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Exponential L^2-Convergence and L^2-Spectral Gap for Markov Processes*)

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Abstract. This paper deals with the exponential L^2-convergence for jump processes. We introduce some reduction methods and improve some previous results. Then we prove that for birth-death processes, exponential L^2- convergence coincides indeed with exponential ergodicity which is widely studied in the Markov chain theory.

1. Introduction

Let $(E, \mathscr{E}, \pi)$ be a probability space, $\{P(t)\}_{t\geqslant 0}$ be a positive, strongly continuous, contractive and Markovian semigroup $(P(t)1=1)$ on $L^2(\pi)$ with an invariant measure π. Denote by Ω and $D(\Omega)$ respectively the infinitesimal generator and its domain induced by $\{P(t)\}_{t\geqslant 0}$. We say that $\{P(t)\}$ converges exponentially in the $L^2(\pi)$ norm if there is a positive ε such that for all $f\in L^2(\pi)$,

$$(1.1)\qquad \|P(t)f-\pi(f)\| \leqslant e^{-\varepsilon t}\|f-\pi(f)\|,$$

where $\|\cdot\|$ denotes the $L^2(\pi)$ norm and $\pi(f)=\int f d\pi$.

Since the constant function $1\in D(\Omega)$ and $\Omega 1=0$, the vector 1 is an eigenvector of Ω with eigenvalue 0. One may seek for the next-to-largest eigenvalue (resp. real part) of the self-adjoint (resp. non- self-adjont) generator Ω. That is, to seek for the infimum of the spectra of $-\Omega$ restricted to the orthogonal complement space $\{f\in L^2(\pi): \pi(f)=0\}\cap D(\Omega)$. This leads us to define

$$(1.2)\qquad \operatorname{gap}(\Omega)=\inf\{-(\Omega f, f): f\in D(\Omega), \pi(f)=0, \|f\|=1\}.$$

We know more or less that (1.1) and (1.2) are closely linked (see the next section for more details).

Exponential convergence in L^2 sense was proved for various classes of stochastic Ising models by Holley and Stroock (1976, 1987), by Holley (1984, 1985a, 1985b) and by Aizenman and Holley (1987). Recently, Liggett (1989) proved that the neatest particle system also exhibits an exponential convergence. He also proved that gap (Ω) coincides indeed with the largest value of ε in (1.1). Thus, as in the large deviation theory, we have a commom rate formula without ergodic assumption. This is especially useful for the study of interacting particle systems.

Motivated by a quantum field's model, Sullivan (1984) studied the spectral gap for jump processes with state space $\mathbb{Z}^+=\{0, 1, 2 \cdots\}$ or $\mathbb{R}^+$. Under some hypotheses,

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he proved the existence of the spectral gap for certain bounded operators.

Estimation of the bound of a spectrum has attracted considerable attention in various branches of mathematics. Motivated by a well-known paper by Cheeger (1970) on the lower bound of the Laplacian on a compact manifold, recently, Lawler and Sokal (1988) obtained a general version of Cheeger's inequality for jump processes with general state space and bounded operator. In their paper, our readers can find much more references.

The main purpose of this paper is to extend the previous results to unbounded generators. Some elementary facts from Dirichlet form theory enable us to obtain a complete formula for the convergence rate. This is done in the next section. Then for jump processes, we reduce the non-symmetric case to the symmetric one and reduce the unbounded case to the bounded one. In Section 5, we first improve two results due to Liggett (1989) and Sullivan (1984) respectively. Then, we prove that for birth-death processes, exponential convergence coincides indeed with exponential ergodicity which is widely studied in the Markov chain theory. Also we introduce a procedure to estimate the lower bound of spectral gap for birth-death processes. Finally, we apply Van Doorn's results (1985) to present some bounds of spectral gap for general positive recurrent Markov chains.

In the last section (§ 6) we briefly discuss the largest eigenvalue of Ω for non-positive recurrent Markov processes by using the techniques developed in the first five sections.

