UNIQUENESS OF REACTION DIFFUSION PROCESSES*

CHEN MU-FA (陈木法)
(*Beijing Normal University, Beijing* 100875, *PRC*)

Received July 7, 1989; revised October 12, 1989.

Keywords: particle system, reaction diffusion process, uniqueness.

I. INTRODUCTION

This note proves the uniqueness of reaction diffusion processes constructed by Chen[1]. Let $\mathbb{Z}_+$ be the set of nonnegative integers, S a countable set and $E=\mathbb{Z}_+^S$. For each $u \in S$, suppose that we are given on $\mathbb{Z}_+$ a function $C_u \geqslant 0$ with $C_u(0)=0$ and a conservative Q-matrix $Q_u=(q_u(i,j))$. For convenience, we set $q_u(i,j)=0$ for $j<0$. Moreover, let $P=(p(u,v))$ be a transition probability matrix on S. The formal generator of the processes considered here is as follows:

$$\begin{aligned}\Omega f(\eta)=&\sum_{u\in S}\sum_{k\neq 0} q_u(\eta(u),\ \eta(u)+k)[f(\eta+ke_u)-f(\eta)]\\&+\sum_{u,v\in S} C_u(\eta(u))\,p(u,v)[f(\eta-e_u+e_v)-f(\eta)]\\&=\Omega_r f(\eta)+\Omega_d f(\eta),\quad \eta\in E,\end{aligned}\tag{1}$$

where e_u is the unit vector in E with value 1 at u. Ω_r and Ω_d are called the reaction part and the diffusion part of Ω respectively. We need the following hypotheses:

(H_1) *Growing condition*

$$C=\sup_{u,k}|C_u(k)-C_u(k+1)|<\infty,\ \sup_u\sum_{k\neq 0} q_u(i,i+k)|k|\leqslant C_1(1+i^m),\quad i\in\mathbb{Z}_+,$$

where m is the minimal natural number so that the above control holds and C_1 is a constant.

(H_2) *Lipschitz condition*

$$C_2=\sup\{g_u(j_1,j_2)+h_u(j_1,j_2): u\in S, j_2>j_1\geqslant 0\}<\infty,$$

where

$$g_u(j_1,j_2)=\sum_{k\neq 0}[q_u(j_2,j_2+k)-q_u(j_1,j_1+k)]\,k/(j_2-j_1),$$

$$\begin{aligned}h_u(j_1,j_2)=2\sum_{k=1}^{\infty}[&(q_u(j_2,j_1-k)-q_u(j_1,2j_1-j_2-k))^+\\&+(q_u(j_1,j_2+k)-q_u(j_2,2j_2-j_1+k))^+]\,k/(j_2-j_1).\end{aligned}$$

* Project supported in part by Ying-Tung Fok Education Foundation and the National Natural Science Foundation of China.

(H_3) *Moment condition*

$$\sup_u \sum_{k\neq 0} q_u(i,i+k)[(i+k)^m - i^m] \leqslant C_3(1+i^m), \quad i\in \mathbb{Z}_+ .$$

(H_4) *Transition condition*

$$\sup_v \sum_u p(u,v) < \infty .$$

The state space of the processes is $E_1 = \{\eta\in E: \|\eta\| = \sum_u \eta(u)\alpha(u) < \infty\}$, where $(\alpha(u): u\in S)$ is a positive summable sequence such that $\sum_v p(u,v)\alpha(v) \leqslant M\alpha(u)$ for some $M>0$ and all $u\in S$. As an example, we take $\alpha(u) = \sum_{n=0}^{\infty} M^{-n}\sum_v p^{(n)}(u,v)d(v)$, $u\in S$, where $M>1$, $(p^{(n)}(u,v)) = P^n$ and $(d(u): u\in S)$ is a positive summable sequence. Set

$$C_4 = \sup\{[C_u(j_1) - C_u(j_2)]/(j_2 - j_1): u\in S, j_2 > j_1 \geqslant 0\},$$

$$E_m = \{\eta\in E: |||\eta||| = \sum_{u\in S}\eta(u)^m\alpha(u) < \infty\},$$

and we denote by $\mathscr{L}_m$ the set of Lipschitz continuous functions on E_1 with respect to $\|\eta-\zeta\|_m = \sum_{u\in S}|\eta(u)-\zeta(u)|\alpha(u)^m$. For $f\in\mathscr{L}_m$, let $L_m(f)$ be the Lipschitz constant of f. But we omit the index m when $m=1$. Finally, let $\{\Lambda_n\}_1^\infty$ be a fixed sequence of finite subsets of S such that $\Lambda_n\uparrow S$. Replacing S with Λ_n in (1) we may define Ω_n, $\Omega_{n,r}$ and $\Omega_{n,d}$. The semigroup corresponding to Ω_n is denoted by $S_n(t)$.

