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On order-preservation and positive correlations for multidimensional diffusion processes*

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Summary. As a continuation of the study by Herbst and Pitt (1991), this note presents two criteria. The first one is on the order-preservation for two (may be different) multidimensional diffusion processes. The second one is on the preservation of positive correlations for a diffusion process.

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1 Introduction

For the background of the study, the related references and applications, the readers are urged to refer to Herbst and Pitt (1991).

Definition 1.1 Let " $\leqq$ " denote the usual semi-order in $\mathbb{R}^n$.
(1) A measurable function f is called *monotone* if

$$f(x) \leqq f(y) \quad \text{for all } x \leqq y .$$

Denote by $\mathscr{M}$ the set of all bounded continuous monotone functions.
(2) For two Markov semigroups $\{P_t\}$ and $\{\bar{P}_t\}$, we write $P_t \geqq \bar{P}_t$ if for all $f \in \mathscr{M}$, all $x \geqq y$ and $t \geqq 0$,

$$P_t f(x) \geqq \bar{P}_t f(y) .$$

If in addition $P_t = \bar{P}_t$, we call $\{P_t\}$ *monotone.*
(3) A probability measure μ is said *to have positive correlations* if

$$\mu(fg) \geqq \mu(f)\mu(g) \quad \text{for all } f, g \in \mathscr{M} ,$$

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where $\mu(f) = \int f d\mu$. Denote by $\mathscr{P}_+$ the set of all such probability measures.
(4) A Markov semigroup $\{P_t\}$ is said *to preserve positive correlations* if

$$\mu P_t \in \mathscr{P}_+ \text{ for all } \mu \in \mathscr{P}_+ \text{ and } t \geqq 0 .$$

Let

$$A = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} .$$

Throughout this paper, assume that (a_{ij}) is nonnegative definite everywhere, a_{ij}, $b_i \in C(\mathbb{R}^n)$ and the martingale problem for A is well posed. Let $\{P_t\}$ be the semigroup generated by A. One of the main result of Herbst and Pitt (1991) can be summarized as follows:

Theorem 1.2 (1) *If*

(a) *for all i and j, $a_{ij}(x)$ depends only on x_i and x_j and*
(b) *for all $i \neq k$, $b_i(x)$ is an increasing function of x_k,*
then $\{P_t\}$ is monotone. The conditions (a) *and* (b) *are necessary in the case that a_{ij} and b_i are all bounded and having bounded continuous derivatives of all orders.*
(2) *If $\{P_t\}$ is monotone and the following condition holds*:

(1.1) $$a_{ij} \geqq 0 \textit{ for all } i \textit{ and } j ,$$

then $\{P_t\}$ preserves positive correlations. Moreover, the condition (1.1) *is necessary in the same case mentioned in* (1).

Our first criterion is a generalization to part (1) of Theorem 1.2. Here, we allow the two semigroups to be different. This is meaningful since it enables us to compare a given diffusion process with a simpler one. Let

$$\bar{A} = \frac{1}{2} \sum_{i,j=1}^{n} \bar{a}_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} \bar{b}_i \frac{\partial}{\partial x_i} ,$$

and let $\{\bar{P}_t\}$ be the semigroup generated by $\bar{A}$.

Theorem 1.3 $P_t \geqq \bar{P}_t$ *if and only if the following two conditions hold*:

(1.2) *for all i and j, $a_{ij} \equiv \bar{a}_{ij}$ and $a_{ij}(x)$ depends only on x_i and x_j.*

(1.3) *for all i, $b_i(x) \geqq \bar{b}_i(y)$ whenever $x \geqq y$ with $x_i = y_i$.*

As for the preservation of positive correlations, we claim that the monotonicity is also necessary in the context of diffusion processes.

Theorem 1.4 *$\{P_t\}$ preserves positive correlations if and only if it is monotone and* (1.1) *holds.*

The proofs of Theorem 1.3 and Theorem 1.4 are given in the following two sections respectively. It is worth to mention that the monotonicity is not necessarily needed for the preservation of positive correlations for other type of Markov processes. For instance, every Markov process on real line (and actually, every probability measure on totally ordered space) has positive correlations but some one-dimensional jump processes are not monotone. A general criterion for order-preservation (in particular, for monotonicity) of jump processes was presented in [1] (also in [2]), from which some idea of this paper follows.