2. Some General Results

Let $(E, \mathscr{E}, \pi)$ be a probability space and $L^2(\pi)$ be the set of all real square integrable functions with respect to π on $(E, \mathscr{E})$. Given a positive, strongly continuous, contractive and Markovian semigroup $\{P(t)\}_{t\geqslant 0}$ on $L^2(\pi)$ $(P(t)1=1)$ with an invariant mesure π, we denote by Ω its generator with domain $\mathscr{D}(\Omega)$. Define gap(Ω) by (1.2). Similarly, we can define gap $(\widetilde{\Omega})$, where $\widetilde{\Omega}$ is the generator in the weak sense. Denote by $\mathscr{D}(\widetilde{\Omega})$ the domain of $\widetilde{\Omega}$ in $L^2(\pi)$. Finally, if the limit

$$(2.1) \qquad \lim_{t\downarrow 0} \frac{1}{t}(f-P(t)f, f) = \lim_{t\downarrow 0} \frac{1}{2t} \int \pi(dx)(P(t)(f-f(x))^2)(x) \geqslant 0$$

exists, we denote it by $D(f)$. Such functions $f\in L^2(\pi)$ with $D(f)<\infty$ constitute the domain $\mathscr{D}(D)$ of D. In the case of $\{P(t)\}_{t\geqslant 0}$ being symmetric on $L^2(\pi)$, as a direct consequence of elementary spectrum theory (cf. Fukushima (1980)), the limit defined by (2.1) always exists for all $f\in L^2(\pi)$. We also use $D(f, f)$ to denote the limit. The bilinear form $D(f, g) = \frac{1}{4}[D(f+g, f+g) - D(f-g, f-g)]$ defined on $\mathscr{D}(D) = \{f\in L^2(\pi): D(f,f)<\infty\}$ is called the Dirichlet form corresponding to the semigroup $\{P(t)\}_{t\geqslant 0}$. Clearly, in this case, $D(f) = D(f,f)$ with the same domain. This explains why we choose the notations $D(f)$ and $\mathscr{D}(D)$.

Now, we define

$$\text{gap}(D) = \inf\{D(f): f\in\mathscr{D}(D),\ \pi(f)=0,\ \|f\| = 1\}.$$

For the symmetric case, we have

$$\text{gap}(D) = \inf\{D(f,f): \pi(f)=0,\ \|f\| = 1\}.$$

Next, following Liggett (1989), we set

$$\sigma(t) = -\sup\{\log \| P(t)f \| : \pi(f)=0 \text{ and } \| f \| = 1\}.$$

By the contraction and semigroup properties, it is easy to see that $\sigma(\cdot)$ is superadditive and $\sigma(0)=0$. Hence, the limit

$$(2.3) \qquad \sigma = \lim_{t \downarrow 0} \frac{\sigma(t)}{t} = \inf_{t>0} \frac{\sigma(t)}{t}$$

is well defined.

The following result is an extension of Liggett's (1989, Theorem (2.3)) in which $\sigma = \text{gap}(\Omega)$ was proved.

(2.4) **Theorem**. *We have*

$$\sigma = \text{gap}(D) = \text{gap}(\widetilde{\Omega}) = \text{gap}(\Omega).$$

Proof. The proof is essentially due to Liggett (1989). Cleary,

$$\text{gap}(D) \leqslant \text{gap}(\widetilde{\Omega}) \leqslant \text{gap}(\Omega) \quad \text{on } \mathscr{D}(\Omega),$$

since

$$D(f) = (-\widetilde{\Omega} f, f) = (-\Omega f, f) \text{ on } \mathscr{D}(\Omega).$$

To prove $\sigma \geqslant \text{gap}(\Omega)$, we simply use the fact:

$$\frac{d}{dt} \| P(t)f \|^2 = 2(P(t)f, \Omega P(t)f) \leqslant -2\, \text{gap}(\Omega) \| P(t)f \|^2,$$
$$f \in \mathscr{D}(\Omega),\ \pi(f)=0 \text{ and } \| f \| = 1,$$

and the density of $\mathscr{D}(\Omega)$ in $L^2(\pi)$. Finally, let $f \in \mathscr{D}(D)$; then

$$D(f) = \lim_{t \downarrow 0} \frac{1}{t} (f - P(t)f, f)$$
$$\geqslant \lim_{t \downarrow 0} \frac{1}{t} (1 - e^{-\sigma t}) = \sigma.$$

