-1).$$

Thus, when $n \geq N = \max\{N_1, N_2\}$, the left hand side of (1.6) equals

$$\begin{aligned}
&\sum_{m=0}^{k-1} n^m[\beta(n)(n+1)^{k-1-m} - \delta(n)(n-1)^{k-1-m}] \\
&= \sum_{m=0}^{k-1} n^{m+1}\beta(n-1)\left[\frac{\beta(n)}{n\beta(n-1)}(n+1)^{k-1-m} - \frac{(n-1)^{k-1-m}}{\lambda}\right] \\
&\leq \sum_{m=0}^{k-1} \frac{n^{m+1}\beta(n-1)}{\lambda}\left[(\varepsilon^{\frac{1}{k-1-m}}(n+1))^{k-1-m} - (n-1)^{k-1-m}\right] \\
&\leq 0.
\end{aligned}$$

So (1.6) holds. The Condition (iii) is technical. The upper bound of δ (and hence of β) is used to control the integration of $\Omega f (f \in \mathcal{L})$ with respect to a stationary measure μ. The lower bound of δ is used to estimate the moment starting from infinity (cf. Section 3 below), it can be slightly weakened but has to be faster than linear.

If $\beta(n) \sim n^\theta$ for some $\theta \geq 0$, particularly, $\beta(n)$ is a polynomial in n, and $\delta(n) = n\beta(n-1)/\lambda$ for some $\lambda > 0$, then β, δ satisfy the conditions of Theorem 2. Therefore, Theorem 2 is a generalization of the result in [6].

Combining Theorem 1 with Theorem 2, we obtain the following result.

Theroem 3. *Let* (H) (i), *and* (1.4) *hold. Suppose that*

(i) $\sup_{n\geq 0}\{\beta(n+1) - \beta(n) - \delta(n+1) + \delta(n)\} < \infty, \overline{\lim}_{n\to\infty} \frac{\beta(n)}{\delta(n)} < 1$;

(ii) *there are constants* $\varepsilon > 0, c > 0, d > 0$ *and a positive integer* k *such that*

$$cn^{1+\varepsilon} \leq \delta(n) \leq dn^k, \quad \textit{for all} \quad n \geq 0.$$

Then $\mathcal{R}(\Omega) \neq \emptyset$ *if and only if there is a positve constant* $\lambda > 0$ *such that* $\delta(n) = n\beta(n-1)/\lambda$ *for all* $n \geq 1$; *and* $p(x, y) = p(y, x)$ *for all* $x, y \in Z$. *Moreover,* $\mathcal{R}(\Omega) = \mathcal{I} = \{\nu\}$, *where* $\mathcal{I}$ *denotes the set of invariance measures for* $\{\eta_t\}_{t\geq 0}$.

The proof of Theorem 2 and Theorem 3 will be completed in Section 4. Some techniques needed for proving these theorems will be given in Section 3. In the next section, we prove Theorem 1.

§2. Reversibility

To prove Theorem 1, we begin with the following lemma.

Lemma 2.1. *Let* (H)(i), (1.4), (1.5) *and* (1.6) *hold. Given* $\mu \in \mathcal{R}(\Omega)$ *with*

$$\int \|\eta\|_k \mu(d\eta) < \infty, \tag{2.1}$$

we have for any bounded cylinder function f *and* $x, y \in Z$,

$$\int \beta(\eta(x)) f(\eta)\mu(d\eta) = \int \delta(\eta(x)) f(\eta - e_x)\mu(d\eta), \tag{2.2}$$

$$\int \eta(y)p(y, x) f(\eta)\mu(d\eta) = \int \eta(x)p(x, y) f(\eta - e_x + e_y)\mu(d\eta). \tag{2.3}$$

Proof. By assumption, μ is concentrated on X_k. $\mu \in \mathcal{R}(\Omega)$ means that

$$\int fP(t)g d\mu = \int gP(t)f d\mu$$

holds for all bounded cylinder functions f and g. From this it follows that

$$\int_{X_k} f(\eta)\frac{P(t)g(\eta) - g(\eta)}{t}\mu(d\eta) = \int_{X_k} g(\eta)\frac{P(t)f(\eta) - f(\eta)}{t}\mu(d\eta).$$