2 Proof of Theorem 1.3

Let

$$C_0^\infty = \{ f \in C^\infty(\mathbb{R}^n) : f \text{ is constant out of a compact set}\}$$

$$C_b^\infty = \{ f \in C^\infty(\mathbb{R}^n) : f \text{ has bounded continuous derivatives of all orders}\} .$$

Since $C_0^\infty \subset \{ f \in C_b^\infty : Af \text{ is bounded}\}$ and the martingale problem for A is well posed, we have

(2.1) $$\lim_{t \to 0} \frac{1}{t} [P_t f(x) - f(x)] = Af(x), \quad x \in \mathbb{R}^n, \quad f \in C_0^\infty .$$

Because $\mathscr{M} \cap C_b^\infty$ is dense in $\mathscr{M}$ under pointwise convergence, we can replace $\mathscr{M}$ by $\mathscr{M} \cap C_b^\infty$ in Definition 1.1.

To prove Theorem 1.3, we need some preparations.

Lemma 2.1 *If $P_t \geqq \bar{P}_t$, then $Af(x) \geqq \bar{A}f(y)$ for all $x \geqq y$ and $f \in \mathscr{M} \cap C_b^\infty$ with $f(x) = f(y)$.*

Proof. Without loss of generality, assume that $f \geqq 0$. Choose $m > 0$ such that $\{z : |z| < m\}$ contains x and y and take $h \in C_b^\infty$ such that

$$1 \geqq h(z) = \begin{cases} 1, & \text{if } |z| \leqq m \\ 0, & \text{if } |z| \geqq m+1 \\ > 0, & \text{otherwise} . \end{cases}$$

Set

$$f_1 = hf + a(1-h), \quad f_2 = hf ,$$

where a is a constant larger than the upper bound of f. Then $f_1, f_2 \in C_0^\infty$, $f_1 \geqq f \geqq f_2$ and $f_1 = f_2 = f$ on the set $\{z : |z| < m\}$. Since

$$P_t f(x) \geqq \bar{P}_t f(y), \quad f(x) = f(y) ,$$

we have

$$\frac{1}{t} [P_t f_1(x) - f_1(x)] \geqq \frac{1}{t} [\bar{P}_t f_2(y) - f_2(y)], \quad t > 0 .$$

The assertion now follows from (2.1) by letting $t \downarrow 0$. □

Lemma 2.2 *If $P_t \geqq \bar{P}_t$, then* (1.3) *holds.*

Proof. For given i, let $u \leqq v$ with $u_i = v_i$. Choose $f \in \mathscr{M} \cap C_b^\infty$ such that in a neighborhood of $\{u, v\}$,

$$f(x) = x_i .$$

Then by Lemma 2.1 we get $b_i(v) \geqq \bar{b}_i(u)$. □

Lemma 2.3 *If $P_t \geqq \bar{P}_t$, then* (1.2) *holds.*

Proof. The proof consists of three steps.

(a) For given i, let $u \leqq v$ with $u_i = v_i$. Choose $f_m \in \mathscr{M} \cap C_b^\infty (m \in \mathbb{N})$ such that in a neighborhood of $\{u, v\}$,

$$f_m(x) = (x_i - u_i + 1)^{2m+1} .$$

By Lemma 2.1, we have

$$a_{ii}(v) - \bar{a}_{ii}(u) \geqq \frac{1}{m} [\bar{b}_i(u) - b_i(v)] ,$$

and so $a_{ii}(v) \geqq \bar{a}_{ii}(u)$ since m is arbitrary. Replacing f_m with $(x_i - u_i - 1)^{2m+1}$ in the neighborhood of $\{u, v\}$, we obtain the inverse inequality. Therefore $a_{ii}(v) = \bar{a}_{ii}(u)$.

(b) For given $i \neq j$ and $u \leqq v$: $u_i = v_i, u_j = v_j$, choose $f_m \in \mathscr{M} \cap C_b^\infty (m \in \mathbb{N})$ such that in a neighborhood of $\{u, v\}$,

$$f_m(x) = (x_i + x_j - u_i - u_j + 1)^{2m+1} .$$

By (a) and Lemma 2.1, we get

$$a_{ij}(v) - \bar{a}_{ij}(u) \geqq \frac{1}{2m} [\bar{b}_i(u) + \bar{b}_j(u) - b_i(v) - b_j(v)] ,$$

and so $a_{ij}(v) \geqq \bar{a}_{ij}(u)$. Similarly we have the inverse inequality and hence $a_{ij}(v) = \bar{a}_{ij}(u)$.

