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COMPARISON THEOREMS FOR GREEN FUNCTIONS OF MARKOV CHAINS

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Abstract

The author presents some straightforward proofs for two comparison theorems for Green functions of Markov chains, which slightly improve the previous results by Varopoulos[6,7], Durrett[3] and Yan and Chen [10]. A recent result by Rogers and Williams[7] about instantaneous Markov chains is also improved by using the same idea.

§1. Introduction

Let E be a countable set and P a transition probability on E. Denote by $\mathscr{E}_+$ the set of all nonnegative functions on E. A measure μ on E is called excessive for P if

$$\mu_i \geqslant \sum_j \mu_j P_{ji},\ i \in E$$

and is called invariant for P if the above inequality becomes equality. The measures μ and ν on E are called equivalent if

$$C^{-1} \leqslant \frac{d\mu}{d\nu} \leqslant C$$

for some constant $0 < C < \infty$.

Since the transience of symmetrizable Markov Chains is well understood (see Griffeath and Liggett [4], Lyons [6] and Varopoulos [8, 9]), it is interesting to know the transience of non-symmetrizable Markov chains by using the criteria for the transience of symmetrizable ones. The following is one of the results of such kind.

Theorem 1. *Let P and Q be two irreducible transition probability which have excessive measure μ and invariant measure ν respectively. Suppose that*

(a) *μ and ν are equivalent;*

(b) *Q is symmetrizable with respect to ν; i. e.*

$$\nu_i Q_{ij} = \nu_j Q_{ji}$$

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for all i, $j\in E$;

(c) *$P\geqslant \varepsilon Q$, i. e. $P_{ij}\geqslant \varepsilon Q_{ij}$ for some $\varepsilon>0$ and all i, $j\in E$.*

Then for all $\lambda<1$ and $f\in \mathscr{E}_{+}$,

$$\sum_{n=0}^{\infty}\left[\frac{\lambda}{2-\lambda}\right]^{n}\langle P^{n}f,\ f\rangle_{\nu}\leqslant K\sum_{n=0}^{\infty}\lambda^{n}\langle Q^{n}f,\ f\rangle_{\nu},$$

where $K=2C^{2}\Big/\left(\frac{1}{2}\wedge\varepsilon\right)$ and the subscript indicates that the inner product is taken with respect to ν. In particular, if Q is transient, then so is P.

In the case of μ being an invariant measure for P and $\nu=\mu$, the above result was obtained by Varopoulos[6,7], and then extended by Durrett [3] to the case where μ and ν are equivalent. The new point here is the use of the excessive measure μ instead of the invariant one for P. This is more reasonable since a non-trivial excessive measure for a transient Markov chain always exists but is not necessarily an invariant one.

A straightforward proof of Theorem 1 is given in the next section. The same proof shows that the theorem can be generalized to the context of an arbitrary Markov process with discrete time parameter and an arbitrary state space.

The next result shows that the converse direction is also possible. That is to get the recurrence of a Markov chain by comparing it with a related simpler Markov chain.

Theorem 2. *Take $E=\{0,1,2,\cdots\}$. Let $X=\{x,y,z,\cdots\}$ be another countable set with a reference point θ. Let $|\cdot|$ be a surjective mapping from X to E such that $|\theta|=0$, $|x|\neq 0$ if $x\neq\theta$ and $\#\{x\colon |x|=k\}$ is finite for each $k\in E$.*

Let P and Q be irreducible transition probabilities on X and E respectively. Suppose that

(a) *$Q_{ij}>0$ if $j=i+1$, $Q_{ij}=0$ if $j>i+1$;*

(b) $\max\{\sum_{y:|y|=j}P(x,y)\colon |x|=i\}\leqslant Q_{ij},\ j>i,$

$\min\{\sum_{y:|y|=j}P(x,y)\colon |x|=i\}\geqslant Q_{ij},\ j<i,\ i,\ j\in E,$

$\sum_{y:|y|=i+1}P(x,y)>0,\ |x|=i,\ i\in E.$

Then

$$\min_{x:|x|=i}\left[\sum_{n=0}^{\infty}P^{n}\right](x,\ \theta)\geqslant\left[\sum_{n=0}^{\infty}Q^{n}\right]_{i,0},\quad i\in E.$$

In particular, if Q is recurrent, then so is P.

The last conclusion was proved by Yan and Chen [10]. In the next section we will present a slightly different proof.