By the assumptions and the dominated convergence theorem, letting $t \to 0$, we get

$$\int f\Omega g d\mu = \int g\Omega f d\mu. \tag{2.4}$$

Noticing that any cylinder function f with base Λ belongs to a uniform closure of span$\{1_{[\{\xi\}\times X(Z\setminus\Lambda)]}$, $\xi \in X(\Lambda)\}$, it suffices to prove (2.2) and (2.3) for those functions having the form $1_{[\{\xi\}\times X(Z\setminus\Lambda)]}, \xi \in X(\Lambda)$. For $\xi \in X(\Lambda), x, y \in Z$, let $f = 1_{[\{\xi\}\times X(Z\setminus\Lambda)]}, g = 1_{[\{\xi-e_x+e_y\}\times X(Z\setminus\Lambda)]}$. By an elementary computation, we get

$$f(\eta)\Omega g(\eta) = \eta(x)p(x,y)f(\eta),$$

$$g(\eta)\Omega f(\eta) = \eta(y)p(y,x)f(\eta - e_x + e_y).$$

Substituting these into (2.4), we see that (2.3) holds. Similarly, let $f = 1_{[\{\xi\}\times X(Z\setminus\Lambda)]}$ and $g = 1_{[\{\xi+e_x\}\times X(Z\setminus\Lambda)]}$. By an elementary computation, we have

$$f(\eta)\Omega g(\eta) = \beta(\eta(x))f(\eta) + \sum_{v\notin\Lambda}\eta(v)p(v,x)f(\eta),$$
$$g(\eta)\Omega f(\eta) = \delta(\eta(x))f(\eta - e_x) + \sum_{u\notin\Lambda}\eta(x)p(x,u)f(\eta - e_x + e_u).$$

Now, (2.2) follows from (2.4) and (2.3).

Proof of Theorem 1. Let μ be a probability measure on X_1. For $x, y \in Z$ and $k, l \in Z_+$, let

$$\mu_x(k) = \mu(\eta : \eta(x) = k), \quad \mu_{x,y}(k,l) = \mu(\eta : \eta(x) = k, \eta(y) = l);$$

and let

$$f_x^k(\eta) = \begin{cases} 1, & \text{if } \eta(x) = k \\ 0, & \text{otherwise.} \end{cases}$$

$$g_{x,y}^{k,l}(\eta) = \begin{cases} 1, & \text{if } \eta(x) = k, \eta(y) = l \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, $f_x^k, g_{x,y}^{k,l}$ are both bounded cylinder functions. Substituting them into (2.2) and (2.3), we get

$$\beta(k)\mu_x(k) = \delta(k+1)\mu_x(k+1), \quad k \geq 0; \tag{2.5}$$

$$\beta(k)\mu_{x,y}(k,l) = \delta(k+1)\mu_{x,y}(k+1,l), \quad k,l \geq 0. \tag{2.6}$$

Making an exchange of x and y, we also have

$$\beta(l)\mu_{x,y}(k,l) = \delta(l+1)\mu_{x,y}(k,l+1), \quad k,l \geq 0. \tag{2.7}$$

and

$$lp(y,x)\mu_{x,y}(k,l) = (k+1)p(x,y)\mu_{x,y}(k+1,l-1), \quad k \geq 0, l \geq 1. \tag{2.8}$$

From (2.5) it follows that for all $k \geq 0$

$$\mu_x(k+1) = \frac{\beta(k)}{\delta(k+1)}\mu(k) = \cdots = \frac{\beta(k)\cdots\beta(0)}{\delta(k+1)\cdots\delta(1)}\mu_x(0).$$

Summing up k from 0 to ∞, we get

$$1 = \sum_{k=0}^{\infty}\mu_x(k) = \left(1 + \sum_{k=0}^{\infty}\frac{\beta(k)\cdots\beta(0)}{\delta(k+1)\cdots\delta(1)}\right)\mu_x(0).$$

So

$$\sum_{k=0}^{\infty} \frac{\beta(k)\cdots\beta(0)}{\delta(k+1)\cdots\delta(1)} < \infty;$$
$$\mu_x(0) = \left\{1 + \sum_{k=0}^{\infty} \frac{\beta(k)\cdots\beta(0)}{\delta(k+1)\cdots\delta(1)}\right\}^{-1}. \tag{2.9}$$

And hence the following expression

$$\mu_x(k+1) = \frac{\beta(k)\cdots\beta(0)}{\delta(k+1)\cdots\delta(1)}\left\{1 + \sum_{k=0}^{\infty} \frac{\beta(k)\cdots\beta(0)}{\delta(k+1)\cdots\delta(1)}\right\}^{-1} \tag{2.10}$$

is independent of x.