(c) By (a) and (b) we get (1.2) immediately. □

Lemma 2.4 *Suppose that* (1.2) *and* (1.3) *hold and $a_{ij}, b_i, \bar{b}_i$ are all bounded. If one of $\{P_t\}$ and $\{\bar{P}_t\}$ is monotone, then $P_t \geqq \bar{P}_t$.*

Proof. (a) Without loss of generality, assume that $\{\bar{P}_t\}$ is monotone. We can also assume that a_{ij}, b_i and $\bar{b}_i$ are smooth. Otherwise, choose $\varphi \in C_0^\infty(\mathbb{R}), \varphi \geqq 0$, $\int_{\mathbb{R}} \varphi = 1$ and define

$$\varphi_m(x) = m^n \prod_{k=1}^{n} \varphi(m x_k) ,$$

$$a_{ij}^m = a_{ij} * \varphi_m + \delta_{ij}/m, \quad b_i^m = b_i * \varphi_m, \quad \bar{b}_i^m = \bar{b}_i * \varphi_m .$$

Note that

$$b_i^m(x) \geqq \bar{b}_i^m(x) \geqq \bar{b}_i^m(y)$$

for all $x \geqq y$ with $x_i \neq y_i$. For the last inequality, we have used the monotonicity of $\bar{P}_t$ and Lemma 2.2. Thus, (1.3) holds for b_i^m and $\bar{b}_i^m$. Clearly, every (a_{ij}^m) satisfies (1.2). Since a_{ij}^m, b_i^m and $\bar{b}_i^m$ converge uniformly to a_{ij}, b_i and $\bar{b}_i$ respectively, by a convergence theorem ([4, Theorem 5.3] or [6, Theorem 11.1.4]), the proof of the lemma is reduced to the smooth case.

(b) By (1.2) and (1.3), we have

$$Af(x) \geqq \bar{A}f(x), \quad x \in \mathbb{R}^n$$

for all $f \in \mathcal{M} \cap C_b^\infty$. On the other hand, since the coefficients of the operators are assumed by (a) to be bounded and smooth, we have the integration by parts formula

$$P_t f(x) - \bar{P}_t f(x) = \int_0^t P_s(A - \bar{A})\bar{P}_{t-s} f(x) ds .$$

From this and the monotonicity of $\{\bar{P}_t\}$, it follows that

$$P_t f(x) \geqq \bar{P}_t f(x), \quad x \in \mathbb{R}^n .$$

Hence

$$P_t f(x) \geqq \bar{P}_t f(x) \geqq \bar{P}_t f(y) ,$$

for all $f \in \mathcal{M} \cap C_b^\infty$ and $x \geqq y$. □

Lemma 2.5 *Suppose that* (1.2) *and* (1.3) *hold and $a_{ij}, b_i, \bar{b}_i$ are all bounded. If in addition, (a_{ij}) is uniformly positive definite and b_i is uniformly continuous for each i. Then $P_t \geqq \bar{P}_t$.*

Proof. Set

$$\tilde{b}_i(x) = \sup_{y: y \leqq x, y_i = x_i} \bar{b}_i(y), \quad x \in \mathbb{R}^n .$$

$$\tilde{A} = \frac{1}{2} \sum_{i,j=1}^n a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n \tilde{b}_i \frac{\partial}{\partial x_i} .$$

By (1.3) we have

$$b_i(x) \geqq \tilde{b}_i(x) \geqq \bar{b}_i(x), \quad x \in \mathbb{R}^n . \tag{2.2}$$

By the assumption, $\tilde{b}_i$ is still uniformly continuous, the martingale problem for $\tilde{A}$ is well posed (see [5]). On the other hand, it is easy to check that, for all $i \neq k$, $\tilde{b}_i(x)$ is an increasing function of x_k. $\tilde{A}$ generates the semigroup $\{\tilde{P}_t\}$, which is monotone by part (1) of Theorem 1.2. Furthermore, by (2.2) and Lemma 2.4 we have

$$P_t \geqq \tilde{P}_t \quad \text{and} \quad \tilde{P}_t \geqq \bar{P}_t .$$

The proof is completed. □

Proof of Theorem 1.3 By Lemmas 2.2 and 2.3, we need only to prove the sufficiency. For each $m \in \mathbb{N}$, take $\eta_m: \mathbb{R}^n \to \mathbb{R}^n$ such that