Now we give another application of the above idea. Recall that for a standard Markov chain $P(t)=(P_{ij}(t)\colon i,\ j\in E)$, we have

$$\lim_{t\downarrow 0}\frac{1-P_{ii}(t)}{t}\equiv q_{i}\equiv -q_{ii}\leqslant+\infty,$$

$$\lim_{t\downarrow 0}\frac{P_{ij}(t)}{t}\equiv q_{ij}<\infty,\ i\neq j$$

and

$$\sum_{j\neq i} q_{ij}\leqslant q_i,\ i\in E.$$

In this note, we restrict ourselves on the conservative case. That is

$$\sum_{j\neq i} q_{ij}=q_i,\ i\in E.$$

Theorem 3. *Let $P(t)$ be a standard Markov chain with matrix $Q=(q_{ij})$ and excessive measure μ.*

$$\mu_j\geqslant\sum_i \mu_i P_{ij}(t),\ t\geqslant 0,\ j\in E. \tag{4}$$

Then $P(t)$ is a strongly continuous and contractive semigroup on $L^2(\mu)$. Denote by L the generator of $P(t)$ on $L^2(\mu)$ and define

$$\mathscr{E}^Q(f,f)=\frac{1}{2}\sum_{i,j}\mu_i q_{ij}(f_j-f_i)^2,$$

$$\mathscr{D}_Q=\{f\in L^2(\mu):\ \mathscr{E}^Q(f,f)<\infty\}.$$

Then $\mathscr{D}(L)\subset\mathscr{D}^Q$. In particular, $\mathscr{D}^Q$ is dense in $L^2(\mu)$.

This result is proved in [7] in the case of $P(t)$ being irreducible and recurrent. Refer to [7] for the application of this result. Certainly, it is more interesting if we require $\mu_i>0$, $i\in E$.

Theorem 5. *There exists a positive excessive measure for $P(t)$ if and only if there is no transient state i such that*

$$P_{ij}(t)>0 \tag{6}$$

for some recurrent state j.

Corollary 7. *If $Q=(q_{ij})$ is irreducible, then for any $P(t)$ (if exists), the condition of Theorem 5 holds.*

§2. Proofs

For the reader's convenience, we restate a result, which is easy to check, due to Baldi, Lohoué and Peyriére[1].

Lemma 8. *Let A and B be two invertible operators on a real Hilbert space with*

$$0\leqslant(Ax,x)\leqslant(Bx,x)$$

for all x and A be symmetric. Then

$$(B^{-1}x,x)\leqslant(A^{-1}x,x)$$

for all x.

Proof of Theorem 1 Let $\varphi=d\nu/d\mu$ and define

$$P'=(I+P)/2,$$

$$Q'=(1-\varphi/2C)I+(\varphi/2C)Q,$$

where I denotes the identity operator. Then it is easy to check that P' and Q' are transition probabilities having μ as their excessive and invariant measures respectively. Since $C^{-1}\leqslant\varphi\leqslant C$, we have $P'\leqslant\left[\frac{1}{2}\wedge\varepsilon\right]Q'$. Set $\delta=\frac{1}{2}\wedge\varepsilon$. Hence $(P'-\delta Q')/(1-\delta)$ is a transition probability and contractive on $L^2(\mu)$, and so

$$0\leqslant\langle(P'-\delta Q')f/(1-\delta),\ f\rangle_\mu\leqslant\langle f,\ f\rangle_\mu.$$

Thus for every $\lambda\leqslant1$, we obtain

$$\langle(I-\lambda P')f,\ f\rangle_\mu\geqslant\delta\langle(I-\lambda Q')f,\ f\rangle_\mu.$$

Similarly from the equality

$$[Q'-(\varphi/2C)Q]/(1-\varphi/2C)=I$$

it follows that

$$\langle(I-\lambda Q')f,\ f\rangle_\mu\geqslant\langle\varphi(I-\lambda Q)f,\ f\rangle_\mu/2C=(1/2C)\langle(I-\lambda Q)f,\ f\rangle_\nu.$$

Therefore

$$(\delta/2C)\langle(I-\lambda Q)f,\ f\rangle_\nu\leqslant\langle(I-\lambda P')f,\ f\rangle_\mu=\left\langle\left(I-\frac{\lambda}{2-\lambda}P\right)f,f\right\rangle_\mu\leqslant C\left\langle\left(I-\frac{\lambda}{2-\lambda}P\right)f,f\right\rangle_\nu.$$

From this and Lemma (8), the conclusion follows immediately.

Proof of Theorem 2 Let

$$G(x)=\left(\sum_{n=0}^{\infty}P^n\right)(x,\ \theta),$$

$$G_i=\left(\sum_{n=0}^{\infty}Q^n\right)_{i,\ 0}.$$

Then $\{G(x):x\in X\}$ and $\{G_i:i\in E\}$ are the minimal nonnegative solutions to the following equations