Similarly, from (2.6) it follows that

$$\mu_{x,y}(k+1,l) = \frac{\beta(k)\cdots\beta(0)}{\delta(k+1)\cdots\delta(1)}\mu_{x,y}(0,l), \quad k \geq 0.$$

Summing up k from 0 to ∞ and noting (2.9), we have

$$\mu_{x,y}(0,l) = \mu_x(0)\mu_y(l).$$

Substituting this into the previous equation, we get

$$\mu_{x,y}(k+1,l) = \mu_x(k+1)\mu_y(l).$$

This means that $\eta(x)$ and $\eta(y)$ are independent under μ, and

$$\mu_{x,y}(k,l) = \mu_x(k)\mu_y(l) > 0, \quad k,l \geq 0, \tag{2.11}$$

here we utilize the fact that $\mu_x(k)$ is independent of x. By induction, we can prove that $\eta(x), x \in Z$ are i.i.d under μ.

Taking $l=1, k=0$ in (2.8), we get

$$p(y,x)\mu_{x,y}(0,1) = p(x,y)\mu_{x,y}(1,0).$$

But $\mu_{x,y}(0,1) = \mu_{x,y}(1,0) > 0$, hence for any $x, y \in Z$,

$$p(x,y) = p(y,x). \tag{2.12}$$

Since $(p(x,y))_{x,y}$ is a transition probability, there are x and y in Z such that $p(x,y) > 0$. For this fixed pair (x,y), reducing (2.8) by $p(x,y) = p(y,x)$, we get

$$l\mu_{x,y}(k,l) = (k+1)\mu_{x,y}(k+1,l-1).$$

Letting $l=1$ and noting (2.5), we have

$$\mu_x(k)\mu_y(1) = (k+1)\frac{\beta(k)}{\delta(k+1)}\mu_x(k)\mu_y(0).$$

Hence

$$\delta(k+1) = \frac{(k+1)\beta(k)}{\lambda}, \tag{2.13}$$

here $\lambda = \frac{\mu_y(1)}{\mu_y(0)} = \frac{\beta(0)}{\delta(1)}$ is independent of y. (2.10), (2.12), (2.13) and the independence of $\eta(x), x \in Z$ imply that μ is a product measure of Poisson distributions with mean λ.

To complete the proof of Theorem 1, we still need to show that ν is a reversible measure of the process whenever $\delta(n+1) = (n+1)\beta(n)/\lambda$ for some $\lambda > 0$ and $p(x,y)$ being symmetric. This can be done in two steps. First, the assumptions imply (2.4) by replacing μ with ν in the finite dimensional case, which implies the reversibility of the finite dimensional Markov chains. Next, the reversibility of the infinite dimensional process follows from the finite dimensional ones in the light of the construction of the process (cf. [1] or [2]).

§3. Moments

Let $\mathcal{I}$ denote the set of all stationary distributions for $\{\eta_t\}$, and $\mathcal{S}$ denote the set of all translation invariant probability measures on X_1. The proof of Theorem 2 can be divided into two steps. The first step is to prove that there exist maximal and minimal stationary distributions $\overline{\mu}$ and $\underline{\mu}$ in the sense of partial order defined in II, [8]. Both $\overline{\mu}$ and $\underline{\mu}$ are in $\mathcal{I} \cap \mathcal{S}$. The second step is to prove $\mu = \nu$ for all $\mu \in \mathcal{I} \cap \mathcal{S}$ by the free energy technique. The ergodicity then follows from the attractivity of the process. These two steps will be completed in this and the next section respectively.

We begin to show that all moments of $\{\eta_t\}_{t\geq 0}$ starting from any configuration are finite, which are then used to construct $\overline{\mu}$ and $\underline{\mu}$. Throughout this section, suppose that the hypotheses of Theorem 3 hold. This means that the hypotheses of Theorem 2 are also satisfied except the Condition (i) there and the symmetry of $p(x,y)$.

