in “Probability Theory and its Applications in China”
edited by S. J. Yan, C. C. Yang and J. G. Wang
Providence, AMS. 118(1991), 23-57.

[1]

JUMP PROCESSES AND PARTICLE SYSTEMS

Mu-Fa Chen and Shi-Jian Yan

Department of Mathematics
Beijing Normal University
Beijing, 100875

Abstract. This article surveys some progress in China on jump procrsses and interacting particle systems, especially those results obtained by our group. It contains three parts as showed below in the content. The first part deals with the jump processes, a quite classical subject. The constributions are both on some classical problems (uniqueness and positive recurrence) and on some fashinable topics: couplings, monotonicity and large deviations. The second part studies the reversibility of some type of particle systems and the phase transitions for some peticule models. The last part introduces most recent works on reaction- diffusion processes, including the construction of the Markov processes, the ergodicity and also the phase transition phenomenon.

[1] 1980 Mathematics Subject Classification (1985 Revision). 60J25, 60K35

The first author was supported in part by Ying-Tung Fok Education Fundation and Natural Science Fundation of China and the second author was supported in part by Natural Science Fundation of China

Content

Introduction

This article presents a survey of our investigations in the past eleven years with the emphasis falling on the two espects expressed by the title and their interactions. However, the survey may be incomplete.

The theory of jump processes, especially Markov chains has a long history of development in the world and some achievements have been obtained in this field in China. Some of our works on Markov chains are alrealy included in this volume. Hence we consider only some special topics here. The uniqueness problem for jump processes is quite classical, we introduce a very general criterion and some practical sufficient conditions. Then, we go to some more fashinable topics: couplings, monotonicity and large deviations. Clearly, these topics are stimulated by the study of particle systems. As usual, the state spaces $\boldsymbol{Z}^d$ and $\boldsymbol{R}^d$ are enough in practice. But we prefer to use the general state space $(E, \mathscr{E})$ without topology. One reason to do so is to simplify our representation and, for another reason, such type of processes does not depend on the topology closely. It is believed that the results of Part I should have wide applications. For instance, they have been used to study the controlled Markov processes which are not involved in the article.

Ten years ago, we peered into a window of the large building, Interacting Particle Systems, by studying the reversibility for some particle systems. We found some very simple criteria for the reversibility, which is due to our experiences on the study of reversible Markov chains and consists of the first two sections of Part II. The idea was then used to study some more general or particular models, quasi-nearest particle systems, for example. The last model, even irreducible, does eshibit an interesting phase transition phenomenon. The last half of Part II reflects the strong influences our colleaques from the probabitists in the United States.

One of our main motive powers to study the particle system comes from the non-equilibrium statistical physics. There are a lot of models, attractive enough but quite difficult to handle. They are now named reaction diffusion processes. The essential progress was not made until 1983 even though we were interested in this area much earlier. A glance at the content of Part III will tell one what we have done. However, many important problems are still unsolved. This is quite natural for a developing subject.

Most of the authors mentioned in the article have worked with us in our unirersity for a period, even for years. In the process of writting the article, we constantly recalled the nice time we had spent together. We do appreciate very much their cooperations.

Notations

$\boldsymbol{N} = \{1, 2, \cdots\}$	$\mathscr{E}_+ = \{f \in \mathscr{E} : \ f \geq 0\}$
$\boldsymbol{Z}_+ = \boldsymbol{N} \cup \{0\}$	$_b\mathscr{E} = \{f \in \mathscr{E} : \|f\| = \sup_{x\in E} \|f(x)\| < \infty\}$
$\boldsymbol{Z} = \boldsymbol{Z}_+ \cup (-\boldsymbol{Z}_+)$,	$\delta(\cdot, A) = I_A =$ the indicator function of A,
$\boldsymbol{R} =$ the real line	S, a finite or countable set
$(E, \mathscr{E})$, a measurable space	$\mathscr{S} =$ the non-empty finite subsets of S
$\mathscr{P}(E) =$ the set of probabilities on $(E, \mathscr{E})$	$\|S\| =$ the cardinate of S
$\mathscr{E} =$ the set of real $\mathscr{E}$-measurable functions	

Part I. Jump Processes

(1) **Definitions**

Let $(E, \mathscr{E})$ be a measurable space such that $\{(x, x) : \ x \in E\} \in \mathscr{E} \times \mathscr{E}$ and $\{x\} \in \mathscr{E}$ for all $x \in E$. It is well-known that for a given sub-Markovian transition function $P(t, x, A)(t \geq 0, x \in E, A \in \mathscr{E})$, if it does satisfy the jump condition

$$\lim_{t\to 0} P(t, x, \{x\}) = 1, \qquad x \in E \tag{1.1}$$

then the limits

$$q(x) \equiv \lim_{t\to 0}[1 - P(t, x, \{x\})]/t \tag{1.2}$$

and

$$q(x, A) \equiv \lim_{t\to 0} P(t, x, A \setminus \{x\})/t \tag{1.3}$$

exist for all $x \in E$ and $A \in \mathscr{R}$, where

$$\mathscr{R} = \{A \in \mathscr{E} : \lim_{t\to 0} \sup_{x\in A} \big(1 - P(t, x, \{x\})\big) = 0\}.$$

Moreover, $q(\cdot) \in \mathscr{E}$ and $q(x, A)$ is a kernel on $(E, \mathscr{E})$ and $0 \leq q(x, A) \leq q(x) \leq \infty$ for all $x \in E$ and $A \in \mathscr{R}$. The pair $(q(x), q(x, A))(x \in E, A \in \mathscr{R})$ is called a q-pair. Throughout this paper, we restrict ourselves to the totally stable case which means that $q(x) < \infty$ for all $x \in E$. Then $q(x, \cdot)$ can be uniquely extended to the whole space $\mathscr{E}$ as a finite measure. Thus, in what follows, we assume that $q(x, A)$ is a kernel on $(E, \mathscr{E})$. Next, the q-pair $\big(q(x), q(x, A)\big)$ is called conservative if $q(x, E) = q(x)$ for all $x \in E$. Because of the above facts, we often call the sub-Markovian transition $P(t, x, A)$ satisfying (1.1) a jump process or a q-process.

A typical case is that E is countable. In this case, conventionally we use the matrices $Q = (q_{ij} : i, j \in E)$ and $P(t) = (p_{ij}(t) : i, j \in E)$ instead of the q-pair and the jump process respectirely. Here $q_{ij} = -q_i, \ i \in E$. We also call $P(t) = (p_{ij}(t))$ a Markov chain or a Q-process.

(2) **Existence**

According to the last section, we have obtained a q-pair from a jump process. However, the study of jump processes is mainly in the converse direction. That is, what can we say about $P(t, x, A)$ from a given q-pair $\big(q(x), q(x, A)\big)$? Certainly, the starting point is only that

$$\lim_{t\to 0} \frac{P(t, x, A) - \delta(x, A)}{t} = q(x, A) - q(x)\delta(x, A), \quad x \in E, A \in \mathscr{R}. \tag{2.1}$$

Along this direction, the first result is due to W.Feller

(2.2) **Theorem** (Feller (1940)). For a given q-pair $\big(q(x), q(x, A)\big)(x \in E, A \in \mathscr{E})$, there exists a minimal jump process $P^{min}(t, x, A)$. More precisely, set

*

$$P^{(0)}(t, x, A) = 0,$$
$$P^{(n+1)}(t, x, A) = \int_0^t e^{-q(x)(t-s)} ds \int q(x, dy) P^{(n)}(s, y, A) + \delta(x, A) e^{-q(x)t},$$
$$t \geq 0, x \in E, A \in \mathscr{E}, n \geq 0.$$

Then for each t, x and A,

$$P^{(n)}(t, x, A) \uparrow P^{min}(t, x, A) \qquad \text{as} \qquad n \to \infty$$

Furthermore, $P^{min}(t, x, A)$ is a jump process and satisfies (2.1). Finally, for every jump process $P(t, x, A)$ satisfying (2.1) we have

$$P(t, x, A) \geq P^{min}(t, x, A)$$

for all t, x and A.

Since there is one-to-one correspondence between a jump process P(t, x, A) and its Laplace transform (cf. Hu (1966))

$$P(\lambda, x, A) = \int_0^\infty e^{-\lambda t} P(t, x, A) dt, \quad \lambda > 0, x \in E, \quad A \in \mathscr{E},$$

we also call $P(\lambda, x, A)$ a jump process without any confusion. By using the Laplace transform, the minimal (or Feller's) jump process has an alternative construction:

$$P^{(0)}(\lambda, x, A) = 0,$$
$$P^{(n+1)}(\lambda, x, A) = \int \frac{q(x, dy)}{\lambda + q(x)} P^{(n)}(\lambda, y, A) + \frac{\delta(x, A)}{\lambda + q(x)}, \ \lambda > 0, x \in E, \ A \in \mathscr{E}, n \geq 0,$$
$$P^{(n)}(\lambda, x, A) \uparrow P^{min}(\lambda, x, A) = \int_0^\infty e^{-\lambda t} P^{min}(t, x, A) dt, \ \text{as } n \to \infty, \ \lambda > 0, x \in E, A \in \mathscr{E}.$$

(3) **Uniqueness**

Now, we turn to the uniqueness problem for jump processes. Set $\dot{\mathrm{E}} = \{x \in E : \ q(x, E) < q(x)\}$, which is the set of all non-conservative points. Put

$\mathscr{U}_\lambda = \{f : \ (\lambda + q(x)) f(x) = \int q(x, dy) f(y) \text{ for all } x \in E\}$,

$\mathscr{V}_\lambda = \{\nu : \nu$ is a non-negative $\sigma-$finite measure on $\mathscr{E}$ such that

$\nu(A) = \int \nu(dx) \int_A q(x, dy)/(\lambda + q(y))$ for all $A \in \mathscr{E}\}$.

(3.1) **Theorem** (Chen and Zheng (1983)). For a given q-pair (q(x), q(x, A)), there exists precisely one jump process iff the following conditions all hold for some (equivalently, for all) $\lambda > 0$:

i) $\inf_{x \in \dot{E}} P^{min}(\lambda, x, E) > 0$,

ii) $\mathscr{U}_\lambda = \{0\}$,

iii) the q-pair (q(x), q(x, A)) is conservative, or although it is not conservative still $\mathscr{V}_\lambda = \{0\}$.

As a simple consequence, we have

(3.2) **Corollary.** If $\sup_{x \in E} q(x) < \infty$, then the jump process is unique.

Let us consider a particular case. Take $E = \boldsymbol{Z}_+$. A Q-matrix $Q = (q_{ij})$ is called a conservative single birth Q-matrix if $q_{ij} = 0$ unless $j \leq i + 1$, $q_{i,i+1} > 0$ and $-q_{ii} = q_i = \sum_{j \neq i} q_{ij}$ for all $i \in \boldsymbol{Z}_+$. Let

$$q_k^{(i)} = \sum_{j=0}^{i} q_{ij}, \qquad 0 \le i < k, \ k \in \boldsymbol{Z}_+$$

$$F_k^{(k)} = 1, \ F_k^{(i)} = \sum_{j=i}^{k-1} q_k^{(i)} F_j^{(i)} / q_{k,k+1}, \qquad 0 \le i < k, \ h \in \boldsymbol{Z}_+$$

and

$$m_k = \sum_{i=0}^{k} F_k^{(i)} / q_{i,i+1}, \qquad k \in \boldsymbol{Z}_+.$$

(3.3) **Corollary** (Yan and Chen (1986)). For a given conservative single birth Q-matrix $Q = (q_{ij})$, the Q-process is unique iff $\sum_{k=0}^{\infty} m_k = \infty$.

The following result is specially useful when $E = \boldsymbol{Z}_+^d (d \ge 2)$. In that case we simply take $E_k = \{(x^1, \cdots, x^d) \in \boldsymbol{Z}_+^d : \ x^1 + \cdots + x^d = k\}$ as the partition stated below.