Hence gap $(D) \geqslant \sigma$. □

At the moment, except for the fact $\mathscr{D}(\widetilde{\Omega}) \subset \mathscr{D}(D)$, our knowledge about $\mathscr{D}(D)$ is quite limited. However, it will be clear later, whenever we have a little more information about the generator, the domain $\mathscr{D}(D)$ is actually manageable.

The following obvious facts will be helpful for our further study.

(2.5) **Lemma**.

(i) $D(f) \geqslant 0$, $f \in \mathscr{D}(D)$;

(ii) $f \in \mathscr{D}(D) \Rightarrow g = cf + d \in \mathscr{D}(D)$ *and* $D(g) = c^2 D(f)$ *for all* $c, d \in \mathbb{R}$;

(iii) $f, g \in \mathscr{D}(D)$ *and* $f+g \in \mathscr{D}(D) \Rightarrow D(f+g) \leqslant 2(D(f)+D(g))$.

As an immediate consequence of Theorem (2.4), we have

(2.6) **Corollary**.

(i) *If* Ω *is bounded, then*

$$\sigma = \inf\{(-\Omega f, f) : \pi(f)=0,\ \| f \| = 1\}.$$

(ii) *If* Ω *is self-adjoint, then*

$$\sigma = \inf\{D(f,f) : \pi(f)=0,\ \| f \| = 1\},$$

where $D(f,f)$ *is the Dirichlet form corresponding to the semigroup* $\{P(t)\}_{t \geqslant 0}$ *(resp. generator* Ω *).*

Finally, we want to show that the non-symmetric case can often be reduced to a symmetric case.

Let E be a locally compact separable space with Borel field $\mathscr{E}$, π be a probability measure on $(E, \mathscr{E})$ with supp $(\pi) = E$. Let $D(f,g)(f, g \in \mathscr{D}(D) \subset L^2(\pi))$ be a generalized Dirichlet form (see Kim (1987) for details). Suppose that the semigroup $\{P(t)\}_{t\geq 0}$ corresponding to $D(f, g)$ has an invariant probability π. Obviously, by Theorem (2.4), we have

(2.7) $$\sigma=\inf\{D(f, f): f\in\mathscr{D}(D),\ \pi(f)=0,\ \|f\|=1\}.$$

Next, define the dual of D as follows:

$$\hat{D}(f,g)=D(g,f),\ f,g\in\mathscr{D}(\hat{D})=\mathscr{D}(D);$$

and set

$$\overline{D}=\frac{1}{2}(D+\hat{D}),\ \mathscr{D}(\overline{D})=\mathscr{D}(D).$$

Then $\overline{D}$ is a symmetric Dirichlet form for which we have

(2.8) $$\overline{\sigma}=\inf\{\overline{D}(f,f): \pi(f)=0,\ \|f\|=1\}.$$

But

$$\overline{D}(f,f)=\frac{1}{2}(D(f,f)+\hat{D}(f,f))=D(f,f),$$
$$f\in\mathscr{D}(\overline{D})=\mathscr{D}(D).$$

Thus, we have proved the following result.

(2.9) **Corollary.** $\sigma=\overline{\sigma}$.

(2.10) **Example.** For the Ornstein-Uhlenbeck proccess in $\mathbb{R}$,

$$\Omega=\frac{1}{2}\left(\frac{d^2}{dx^2}-x\frac{d}{dx}\right),$$

we have

$$\sigma=\text{gap}(\Omega)=1/2,$$

since the eigenvalues of Ω are

$$\lambda_n=n/2,\ n\geq 0,$$

and the associated eigenvectors belong to $\mathscr{D}(\Omega)$. By the independence of components, this conclusion is also correct in the multidimensional case. Moreover, for the infinite dimensional Ornstein-Uhlenbeck process in Wiener space, we still have

$$\sigma=\text{gap}(\Omega)=1/2.$$

Cf. Stroock (1981) for details.