Lemma 3.1. *There is a constant C such that*

$$E_\eta \eta_t(x) \leq e^{-Ct} \sum_y \eta(y) p(t,y,x) + \frac{K}{C}(1 - e^{-Ct}), \tag{3.1}$$

for any $\eta \in X_1, x \in Z$, here $p(t,y,x) = e^{-t} \sum_{m=0}^{\infty} \frac{t^m}{m!} p^{(m)}(y,x)$.

Proof. By (i), (ii) of Theorem 3, there is a positive constant C such that for a sufficiently large n,

$$\frac{\beta(n)}{\delta(n)} - 1 \leq -C.$$

Therefore

$$\beta(n) - \delta(n) = \delta(n)\left[\frac{\beta(n)}{\delta(n)} - 1\right] \leq -C\delta(n) \leq -Cn^{1+\varepsilon}.$$

Thus, we can choose a positive constant K so that

$$\beta(n) - \delta(n) \leq K - Cn^{1+\varepsilon}, \tag{3.2}$$

for all $n \geq 0$. Now, the assertion follows by a standard argument (cf. [1]).

The following result is also standard.

Lemma 3.2. *Let $\{\eta_t\}, \{\zeta_t\}$ denote the processes starting from η, ζ respectively with the same generator Ω. For $\eta, \zeta \in X_1$ with $\eta \le \zeta, \{\eta_t\}$ and $\{\zeta_t\}$ can be constructed on the same probability space in such a way that $\eta_t \le \zeta_t$ for all $t \ge 0$.*

This lemma means that the process is attractive. Particularly, let η_t^n denote the process starting from $\eta_0^n(x) \equiv \vec{n}\ (x) = n$, according to Lemma 3.2 when $m \le n, \eta_t^m \le \eta_t^n$ for all $t \ge 0$. This implies that for each $t > 0, \eta_t^n$ increases to a limit η_t^∞. We will show that η_t^∞ is also finite for all $t > 0$.

Lemma 3.3. *Let E^n denote the expectation of η_t^n. There is a nonnegative, decreasing function $\varphi(t)$ on $[0, \infty)$ which is independent of n and finite in $(0, \infty)$ so that*

$$E^n \eta_t(x) \le \varphi(t), \quad \text{for all} \quad t \ge 0. \tag{3.3}$$

Proof. By Lemma 3.1,

$$E^n \eta_t(x) \le e^{-Ct} \sum_y np(t, y, x) + \frac{K}{C}(1 - e^{-Ct}) \le ne^{-Ct} + \frac{K}{C} < \infty.$$

Next, by Theorem 0,

$$\frac{d}{dt} E^n \eta_t(x) = E^n[\beta(\eta_t(x)) - \delta(\eta_t(x)) - \eta_t(x) + \sum_y \eta_t(y) p(y, x)].$$

Since $\delta_{\vec{n}}$ is translation invariant, so is the distribution of η_t^n. Hence the last two terms cancel each other. Thus we get

$$\begin{aligned} \frac{d}{dt} E^n \eta_t(x) &= E^n[\beta(\eta_t(x)) - \delta(\eta_t(x)) = K - CE^n[\eta_t(x)^{1+\epsilon}] \\ &\le K - C(E^n \eta_t(x))^{1+\epsilon}. \end{aligned}$$

Here the first inequality comes from (3.2), and the last inequality comes from Hölder's inequality. Write $g_n(t) = E^n \eta_t(x)$. Combining the previous arguments gives the differential inequality:

$$g_n'(t) \le K - C g_n^{1+\epsilon}(t). \tag{3.4}$$

here $g_n(0) = n$. Repeating the last part of the proof of Lemma 2.3 in [6] gives the desired conclusion.

Lemma 3.4. *For each $m > 1$, there is a nonnegative, decreasing function $\varphi_m(t)$ on $[0, \infty)$ which is independent of n and finite in $(0, \infty)$ so that*

$$E^n(\eta_t(x))^m \le \varphi_m(t), \quad \text{for all} \quad t \ge 0. \tag{3.5}$$

Proof. At present $f_x^m(\eta) = \eta(x)^m$ is not in $\mathcal{L}$, so we cannot use Theorem 0 directly. Fortunately, we can utilize the truncation technique mentioned at the end of the proof of Lemma 2.3 in [6]. Here we only give the computations in brief and leave the justification to our readers.