(3.4) **Corollary** (Yan and Chen (1986)). Let E be a countable set and $Q = (q(x,y) : x, y \in E)$ be a conservative Q-matrix. Suppose that there is a countable partition $\{E_k\}_0^\infty$ of E such that sup $\{q(x) : x \in E_k\} < \infty$, and that for every $x \in E_k, \sum_{y \in E_{k+1}} q(x,y) > 0$ and $q(x,y) > 0$ only if $y \in \sum_{l=0}^{k+1} E_l$. Introduce a conservative single birth Q-matrix $Q = (q_{ij})$ as follows:
*

$$q_{ij} = \begin{cases} \sup\{\sum_{y \in E_j} q(x,y) : \ x \in E_i\}, & \text{if} \quad j = i+1 \\ \inf\{\sum_{y \in E_j} q(x,y) : \ x \in E_i\}, & \text{if} \quad j < i \\ 0, & \text{if } j \ge i+2. \end{cases}$$

If the $(q_{ij})-$process is unique then so is the $(q(x,y))-$process.

In the paper quoted above, the Corollary (3.4) is applied to the following models.

(3.5) **Autocalalytic reaction model.**

The state space is $E = \boldsymbol{Z}_+^S$, where S is a finite set. The Q-matrix is as follows:

$$q(x,y) = \begin{cases} \lambda_1 x(u), & \text{if} \quad y = x + e_u \\ \lambda_2 \binom{x(u)}{2} = \lambda_2 x(u)(x(u)-1)/2, & \text{if } y = x - 2e_u \\ x(u)p(u,v), & \text{if } y = x - e_u + e_v \text{ and } u \ne v \\ 0 & \text{the other cases of } y \ne x, \quad x, y \in E, \ u, v \in S \end{cases}$$

$$q(x) = \sum_{y \ne x} q(x,y), \ x \in E,$$

where $e_u \in E$ has $e_u(u) = 1$ and $e_u(v) = 0$ for $v \ne u$, λ_1 and λ_2 are positive constant and $(p(u,v) : u, v \in S)$ is a transition probability matrix on S. The similar notations will be used below.

(3.6) **Schlőgl's second model.**

$E = \boldsymbol{Z}_+^S, \ x = (x(u) : u \in S) \in E.$

$$q(x,y) = \begin{cases} \lambda_1 \binom{x(u)}{2} + \lambda_4, & \text{if} \quad y = x + e_u \\ \lambda_2 \binom{x(u)}{3} + \lambda_3 x(u), & \text{if } y = x - e_u \\ x(u)p(u,v), & \text{if } y = x - e_u + e_v \text{ and } u \ne v \\ 0 & \text{the other cases of } y \ne x, \end{cases}$$

$$q(x) = \sum_{y \ne x} q(x,y)$$

(3.7) **Lotka-Volterra model.**

$E = (\boldsymbol{Z}_+^2)^S. \qquad x = ((x_1(u), x_2(u)) : \ u \in S) \in E.$

$$q(x,y) = \begin{cases} \lambda_1 x_1(u), & \text{if} \quad y = x + e_{u1} \\ \lambda_3 x_2(u), & \text{if} \quad y = x - e_{u2} \\ \lambda_2 x_1(u) x_2(u), & \text{if} y = x - e_{u1} + e_{u2} \\ x_i(u) p_i(u,v), & \text{if } y = x - e_{ui} + e_{vi}, \ i = 1,2, \ u \ne v \\ 0, \quad \text{the other cases of } y \ne x \end{cases}$$

$$q(x) = \sum_{y \ne x} q(x,y)$$

where $e_{ui}(v,j) = 1$ if v=u and j=i, =0 otherwise.

(3.8) **Brusseltor model.**

$E = (\boldsymbol{Z}_+^2)^S, \ x = ((x_1(u), x_2(u)) : u \in S).$

$$q(x,y) = \begin{cases} \lambda_1, & \text{if} \quad y = x + e_{u1} \\ \lambda_4 x_1(u), & \text{if} \quad y = x - e_{u1} \\ \lambda_2 x_1(u), & \text{if} \quad y = x - e_{u1} + e_{u2} \\ \lambda_3 \binom{x_1(u)}{2} x_2(u), & \text{if} \quad y = x - e_{u2} + e_{u1} \\ x_i(u) p_i(u,v), & \text{if } y = x - e_{ui} + e_{vi}, \ i = 1,2, \ u \ne v \\ 0, \quad \text{the other cases of } y \ne x \end{cases}$$

$$q(x) = \sum_{y \ne x} q(x,y)$$

All the above Q-matrices are conservative. In this case, it is obvious that the criterion (3.1) is reduced to $\mathscr{U}_\lambda = \{0\}$. However, if $| \ S \ | \ge 2$, then it is hard to check directly the condition that $\mathscr{U}_\lambda = \{0\}$. The key point of Corollary (3.4) is to reduce our problem to the single birth case for which we have had a complete solution. (Corollary (3.3)).

Though we have just seen that the Corollary (3.4) is effective, yet it has no use in the case that $E = \boldsymbol{R}^d$. This leads us to search some more general and still practical conditions.

(3.9) **Theorem** (Chen (1986a)). Let $(q(x), q(x,A))$ be a conservative q-pair. Suppose that there exist a sequence $\{E_n\}_1^\infty \subset \mathscr{E}$ and a $\varphi \in \mathscr{E}_+$ such that

$$E_n \uparrow E, \ \sup_{x \in E_n} q(x) < \infty, \ \lim_{n \to \infty} \inf_{x \notin E_n} \varphi(x) = \infty$$

and

$$\int q(x,dy)\varphi(y) \le (c + q(x))\varphi(x), \quad x \in E \tag{3.10}$$

for a constant $c \in \boldsymbol{R}$. Then the jump process is unique. Moreover, if $E = \boldsymbol{Z}_+$ and the given q-pair (i.e., Q-matrix) is a conservative single birth Q-matrix, then the above condition is also necessary for the uniqueness of jump processes.

(3.11) **Corollary.** Let $(q(x), q(x,A))$ be a conservative q-pair. Suppose that there exist a $\varphi \in \mathscr{E}_+$: $\varphi(x) \ge q(x)$, $x \in E$ and a constant $c \in \boldsymbol{R}$ such that (3.10) holds. Then the jump process is unique.

q(x)

Of course, these results are applicable to the above models. To see this, let us consider example (3.6) for instance. To use Theorem (3.9), we take $\varphi(x) = c[1 + \sum_{u \in S} x(u)]$. for some $c > 0$. But to apply Corollary (3.11), we take $\varphi(x) = c[1 + (\sum_{u \in S} x(u))^3]$.

(4) **Stationary distributions.**

In this section, we suppose that $(E, \rho, \mathscr{E})$ is a complete separable metric space with metric ρ and topological $\sigma-$field $\mathscr{E}$.

(4.1) **Definition.** A function $h \in \mathscr{E}_+$ is called compact if for each $0 \le c < \infty$, the set $\{x \in E : h(x) \le c\}$ is compact.
(4.2) **Definition.** A q-pair is said to be reqular if it is conservative and it determines uniquely a jump process.
(4.3) **Theorem** (Chen (1986b)). Let $(q(x), q(x, A))$ be a regular q-pair. Denote by $P(t, x, A)$ the corresponding jump process. Suppose that for every bounded Lipschitz continuous function $f, x \to P(t)f(x) = \int P(t, x, dy)f(y)$ is continous and that there exists a compact function h and constants $K \in [0, \infty), \eta \in (0, \infty)$ such that

$$\begin{aligned}
&\rho(x,\theta) \le h(x), \ x \in E \\
&\int q(x, dy)(h(y) - h(x)) \le K - \eta h(x), \ x \in E
\end{aligned}$$

where θ is an arbitrary reference point in E. Then there exists at least one stationary distribution π for the process:

$$\pi = \pi P(t).$$

Moreover, for each stationary distribution π of $P(t, x, A)$, we have *

$$\int \pi(dx)h(x) \le K/\eta.$$

Applying the above result to the model (3.6), we take

$$\rho(x, y) = \sum_{u \in S} |x(u) - y(u)| \ \text{ and } \ h(x) = \rho(x, \theta),$$

where $\theta = (\theta_u : \theta_u = 0 \text{ for all } u \in S)$.

For Markov chains, the above problems are also discussed in * Chen (1991).

(5) **Couplings**

Suppose that we are given two jump processes $P_k(t, x_k, A_k)$ with regular q-pair $(q_k(x), q_k(x_k, A_k))$ on state space $(E_k, \mathscr{E}_k), k = 1, 2$ respectively. We want to find some coupling jump process $\widetilde{P}(t; x_1, x_2; dy_1, dy_2)$ with q-pair $(q(x_1, x_2),\ q(x_1, x_2; dy_1, dy_2))$ on the product state space $(E_1 \times E_2, \mathscr{E}_1 \times \mathscr{E}_2)$ having the marginality:

$$\begin{aligned}
&\widetilde{P}(t; x_1, x_2; A_1 \times E_2) = P_1(t, x_1, A_1) \\
&\widetilde{P}(t; x_1, x_2; E_1 \times A_2) = P_2(t, x_2, A_2), \ t \ge 0, x_k \in E_k, A_k \in \mathscr{E}_k, \ k = 1, 2.
\end{aligned}$$

Define

$$\Omega_1 f(x_1) = \int q_1(x_1, dy_1)(f(y_1) - f(x_1)), \quad f \in {}_b\mathscr{E}_1$$

Similarly, we can define Ω_2 and $\widetilde{\Omega}$. Regarding $f \in {}_b\mathscr{E}_1$ (resp. $f \in {}_b\mathscr{E}_2$) as a function in ${}_b(\mathscr{E}_1 \times \mathscr{E}_2)$, it is not difficult to prove that the condition (5.1) implies

$$\begin{aligned}
&\widetilde{\Omega} f(x_1, x_2) = \Omega_1 f(x_1), \quad f \in {}_b\mathscr{E}_1 \\
&\widetilde{\Omega} f(x_1, x_2) = \Omega_2 f(x_2), \quad f \in {}_b\mathscr{E}_2, x_k \in E_k, k = 1, 2.
\end{aligned} \tag{5.2}$$

Conversely, if $\widetilde{\Omega}$ is regular, then (5.2) implies (5.1). Any $\widetilde{\Omega}$ satisfying (5.2) is called a coupling operator.

Before going further, let us give some examples of coupling operators. *

(5.3) **Independent coupling**

$$\widetilde{\Omega}_0 f = \Omega_1 f + \Omega_2 f, \quad f \in {}_b(\mathscr{E}_1 \times \mathscr{E}_2)$$

This trivial example already shows that a coupling operator always exists.

In the following examples, we assume that $E_1 = E_2 = E, \mathscr{E}_1 = \mathscr{E}_2 = \mathscr{E}$ and write $(E^2, \mathscr{E}^2) = (E \times E, \mathscr{E} \times \mathscr{E})$.

(5.4) **Classical coupling.** Let the two marginal q-pairs be the same $(q(x), q(x, A))$. Set

$$\widetilde{\Omega}_c f(x_1, x_2) = I_{\triangle^c}(x_1, x_2)\widetilde{\Omega}_0 f(x_1, x_2) + I_{\triangle}(x_1, x_2)\Omega g(x_1),$$
$$x_1, x_2 \in E, \ f \in b\mathscr{E}^2,$$

where $\triangle = \{(x_1, x_2) \in E^2 : x_1 = x_2\}, \ g(x) = f(x, x)$ and

$$\Omega g(x) = \int q(x, dy)(g(y) - g(x))$$

as defined above.

(5.5) **Basic coupling.**

$$\begin{aligned}\widetilde{\Omega}_b f(x_1, x_2) = &\int (q_1(x_1, \cdot) - q_2(x_2, \cdot))^+(dy)(f(y, x_2) - f(x_1, x_2))\\ &+ \int (q_2(x_2, \cdot) - q_1(x_1, \cdot))^+(dy)(f(x_1, y) - f(x_1, x_2))\\ &+ \int (q_1(x_1, \cdot) \wedge q_2(x_2, \cdot))(dy)(f(y, y) - f(x_1, x_2)), \quad x_1, x_2 \in E, \ f \in {}_b\mathscr{E}^2,\end{aligned}$$

where for two measures μ_1 and $\mu_2, (\mu_1 - \mu_2)^{\pm}$ are the Jordan-Hahn decomposition of $\mu_1 - \mu_2$ and $\mu_1 \wedge \mu_2 = \mu_1 - (\mu_1 - \mu_2)^+$.