More examples for diffusion processes can be found from Karlin and Taylor (1981), Chapter 15, Section 13. Also see Holley and Stroock (1987) and Korzeniowski (1987).

3. Spectral Gap for Jump Processes : General Case

Let (E, ρ) be a separable locally compact space, $P(t, x, dy)$ be a jump process on $(E, \rho, \mathscr{E})$. That is,

(3.1) $$\lim_{t\downarrow 0} P(t, x, A)=P(0, x, A)=I_A(x),\ x\in E,\ A\in\mathscr{E}.$$

Associated with each jump process $P(t, x, dy)$, we have a q-pair $q(x)-q(x, dy)$:

(3.2) $$\frac{d}{dt}P(t,x,A)\Big|_{t=0}=q(x, A)-q(x)I_A(x).$$

Unless otherwise stated, we assume that the q-pair is regular. That is, the q-pair is conservative:

$$0\leq q(x, A)\leq q(x, E)=q(x)<\infty,\ x\in E,\ A\in\mathscr{E},$$

and there is precise one jump process $P(t, x, dy)$ satisfying (3.2). Moreover, assume that π is an invariant measure of $P(t, x, dy)$.

Under the above conditions, it is known that the semigroup $\{P(t)\}_{t\geq 0}$ induced by jump process $P(t, x, dy)$ satisfies the hypotheses given at the begining of the previous section (cf. Chen (1987)).

Define

$$\pi_q(dx, dy) = \pi(dx) q(x, dy) \quad \text{on } \mathscr{E}\times\mathscr{E},$$

$$D^*(f) = \frac{1}{2}\int \pi_q(dx, dy)(f(y)-f(x))^2,$$

$$\mathscr{D}(D^*) = \{f\in L^2(\pi):\ D^*(f)<\infty\};$$

$$\mathscr{K} = \{f\in L^\infty(\pi):\ \overline{\operatorname{supp}(f)} \text{ is compact}\}$$

and

$$\mathscr{K}_L = \{g = cf + d : f\in\mathscr{K}, c, d\in\mathbb{R}\}.$$

Suppose that

(3.3) $q(x)$ is locally bounded.

Then we have

(3.4) **Lemma.** *Under* (3.3), $\mathscr{K}_L\subset\mathscr{D}(D)$.

Proof. By the regularity of the q-pair, it follows that (3.2) holds for all indicators I_A, $A\in\mathscr{E}$, and hence for all bounded $\mathscr{E}$-measurable functions. Thus, we have

$$\lim_{t\downarrow 0}\frac{1}{t}\int_{E\setminus\{x\}} P(t, x, dy) f(y) = \int q(x, dy) f(y). \tag{3.5}$$

On the other hand, since

$$\left|\int_{E\setminus\{x\}}\frac{P(t, x, dy)}{t} f(y)\right|$$
$$\leq (\sup_x |f(x)|)(1-P(t, x, \{x\}))/t$$
$$\leq (\sup_x |f(x)|)\, q(x),$$

it follows that

$$\int \pi(dx) f(x)\frac{1}{t}[f(x)-P(t)f(x)]$$

$$= \int_{\operatorname{supp}(f)} \pi(dx) f(x)^2 (1-P(t, x, \{x\}))/t$$

$$- \int_{\operatorname{supp}(f)} \pi(dx) f(x)\int_{E\setminus\{x\}}\frac{P(t, x, dy)}{t} f(y)$$

$$\to \int_{\operatorname{supp}(f)} \pi(dx) q(x) f(x)^2$$

$$- \int \pi(dx) f(x)\int q(x, dy) f(y) \quad \text{as } t\downarrow 0$$

(cf. Chen (1986)). Note that π is an invariant measure of $\{P(t)\}_{t\geq 0}$.

$$\int \pi(dx) q(x) f^2(x) = \int \pi(dx) \int q(x, dy) f(y)^2 . \tag{3.6}$$

Combining the above facts, we arrive at

$$\left(\frac{f-P(t)f}{t}, f\right) \to D(f) = D^*(f) < \infty \quad \text{as } t \downarrow 0$$

for $f \in \mathscr{K}$. Now, the conclusion follows from Lemma (2.5). □

This simple result already enables us to get an upper bound for gap (D).