II. Main Results

Theorem 1. *If* (H_1)—(H_4) *hold, then there exists uniquely a semigroup of positive operators $S(t)$ on $\mathscr{L}$ such that $S(0)=I$; $S(t)$ is strong contraction on the uniform closure $\overline{\mathscr{L}}$ of $\mathscr{L}$; $\lim_{n\to\infty} S_n(t)f(\eta) = S(t)f(\eta)$ for $f\in\mathscr{L}$, $\eta\in E_m$ and $t\geqslant 0$. Moreover,*

(i) *$S(t)f\in\mathscr{L}$ and $L(S(t)f)\leqslant L(f)\exp[t(C_2+C_4+C(M+1))]$ for $f\in\mathscr{L}$ and $t\geqslant 0$;*

(ii) *$S(t)(|||\cdot|||)(\eta)\leqslant C(t)(1+|||\eta|||)$ for $\eta\in E_m$ and $t\geqslant 0$, where $C(t)$ is a constant depending on t only;*

(iii) *$\frac{d}{dt}S(t)f = \Omega S(t)f = S(t)\Omega f$ for $f\in\mathscr{L}$, $\eta\in E_m$ and $t\geqslant 0$.*

Finally, there exists a Markov process $(\{\eta_t\}_{t\geqslant 0}, P^\eta)$ on E_1 such that

$$S(t)f(\eta) = \int f(\xi)P^\eta(\eta_t\in d\xi) = \int f(\xi)P(t,\eta,d\xi),$$

where $P(t,\eta\, d\xi)$ is the transition function of the process.

To state another uniqueness result, we replace (H_3) and (H_4) respectively with

(H_3') $\Omega_{n,r}((\|\cdot\|)^m)(\eta)\leqslant C_3'(1+\|\eta\|^m)$, $\eta\in\mathbb{Z}_+^n$, $n\geqslant 1$,

where C_3' is a constant independent of n,

(H_4') there is a positive summable sequence $(\alpha(u))$ and a constant $M(m)>0$ such that

$$\sum_v p(u,v)\alpha(v)^m \leqslant M(m)\alpha(u)^m,\ u\in S.$$

Theorem 2. *Let* (H_1), (H_2), (H_3') *and* (H_4') *be satisfied. Then the assertions of Theorem* 1 *hold provided* (ii) *and* (iii) *are replaced by*

(ii)′ $S(t)(\|\cdot\|^m)(\eta)\leqslant C(t)'(1+\|\eta\|^m)$ *for* $\eta\in E_1$ *and* $t\geqslant 0$;

(iii)′ *the above* (iii) *holds for* $f\in\mathscr{L}_m$ *and* $\eta\in E_1$.

Moreover, the above (i) *can be stressed as follows*:

(i)′ $S(t)f\in\mathscr{L}_m$ *for* $f\in\mathscr{L}_m$ *and* $t\geqslant 0$.

Remark. Note that (H_4) plus

$M_1=\sup\{p(u,v)^{1-m}: u,v\in s \text{ and } p(u,v)>0\}<\infty$

imply (H_4'). Indeed, if we take $(\alpha(u))$ as before, then

$$\sum_v p(u,v)\alpha(v)^m\leqslant M_1\sum_v\{p(u,v)^m\alpha(v)^m\}\leqslant M_1\Big(\sum_v p(u,v)\alpha(v)\Big)^m\leqslant (M_1M^m)\alpha(u)^m,\ u\in S.$$

Corollary 1. *Take* $C_u(k)=k$ *and let the reaction be the type of birth-death*:

$q_u(i,i+1)=b(i),\ i\geqslant 0;\ q_u(i,i-1)=a(i),\ i\geqslant 1,\ u\in S.$

Suppose that for some $c\in(0,1)$, *we have* $\overline{\lim\limits_{i\to\infty}}\,[b(i)-c^m a(i)]/i<\infty$. *Then* (H_3) *and* (H_3') *are satisfied. Furthermore, if* (H_2) *and* (H_4)(*resp.* (H_4')) *hold, then Theorem* 1 (*resp. Theorem* 2) *is applicable.*

Proof. Here, we check (H_3') only. Choose $N^1=N^1(m)$ so that $\|\eta-e_u\|\geqslant c\,\|\eta+e_u\|$ whenever $\eta(u)\geqslant N^1$. Next, choose N^2 so that $[b(i)-c^m a(i)]/i\leqslant A$ for some $A\in(0,\infty)$ and all $i\geqslant N^2$. Put $N=N^1\vee N^2$. Then, for each $n\geqslant 1$, we have

$$\begin{aligned}\Omega_{n,r}(\|\cdot\|^m)(\eta)&=\sum_{u\in\Lambda_n}\{b(\eta(u))[\|\eta+e_u\|^m-\|\eta\|^m]+a(\eta(u))[\|\eta-e_u\|^m-\|\eta\|^m]\}\\&=\sum_{u\in\Lambda_n}\alpha(u)\sum_{l=0}^{m-1}\|\eta\|^l\|\eta+e_u\|^{m-1-l}\left[b(\eta(u))-a(\eta(u))\left(\frac{\|\eta-e_u\|}{\|\eta+e_u\|}\right)^m\right]\\&=\sum_{u\in\Lambda_n,\,\eta(u)\geqslant N}+\sum_{u\in\Lambda_n,\,\eta(u)\leqslant N-1}\\&\leqslant m\big(A+\max_{0\leqslant i\leqslant N-1}b(i)\big)\,(\|\eta\|+|\alpha|)^m,\ \eta\in\mathbb{Z}_+^{\Lambda_n},\ n\geqslant 1,\end{aligned}$$

where $|\alpha|=\sum_u\alpha(u)$.

Corollary 2. *For the autocatalytic model*: $C_u(k)=k$, $q_u(i,i+1)=\beta_1 i$, $q_u(i,i-2)=\delta_2 i(i-1)$, $k,i\in\mathbb{Z}_+$, $u\in S$, *the same conclusions of Corollary* 1 *hold.*

When $S=\mathbb{Z}$ and P is the simplest random walk, the uniqueness conclusion in the sense of Theorem 2 for the last model was proved by Zheng[2].

III. Proofs

We first prove Theorem 1 briefly.

a) It follows from [3, Theorem 16] or [4, Theorem 2.3.7] that $S_n(t)$ is uniquely determined by Ω_n. For $m\geqslant 2$, by (H_1), (H_4) and the Hölder inequality, we have

$$\Omega_{n,d}(|||\cdot|||)(\eta)=\sum_{u,v\in\Lambda_n} C_u(\eta(u))\, p(u,v)[\,|||\eta-e_u+e_v|||-|||\eta|||\,]$$

$$\leqslant(2^m-2)\sum_{u,v\in\Lambda_n} C\eta(u)[p(u,v)\alpha(v)]^{1/m}[p(u,v)\alpha(v)]^{(m-1)/m}\eta(v)^{m-1}+CM\|\eta\|$$

$$\leqslant(2^m-2)C\Big[\sum_{u,v\in\Lambda_n}\eta(u)^m p(u,v)\alpha(v)\Big]^{1/m}\Big[\sum_{u,v\in\Lambda_n}p(u,v)\alpha(v)\eta(v)^m\Big]^{(m-1)/m}+CM\|\eta\|$$

$$\leqslant(2^m-2)CM^{1/m}|||\eta|||^{1/m}\Big(\sup_v\sum_u p(u,v)\Big)^{(m-1)/m}\cdot|||\eta|||^{(m-1)/m}+CM\|\eta\|$$

$$\leqslant \text{const.}\,|||\eta|||.$$

This inequality holds even for $m=1$. From this and (H_3), we see that there is a constant $\overline{C}_3$ so that $\Omega_n(|||\cdot|||)(\eta)\leqslant\overline{C}_3(1+|||\eta|||)$. Thus, by [2, Lemma 6.7.19], we have

$$S_n(t)(|||\cdot|||)(\eta)\leqslant\exp[\overline{C}_3 t]\,(1+|||\eta|||),\quad t\geqslant 0,\ \eta\in\mathbb{Z}_+^{\Lambda_n},\ n\geqslant 1.$$