(5.6) **March coupling.** Let E be an additive group. Define

$$\begin{aligned}\widetilde{\Omega}_m f(x_1, x_2) = &\int (q_1(x_1, x_1 + \cdot) - q_2(x_2, x_2 + \cdot))^+(dy)(f(x_1 + y, x_2) - f(x_1, x_2))\\ &+ \int (q_2(x_2, x_2 + \cdot) - q_1(x_1, x_1 + \cdot))^+(dy)(f(x_1, x_2 + y) - f(x_1, x_2))\\ &+ \int (q_1(x_1, x_1 + \cdot) \wedge q_2(x_2, x_2 + \cdot))(dy)(f(x_1 + y, x_2 + y) - f(x_1, x_2)),\\ &x_1, x_2 \in E, \ f \in {}_b\mathscr{E}^2.\end{aligned}$$

This coupling is also meaningful even E is only a subset of an additive group. * Let us now consider a birth-death process with regular Q-matrix:

$q_{i,i+1} = b_i, \quad i \geq 0$
$q_{i,i-1} = a_i, \quad i \geq 1.$

Then for the two copies of the process starting from different points, we have

(5.7) **Inner reflection coupling.** For $i_1 \leq i_2$, we take

$$\begin{aligned}\widetilde{\Omega}_r f(i_1, i_2) = &I_{\{i_2 - i_1 \leq 1\}}\widetilde{\Omega}_c f(i_1, i_2)\\ &+ I_{\{i_2 - i_1 \geq 2\}}\Big[(b_{i_1} \wedge a_{i_2})(f(i_1 + 1, i_2 - 1) - f(i_1, i_2))\\ &+ (b_{i_1} - a_{i_2})^+(f(i_1 + 1, i_2) - f(i_1, i_2))\\ &+ (b_{i_2} - a_{i_1})^+(f(i_1, i_2 - 1) - f(i_1, i_2))\\ &+ a_{i_1}(f(i_1 - 1, i_2) - f(i_1, i_2)) + b_{i_2}(f(i_1, i_2 + 1) - f(i_1, i_2))\Big]\end{aligned}$$

By exchanging i_1 and i_2, we can get the expression of $\widetilde{\Omega}_r$ for the case that $i_1 \geq i_2$.

* One* reason* we* introduce so many* examples* of* coupling * operator is for the later use. For the other reason, we have shown that there are many choices of the coupling operator $\widetilde{\Omega}$. Indeed, there are infinite many choices! For instance, for every $\Gamma \in \mathscr{E}^2$,

$$\widetilde{\Omega} f(x_1, x_2) = I_\Gamma(x_1, x_2)\widetilde{\Omega}_c f(x_1, x_2) + I_{\Gamma^c}(x_1, x_2)\widetilde{\Omega}_b f(x_1, x_2)$$

is a coupling operator. Now, in order to use the coupling technique, we should study the regularity of coupling operators.

(5.8) **Theorem (Chen (1986a)).** If the given two marginal q-pairs are regular, then any coupling q-pair (resp. operator) is regular. Conversely, if a coupling q-pair is regular then so are its two marginals.

In what follows, we will meet several times the applications of coupling method. Let us now mention a typical application here. Let $\widetilde{X}_t = (X_t^1, X_t^2)$ $(t \geq 0)$ be the path of a coupling jump process, set

$$T = \inf\{t \geq 0 : \ X_t^1 = X_t^2\}.$$

A coupling is called successful if

$$\boldsymbol{P}^{x_1, x_2}[T < \infty] = 1, \ x_1 \neq x_2$$

and

$$\boldsymbol{P}^{x_1, x_2}[X_t^1 = X_t^2 \text{ for \ all } t \geq T] = 1, x_1 \neq x_2.$$

Suppose that a successful coupling does exist, then

$$\|P(t, x_1, \cdot) - P(t, x_2, \cdot)\|_{Var} \leq 2\boldsymbol{P}^{x_1, x_2}[T > t] \longrightarrow 0.$$

Furthermore,* if the process has a stationary distribution π * , then

$$\begin{aligned}\|P(t, x, \cdot *) - \pi * \|_{Var} &= \|P(t, x, \cdot) - \int \pi * (dy) P(t, y, \cdot)\|_{Var} \\ &\leq \int \pi(dy) \|P(t, x, \cdot) - P(t, y, \cdot)\|_{Var} \\ &\leq *2 \int *\pi * (dy) P^{x,y}[T * > *t] * \longrightarrow *o, \quad *t \to \infty.\end{aligned}$$

and so the process is ergodic. For more details, refer to Chen and Li (1989) and Chen (1987a).

(6) **Monotonicity.**

Suppose that our state space E is endowed with a measurable semi-order“ $\prec$ ”.

(6.1) **Definition.** An $f \in \mathscr{E}$ is called monotone if

$$x_1 \prec x_2 \Longrightarrow f(x_1) \leq f(x_2).$$

A set $A \in \mathscr{E}$ is called monotone, if so is the function I_A. We say that $P_1(t) \prec P_2(t)$ if for every monotone function f,

$$x_1 \prec x_2 \Longrightarrow P_1(t) f(x_1) \leq P_2(t) f(x_2), \quad t \geq 0$$

where $P_k(t)$ is the sub-Markovian semifroup induced by $P_k(t, x, A)$ $(x \in E, A \in \mathscr{E})$, $k = 1, 2$. If $P_1(t) = P_2(t)$, we call $P_1(t)$ itself monotone.

One way to prove the monotonicity is by using the coupling method. For example, applying the basic coupling to a Markov chain with regular Q-matrix $Q = (q_{ij})$ on $\boldsymbol{Z}_+$, we find that the conditions:

$$q_{i_1 k} \leq q_{i_2 k} \quad \text{for} \quad i_1 \leq i_2 < k$$

and *

$$q_{i_1 k} \geq q_{i_2 k} \quad \text{for} \quad k < i_1 \leq i_2$$

are sufficient for the monotonicity of $P(t)$. However, the above conditions are not necessary. The complete answer is as follows:
(6.2) **Theorem** (Chen (199?a)). Let $E = \boldsymbol{R}^d$, $\boldsymbol{Z}^d$, $\boldsymbol{R}^d_+$ or $\boldsymbol{Z}^d_+$ with the ordinary semi-order. Suppose that $(q_k(x), q_k(x,A))(k=1,2)$ are regular q-pairs and locally bounded (i.e., $q_k(x)(k=1,2)$ are locally bounded). Then $P_1(t) \prec P_2(t)$ iff for every monotone set A,

$$x_1 \prec x_2,\ x_1, x_2 \notin A \Longrightarrow q_1(x_1, A) \le q_2(x_2, A).$$

and

$$x_1 \prec x_2,\ x_1, x_2 \in A \Longrightarrow q_1(x_1, A^c) \ge q_2(x_2, A^c).$$

Bearing these results in mind, it is not difficult to check that the Schlőgl's second model is monotone and ergodic (an alternative proof was presented in Yan and Chen (1986)) and the Lotka-Volterra model is not monotone. Howoever, it remains open whether the latter model is positive recurrent or not.

(7) **Reversibility.**
An important subclass of jump processes is the reversible ones.
(7.1) **Definition.** A jump process $P(t,x,A)$ is called reversible (resp. symmetrizable) if there exists a probability (resp. $\sigma-$finite) measure π such that

$$\int_A \pi(dx)P(t,x,B) = \int_B \pi(dx)P(t,x,A), \qquad t \ge 0,\ A, B \in \mathscr{E}. \tag{7.2}$$

Similarly, we can define reversible (resp. symmetrizable) q-pair:

$$\int_A \pi(dx)q(x,B) = \int_B \pi(dx)q(x,A), \quad A, B \in \mathscr{E}. \tag{7.3}$$

Certainly, (7.2) implies (7.3). But the inverse is not necessarily true. In general, we have
(7.4) **Theorem** (Chen (1980)). The minimal jump process is reversible (resp. symmetrizable) with respect to π iff so is its q-pair.
(7.5) **Theorem** (Chen (1986b)) with respect to a probability measure π, the reversible jump process is unique iff * i) Its q-pair is reversible with respect to π,
ii) $\int \pi(dx)[q(x) - q(x,E)] < \infty$,
iii) $\mathscr{U}_\lambda = \{0\}$
all hold. Furthermore, if i) holds but not one of ii) and iii), then there exist infinite many reversible jump processes with respect to π.
For a $\sigma-$finite measure π, we only have the following result.
(7.6) **Theorem** (Chen (199?b)). With respect to the measure π, there exists precisely one symmetrizable jump process of the following three conditions all hold:
i) Its q-pair $(q(x), q(x,A))$ is symmetrizable with respect to π,
ii) $\int \pi(dx)[q(x) - q(x,E)] < \infty$ or $\inf_{x \in \dot{E}} P^{min}(\lambda, x, E) > 0$
iii) $\mathscr{U}_\lambda \cap L^1(\pi) = \{0\}$.
Moreover, if the symmetrizable process is unique, then the corresponding Dirichlet form is as follows:

$$\begin{aligned} D(f,g) = \frac{1}{2}\int \pi(dx) \int q(x,dy)\big(f(y)-f(x)\big)\big(g(y)-g(x)\big) \\ + \int \pi(dx)[q(x) - q(x,E)]f(x)g(x) \end{aligned}$$

with domain $\mathscr{D}(D) = \{f \in L^2(\pi):\ D(f,f) < \infty\}$.

We have known that the condition ii) is still stronger than to be necessary. Thus, a complete criterion for the uniqueness of symmetrizable jump processes is still open. Besides, the next problem seems quite hard.

(7.7) **Open problem.** What is the uniqueness criterion for honest reversible (resp. symmetrizable) jump processes? Here, "honest" means that

$$P(t,x,E)=1, \qquad t\ge 0, \quad x\in E.$$

There are quite a number of papers on symmetrizable jump processes. Most of them are collected in Qian,* Hou et al.* (1979),* Chen (1986b) * and (199?b).

(8) **Large deviations for Markov chains.**

For simplicily, we restrict ourselves to the Markov chain $P(t)=(P_{ij}(t))$ on $E=\boldsymbol{Z}_+$ with regular Q-marix $Q=(q_{ij})$. * In the Donsker-Varadhan's large deviation theory, we are interested in the entropy (rate function):

$$I(\mu)=-\inf_{f\in\mathscr{D}^+(L)}\int_E\frac{Lf}{f}d\mu$$

and

upper estimate: $\overline{\lim}_{t\to\infty}\frac{1}{t}Q_{t,i}(C)\le-\inf_{\mu\in C}I(\mu), C$ is closed

lower estimate: $\underline{lim}_{t\to\infty}\frac{1}{t}Q_{t,i}(G)\ge-\inf_{\mu\in G}I(\mu),\ G$ is open.

We should explain the notations used here. Let $\{X_t\}_{t\ge 0}$ be the Markov chain with transition probability P(t) and P_i be the probability that the chain starts from $i\in E.\mathscr{P}(E)$ is endowed with the weak topology. Set

$$L_t(\cdot,A)=\frac{1}{t}\int_0^t I_A(X_s)ds$$

and $Q_{t,i}=P_i\circ L_t^{-1}$. Consider ${}_b\mathscr{E}$ as a Banach space with the uniform norm. L is the infinitesmal generator of the process (resp. semigroup) on ${}_b\mathscr{E},\mathscr{D}(L)$ is the domain of L and

$$\mathscr{D}^+(L)=\{f\in\mathscr{D}(L):\ f\ge\epsilon>0 \quad\text{for some}\quad \epsilon>0\}.$$

What we are interested in is to find some explicit expression for the entropy and sufficient conditions for the estimates in the present context. We certainly need some work since, for example, $\mathscr{D}(L)$ is quite poor, even $I_{\{i\}}$ $(i\in E)$ is usually not in $\mathscr{D}(L)$. However, the lower estimate is usually satisfied, we need only to consider the upper estimate.

(8.1) **Theorem** (Chen and Lu (1989,* 1990),* Chen * (199?b)).