(3.7) **Theorem.** *Under* (3.3),

$$\text{gap}(D) \leqslant \frac{1}{2} \inf\left\{ \frac{\pi_q(K \times K^c + K^c \times K)}{\pi(K)\pi(K^c)} : 0 < \pi(K) < 1, K \text{ is compact} \right\}.$$

Proof. For a compact K, $0 < \pi(K) < 1$, set $f = cI_K + d$. Choose c and $d \in \mathbb{R}$ such that $\pi(f) = 0$ and $\|f\| = 1$. Compute $D^*(f)$. The assertion follows from Lemma (3.4). □

(3.8) **Definition.** We say that $\mathscr{C} \subset \mathscr{D}(D^*)$ is a core of D^*, if for every $f \in \mathscr{D}(D^*)$, there exists a sequence $\{f_n\}_1^\infty \subset \mathscr{C}$ such that

$$D_1^*(f_n - f) = D^*(f_n - f) + \|f_n - f\|^2 \to 0 \qquad \text{as } n \to \infty .$$

(3.9) **Lemma.** *If*

$$\int \pi(dx)\, q(x) < \infty , \tag{3.10}$$

then $\mathscr{K}_L$ *is a core of* D^*.

Proof. We need only to show that $\mathscr{K}$ is a core of D^*. Take a sequence of compacts $E_n \uparrow E$ and let $f \in \mathscr{D}(D^*)$. Set $f_n = f I_{E_n}$. Then

$$D_1^*(f_n - f) = \frac{1}{2} \int \pi_q(dx, dy)(f(y) - f(x) - f_n(y) + f_n(x))^2 + \|f_n - f\|^2$$

$$\leqslant \int \pi_q(dx, dy)[(f_n(y) - f_n(x))^2 + (f(x) - f_n(x))^2] + \|f_n - f\|^2$$

$$= \int_{[f(y)>n]} \pi_q(dx, dy) f(y)^2 + \int_{[f(x)>n]} \pi_q(dx, dy) f(x)^2 + \|f_n - f\|^2$$

$$\leqslant (\sup_x |f(x)|)^2 \left[\int \pi(dx) q(x, [f>n]) + \int_{[f(x)>n]} \pi(dx) q(x) \right] + \|f_n - f\|^2$$

$$\to 0 \quad \text{as } n \to \infty .$$

□

(3.11) **Theorem.** *If* $\mathscr{K}$ *is a core of* D^*, *in particular, if* (3.10) *holds, then*

$$\text{gap}(D) = \frac{1}{2} \inf\left\{ \int \pi_q(dx, dy)(f(y) - f(x))^2 : \pi(f) = 0, \|f\| = 1 \right\}$$

$$= \frac{1}{2} \inf\left\{ \int \pi_q(dx, dy)(f(y) - f(x))^2 : f \in \mathscr{K}_L, \pi(f) = 0, \|f\| = 1 \right\}.$$

Proof. First, $D^*(f_n - f) \to 0$ implies that

$$\int \pi_q(dx, dy)(f_n(y)-f_n(x))^2$$

is bounded with respect to n. On the other hand,

$$\left|\int \pi_q(dx, dy)(f_n(y)-f_n(x))^2 - \int \pi_q(dx, dy)(f(y)-f(x))^2\right|$$

$$\leqslant \frac{1}{2}\int \pi_q(dx, dy)(f_n(y)-f_n(x)-f(y)+f(x))^2$$

$$\cdot \int \pi_q(dx, dy)(f_n(y)-f_n(x)+f(y)-f(x))^2$$

$$\leqslant CD^*(f_n-f)\to 0 \qquad \text{as } n\to\infty,$$

and so

$$D(f_n)=D^*(f_n)\to D^*(f) \qquad \text{as } n\to\infty.$$

Our assertion follows from Theorem (2.4), Lemma (3.4) and (3.9) immediately. □

The next result is roughly a special case of Corollary (2.9). It shows that if π and $q(x,\cdot)$ have a density with respect to a reference measure, we can avoid the condition (3.10).