$$\begin{aligned} \frac{d}{dt} E^n(\eta_t(x))^m = E^n\Big\{ &\Big[\beta(\eta_t(x)) + \sum_y \eta_t(y) p(y, x)\Big] \Delta_m\ (\eta_t(x)) \\ &-[\delta(\eta_t(x)) + \eta_t(x)]\ \Delta_m\ (\eta_t(x) - 1)\Big\}, \end{aligned}$$

where $\Delta_m(x) = (x+1)^m - x^m$. We use the fact that $-\eta_t(x)\,\Delta_m\,(\eta_t(x)-1) \le 0$ to drop out the last term. Using Hölder's inequality and the translation invariance, we know that

$$E^n\left[\sum_y \eta_t(y)p(y,x)\right]\Delta_m\,(\eta_t(x)) \le CE^n(\eta_t(x))^m$$

for some constant C depending on m only. To deal with the remaining terms, notice that

$$\lim_{n\to\infty}\frac{(n+1)^m - n^m}{(n+1)^{m-1}} = m > 0.$$

Taking a sufficiently small $\Delta > 0$, which will be fixed later on, gives for a sufficiently large n,

$$(m-\Delta)(n+1)^{m-1} < (n+1)^m - n^m < (m+\Delta)(n+1)^{m-1}. \tag{3.6}$$

Next,

$$\begin{aligned}&(m-\Delta)\beta(n)(n+1)^{m-1} - (m-\Delta)\delta(n)n^{m-1} + n^{m+\epsilon/2}\\ &= n^{m-1}\delta(n)\left[(m-\Delta)\left(1+\frac{1}{n}\right)^{m-1}\frac{\beta(n)}{\delta(n)} - (m-\Delta) + \frac{n^{1+\epsilon/2}}{\delta(n)}\right].\end{aligned}$$

but

$$\begin{aligned}&\overline{\lim}_{n\to\infty}\left[(m-\Delta)\left(1+\frac{1}{n}\right)^{m-1}\frac{\beta(n)}{\delta(n)} - (m-\Delta) + \frac{n^{1+\epsilon/2}}{\delta(n)}\right]\\ &= (m-\Delta)\alpha - (m-\Delta),\end{aligned}$$

where $\alpha = \overline{\lim}_{n\to\infty}\dfrac{\beta(n)}{\delta(n)} < 1$. When $\Delta = \dfrac{(m-1)(1-\alpha)}{2(1+\alpha)}$, there is a positive integer N such that (3.6) holds for all $n \ge N$ and

$$(m-\Delta)\beta(n)(n+1)^{m-1} - (m-\Delta)\delta(n)n^{m-1} + n^{m+\epsilon/2} \le 0.$$

Therefore, there is a constant a such that

$$(m-\Delta)\beta(n)(n+1)^{m-1} - (m-\Delta)\delta(n)n^{m-1} \le a - n^{m+\epsilon/2}$$

for all n. Now

$$\begin{aligned}&E^n\{\beta(\eta_t(x))\,\Delta_m\,(\eta_t(x)) - \delta(\eta_t(x))\,\Delta_m\,(\eta_t(x)-1)\}\\ &= E^n\{\beta(\eta_t(x))\,\Delta_m\,(\eta_t(x)) - \delta(\eta_t(x))\,\Delta_m\,(\eta_t(x)-1), \eta_t(x) \le N\}\\ &+E^n\{\beta(\eta_t(x))\,\Delta_m\,(\eta_t(x)) - \delta(\eta_t(x))\,\Delta_m\,(\eta_t(x)-1), \eta_t(x) > N\}\\ &\le a' + E^n\{(m+\Delta)\beta(\eta_t(x))(\eta_t(x)+1)^{m-1}\\ &-(m-\Delta)\delta(\eta_t(x))\eta_t(x)^{m-1}, \eta_t(x) > N\}\\ &\le a' + E^n\{a - (\eta_t(x))^{m+\epsilon/2}, \eta_t(x) > N\}\\ &\le A - CE^n(\eta_t(x))^{m+\epsilon/2},\end{aligned}$$

here a', A and C are constants. Combining these estimates gives

$$\frac{d}{dt}E^n(\eta_t(x))^m \le A + BE^n(\eta_t(x))^m - CE^n(\eta_t(x))^{m+\epsilon/2}. \tag{3.7}$$

Using

$$(E^n(\eta_t(x))^m)^{\frac{m+\epsilon/2}{m}} \le E^n(\eta_t(x))^{m+\epsilon/2}$$

and writting $u(t) = E^n(\eta_t(x))^m$ in (3.7), we have

$$u'(t) \le A + Bu(t) - Cu(t)^{1+\delta},$$

where $\delta = \frac{\epsilon}{2m} > 0$. So the rest of the proof is the same as before.