Suppose that $Q=(q_{ij})$ is a regular Q-matrix on $E=\boldsymbol{Z}_+$.

i) If $\mu\in\mathscr{P}(E)$ satisfies $\sum_i\mu_iq_i<\infty$, then

$$\begin{aligned}I(\mu)=&\frac{1}{2}\sum_{i,j}(\sqrt{\mu_iq_{ij}}-\sqrt{\mu_jq_{ji}})^2\\&-\frac{1}{2}\inf_{f\in\mathscr{C}}\sum_{i,j}(\sqrt{\mu_iq_{ij}f_j/f_i}-\sqrt{\mu_jq_{ji}f_i/f_j})^2<\infty,\end{aligned}$$

where $\mathscr{C}=\mathscr{D}^+(L)$ or one of the following set:

$$\begin{aligned}\mathscr{E}^+&=\{f\in\mathscr{E}:\ f\ge\epsilon>0\text{ for }\ \text{some }\epsilon>0\}\\\mathscr{E}^0&=\{f\in\mathscr{E}:\ 0<f<\infty\}\\{}_b\mathscr{E}^+&={}_b\mathscr{E}\cap\mathscr{E}^+,\ {}_b\mathscr{E}^0={}_b\mathscr{E}\cap\mathscr{E}^0.\end{aligned}$$

Moreover, if (q_{ij}) is reversible with respect to some $\pi\in\mathscr{P}(E)$, then for every $\mu\in\mathscr{P}(E)$, we have

$$I(\mu) = \frac{1}{2}\sum_{i,j}(\sqrt{\mu_i q_{ij}} - \sqrt{\mu_j q_{ji}})^2.$$

ii) The upper estimate holds in the following cases:

a) There exists an $\alpha > 1$ such that $\sum_{j<i} q_{ij} \geq \alpha q_{i,i+1} > 0, i \in E, \max\{i : q_i = 0\} < \infty$ and $\sum_{j<i} q_{ij} \longrightarrow \infty$ as $i \longrightarrow \infty$.

In the following three cases, we assume that there are no absorbing states (i.e., $q_i > 0$ for all $i \in E$).

b) There is an $n \in \boldsymbol{N}$ such that $q_{ij} = 0,\ j > n + i,\ i \in E$: there also exists an $\alpha > n$ and an $N \in E$ such that $\sum_{j<i} q_{ij} \geq \alpha \sum_{j>i} q_{ij}$ for all $i \geq N$; $\sum_{j<i} q_{ij} \longrightarrow \infty$ as $i \longrightarrow \infty$.

c) There exist $\{c_i\} \subset (0, \infty)$ and $\{d_i\} \subset [0, \infty)$ such that $q_{ij} \leq c_i d_j$ for all large enough i and all $j > i$. Moreover, $\sum_j \overline{c}_j d_j < \infty$ and $\sum_{j<i}(1 - \overline{c}_j/\overline{c}_i)q_{ij} \longrightarrow \infty$ as $i \longrightarrow \infty$, where $\overline{c}_i = \max\{c_k : k \leq i\}$.

d) There exists an $\overline{\alpha} \in (0, \infty)$ and an $\alpha \in (0, \overline{\alpha})$ such that $\sum_{j<i} q_{ij} \geq \overline{\alpha} \sum_{j>i} q_{ij}$ for all large enough i and $(\frac{\alpha}{1+\alpha})^i \sum_{j<i} q_{ij} \longrightarrow \infty$ as $i \longrightarrow \infty$.

(8.2) **Remark.** Consider the linear birth-death matrix : $q_{i,i+1} = \beta_0 + \beta_1 i,\ q_{i,i-1} = \delta_1 i,\ \beta_0 \geq 0,\ \beta_1, \delta_1 > 0$. If $\beta_0 = 0$, then the conditions in a) hold in the case that $\beta_1 < \delta_1$. On the other hand, the process is ergodic iff $\beta_1 \leq \delta_1$. If $\beta_0 > 0$, then we can use b) whenever $\beta_1 < \delta_1$ which is exactly the same case that the process* being ergodic.

Part II. Interacting Particle Systems with Compact State Spaces

Throughout this part, we assume that S is a countable set. For each $u \in S$, suppose that E_u is a finite set with the discrete topology and Bord field $\mathscr{E}_u$. For $\wedge \subset S$, define $E(\wedge) = \Pi_{u\in\wedge} E_u$ and $\mathscr{E}(\wedge) = \Pi_{u\in\wedge}\mathscr{E}_u$ as the usual product space. Set $(E, \mathscr{E}) = (E(S), \mathscr{E}(S))$. Finally, let $\mathscr{F}(\wedge) = \mathscr{E}(\wedge) \times E(S \setminus \wedge)$ and let $\mathscr{C}(E)$ be the set of all continuous function on E.

(1) **Spin-flip processes**

Take $E_u = \{0, 1\},\ u \in S$. Suppose that we are given a rate function $c(u, x)(u \in S, x \in E)$ satisfying

$$c(u, x) > 0, \quad u \in S, \quad x \in E \tag{1.1}$$

$$c(u, \cdot) \in \mathscr{C}(E), \quad u \in S. \tag{1.2}$$

Define

$$\Omega f(x) = \sum_{u\in S} c(u, x)(f({}_u x) - f(x))$$

where

$${}_u x(v) = \begin{cases} 1 - x(u) & \text{if } \ v = u \\ x(v) & \text{if } \ v \neq u. \end{cases}$$

Assume that $c(u, x)$ satisfies the Liggett's uniqueness conditions (see Liggett (1985)) and hence generates a unique Feller's process. This process is called a spin-flip process.

Now we are interesting to know when the process is reversible (with respect to some probability measure). For this, the following condition is essential:

$$\begin{aligned} &c(u, x)c(v, {}_u x)c(u, {}_u({}_v x))c(v, {}_v x) \\ &= c(v, x)c(u, {}_v x)c(v, {}_v({}_u x))c(u, {}_u x) \quad u, v \in S,\ x \in E. \end{aligned} \tag{1.3}$$

Next, define an equivalent relation $\sim$ on E as follows: $x \sim y$ iff x=y or there are some $u_i, i = 0, \cdots, k$, such that $x^{(i+1)} = {}_{u_i}x^{(i)}, i = 0, \cdots, k$, where $x^{(k+1)} = y,\ x^{(0)} = x$. According to this relation, E is devided into some equivalent classes $\{E_l : l \in D\}$. For each $l \in D$, choose an arbitrary reference point $\triangle_l$, then for each $x \in E_l, x \neq \triangle_l$, choose an arbitrary reference path: *

$$L(\triangle_l, x) = (\triangle_l, x^{(1)}, \cdots, x^{(k)}, x)$$

where $x^{(i+1)} = {}_{u_i}x^{(i)}$ for some $u_i \in S, i = 0, \cdots, k, x^{(0)} = \triangle_l,\ x^{(k+1)} = x$. Define

$$V(x) = \sum_{i=0}^{k} [\log c(u_i, x^{(i+1)}) - \log c(u_i, x^{(i)})].$$

Of course, the definition of function $\vee$ depends on the choices of the reference point $\triangle_l$ and path $L(\triangle_l, x)$. However, the condition (1.3) guarantees that the functions

$$f^{\wedge}(x) = \exp[V(x)] / \sum_{y \in E(\wedge)} \exp[V(y \times x_{S \setminus \wedge})], \quad \wedge \in \mathscr{S}, x \in E$$

do not depend on these choices, where $x_\wedge$ denotes the restriction of x on $E(\wedge)$.
(1.4) **Definition.** We say that $\mu \in \mathscr{P}(E)$ is a Gibbs state if for every $\wedge \in \mathscr{S}$ and $y \in E(\wedge)$,

$$\mu\Big(\{y\} \times E(S \setminus \wedge) \mid \mathscr{F}(S \setminus \wedge)\Big) = f^{\wedge}(y \times (\cdot)_{S \setminus \wedge}),\ \mu - a.s.$$

(1.5) **Theorem** (Ding and Chen (1981), Tang (1982)).
Given a spin-flip process with generator Ω defined as above. Suppose that (1.1) and (1.2) hold. Then the process is reversible iff (1.3) holds. If so, the set of all reversible probability measures for the process is non-empty and coincides with the set of all Gibbs states.

(2) **Exclusion processes.**
Again, we take $E_u = \{0, 1\}, u \in S$. Given a speed function $c(u, v, x)$:

$$c(u, v, x) > 0, \quad u, v \in S,\ u \neq v;\ x \in E,\ x(u) \neq x(v) \tag{2.1}$$

$$c(u, v, \cdot) \in \mathscr{C}(E), \quad u, v \in S, \tag{2.2}$$

we define

$$\Omega f(x) = \sum_{u,v \in S} c(u, v, x)\big(f({}_{(u,v)}x) - f(x)\big),\ x \in E$$

where

$$\big({}_{(u,v)}x\big)(w) = \begin{cases} x(w), & \text{if } w \neq u, v \\ x(v), & \text{if } w = u \\ x(u), & \text{if } w = v. \end{cases}$$

Now, our essential condition becomes

$$\begin{aligned} &c(u, v, x)c(v, w, {}_{(u,v)}x)c(w, u, {}_{(w,u)}x)) \\ &= c(u, w, x)c(w, u, {}_{(w,u)}x)c(v, u, {}_{(v,u)}x) \qquad u, v, w \in S,\ x \in E. \end{aligned} \tag{2.3}$$

By a similar provedure as given in the last section, me may introduce a potential function V on E. Next, for every $\wedge \in \mathscr{S}, k \in \boldsymbol{Z}_+, k \leq |\wedge|$ and $y \in E(\wedge)$, define

$$f_k^{\wedge}(y \times z) = I_{E_k(\wedge)}(y) \exp[V(y \times z)] / \sum_{y\prime \in E_k(\wedge)} \exp[V(y\prime \times z)],$$

where

$$E_k(\wedge) = \{y \in E(\wedge) :\mid y \mid \equiv \sum_{u \in \wedge} y(u) = k\}.$$

(2.4) **Definition.** We say that $\mu \in \mathscr{P}(E)$ is a canonical Gibbs state if for every $\wedge \in \mathscr{S}$ and $y \in E(\wedge)$,

$$\mu(\{y\} \times E(S \setminus \wedge) \mid \mathscr{A}(S \setminus \wedge)) = f^{\wedge}_{|(\cdot)_\wedge|}(y \times (\cdot)_{S\setminus\wedge}),\ \mu - a.s.$$

(2.5)**Theorem** (Yan, Chen and Ding (1982a,b)).

Given an exclusion process with generator Ω defined above. Suppose that the continuous condition (2.2) holds, then the process is always reversible. Next, suppose that (2.1), (2.2) and (2.3) all hold, then the set of reversible measures coincides with the set of canonical Gibbs states for the process. Finally, if (2.1) and (2.2) hold and there exists a reversible measure π which is positive in the sense:

*

$$\pi(\{y\} \times E(S \setminus \wedge)) > 0 \text{ for all } y \in E(\wedge) \text{ and } \wedge \in \mathscr{S},$$

then (2.3) holds and the set of positive reversible measures coincides with the set of positive Gibbs states.

In the last quoted paper, a proof of Theorem (1.5) is also included. One of the key points in proving the above results goes back to Hou and Chen (1980). Along this direction, some more general models have been studied by Dai (1986), Li(1983), Ren(1983) and Zeng (1983).

(3) **Generalized simple exclusion processes.**

Take $E_u = \{0, 1, \cdots, m\},\ u \in S$, where $m \in \boldsymbol{N} = \{1, 2, \cdots\}$ is fixed. Suppose that $(p(u,v) : u, v \in S)$ is an irreducible transition probability having the properties:

$$\sup_v \sum_u p(u,v) < \infty$$

and

$$\pi(u)p(u,v) = \pi(v)p(v,u),\ u,v \in S$$

for some $(\pi(u) : u \in S)$. Let $g : \{0, 1, \cdots, m\} \longrightarrow [0,\infty)$ be a strictly increasing function with $g(0) = 0$. For $u,v \in S$ and $x \in E$, define ${}_{(u,v)}x \in E$ as follows: if $x(u) = 0$ or $x(v) = m$, then ${}_{(u,v)}x = x$; otherwise

$${}_{(u,v)}x(w) = \begin{cases} x(w), & \text{if } w \neq u,v \\ x(u) - 1, & \text{if } w = u \\ x(v) + 1, & \text{if } w = v \end{cases}$$

Consider the Markov process generated by

$$\Omega f(x) = \sum_{u,v \in S} g(x(u))p(u,v)[f({}_{(u,v)}x) - f(x)].$$

We want to describe the set $\mathscr{I}_e$ of the extremal invariant probability measures of the process. To do this, let ν_∞ denote the point mass at $\{x_u :\ x_u = m,\ u \in S\}$ and let $\nu_{\rho\pi}(0 < \rho < \infty)$ denote the product measure with the marginal distributions:

$$\nu_{\rho\pi}(x(u) = k) = \frac{(\rho\pi(u))^k}{\Pi_{j=1}^k g(j)} / \sum_{i=0}^m \frac{(\rho\pi(u))^i}{\Pi_{j=1}^i g(j)},\quad 0 \leq k \leq m.*$$

Here we have used a convertion : $\Pi_{j=1}^{k} g(j) = 1$ if $k = 0$. In particular, we can define $\nu_\pi = \nu_{\rho\pi}$ with $\rho = 1$ and $\nu_\rho = \nu_{\rho\pi}$ if $\pi(u) = 1$ for all $u \in S$.