$$u(x)=\sum_y P(x,\ y)u(y)+P(x,\ \theta),\ x\in X$$

and

$$u_i=\sum_j Q_{ij}u_j+Q_{i0},\ i\in E$$

respectively. Put

$$v_i=\min_{x:|x|=i}G(x).$$

As usual (cf. [2]) we need only to show that

$$v_i\geqslant\sum_j Q_{ij}v_j+Q_{i0},\ i\in E.$$

Choose $x^{(k)}\in\{x:\ |x|=k\}$ such that

$$v_k=G(x^{(k)}).$$

Without loss of generality, we may assume that $0<v_k<\infty$. Then

$$v_0=g(x^{(0)})=g(\theta)=\sum_{y:|y|=1}P(\theta,\ y)G(y)+P(\theta,\ \theta)$$

$$\geqslant\sum_{y:|y|=1}P(\theta,\ y)v_1+(1+v_0)P(\theta,\ \theta).$$

This can be rewritten as

$$\sum_{y:|y|=1} P(\theta, y)(v_0 - v_1) \geqslant 0$$

and so $v_0 \geqslant v_1$. In general

$$v_k = G(x^{(k)}) = \sum_{j=0}^{k} \sum_{y:|y|=j} P(x^{(k)}, y)G(y) + \sum_{y:|y|=k+1} P(x^{(k)}, y)G(y) + P(x^{(k)}, \theta).$$

$$\geqslant \sum_{j=0}^{k} \sum_{y:|y|=j} P(x^{(k)}, y)v_j + \sum_{y:|y|=k+1} P(x^{(k)}, y)v_{k+1} + P(x^{(k)}, \theta).$$

Hence

$$\sum_{y:|y|=k+1} P(x^{(k)}, y)(v_k - v_{k+1}) \geqslant \sum_{j=0}^{k} \sum_{y:|y|=j} P(x^{(k)}, y)(v_j - v_k) + P(x^{(k)}, \theta).$$

This gives us not only $v_k \downarrow$ but also

$$Q_{k,k+1}(v_k - v_{k+1}) \geqslant \sum_{j=0}^{k} Q^t{}_j(v_j - v_k) + Q_{k0},$$

which is just what we required.

Proof of Theorem 3 To prove the strong continuity, let $f \in L^2(\mu)$. By (4), we have

$$\sum_{i \neq j} \mu_i P_{ij}(t) \leqslant \mu_j(1 - P_{jj}(t))$$

and so

$$\|P(t)f - f\|^2 = \sum_i \mu_i \Big(\sum_j P_{ij}(t) f_j - f_i\Big)^2 \leqslant 2\Big[\sum_i \mu_i (1 - P_{ii}(t))^2 f_i^2 + \sum_i \mu_i \Big(\sum_{j \neq i} P_{ij}(t) f_j\Big)^2\Big]$$

$$\leqslant 2\Big[\sum_i \mu_i (1 - P_{ii}(t)) f_i^2 + \sum_i \mu_i \sum_{j \neq i} P_{ij}(t) f_j^2\Big]$$

$$\leqslant 4 \sum_i \mu_i f_i^2 (1 - P_{ii}(t)) \to 0, \text{ as } t \downarrow 0.$$

More easily, we may prove the contraction. Now, let $f \in \mathscr{D}(L)$. Again, by (4), we may have

$$\langle 1, P(t)f \rangle^2 = \sum_j f_j^2 \sum_i \mu_i P_{ij}(t) \leqslant \langle f, f \rangle.$$

Hence

$$\frac{1}{2} \sum_{i,j} \mu_i P_{ij}(t)\ (f_j - f_i)^2 = \frac{1}{2}[\langle 1, P(t)f^2 \rangle + \langle f^2, P(t)1 \rangle - 2\langle f, P(t)f \rangle]$$

$$\leqslant \langle f - P(t)f, f \rangle.$$

Thus

$$\infty > -\langle Lf, f \rangle = \lim_{t \downarrow 0} \left\langle \frac{f - P(t)f}{t}, f \right\rangle \geqslant \frac{1}{2} \sum_{i,j} \mu_i \lim_{t \downarrow 0} \frac{P_{ij}(t)}{t} (f_j - f_i)^2$$

$$= \mathscr{E}^Q(f, f).$$

Proof of Theorem 5 The necessity can be deduced to the discrete time case which was proved in [5, Theorem 8.3.1]. Now, assume that the condition holds. By the ordinary procedure we can decompose the state space into some irreducible recurrent classes and a transient class. On each recurrent class we have uniquely a positive invariant measure. Once we construct a positive excessive measure on the transient class, combining these measures together in a natural way we will get the required measure for $P(t)$. Hence, without loss of generality, we assume that $P(t)$ is transient (here we have used our assumption). In this case, we simply choose an arbitrary positive probability measure α and set

$$\mu_j=\sum_i \alpha_i\int_0^\infty P_{ij}(t)\,dt,\ j\in E.$$

Then

$$0<\alpha_j\int_0^\infty P_{jj}(t)\leqslant\mu_j=\sum_i \alpha_i\int_0^\infty dt\int_0^t f_{ij}(s)\,ds\,P_{jj}(t-s)$$
$$=\sum_i \alpha_i\int_0^\infty f_{ij}(s)\,ds\int_0^\infty P_{jj}(t)\,dt\leqslant\int_0^\infty P_{jj}(t)\,dt<\infty,$$

where $f_{ij}(s)ds$ is the probability that the process $(X_t)_{t>0}$ starting at i first hits j between times s and $s+ds$.

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