By using an approximating argument, we obtain

$$S(t)(|||\cdot|||)(\eta)\leqslant\exp[\overline{C}_3 t]\,(1+|||\eta|||),\quad t\geqslant 0,\ \eta\in E_m. \tag{2}$$

This proves not only (ii) but also that E_m is a closed set of the process.

b) By [1], [4] and [5] we know that there exists a semigroup $S(t)$ having all properties in Theorem 1 except the last equality of (iii). However, it follows from (H_1) that

$$|\Omega f(\eta)|\leqslant\overline{C}_1 L(f)(1+|||\eta|||),\quad \text{for } f\in\mathscr{L} \text{ and } \eta\in E_m.$$

Hence $|\Omega S(t)f(\eta)|\leqslant\overline{C}_1 L(S(t)f)(1+|||\eta|||)$ and so

$$|S(t)f(\eta)-f(\eta)|/t\leqslant\frac{1}{t}\int_0^t|\Omega S(s)f(\eta)|ds\leqslant\overline{C}_1(1+|||\eta|||)L(f)\exp[C_2+C_4+C(M+1)]$$

for $t\leqslant 1$, $f\in\mathscr{L}$ and $\eta\in E_m$. Therefore, by (2) and the dominated convergence theorem, we get

$$\lim_{s\to 0}S(t)[S(s)f(\eta)-f(\eta)]/s=S(t)\Omega f(\eta),\quad t\geqslant 0,\ f\in\mathscr{L},\ \eta\in E_m.$$

This proves the last equality of (iii).

c) Now, let $S_k(t)$, $k=1,2$ be two semigroups having the properties in Theorem 1. To prove the uniqueness, we need only to show that $S_1(t)f=S_2(t)f$, $t\geqslant 0$ for all bounded $f\in\mathscr{L}$. Since E_m is dense in E_1 with respect to $\|\cdot\|$, by the Lipschitz property of the semigroups, it suffices to show that $S_1(t)f(\eta)=S_2(t)f(\eta)$ for all $\eta\in E_m$ and $t\geqslant 0$. On the other hand, E_m is a closed set of $S_k(t)$, $k=1,2$, the required fact is a straightforward consequence of (iii) (see [4, Corollary 6.4.22] for details).

Remark. In view of the above proof, we can restrict ourselves to E_m in the study of the process.

Now, we turn to prove Theorem 2.

a) By (H_4') and [1], [5], we can use $(\alpha(u)^m)$ and $M(m)$ instead of $(\alpha(u))$ and M respectively to construct a semigroup $S(t)$ having the Lipschitz property with respect to $\|\cdot\|_m$. So (i)′ holds.

b) Because of

$$\Omega_{n,d}(\|\cdot\|^m)(\eta)\leqslant mCM\|\eta\|(\|\eta\|+|\alpha|)^{m-1},\ \eta\in\mathbb{Z}_+^{\Lambda_n},\ n\geqslant 1$$

and (H_3'), there is a constant $\overline{C}_3'$ such that

$$\Omega_n(\|\cdot\|^m)(\eta)\leqslant\overline{C}_3'(1+\|\eta\|^m),\ \eta\in\mathbb{Z}_+^{\Lambda_n},\ n\geqslant 1.$$

A similar argument as above gives us

$$S(t)(\|\cdot\|^m)(\eta)\leqslant\exp[\overline{C}_3't](1+\|\eta\|^m),\ t\geqslant 0,\ \eta\in E_1.$$

This is just (ii)′.

c) For $f\in\mathscr{L}_m$, we have

$$|\Omega f(\eta)|\leqslant L_m(f)[\sum_u C_1(1+\eta(u)^m)\alpha(u)^m+C(1+M(m))\sum_u\eta(u)\alpha(u)^m]$$
$$\leqslant\text{const.}\,L_m(f)(1+\|\eta\|^m),\ \eta\in E_1.$$

Now, the remainder of the proof is similar to the previous one.

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