(3.1) **Theorem** (Zheng and Zeng (1986, 1987)). Under the above hypotheses,

i) if $\sum_u \pi(u) < \infty$, equivalently, $(p(u,v))$ is positive recurrent, then

$$\mathscr{I}_e = \{\nu_n : \ 0 \le n \le \infty\}$$

where

$$\nu_n = \nu_\pi(\ \cdot \ \mid \{x \in E : \ \sum_{u \in S} x(u) = n\}).$$

Moreover, for each $x \in E$ with $\sum_{u \in S} x(u) = \infty$,

$$\lim_{t \to \infty} \boldsymbol{P}^x [X_t(u) = m] = 1, \quad u \in S.$$

ii) if $S = \boldsymbol{Z}^d$ and $(p(u,v))$ is translation invariant, then

$$\mathscr{I}_e = \{\nu_\rho : 0 < \rho \le \infty\}.$$

For a generalized long-range exclusion model, see Zheng (1988).

(4) **Quasi-nearest particle systems.**

Take $E_u = \{0, 1\}$, $u \in S = \boldsymbol{Z}$. Set

$$E_0 = \{x \in E : \ \sum_{u > 0} (1 - x(u)) < \infty \text{ and } \sum_{u < 0} (1 - x(u)) < \infty\},$$

$$\overline{*}{\boldsymbol{*}}N* = \boldsymbol{N} \cup *\{\infty*\}, * \quad * \boldsymbol{*}N\prime = \boldsymbol{N} \setminus \{1\}, \quad \overline{\boldsymbol{N}}\prime = \boldsymbol{N}\prime \cup \{\infty\}.$$

For $x \in E$ and $u \in \boldsymbol{Z}$, define

$$\begin{aligned} l(u,x) &= u - \max\{v < u : \ x(v) = 1\} \in \overline{\boldsymbol{N}}, \\ r(u,x) &= \min\{v > u : x(v) = 1\} - u \in \overline{\boldsymbol{N}}, \\ m(u,x) &= x - l(u,x) - \max\{v < u - l(u,x) : x(v) = 0\} + 1 \in \overline{\boldsymbol{N}}\prime \\ n(u,x) &= \min\{v > u + r(u,x) : \ x(v) = 0\} - u - r(u,x) + 1 \in \overline{\boldsymbol{N}}\prime \end{aligned}$$

and set $m = \infty$ if $l = \infty$ and $n = \infty$ if $r = \infty$. Given a function $\beta(m,l,r,n) : \overline{\boldsymbol{N}}\prime \times \overline{\boldsymbol{*}N} * \times * \overline{*}\boldsymbol{*}N * \times * \overline{*}\boldsymbol{*}N\prime \longrightarrow [0, \infty)$, we define

$$c(u,x) = \begin{cases} \beta(m(u,x), l(u,x), r(u,x), n(u,x)), \text{if } \ x(u) = 0 \\ 1, \qquad\qquad\qquad\qquad\qquad\qquad \text{if } x(u) = 1 \end{cases}$$

and study the spin-flip process corresponding to the above rate function. This process is called a quasi-nearest particle system.

(4.1) **Theorem** (Dai and Liu (1986)).

Let $\beta(\infty, \infty, \infty, \infty) = 0$ and β be positive on $\overline{\boldsymbol{N}}\prime \times \overline{\boldsymbol{*}N*} * \times * \overline{\boldsymbol{N}} * \times * \overline{*}\boldsymbol{*}N\prime \setminus \{(\infty, \infty, \infty, \infty)\}$.

1) In order for the process to be reversible, it is necessary that there exist $f, g : \overline{\boldsymbol{N}}\prime \longrightarrow (0, \infty)$ such that $f(2)g(2) = 1$ and

$$\beta(m,l,r,n) = \begin{cases} \frac{f(l)f(r)}{f(l+r)}, & \text{if } \ l, r \ge 2 \\ \frac{g(m)g(n)}{g(m+n)}, & \text{if } \ l = r = 1 \\ \frac{g(n)f(l)}{g(n+1)f(l+1)}, & \text{if } \ l \ge 2, \ r = 1 \\ \frac{g(m)f(r)}{f(m+1)f(r+1)}, & \text{if } \ l = 1, \ r \ge 2 \end{cases}$$

ii) Suppose in addition that the limits

$$\lim_{r\to\infty}\frac{f(r)}{f(r+1)}=c_1,\quad \lim_{n\to\infty}\frac{g(n)}{g(n+1)}=c_2,$$
$$\lim_{r\to\infty} f(r)c_1^r=c_3,\quad \lim_{n\to\infty} g(n)c_2^n=c_4$$

all exist in $(0,\infty)$, then the process is reversible and the set of the reversible measures coincides with the set of Gibbs states.

iii) Furthermore, if

$$\frac{f(r)}{f(r+1)}\downarrow,\quad \frac{g(n)}{g(n+1)}\uparrow \quad\text{and}\quad g(2)c_2\le c_1,$$

then the process is monotone (attractive).

iv) Under the above hypotheses, take $f(2)=g(2)=1$ and $c_1=1$. Let

$$c^k(u,x)=\begin{cases}\beta(k\wedge m(u,x),l(u,x),r(u,x),k\wedge n(u,x)), & \text{if } x(u)=0\\ 1, & \text{if } x(u)=1\end{cases}$$

and suppose that

$$\sup_{x\in E}\mid c(0,x)-c^k(0,x)\mid=O(e^{-\delta k}),\ \text{as } k\longrightarrow\infty$$

for some $\delta>0$. Replacing β by $\beta_\lambda=\lambda\beta$ $(\lambda>0)$, then $\delta_1 P_\lambda(t)\Longrightarrow\nu^\lambda$ as $t\longrightarrow\infty$ and there exists a critical value λ_c such that

$$\nu^\lambda=\delta_0 \text{ (dies out) },\ \text{ for } \lambda<\lambda_c$$
$$\nu^\lambda\ne\delta_0 \text{ (survives) },\ \text{ for } \lambda<\lambda_c,$$

where δ_0 and δ_1 denote the probabilifies with point mass at $\{x_u=0: u\in S\}$ and $\{x_u=1:u\in S\}$ respectively. Moreover, if $\sum_{r=2}^\infty f(r)<\infty$, then λ_c is the unique positive root $(<c_2^{-1})$ of the equation

$$\sum_{r=2}^\infty f(r)\sum_{n=2}^\infty \lambda^{n-1}g(n)^{-1}=1,$$

and $\nu^{\lambda_c}=\delta_0$ or $\ne\delta_0$ according to $\sum_{r=1}^\infty rf(r+1)=\infty$ or $<\infty$ respectively.

Next, we consider the finite quasi-nearest particle systems. That is, the state space is replaced by

$$\{x\in E:\sum_{u\in\boldsymbol{Z}}x(u)<\infty\}.$$

We use the notations in front of (4.1).

(4.2) **Theorem** (Liu (1987)). Suppose that * i) $\beta(m,l,r,n)=\beta(n,r,l,m)$

ii) $\beta(m,1,\infty,\infty)=\beta(\infty,\infty,1,m)>0$

iii) $\beta(\infty,\infty,\infty,\infty)=0$

iv) $\sum_{l+r=k+1}\beta(m,l,r,n)=b(m+n)/2$ is independent of $k\in\boldsymbol{N}$ and

$$\sum_{l=1}^\infty\beta(m,l,\infty,\infty)+\sum_{r=1}^\infty\beta(\infty,\infty,r,n)=\frac{b}{2}(m+n)+\alpha$$

for some constants $b,\alpha>0$. Then the process (Markov chain) dies out in the case either $b<1$ or $b=1$ still $\alpha\le1$. In the other cases, the process surrives.

It is worth to point out that the last model is irreversible. For the reversible finite systems, some more general results are included in Dai and Liu (1985).

Part III. Interacting Particle Systems with Non-Compact State Spaces

Throughout this part, we again suppose that S is a countable set. For each $u \in S$, let $(E_u, \rho_u, \mathscr{E}_u)$ be a complete separable metric space, where $\mathscr{E}_u$ is the $\sigma-$algebra generated by the metric ρ_u. Let $(E, \mathscr{E})$ be the usual product space of $(E_u, \mathscr{E}_u),\ u \in S$. Choose an arbitrary reference point $\theta = (\theta_u : u \in S)$ and suppose that we are given a positive summable sequence $(\alpha_u :\ u \in S)$.

For $x = (x_u : u \in S),\ y = (y_u : u \in S)$ and $\wedge \subset S$, define

$$p_\wedge(x,y) = \sum_{u\in\wedge} \rho_u(x_u, y_u)\alpha_u.$$

For simplicity, we also use the notations :

$$\rho_u(x) = \rho_u(x_u, \theta_u), \quad p_\wedge(x) = p_\wedge(x, \theta).$$

Set $E^\wedge = \{x \in E : p_{S\setminus\wedge}(x,\theta) = 0\}$. The $^*\sigma-$algebra $\mathscr{E}^\wedge$ is induced on $E^\wedge$ by the $\sigma-$algebra $\mathscr{E}$. Finally, let $x^\wedge$ to denote the projection of x on $E^\wedge$:

$$p_\wedge(x^\wedge, x) + p_{S\setminus\wedge}(x^\wedge, \theta) = 0.$$

(1) **Constructions of the processes. Analytic Approach.**

Suppose that there is fixed a sequence $\{\wedge_n\}_1^\infty \subset \mathscr{S}$ such that $\wedge_n \uparrow S$. For each $n \geq 1$, there is also fixed a regular q-pair $(q_n(x), q_n(x,\cdot))$ on $(E^{\wedge_n}, \mathscr{E}^{\wedge_n})$ (cf. Part I). The problem we are interested in is to find a limit process of those jump processes $P_n(t,x,\cdot)$ determined by the q-pair $(q_n(x), q_n(x,\cdot))(n \geq 1)$. To this end, let p be an $\mathscr{E}-$measurable function
(may be valued $+\infty$) satisfying:

1^0 $0 \leq p(x) < +\infty$ for each $x \in E^{\wedge_n}$ and $n \geq 1$,

2^0 For each $0 \leq d < +\infty$ and $n \geq 1$, the set $\{x \in E : p(x^{\wedge_n}) > d\}$ is an open set in E,

3^0 For each $x \in E, p(x^{\wedge_n}) \uparrow p(x)$ as $n \uparrow \infty$.

Put $E_0 = \{x \in E : p(x) < \infty\}$ the $\sigma-$algebra $\mathscr{E}_0$ is also induced on E_0 by $\mathscr{E}$.

One of our main tools in the study is the Kantorovich distance of probability measures:

$$R_\wedge(P,Q) = \inf_\mu \int_{E^\wedge\times E^\wedge} p_\wedge(x,y)\mu(dx,dy), \quad \wedge \in \mathscr{S}$$

where the greatest lower bound is computed over all measures μ on $\mathscr{E}^\wedge \times \mathscr{E}^\wedge$ satisfying

$$\mu(A \times E^\wedge) = P(A),\ \mu(E^\wedge \times A) = Q(A),\ A \in \mathscr{E}^\wedge.$$

The measure μ having the above marginality in called a coupling measure of P and Q.

Recall that every conservatiove q-pair $(q_n(x), q_n(x,\cdot))$ corresponds an operator

$$\Omega_n f(x) = \int q_n(x,dy)(f(y) - f(x)), \quad n \geq 1$$

and vice versa.

Now, we are at the position to state our first construction for the limit process.

(1.1) **Theorem** (Chen (1986b) or (1987)). Suppose that the following conditions hold:

1) There exists a constant $c \in \boldsymbol{R}$ such that $\Omega_n p(x) \leq c(1 + p(x)),\quad x \in E_0,\ n \geq 1$.