Now, we are in a position to show that $\eta_t^\infty \in X_1$, a.e. for all $t > 0$ and there exist maximal and minimal stationary distributions $\overline{\mu}$ and $\underline{\mu}$. In fact, by Lemma 3.3,

$$E\eta_t^n(x) \le \varphi(t) < \infty, \quad \text{for all} \quad t > 0,$$

so

$$E\eta_t^\infty(x) \le \varphi(t) < \infty.$$

Thus

$$E\left\{\sum_x \eta_t^\infty(x)\rho(x)\right\} \le \varphi(t)\sum_x \rho(x) < \infty.$$

This implies that $\eta_t^\infty \in X_1$, a.s.

Next, by Lemma 3.2, we can adopt the way mentioned at the end of Section 2 in [6] to obtain the required conclusions. Lemmas 3.3 and 3.4 imply that all moments of $\overline{\mu}$ are finite. Therefore all moments of any stationary distributions also are finite.

§4. Free Energy

Assume $\mu \in \mathcal{I} \cap \mathcal{S}$ and let $\Lambda \subset Z$ be finite. Define $\mu(\xi) = \mu(\eta : \eta = \xi \text{ on } \Lambda), \xi \in X(\Lambda)$. When $\Lambda = \{x\}$, and $\xi(x) = j$, write $\mu_x(j) = \mu(\xi)$.

Lemma 4.1. *Assume $\mu \in \mathcal{I} \cap \mathcal{S}$. Then for any integer $m \ge 1$ and $x \in \Lambda$,*

$$-\sum_\xi \delta^m(\xi(x))\mu(\xi)\log\mu(\xi) < \infty, \tag{4.1}$$

where the summation extends over $X(\Lambda)$.

Proof. By assumption (iii) of Theorem 2,

$$-\delta^m(\xi(x))\mu(\xi)\log\mu(\xi) \le -d^m\xi(x)^{km}\mu(\xi)\log\mu(\xi). \tag{4.2}$$

On the basis of Lemma 3.4,

$$\sum_{j=0}^\infty j^{mk+1}\mu(j) = \int \eta(x)^{mk+1}\mu(d\eta) < \infty.$$

Combining this with (4.2) and Lemma 4.2 in [6], we obtain (4.1).

Lemma 4.2. *Let $\mu \in \mathcal{I}$. Then for any finite $\Lambda \subset Z$ and $\xi \in X(\Lambda)$, we have $\mu(\xi) > 0$.*

Proof. If the conclusion were not true, then there would be some $\zeta \in X(\Lambda)$ so that $\mu(\zeta) = 0$. Let $f_\zeta(\eta) = 1$ if $\eta = \zeta$ on Λ, and $= 0$ otherwise. $\mu \in \mathcal{I}$ implies

$$\int \Omega f_\zeta(\eta)\mu(d\eta) = 0. \tag{4.3}$$

By an elementary computation, under $\mu(\zeta) = 0$, (4.3) is equivalent to

$$\begin{aligned}&\sum_{x\in\Lambda}\{\beta(\zeta(x)-1)\mu(\zeta-e_x)+\delta(\zeta(x)+1)\mu(\zeta+e_x)\\ &\quad+(\zeta(x)+1)\sum_{y\in\Lambda}p(x,y)\mu(\zeta+e_x-e_y)\\ &\quad+(\zeta(x)+1)\sum_{y\notin\Lambda}p(x,y)\mu(\zeta+e_x)\ .\\ &\quad+\sum_{y\notin\Lambda}\int\eta(y)p(y,x)f_{\zeta-e_x}(\eta)\mu(d\eta)\}\\ &\quad=0.\end{aligned}$$

But $\beta(n) > 0$ for all $n \geq 0, \delta(n) > 0$ for all $n \geq 1$, it follows that $\mu(\zeta \pm e_x) = 0$. Using induction gives $\mu(\zeta \pm ke_x) = 0$ for all $x \in \Lambda$ and $k \geq 1$, provided that $\zeta \pm ke_x \in X(\Lambda)$. This means that $\mu(\xi) = 0$ for all $\xi \in X(\Lambda)$ which contradicts $\mu(X(\Lambda)) = 1$.