(3.12) **Theorem.** *Let E be countalbe and $Q=(q_{ij})$ be an irreducible regular Q-matrix. The Markov chain $(P_{ij}(t))$, determined by the Q-matrix $Q=(q_{ij})$, has an invariant probability measure (π_i). Then*

$$\text{gap}(D)=\frac{1}{2}\inf\left\{\sum_{i,j}\pi_i q_{ij}(f_j-f_i)^2 : \pi(f)=0,\ \|f\|=1\right\}$$

$$=\frac{1}{2}\inf\left\{\sum_{i,j}\pi_i q_{ij}(f_j-f_i)^2 : f\in\mathscr{K}_L,\ \pi(f)=0,\ \|f\|=1\right\}.$$

Proof. By Theorem (3.11), we need only to prove that $\mathscr{K}_L$ is a core of D^*. Define

$$\hat{q}_{ij}=\pi_j q_{ji}/\pi_i,\ i,\ j\in E,$$

$$\overline{q}_{ij}=\frac{1}{2}(q_{ij}+\hat{q}_{ij}),\ i,\ j\in E.$$

It is easy to check that $(\hat{q}_{ij})$ is a conservative Q-matrix with stationary measure (π_i), and so is $(\overline{q}_{ij})$. Moreover, $(\overline{q}_{ij})$ is a reversible Q-matrix with respect to the same probability measure (π_i), and so its Dirichlet form is regular (cf. Chen (1989); Theorem (3.10)). That is, $\mathscr{K}$ is a core of $\overline{D}$. However,

$$\overline{D}(f,f)=\frac{1}{2}\sum_{i,j}\pi_i\overline{q}_{ij}(f_j-f_i)^2$$

$$=\frac{1}{4}\sum_{i,j}\pi_i(q_{ij}+\hat{q}_{ij})(f_j-f_i)^2$$

$$=\frac{1}{2}\sum_{i,j}\pi_i q_{ij}(f_j-f_i)^2$$

$$=D^*(f);$$

hence claim that $\mathscr{K}$ is also a core of D^*. $\square$

The above result is due to a simple observation:

$$\text{Re spec.}\,(\Omega) = \text{spec.}\left(\frac{1}{2}(\Omega + \hat{\Omega})\right),$$

where $\hat{\Omega}$ is the adjoint operator of Ω.

(3.13) **Example.** Take

$$Q = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}.$$

Then gap $(D) = 3/2$ and the eigenvalues of Ω are 0, $-3/2 \pm \sqrt{3}\, i/2$.

(3.14) **Example.** Take $E = \{0, 1, 2 \cdots\}$,

$$q_{0i} = b_i,\ i \geqslant 1,\quad q_0 = \sum_{i=1}^{\infty} b_i < \infty,$$

$$q_{i0} = q_i,\ i \geqslant 1,\quad q_{ij} = 0 \quad \text{otherwise}.$$

Then

$$\pi_i = \mu_i / \rho,\quad i \geqslant 0,$$

where

$$\mu_0 = 1,\ \mu_i = b_i / q_i,\ i \geqslant 1;\ \rho = \sum_{i=1}^{\infty} b_i / q_i + 1.$$

For every $f \in L^2(\pi)$, $\pi(f) = 0$, $\|f\| = 1$, we have

$$\begin{aligned} 1 &= \frac{1}{2}\sum_{i,j} \pi_i \pi_j (f_j - f_i)^2 \\ &\leqslant \sum_{i,j} \pi_i \pi_j [(f_j - f_0)^2 + (f_i - f_0)^2] \\ &= 2\sum_{i \neq 0} \pi_i (f_i - f_0)^2 \\ &\leqslant 2\sum_{i \neq 0} \pi_i q_{i0} (f_i - f_0)^2 / \inf_{j \geqslant 1} q_j \\ &= \frac{1}{2}\sum_{i,j} \pi_i q_{ij} (f_j - f_i)^2 \cdot 2 / \inf_{k \geqslant 1} q_k. \end{aligned}$$

Thus,

$$\text{gap}\,(D) \geqslant \frac{1}{2}\inf_{i \geqslant 1} q_i.$$

Now, we study two comparison theorems to close this section.