2) For all $m \geq n \geq 1$ there exists a coupling operator $\Omega_{n,m}$ of Ω_n and Ω_m such that

$$\Omega_{n,m} p_w(x_1,x_2) \leq \sum_{u\in\wedge_n} c_{uw} p_u(x_1,x_2) + c_w(n,m)(1 + p(x_1) + p(x_2)),$$

$$w \in \wedge_n, \quad x_1, x_2 \in E_0$$

where the non-diagonal elements of $(c_{uw} : u,\ w \in S)$ and the elements of $c_w(n,m)(w \in \wedge_n,\ m \geq n \geq 1)$ are non-negative and satisfying

$$c_w(t,n,m) \equiv \sum_{k=0}^{\infty} \frac{t^{k+1}}{(k+1)!}\big[(B_n^*)^k c.(n,m)\big](w) \longrightarrow 0 \quad \text{as } m \geq n \longrightarrow \infty,\quad t \geq 0$$

where B_n is the matrix $(c_{uv} : u, v \in \wedge_n)$ and B_n^* is the transpose of B_n.

Then there exists a Markov process with transition probability function $P(t,x,\cdot)$ on state space $(E_0, \mathscr{E}_0)$ such that for each $\wedge \in \mathscr{S}$,

$$\lim_{n\to\infty} R_\wedge(P_n(t,x,\cdot),\ P(t,x,\cdot)) = 0,\ x \in E_0,\ t \geq 0.$$

Moreover, the convergence is uniform in $x \in E_0^N \equiv \{x \in E_0 :\ p(x) \leq N\}$. Finally, for fixed $t, P(t,x,\cdot)$ is continuous in the following sense : if $x, x_n \in E_0,\ n \geq 1, \sup_n p(x_n) < \infty$ and $\lim_{n\to\infty} \rho_u(x_n, x) = 0$ for every $u \in S$, then

$$\lim_{n\to\infty} R_\wedge(P(t,x_n,\cdot),\ P(t,x,\cdot)) = 0$$

for every $\wedge \in \mathscr{S}$.

(1.3) **Remak.** If

$$\lim_{m\geq n\to\infty} c_u(n,m) = 0, \quad u \in S$$
$$\sup_{m\geq n, u\in\wedge_n} c_u(n,m) + \sup_u \sum_v \mid c_{uv} \mid < \infty$$

then the condition (1.2) holds.

The condition (1.2) means that the interactions are rapidly decreasing when the distance between the components increases. The next theorem relaxes the restriction for the special p defined by

$$p(x) = \sum_{u\in S} \rho_u(x,\theta)\alpha_u, \quad x \in E. \tag{1.4}$$

Let $\mathscr{L}$ denote the set of all Lipschitz continuous functions with respect to the above metric p . For $f \in \mathscr{L}$, let $L(f)$ denote the Lipschitz constant of f.

(1.5) **Theorem** (Chen (1986b, 1987)). Let p be the function given by (1.4). Suppose that the following conditions hold:

1) There exist $c_1 \in \boldsymbol{R}$ and a non-negative matrix $(b(u,v) :\ u, v \in S)$ such that

$$\Omega_n\rho_v(x) \leq \beta_v + c_1\rho_v(x) + \sum_{u\in\wedge_n} \rho_u(x)b(u,v), \quad v \in \wedge_n,\ x \in E_0,\ n \geq 1$$

where $\rho_v = \rho_v(\cdot,\theta),\ \beta_u \geq 0(u \in S), \sum_u \beta_u\alpha_u < \infty$ and

$$\sum_v b(u,v)\alpha_v \leq M\alpha_u, \quad u \in S$$

for some $M > 0$.

2) For $m \geq n \geq 1$ there exists a coupling operator $\Omega_{n,m}$ of Ω_n and Ω_m such that

$$\Omega_{n,m}p_w(x_1,x_2) \leq \sum_{u\in\wedge_n} c_{uw}p_u(x_1,x_2) + \sum_{u\in\wedge_m\setminus\wedge_n} p_u(x_2)g_{uw} + p_w(x_2)c_w(n,m),$$
$$w \in \wedge_n,\ x_1, x_2 \in E_0,$$

where $(c_{uv}), (g_{uv})$ and $(c_w(n,m))$ are all non-negative, they satisfy the conditions in (1.3) and

$$c_w(n,n)=0,\quad w\in\wedge_n,\ n\ge 1;\quad \sup_u\sum_v g_{uv}<\infty.$$

Then there exists a Markov process with transition probability function $P(t,x,\cdot)$ and state space $(E_0,\mathscr{E}_0)$ such that
*

$$R_{\wedge_n}(P_n(t,x,\cdot),P(t,x,\cdot))\longrightarrow 0,\ \text{as } n\longrightarrow\infty,\ x\in E_0.$$

Moreover, the convergence is uniform in t in finite intervals. Finally, the semigroup $\{P(t)\}_{t\ge 0}$ on $\mathscr{L}$ induced by $P(t,x,\cdot)$ has properties: $P(0)=I$, $P(t)$ is contractive in the uniform norm, there is a constant $c_2\in\boldsymbol{R}$ such that

$$|P(t)f(x)-P(t)f(y)|\le L(f)p(x,y)\exp[c_2t].$$

Before going to the descrete spin spaces, let us mention two models for which the above results are suitable. The first model is a Gaussian system: $S=\boldsymbol{Z}^d,\ E_u=\boldsymbol{R}\ (u\in S)$. $q_n(x,\cdot)$ is given by $\sum_{u\in\wedge_n}q_{n,u}(x,\cdot)$, where $q_{n,u}(x,\cdot)(u\in\wedge_n)$ is a probability measure on $(E^{\wedge_n},\mathscr{E}^{\wedge_n})$ the projection of which on $(E_u,\mathscr{E}_u)$ is a Gaussian measure. This model was studied by Basis (1980). The second moded is the generalized Potlatch process. For which, $S=\boldsymbol{Z}^d,\ E_u=\boldsymbol{R}_+\ (u\in S)$ and

$$\Omega_nf(x)=\sum_{u\in\wedge_n}\int_0^\infty[f(x-e_ux_u+\xi\sum_{v\in\wedge_n}x_up(u,v)e_v)-f(x)]dF(\xi)$$

where $(p(u,v))$ is a random walk on $\boldsymbol{Z}^d$, and ξ is a non-negative random variable with distribution $F(\xi)$ and mean value one. This model was studied by Holley and Liggett (1981).

Now, we start to discuss the reaction diffusion processes. Imagining each $u\in S$ as a small ressel in which there is a reaction. The rates of reaction are given by a Q-matrix $Q_u=(q_u(i,j):i,j\in E_u)$. If there are $d(\ge 1)$ different reactors, then the numbers of the reactors consist of the spin space $E_u=\boldsymbol{Z}_+^d$. For $d\ge 2$, some models are covered by the above results and we will also come back to this situation in the next section. Thus, in the remainder of this section, we may assume that $d=1$ and so $E_u=\boldsymbol{Z}_+\ (u\in S)$. We further allow some diffusions between the vessels. We use a transition probability matrix $P=(p(u,v):u,v\in S)$ to describe the diffusions. Thus if there are k particles in a vessel u, then the rate function of the diffusion from u to v is given by $C_u(k)p(u,v)$, where

$$C_u\ge 0,\quad C_u(0)=0.$$

Suppose that $(\alpha_u:u\in S)$ is summable and

$$\sum_v p(u,v)\alpha_v\le M\alpha_u,\quad u\in S$$

for some $M>0$. Set

$$E_0=\{x\in E:\|x\|=\sum_{u\in S}x_u\alpha_u<\infty\}.*$$

Finally, we may write the formal generator of the reaction diffusion process as follows:

$$\begin{aligned}\Omega f(x)=&\sum_{u\in S}\sum_{k\ne 0}q_u(x_u,x_u+k)[f(x+ke_u)-f(x)]\\&+\sum_{u\in S}C_u(x_u)\sum_{v\in S}p(u,v)[f(x-e_u+e_v)-f(x)],\quad x\in E_0.\end{aligned}$$

Here and hereafter, we use the convention:

$$q_u(i,j)=0,\ i\notin\boldsymbol{Z}_+,\quad u\in S.$$

In the present case, we can find more explicit conditions than those given above for the construction of the process. To do this, for simplicity, suppose that $Q_u = Q = (q_{ij})$ is independent of $u \in S$. Next suppose that

$$K = \sup_{u,k} |C_u(k) - C_u(k+1)| < \infty$$
$$\|\beta\| \equiv \sum_u \beta_u \alpha_u = \sum_u (\sum_{k=1}^{\infty} q(0,k)k)\alpha_u < \infty$$
$$\sum_{k\neq 0} q(i,i+k) \mid k \mid < \infty, \quad i \in \boldsymbol{Z}_+.$$

Put

$$g(j_1,j_2) = \sum_{k\neq 0}(q(j_2,j_2+k) - q(j_1,j_1+k))k(j_2-j_1)^{-1}$$
$$h(j_1,j_2) = 2\sum_{k=1}^{\infty}[(q(j_2,j_1-k) - q(j_1,2j_1-j_2-k))^+$$
$$+ (q(j_1,j_2+k) - q(j_2,2j_2-j_1+k))^+]k(j_2-j_1)^{-1}, \quad j_2 > j_1 \geq 0.$$

Now, we can introduce our main condition

$$K_{2\prime} = \sup\{g(j_1,j_2) + h(j_1,j_2) : j_2 > j_1 \geq 0\} < \infty.$$

This is due to the fact that at each $u \in S$, we use the coupling $\widetilde{\Omega}_m$ (cf. Part I, Section 5) for the reactions. Finally, set

$$K_{2\prime\prime} = \sup\{(C_u(j_1) - C_u(j_2))(j_2-j_1)^{-1} : j_2 > j_1 \geq 0\}$$
$$K_2 = K_{2\prime} + K_{2\prime\prime}$$

and

$$K_1 = \sup\{g(0,j) : \ j \geq 1\}.$$

Obviously,

$$K_1 * \leq K_{2\prime} < \infty \quad \text{and} * \quad *K_{2\prime} * \prime \leq K < \infty.$$

(1.6) **Theorem** (Chen (1985))

Under the above hypotheses, there exists a semigroup $P(t)$ of operators on $\mathscr{L}$ (the set of Lipschitz continuous functions with respect to $\|\cdot\|$), such that $P(0) = I$ and $P(t)$ is a strongly constraction on the uniform closure $\overline{\mathscr{L}}$ of $\mathscr{L}$. Moreover, for every $f \in \mathscr{L}$, the semigroup $P(t)$ possesses the following properties:

$$| P(t)f(x) - P(t)f(y) | \leq L(f) \|x-y\| \exp[t(K_2 + K(M+1)] \quad x,y \in E_0$$
$$\lim_{t\to 0} \frac{P(t)f(x) - f(x)}{t} = \Omega f(x), \quad x \in \mathscr{D}$$

where

$$\mathscr{D} = \{x \in E_0 : \||\, x \,\|| = \sum_{\substack{u\in S \\ x(u)\neq 0}} \sum_{k\neq 0} q(x_u, x_u+k) \mid k \mid \alpha_u < \infty\}.$$

Finally, there exists a Markov process $(\{X_t\}_{t\geq 0}, \ \boldsymbol{P}^x)$ evaluated in E_0 such that

$$P(t)f(x) = \boldsymbol{E}^x f(X_t) = \int f(\xi)\boldsymbol{P}^x[X_t \in d\xi], \quad f \in \mathscr{L}, \ x \in E_0.$$

Let us now consider some special cases of the reaction diffusion processes.

(1.7) **Zero range processes.** It is the case that the reaction vanishes. That is, $Q = (q_{ij})$ is zero. These processes are well-studied. See Liggett (1985) in the references. Some generalized models are treated by Wu (1983) and Wang (1987).

In the following cases, we take

$$C_u(k) = k, \qquad k \in \boldsymbol{Z}_+$$

and suppose that $Q = (q_{ij})$ is a birth−death Q-matrix with birth rate $b(k)$ and death rate $a(k)$.

(1.8) **Linear growth model.** Take *

$$b(k) = \beta_0 + \beta_1 k, \quad k \geq 0 \quad \text{and} \quad a(k) = \delta_1 k, \quad k \geq 1.$$

where $\beta_1, \delta_1 > 0$ and $\beta_0 \geq 0$. We will return to this model in Section 5.