Now, we are in a position to show that $\mu = \nu$ for all $\mu \in \mathcal{I}\cap\mathcal{S}$. To do this, let $\Lambda_n = [-n, n]$. For each $\zeta \in X(\Lambda_n)$ since $f_\zeta(\cdot)$ is a bounded cylinder function and $\mu \in \mathcal{I}$, we have

$$\int \mu(d\eta)\Omega f_\zeta(\eta) = 0.$$

Therefore

$$\sum_\zeta \int \mu(d\eta)\Omega f_\zeta(\eta)[\log\mu(\zeta) - \log\nu(\zeta)] = 0, \tag{4.4}$$

here the summation extends over $X(\Lambda)$. Repeating almost the same computations done in Section 3 of [6], we get

$$\begin{aligned}0=\ &\sum_{x\in\Lambda_n}\sum_\zeta\frac{\beta(\zeta(x))}{\lambda}[\lambda\mu(\zeta)-(\zeta(x)+1)\mu(\zeta+e_x)]\log\frac{\mu(\zeta+e_x)}{\mu(\zeta)}\frac{\zeta(x)+1}{\lambda}\\ &+\sum_{x\in\Lambda_n}\sum_{y\in\Lambda_n}\sum_\zeta p(x,y)(\zeta(x)+1)\mu(\zeta+e_x)\log\frac{\mu(\zeta+e_y)}{\mu(\zeta+e_x)}\frac{\zeta(y)+1}{\zeta(x)+1}\\ &+\sum_{x\in\Lambda_n}\sum_{y\notin\Lambda_n}\sum_\zeta p(x,y)(\zeta(x)+1)\mu(\zeta+e_x)\log\frac{\mu(\zeta)}{\mu(\zeta+e_x)}\frac{\lambda}{\zeta(x)+1}\\ &+\sum_{x\notin\Lambda_n}\sum_{y\in\Lambda_n}\sum_\zeta p(x,y)\sum_{k=0}^{\infty}k\mu(\zeta+k_x)\log\frac{\mu(\zeta+e_y)}{\mu(\zeta)}\frac{\zeta(y)+1}{\lambda}.\end{aligned} \tag{4.5}$$

Here Lemmas 3.3 and 3.4 guarentee that all series appearing in the computation are absolutely convergent; hence order exchange is permitted. Rearranging, using symmetry of

$p(x,y)$ in the second sum of (4.5), exchanging x and y in the fourth sum, and writting, for any $x, y \in \Lambda_n$,

$$D_n(x) = \sum_{\zeta} \frac{\beta(\zeta(x))}{\lambda}[\lambda\mu(\zeta) - (\zeta(x)+1)\mu(\zeta+e_x)] \log \frac{\mu(\zeta+e_x)}{\mu(\zeta)} \frac{\zeta(x)+1}{\lambda},$$

$$\begin{aligned} D_n(x,y) &= 1/2 \sum_{\zeta} p(x,y)[(\zeta(x)+1)\mu(\zeta+e_x) \\ &\quad -(\zeta(y)+1)\mu(\zeta+e_y)] \log \frac{\mu(\zeta+e_x)}{\mu(\zeta+e_y)} \frac{\zeta(x)+1}{\zeta(y)+1} \\ R_n &= -\sum_{x\in\Lambda_n} \sum_{y\notin\Lambda_n} \sum_{\zeta} p(x,y)[(\zeta(x)+1)\mu(\zeta+e_x) \\ &\quad - \sum_{k=0}^{\infty} k\mu(\zeta \times k_x)] \log \frac{\mu(\zeta+e_x)}{\mu(\zeta)} \frac{\zeta(x)+1}{\lambda}, \end{aligned} \tag{4.6}$$

we get

$$\sum_{x\in\Lambda_n} D_n(x) + \sum_{x,y\in\Lambda_n} D_n(x,y) = R_n. \tag{4.7}$$

Lemma 4.3. *For any integers m, n with $0 < m < n$ and $x, y \in [-m, m]$, we have*

$$0 \le D_m(x) \le D_n(x); \quad 0 \le D_m(x,y) \le D_n(x,y). \tag{4.8}$$

Proof. The idea of the proof is similar to that of Lemma 3.8 in [6].