$$(\eta_m(x))_i = \begin{cases} x_i, & \text{if } |x_i| \leqq m \\ -m, & \text{if } x_i < -m \\ m, & \text{if } x_i > m \ , \end{cases}$$

and choose $h \in C(\mathbb{R})$: $h(r) = (1 - |r|)^+$. Set

$$a_{ij}^m(x) = h(x_i/m) h(x_j/m) a_{ij}(x) + \delta_{ij}/m ,$$

$$b_i^m(x) = b_i(\eta_m(x)), \quad \bar{b}_i^m(x) = \bar{b}_i(\eta_m(x)) ,$$

$$A_m = \frac{1}{2} \sum_{i,j=1}^n a_{ij}^m \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i^m \frac{\partial}{\partial x_i}$$

$$\bar{A}_m = \frac{1}{2} \sum_{i,j=1}^n a_{ij}^m \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n \bar{b}_i^m \frac{\partial}{\partial x_i} .$$

Then

$$\lim_{m\to\infty}\left\{\sum_{i,j=1}^{n}|a_{ij}(x)-a_{ij}^m(x)|+\sum_{i=1}^{n}(|b_i(x)-b_i^m(x)|+|\bar{b}_i(x)-\bar{b}_i^m(x)|)\right\}=0$$

locally uniformly in x. Since A_m, $\bar{A}_m$ satisfy the conditions of Lemma 2.5, we have $P_t^m \geqq \bar{P}_t^m$. Finally, the assertion follows from a convergence theorem mentioned in the proof (a) of Lemma 2.4. □

3 Proof of Theorem 1.4

Because of part (2) of Theorem 1.2, we need only to study the necessity of the monotonicity for preservation of positive correlations. The intuitive idea of the proof is as follows: Due to the fact that the operator consists of two parts: diffusion and drift, by using some probability measure supported at two points, we are able to compare these two parts respectively.

Lemma 3.1 *For $x \leqq y$, let $\mu \in \mathscr{P}$: $\mu(\{x\}) = \mu(\{y\}) = \frac{1}{2}$. Then $\mu \in \mathscr{P}_+$.*

Proof. Simply note that $\{x, y\}$ is a totally ordered space. □

Lemma 3.2 *Suppose that $\{P_t\}$ preserves positive correlations and let μ be the same as in the previous lemma. Then for all $f, g \in \mathscr{M} \cap C_b^\infty$ with*

$$\mu(fg) = \mu(f)\mu(g)\,,$$

we have

$$\begin{aligned}&2[A(fg)(x)+A(fg)(y)]\\&\quad\geqq (f(x)+f(y))[Ag(x)+Ag(y)]+(g(x)+g(y))[Af(x)+Af(y)]\,.\end{aligned}$$

Equivalently,

$$\begin{aligned}&2[\Gamma_1(f,g)(x)+\Gamma_1(f,g)(y)]\\&\quad\geqq (f(x)-f(y))[Ag(y)-Ag(x)]+(g(x)-g(y))[Af(y)-Af(x)]\,,\end{aligned}$$

where

$$\begin{aligned}\Gamma_1(f,g)(x)&=A(fg)(x)-f(x)Ag(x)-g(x)Af(x)\\&=\sum_{i,j}a_{ij}(x)\frac{\partial}{\partial x_i}f(x)\frac{\partial}{\partial x_j}g(x)\,.\end{aligned}$$

Proof. A simple computation shows the equivalence of the two inequalities given above. Thus, to prove the first assertion, it suffices to consider $f, g \geqq 0$ only. Since $\{P_t\}$ preserves positive correlations, we have

$$\mu(P_t(fg)) \geqq \mu(P_t f)\mu(P_t g), \quad t \geqq 0\,. \tag{3.1}$$

Define f_1 and f_2 as in the proof of Lemma 2.1. The only difference from there is that in the present situation, the constant a is chosen to be larger than the upper bound of $f \vee g$. Replacing f with g, we can define g_1 and g_2. Because $f_1, f_2, g_1, g_2 \in C_0^\infty$, $f_1 \geqq f \geqq f_2$ and $g_1 \geqq g \geqq g_2$, by (3.1), we have

$$\mu(P_t(f_1 g_1)) \geqq \mu(P_t f_2)\mu(P_t g_2), \quad t \geqq 0 .$$

On the other hand, $f_1 = f = f_2$, $g_1 = g = g_2$ on the set $\{|z| < m\}$. Hence

$$\frac{2}{t}\{P_t(f_1 g_1)(x) + P_t(f_1 g_1)(y) - (fg)(x) - (fg)(y)\}$$

$$\geqq \frac{1}{t}\{(P_t f_2(x) + P_t f_2(y))(P_t g_2(x) + P_t g_2(y))$$

$$-(f(x) + f(y))(g(x) + g(y))\}, \quad t > 0 .$$

Now, the required assertion follows from (2.1) by setting $t \downarrow 0$. □

Lemma 3.3 *If $\{P_t\}$ preserves positive correlations, then $a_{ij} \geqq 0$ for all i, j and moreover for all $k \neq i$, $b_i(x)$ is an increasing function of x_k.*