(1.9) **Polynomial reaction model.**

$$b(k) = \sum_{j=0}^{m} \beta_j k^{(j)}, \quad a(k) = \sum_{j=1}^{m+1} \delta_j k^{(j)}$$

where* $*k^{(j)} = k(k-1)\cdots(k-j+1),*$ the coefficients* $*\beta_j\prime s$ and $\delta_j\prime s$ one non-negative, $m \geq 1$ and $\beta_0, \delta_1, \delta_{m+1} > 0$. If $m = 2$ and $\beta_1 = \delta_2 = 0$, then it is just the Schlőgl's second model, If $m = 1$, it is called the Schlőgl's first model.

At the first, one may think (1.8) is a special case of (1.9). However, these two models are quite different and have to be treated separately. For example, the latter one has finite mements of all orders but not the former one.

(1.10) **Theorem** (Chen (199? b)). The Markov provesses corresponding to (1.7), (1.8) and (1.9) constructed above are unique.

(2) **Construction of the processes. Martingale approach.**

The semigroup constructed in the last section is Lipschitz. Neverthless, the Lotka-Velterra model (Part I, (3.7)) and the Brusseltor model (Part I, (3.8)) may not have this property since the reactions between the different reactors are too strong (non-linear). Hence, we need a different approach to construct the processes.

Take $E_u = \boldsymbol{Z}_+^d (d \geq 1)$. Suppose that $(p_i(u, v) : u, v \in S)$, $i = 1, 2, \cdots, d$ are given transition probability matrices and $(\alpha_u : u \in S)$ is a summable sequence such that

$$\sum_{v \in S} \sum_{i=1}^{d} p_i(u, v)\alpha_v \leq M d \alpha_u, \quad u \in S$$

for some constant $M > 0$. The state space is

$$E_0 = \{x \in E : \sum_{u \in S} \sum_{i=1}^{d} x_{ui}\alpha_u < \infty\}.$$

Next, suppose that the reaction Q-matrix $Q = (q(i, j) : i, j \in \boldsymbol{Z}^d)$ satisfies

$$\sum_{k \neq 0} q(i, i+k) \sum_{j=1}^{d} k_j \leq A \sum_{j=1}^{d} i_j + \beta_u, \quad u \in S,\ i \in \boldsymbol{Z}_+^d,\ k \in \boldsymbol{Z}^d \tag{2.1}$$

where A is a constant and $\beta_u \geq 0, \|\beta\| = \sum_u \beta_u \alpha_u < \infty$. Suppose that the diffusion coefficients satisfy

$$C_{ui}(k) \geq 0,\ k \in \boldsymbol{Z}_+^d,\ C_{ui}(0) = 0,\ i = 1, \cdots, d,\ u \in S$$

and

$$\sup\{|\ C_{ui}(k+e_i)-C_{ui}(k)\ |: k\in \boldsymbol{Z}_+^d, u\in S,\ 1\le i\le d\}<\infty, \tag{2.2}$$

where e_i is the i-th unit vector in $\boldsymbol{Z}_+^d$. Let

$$\mathscr{D}_0=\{I_{\{x_\wedge\}}:\wedge\in\mathscr{S} \text{ and } x_\wedge\in(\boldsymbol{Z}_+^d)^\wedge\}$$

and define

$$\Omega f(x)=\sum_{u\in S}\sum_{k\in\boldsymbol{Z}_+^d\setminus\{0\}} q(x_u,x_u+k)\big[f(x+ke_u)-f(x)\big]$$
$$+\sum_{u\in S}\sum_{i=1}^d C_{ui}(x_u)\sum_{v\in S}p_i(u,v)\big[f(x-e_{ui}+e_{vi})-f(x)\big],\quad f\in\mathscr{D}_0,\ x\in E_0$$

where e_{ui} is the unit vector in $(\boldsymbol{Z}_+^d)^S=E$:

$$e_{ui}(v,j)=\begin{cases}1, & \text{if } v=u \text{ and } j=i\\ 0, & \text{otherwise.}\end{cases}$$

and $ke_u=\sum_{i=1}^d k_ie_{ui}\in E_0,\ k\in\boldsymbol{Z}^d$. Set

$$\|x-y\|=\sum_{u\in S}(\sum_{i=1}^d|\ x_{ui}-y_{ui}\ |)\alpha_u,\ x,y\in E_0$$

and let $(\boldsymbol{D},\mathscr{B}(\boldsymbol{D}))$ be the Skorohod space of the paths from $[0,\infty)$ to the complete separable metric space $(E_0,\|\cdot\|)$. As usual, we have the flow of $\sigma-$algebras $\{\mathscr{M}_t\}_{t\ge0}$.

(2.3) **Definition.** Let $x\in E_0$. A probability P^x on $(\boldsymbol{D},\mathscr{B}(\boldsymbol{D}))$ is called a solution to the martingale problem for Ω starting from x if

i) $\boldsymbol{P}^x[w\in\boldsymbol{D}:X(0,w)=x]=1$,

ii) For each $f\in\mathscr{D}_0$,

$$f(X(t))-\int_0^t\Omega f(X(s))ds$$

is a $(\boldsymbol{D},\{\mathscr{M}_t\}_{t\ge0},\mathscr{B}(\boldsymbol{D}),\boldsymbol{P}^x)-$martingale.

(2.4) **Theorem** (Han (1989))

Under (2.1) and (2.2), the martingale problem for Ω is well-posed. Moreover, the resulting solution is a Feller's process.

This theorem covers all the models of reaction diffusion processes mentioned above.

(3) **Existence and uniqueness of stationary distributions**

In this section, we study the existence and uniqueness of stationary distributions for the processes constructed in the Section 1.

(3.1) **Theorem** (Chen (1986b) (1989a)). Under the assumptions of Theorem (1.1) with $p(x)=\sum_u\rho_u(x)\alpha_u$, if for each $u\in S$ there is a compact function h_u (see Part I, (4.1)) such that

$$\rho_u(x)\le h_u(x_u),\ x_u\in E_u;\qquad \sup_u h_u(\theta_u)\alpha_u<\infty$$

and there are constants $K\in[0,\infty)$ and $\eta\in(0,\infty)$ such that

$$\Omega_nh_n(x)\le K-\eta h_n(x),\quad x\in E_0$$

where $h_n(x)=\sum_{u\in\wedge_n}h_u(x_u)\alpha_u$. Then

i) for each $n\ge1$, the jump process $P_n(t,x,\cdot)$ has at least one stationary distribution π_n satisfying

$$\int \pi_n(dx)h_n(x) \le K/\eta;$$

ii) the limit process $P(t,x,\cdot)$ constructed in Theorem (1.1) has at least one stationary distribution π, which can be obtained as a weak limit of a subsequence of $\pi_n\prime s$ and satisfies

$$\int \pi(dx)h(x) \le K/\eta$$

where $h(x) = \sum_{u\in S} h_u(x_u)\alpha_u$.

(3.2) **Theorem** (Huang (1987), Chen(1989)). Under the assumptions of Theorem (1.5), if

$$c_1 + M < 0$$

then the conclusions of Theorem (3.1) hold for the jump process $P_n(t,x,\cdot)$ and the limit process $P(t,x,\cdot)$ constructed in Theorem (1.5).

(3.3) **Theorem** (Chen (1986b) (1989 a)). Under the assumptions of Theorem (1.1) (resp. Theorem (1.5)) with $p(x) = \sum_{u\in S} \rho_u(x)\alpha_u$, if the coefficients (c_{uv}) given there also satisfy

$$\sum_{u\in S} c_{uw} \le -\eta < 0$$
$$\sum_{u\in S} \mid c_{uw} \mid \le K < \infty$$

Then

i) the process limit $P(t,x,\cdot)$ constructed in Theorem (1.1) (resp. Theorem (1.5)) has at most one stationary distribution π satisfuing

$$\int_{E_0} \pi(dx)p(x) < \infty.$$

If π is such a distribution, then

$$R_\alpha(P(t,x,\cdot),\pi) \le K(\alpha,x)e^{-\eta t}, \quad x \in E_0,\ \alpha \in \mathscr{S}$$

where $K(\alpha,x)$ is a constant independent of t;

ii) for a fixed $n \ge 1$, if in addition the coefficents $c_w(n,n)$ given in Theorem (1.1) (resp. (1.5)) vanish, then stationary distribution π_n satisfying

$$\int_{E^{\wedge n}} p_{\wedge_n}(x)\pi_n(dx) < \infty.$$

If π_n is such a distribution, then

$$R_{\wedge_n}(P_n(t,x,\cdot),\pi_n) \le K_n(x)e^{-\eta t}, \quad x \in E^{\wedge_n}$$

where $K_n(x)$ is a constant independent of t.

Now we apply the above results to the polynamial reaction model (1.9). It is not difficult to check that a stationary distribution always exists even for the general $C_u(k)$ satisfying $\sup_{u,k} \mid C_u(k) - C_u(k+1) \mid < \infty$. In the special case that $C_u(k) = k$, this result can be also obtained by using the monotonicity. To discuss the uniqueness of the distributions, we use the following coupling : for the reaction part, in each vessel we take the march coupling (Part I, (5.6)); for the diffusion part, we take

$$\begin{aligned}\widetilde{\Omega}_d f(x_1,x_2) = \sum_{u,v} \Big\{ &(x_1(u) \wedge x_2(u))p(u,v)[f(x_1 - e_u + e_v, x_2 - e_u + e_v) - f(x_1,x_2)] \\ &+ (x_1(u) - x_2(u))^+[f(x_1 - e_u + e_v, x_2) - f(x_1,x_2)] \\ &+ (x_2(u) - x_1(u))^+[f(x_1, x_2 - e_u + e_v) - f(x_1,x_2)] \Big\}.\end{aligned}$$

Then the condition in Theorem (3.3) becomes

$$c + M - 1 < 0$$

where

$$c = \sup_{k \geq 0, l \geq 1} \big[b(k+l) - b(k) - a(k+l) + a(k)\big]/l.$$

If in addition, $(p(u,v))$ is transtation invariant in $\boldsymbol{Z}^d = S$, then the constant M can be choosen as clese to one as required. Hence the condition $c < 1$ is indeed enough for the uniqueness. Thus, for the first Schlőgl model, we require that $\beta_1 < \delta_1$; for the second one,

$$\delta_1 > \beta_2 + \frac{3}{4}\delta_3 + \frac{\beta_2^2}{3\delta_3} \quad (\geq 2\beta_2)$$

is sufficient for the uniqueness of the distributions.

(4) **Ergodic theorem for the reaction diffusion processes.**

In this section, we improve the ergodic result obtained above for the polynomial reaction model (1.9). We suppose, in addition, that

(4.1) $(p(u,v))$ is translation invariant in $\boldsymbol{Z}^d = S$ and $p(u,u) = 0, u \in \boldsymbol{Z}^d$.

The main different point is that we replace the march coupling with the inner reflection coupling (Part I, (5.7)), which is optimal in some sense. Define

$$\begin{aligned}
&u_0(\epsilon) = 1,\\
&u_1(\epsilon) = \Big(\inf_{k \geq 0} \frac{b(k) + a(k+1) - \epsilon}{a(k) + b(k+1) + \epsilon}\Big) \vee 0,\\
&u_l(\epsilon) = \Big(\inf_{k \geq 0} \frac{[b(k) \vee a(k+l) + l]u_{l-1}(\epsilon) + (b(k) \wedge a(k+l))u_{l-1}(\epsilon) - l - \epsilon \sum_{j=0}^{l-1} u_j(\epsilon)}{a(k) + b(k+1) + \epsilon}\Big) \vee 0\\
&\qquad l \geq 2.
\end{aligned}$$

(4.2) **Theorem** (Chen (1990 b)). Under the above hypotheses, if there exists an $\epsilon > 0$ such that $u_l(\epsilon) > 0$ for all $l \geq 1$, then the process is ergodic. In particular, for fixed $\beta_1, \cdots, \beta_m, \ \delta_1, \cdots, \delta_{m+1}$ and large enough β_0, we have $u_l(\epsilon) \geq (1 + \alpha l)^{-1}$ for some $\epsilon, \alpha > 0$ and all $l \geq 0$, and so the process is ergodic whenever β_0 is large enough.

This result also improves a recent theorem due to C. Neuhauser (198?).