If $D_n(x) \ne 0$ for some x and n, Lemma 3.7 and the translation invariance imply that the left hand side of (4.7) is greater than Cn for large n. On the other hand, using inequality $\log a \le a^{1/2}$ and Hölder's inequality, we have

$$\begin{aligned} &\sum_{\zeta} \sum_{k=0}^{\infty} k\mu(\zeta \times k_x) \log \frac{\mu(\zeta+e_x)}{\mu(\zeta)} \\ &\le \sum_{\zeta} \sum_{k=0}^{\infty} k\mu(\zeta \times k_x) \left(\frac{\mu(\zeta+e_x)}{\mu(\zeta)}\right)^{1/2} \\ &\le \left[\sum_{\zeta} \sum_{k=0}^{\infty} k\mu(\zeta \times k_x)\right]^{1/2} \left[\sum_{\zeta} \sum_{k=0}^{\infty} \mu(\zeta \times k_x) \frac{\mu(\zeta+e_x)}{\mu(\zeta)}\right]^{1/2} \\ &= [E^{\mu}(\zeta(y))^2]^{1/2} \left[\sum_{\zeta} \mu(\zeta+e_x)\right]^{1/2} \le E^{\mu}[(\zeta(y))^2]^{1/2} < \infty. \end{aligned}$$

$$\begin{aligned} \sum_{\zeta} \sum_{k=0}^{\infty} k\mu(\zeta \times k_x) \log \frac{\zeta(x)+1}{\lambda} &\le \sum_{\zeta} \sum_{k=0}^{\infty} k\mu(\zeta \times k_x) \log(\zeta(x)+1) \\ &\quad + |\log \lambda| \sum_{\zeta} \sum_{k=0}^{\infty} k\mu(\zeta \times k_x) \\ &\le E^{\mu}(\zeta_0(x))^2 + |\log \lambda| E^{\mu}\zeta_0(x) < \infty. \end{aligned}$$

Next, using inequality $-x\log x \le e^{-1}$ for $x > 0$, we have

$$\begin{aligned}
&-\sum_{\zeta} \ (\zeta(x)+1)\mu(\zeta+e_x)\log\frac{\mu(\zeta+e_x)}{\mu(\zeta)} \\
\le\ & \sum_{\zeta}(\zeta(x)+1)\mu(\xi)\left[-\frac{\mu(\zeta+e_x)}{\mu(\zeta)}\right]\log\frac{\mu(\zeta+e_x)}{\mu(\zeta)} \\
\le\ & e^{-1}(1+E^{\mu}\zeta_0(x)) < \infty,
\end{aligned}$$

and

$$-\sum_{\zeta}(\zeta(x)+1)\mu(\zeta+e_x)\log\frac{\zeta(x)+1}{\lambda} \le \lambda e^{-1}.$$

Combining the above inequalities gives

$$R_n \le C\sum_{x\in\Lambda_n}\sum_{y\notin\Lambda_n} p(x,y) = o(n). \tag{4.9}$$

The last equality holds since

$$\sum_{x\in\Lambda_n}\sum_{y\notin\Lambda_n} p(x,y) = \sum_{x=1}^{2n+1}\sum_{y=x}^{\infty}[p(0,y)+p(0,-y)].$$

For an arbitray $\varepsilon > 0$, pick N such that $\sum_{|y|>N} p(0,y) \le \varepsilon$, and then

$$\sum_{x=1}^{2n+1}\sum_{y=x}^{\infty}[p(0,y)+p(0,-y)] \le (2n+1)\varepsilon + (2N+1).$$

Thus, (4.9) contradicts the fact that the left hand side of (4.7) $\ge Cn$. This implies that $D_n(x) = 0$ for all x and n. Therefore, for all n,

$$\mu(\zeta+e_x) = \frac{\lambda}{\zeta(x)+1}\mu(\zeta), \quad \text{for all} \quad \zeta\in X(\Lambda_n), x\in\Lambda_n.$$

It follows that $\mu = \nu$.

Proof of Theorem 3. The proof of Theorem 2 shows that all invariance measures possess finite moments. Thus Theorem 1 gives the "only if" part and Theorem 2 gives the "if" part.

Note. This paper had been completed before the first author wrote his book "From Markov Chains to Non-Equilibrium Particle Systems" (World Scientific, 1992). Part IV of that book is mainly devoted to the subject of reaction-diffusion processes. Slightly different treatment on the reversibility and the estimate of the moments are also included in that book.

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