Proof. (a) Let $x = y$, then Lemma 2.1 gives us

$$\Gamma_1(f, g)(x) \geqq 0, \quad f, g \in \mathscr{M} \cap C_b^\infty .$$

Hence, for given i and j, we get $a_{ij} \geqq 0$ by choosing $f, g \in \mathscr{M} \cap C_b^\infty$ such that in a neighborhood of x,

$$f(z) = z_i, \quad g(z) = z_j .$$

(b) For given i and $k \neq i$, let $u \leqq v \in \mathbb{R}^n$, $u_k < v_k$, $u_j = v_j$ for $j \neq k$ and $\mu \in \mathscr{P}$ such that $\mu(\{u\}) = \mu(\{v\}) = \frac{1}{2}$. Choose an increasing function $h \in C^\infty(\mathbb{R})$ such that

$$h(r) = \begin{cases} 0, & \text{if } r \leqq u_k + (v_k - u_k)/3 \\ 1, & \text{if } r \geqq u_k + (v_k - u_k)/2 . \end{cases}$$

Take $f, g \in \mathscr{M} \cap C_b^\infty$ such that in a neighborhood of $\{u, v\}$,

$$f(x) = x_i, \quad g(x) = h(x_k) .$$

As an application of Lemma 3.2, we obtain

$$2b_i(v) \geqq b_i(u) + b_i(v) .$$

Hence $b_i(v) \geqq b_i(u)$. □

Lemma 3.4 *If $\{P_t\}$ preserves positive correlations, then for all i and j, $a_{ij}(x)$ depends only on x_i and x_j.*

Proof. (a) For given $i \neq k$, let $u \leqq v$, $u_k < v_k$, $u_i = v_i$ and let μ be the same as above. Choose $f_m, g \in \mathscr{M} \cap C_b^\infty (m \in \mathbb{N})$ such that in a neighborhood of $\{u, v\}$,

$$f_m(x) = (x_i - u_i + 1)^{2m+1}, \quad g(x) = (x_k - u_k)^3 .$$

By Lemma 3.2 we have

$$a_{ii}(v) - a_{ii}(u) \geqq \frac{1}{m}\left[b_i(u) - b_i(v) - \frac{6a_{ik}(v)}{(v_k - u_k)} \right].$$

Letting $m \uparrow \infty$, we get

$$a_{ii}(v) - a_{ii}(u) \geqq 0 .$$

Similarly, replacing f_m with $(x_i - u_i - 1)^{2m+1}$ near u and v, we obtain the inverse inequality. Therefore $a_{ii}(x)$ depends only on x_i.

(b) For given $i \neq j$, $k \neq i,j$, let $u \leqq v, u_k < v_k, u_i = v_i, u_j = v_j$, take $\mu(\{u\}) = \mu(\{v\}) = \frac{1}{2}$. Choose $f_m, g \in \mathscr{M} \cap C_b^\infty(m \in \mathbb{N})$ such that in a neighborhood of $\{u, v\}$,

$$f_m(x) = (x_i + x_j - u_i - u_j + 1)^{2m+1}, \quad g(x) = (x_k - u_k)^3 .$$

By Lemma 3.2 and (a) we have

$$a_{ij}(v) - a_{ij}(u) \geqq \frac{1}{2m}\left[b_i(u) + b_j(u) - b_i(v) - b_j(v) - \frac{6(a_{ik}(v) + a_{jk}(v))}{v_k - u_k} \right].$$

Letting $m \uparrow \infty$, we get

$$a_{ij}(v) \geqq a_{ij}(u) .$$

Similarly, we can prove the inverse inequality. Therefore $a_{ij}(u) = a_{ij}(v)$. □

Proof of Theorem 1.4 The sufficiency is given by part (2) of Theorem 1.2. The necessity follows from Lemmas 3.3, 3.4 and Theorem 1.3. □

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