(4.3) **Corrollary.** For the first Schlőgl model, suppose that

$$\delta_1 + \delta_2 \cdot \frac{\delta_1 + \beta_0}{1 + \delta_1} > \beta_1$$

and one of the following conditions holds:

$$3\Big[\big(\frac{\delta_1\delta_2}{2(1+\delta_1)}\big)^2 \beta_0\Big]^{1/3} + \frac{\delta_1(\delta_1 + 1 + \delta_2)}{2(1+\delta_1)} > \beta_1$$

or

$$\delta_1 + \frac{2\delta_1\delta_2}{2(1+\delta_1)} \geq \beta_1.$$

Then the process is ergodic.

For the model discussed in this section, a completely known case is only the reversible one :

$$\delta_j = \alpha\beta_{j-1}, \quad 1 \leq j \leq m+1 \ \text{ for some } \ \alpha > 0.$$

(4.4) **Theorem** (Ding, Durrett and Liggett (198?)).

Under the above hypotheses, suppose further that $(p(u,v))$ is symmetric and $\sum_u p(o,u) \mid u \mid < \infty$. Then, the reversible reaction diffusion process is ergodic. Furthermore, the unique stationary distribution is the product measure in which each coordinate has Poisson distribution with mean α.

(5) **Linear growth model.**

Starting from this section, we show some models of reaction diffusion processes or related models which exhibit phase transition. Firstly, we return to the linear growth model (1.8). Recall that

$$\begin{aligned} b(k) &= \beta_0 + \beta_1 k, \qquad \beta_0, \beta_1 > 0, \\ a(k) &= \delta_1 k, \qquad \delta_1 > 0, \\ C_u(k) &= k, \qquad u \in S,\ k \in \boldsymbol{Z}_+. \end{aligned}$$

*

(5.1) **Theorem** (Ding and Zheng (1989)). Suppose that

$$K \equiv \sup_v \sum_u p(u,v) < \infty$$

and let $p(t,u,v)$ denote the Markov chain generated by the Q-matrix $Q = P - I$, where $P = (p(u,v) : u, v \in S)$ and I is the unit matrix on S.

i) If $\beta_1 - \delta_1 + K - 1 < 0$, then the process is ergodic in the sense : there exists a $\mu \in \mathscr{P}(E)$ such that for every $\nu \in \mathscr{P}(E)$ and every bounded cylinder function f,

$$\lim_{t\to\infty} \nu P(t) f = \int f d\mu.$$

ii) If $\beta_0 = 0$, $\beta_1 > \delta_1$ and $\inf_{t>0} \sum_u p(t,u,v) > 0$ for some $v \in S$, then the process is not ergodic.

iii) If $\beta_0 > 0$, $\beta_1 \geq \delta_1$ and

$$0 < \inf_{t>0} \sum_u p(t,u,v) \leq \sup_{t>0} \sum_u p(t,u,v) < \infty$$

for some $v \in S$, then the process is non-ergodic.

In particular, if P is doubly stochastic and $\beta_0 = 0$, then $\beta_1 = \delta_1$ is the critical line of the ergodic and non-ergodic regions. Furthermore, for $S = \boldsymbol{Z}^d$, P being translation invariant and $\overline{P} = \frac{1}{2}(P + P^*)$ being transient, on the critical line, the above authors also presented a construction of the translation invariant stationary distributions with finite moment of the first order.

(6) **Reaction diffusion processes with an absorbing state.**

Take $S = \boldsymbol{Z}^d$ and suppose that the reaction Q-matrix $Q = (q_{ij})$ satisfies the assumptions in front of Theorem (1.6). In addition, suppose that $q(0,j) = 0$ for all $j \in \boldsymbol{N}$, $q(2,0) = 0$ and $q(i,j) = 0$ for all $i > 2$ and $j < 2$.

(6.1) **Theorem** (Li and Zheng (1988))

Under the above hypotheses, consider the reaction diffusion process generated by

$$\begin{aligned} \Omega f(x) = \sum_{u \in \boldsymbol{Z}^d} \Big\{ &\sum_{k \neq 0} q(x_u, x_u + k)[f(x + k e_u) - f(x)] \\ &+ \sum_{\substack{v \in \boldsymbol{Z}^d \\ |v-u|=1}} \frac{x_u}{2d} [f(x - e_u + \delta(x_v, 2) e_v) - f(x)] \Big\} \end{aligned}$$

If

$$q(1,2) > 2^{19}(c+3)(2c+5)(2c+11)$$

where $c = q(1,0) \vee q(2,1)$, then the process is not ergodic. In particular, there exist at least two stationary distributions.

(6.2) **Corollary.** If

$$\beta_1 > 2^{19}(2\delta_1 + 4\delta_2 + 3)(4\delta_1 + 8\delta_2 + 5)(4\delta_1 + 8\delta_2 + 11),$$

then the process generated by

$$\begin{aligned}\Omega f(x) = \sum_{u \in \boldsymbol{Z}^d} \{&\beta_1 x_u[f(x+e_u) - f(x)] + (\delta_1 x_u + \delta_2 x_u^2)[f(x-e_u) - f(x)] \\ &+ \sum_{\substack{v \in \boldsymbol{Z}^d, \\ |v-u|=1}} \frac{x_u}{2d}[f(x - e_u + e_v) - f(x)]\}\end{aligned}$$

is not ergodic and so there exist at least two stationary distributions.

The last result and Theorem (4.2) are the two extremal cases of reaction diffusion processes. Here we claim that the process exhbits a phase transition when $\beta_0 = 0$ and what we claimed there is the ergodicity for a large enough β_0. From this point of view, we may conjecture that there exists a phase transition for the processes when $\beta_0 > 0$. However, as we have seen before, the reversible case does not support this conjecture. Then, what can you say?

To conclude this paper, we mention a related model which may be considred as the mean field approximation of the reaction diffusion processes. For this model, there does exist a phase transition !

(7) **Non-linear Master equations.**

Here is a one-dimensional process. The state space is $E = \boldsymbol{Z}_+$. For simplicity, we consider only the Q-matrix $q(i,j)$ given in (1.9). We define the Skorohod space $\boldsymbol{D} = \boldsymbol{D}([0,\infty), E), \mathscr{B}(\boldsymbol{D})$ and $\{\mathscr{M}_t\}_{t \ge 0}$ as usual. Let

$$\mathscr{P}_1(E) = \{\mu \in \mathscr{P}(E) : \sum_{i=0}^{\infty} i\mu(i) < \infty\}.$$

(7.1) **Definition.** Given $\mu \in \mathscr{P}_1(E)$. We say that $P_\mu \in \mathscr{P}(\boldsymbol{D})$ is a solution to the martingale problem for the Q-matrix $q(i,j)$ if

i) $P_\mu \circ X_0^{-1} = \mu$,

ii) $\mu(t) \equiv P_\mu \circ X_t^{-1} \in \mathscr{P}_1(E), \quad t > 0$,

iii) for every $j \in E$,

$$I_j(X_t) - \int_0^t \Omega_{\mu(s)} I_j(X_s) ds$$

is a $(\boldsymbol{D}, \mathscr{M}_t, P_\mu)-$ martingale, where

$$\Omega_{\mu(t)} f(i) = \sum_{k=\pm 1} q(i, i+k)[f(i+k) - f(i)] + (E_\mu X_t)(f(i+1) - f(i)), \qquad i \in E,\ t \ge 0.$$

(7.2) **Theorem** (Feng and Zheng (1988)).

i) The solution to the martingale problem for $(q(i,j))$ is well-posed. The unique solution is Markovian (may be time-inhomogeneous).

ii) If the constant $K_{2'}$ defined just before (1.6) is negative, then the stationary distribution of the process in unique.

iii). For the second Schlögl model, if

$$(1+\delta_1)^2 \le \frac{1}{2} + \frac{1 + 2\beta_2}{3(1 + \delta_1 + 2\delta_3)}$$

then there is a constant $c > 0$ such that for each $\beta_0 \in [0, c)$, there exist at least three stationary distributions for the process.
(7.3) **Remark** By Chen (1990), it follows that the above assertion ii) can be improved as follows: if the sequence $\{u_l(\epsilon)\}_{l\geq 0}$ defined in Section 4 is positive for some $\epsilon > 0$, then the stationary distribution is unique.
(7.4) **Remark.** For the reversible second Schlőgl model, it happens that the coefficients β/s and δ/s satisfy the condition given in the above iii). Hence the mean field approximation of an infinte dimensional reaction diffusion process is usually not the same as the original one.

References

*

Basis, V. Ya (1980). Multicomponent Random Systems, edited by R.L. Dobrushin and Ya, G. Sinai, Marcel Dekker, Inc. 37-58.
Chen, M. F. (1980). Chin. Ann. Math. 1, 437-451.
(1985). Acta Math. Sinica 1, 261-273.
(1986a). Acta Math. Sinica 2, 123-136.
(1986b). Jump Processes and Particle Systems, Beijing Normal Univ. Press.(1987). Sci. Sinica 30, 148-156.
(1989a). Acta Math. Sci. 9:1, 7-19.
(1989b). Quart. J. Math. 4,
(1990). J. Statis. Phys. 58, 58:5/6, 939-966.
(1991). J. Applied Probab. 28, 305-320.
(1991a). On Coupling of Jump Processes (II), Chin, Ann. Math.., 1991, 12(B): 4, 385-399.
(1991b). Uniqueness of reaction diffusion processes, Kexue Tongbao 36:12, 969-973.
Chen, M. F. and Li, S. F. (1989). Ann. Probab. 17, 151-177.
Chen, M. F. and Lu, Y. G.
(1989) Large deviations for Markov chains, Acta Math. Sci.
(1990) On evaluating the rate function of large deviations for jump processes, Acta Math. Sinica.
Chen, M. F. and Zheng, X. G.(1983). Sci. Sinica 1, 11-24.
Dai, Y. L. (1986). Acta Math. Sinica 29, 103-111.
Dai, Y. L. and Liu, X. J. (1985). Chin, J. Appl. Probab. and Statis. 1, 31-42.
Dai, Y. L. and Liu, X. J. (1986). Acta Math. Sinica 2, 92-104.
Ding, W. D. and Chen, M. F. (1981). Chin. Ann. Math. 2, 47-51.
Ding, W. D., Durrett, R. and Liggett, T. M. (1990). Ergodicity of reversible reaction diffusion processes. 《Probability Theory & Related Fields》 85 (1) :13-26 .
Ding,* W.* D.* and Zheng,* X.* G. (1989). Chin. Ann. of * Math. 10B(3), 386-402.
Feller, W. (1940). Trans. Amer. Math. Soc. 48, 488-515.
Feng, S. and Zheng, X. G. (1988). Carleton Univ. Dreprint No. 115.
Han, D. (1989). Ph. D. Thesis at Beijing Normal Univ.
Holley, R. and Liggett, T.M. (1981). Z.Wahrs. 55, 165-195.
Hou, Z. T. and Chen, M. F. (1980). Kexue Tongbao 25, 807-811.
Hu, D. H. (1966). Acta Math. Sinica 16, 150-165.
Huang, L. P. (1987). Chin. J. Appl. Probab. and Statis. 3, 152-158.
Li, S. Q. (1983). Chin. Ann. Math. 4, 773-780.
Li, Y. and Zheng, X. G. (1988). Non-ergodicity for a class of reaction diffusion processes, Master thesis at Beijing Normal Univ.
Liggett, T. M. (1985). T. M. (1985). Interacting Particle Systems, Springer-Verlag.
Liu, X. J. (1987). Chin. J. Appl. Probab. and Statis. 3, 38-45.
Ren, K. L. (1983). Acta Math. Sci. 3, 300-320.
Tang, S. Z. (1982). Acta Math. Sinica 25, 306-314.

Wang, H. X. (1987). Acta Math. Sci. 7, 55-69.
Wu, Y. D. (1983). Acta Math. Sci. 3, 180-196.
Yan, S. J. and Chen, M. F. (1986). Chin. Ann. Math. 7, 90-110.
Yan, S. J., Chen, M. F. and Ding, W. D. (1982 a). Chin. Ann. Math. 3, 572-586.
(1982 b). Chin. Ann. Math. 3, 705-720.
Zeng, W. Q. (1983). Chin. Ann. Math. 4, 763-772.
Zheng, X. G. (1988). Acta Math. Sinica 4, 193-209.
Zheng, X. G. and Zeng, W. Q. (1986). Chin. J. Appl. Probab. and Statis. 2, 334-340.
(1987). Acta Math. Sci. 7, 169-175.