(3.15) **Theorem.** *Let $Q = (q_{ij})$ and $\widetilde{Q} = (\widetilde{q}_{ij})$ be two Q-matrices as in* (3.12). *Denote by (π_i) and $(\widetilde{\pi}_i)$ their invariant probability measures respectively. Suppose that*

$$\widetilde{q}_{ij} \geqslant b q_{ij},\ i \neq j,\ b > 0$$

and

$$c \leqslant \widetilde{\pi}_i / \pi_i \leqslant c^{-1},\ i \in E;$$

for a constant $c \in (c_0, 1]$,

$$c_0 = \frac{1}{3}\left[1 + \left(\frac{3\sqrt{69} + 11}{2}\right)^{1/3} - \left(\frac{3\sqrt{69} - 11}{2}\right)^{1/3}\right] \approx 0.56984.$$

Then

$$\mathrm{gap}(\widetilde{D}) \geqslant \frac{b}{c}\,(c^3-(1-c)^2)\,\mathrm{gap}(D).$$

Proof. Let $f \in \mathscr{K}_L$, $\widetilde{\pi}(f)=0$, $\|f\|_{\widetilde{\pi}}=1$. Then

$$\begin{aligned} |\pi(f)|^2 &= \Big|\sum_j \widetilde{\pi}_j\,(1-\pi_j/\widetilde{\pi}_j)f_j\Big|^2 \\ &\leqslant \sum_j \widetilde{\pi}_j(1-\pi_j/\widetilde{\pi}_j)^2 \sum_k \widetilde{\pi}_k f_j^2 \\ &= \sum_j \widetilde{\pi}_j(1-\pi_j/\widetilde{\pi}_j)^2 \leqslant \sup_j (1-\pi_j/\widetilde{\pi}_j)^2 \\ &\leqslant (c^{-1}-1)^2. \end{aligned}$$

Hence

$$\begin{aligned} \frac{1}{2}\sum_{i,j}\widetilde{\pi}_i\,\widetilde{q}_{ij}(f_j-f_i)^2 &\geqslant \frac{bc}{2}\sum_{i,j}\pi_i q_{ij}(f_j-f_i)^2 \\ &\geqslant bc\,\mathrm{gap}(D)\,\|f-\pi(f)\|_\pi^2 \\ &= bc\ \mathrm{gap}(D)\,[\ \|f\|_\pi^2-\pi(f)^2] \\ &\geqslant \frac{b}{c}\,(c^3-(1-c)^2)\,\mathrm{gap}(D). \end{aligned}$$ □

For Markov chains, a problem — exponential ergodicity has been well studied. It is known that for every irreducible Markov chain $(P_{ij}(t))$, there is an $\alpha \geqslant 0$ such that

(3.16) $$|P_{ij}(t)-\pi_j| = O(\exp(-\alpha t)) \ \text{as}\ t\to\infty,$$

where $\pi_j = \lim_{t\to\infty} P_{ij}(t)$. Set

(3.17) $$\hat{\alpha} = \sup\{\alpha : (3.16)\ \text{holds for all}\ i\ \text{and}\ j\}.$$

If $\hat{\alpha} > 0$, the process is called exponentially ergodic.

(3.18) **Theorem**. *Let $(P_{ij}(t))$ be an irreducible positive recurrence Markov chain with stationary distribution (π_i) and Q-matrix $Q=(q_{ij})$. Then*

(3.19) $$\mathrm{gap}(D) \leqslant \hat{\alpha}.$$

Proof. Fix $i_0, j_0 \in E$ and take

$$f_j = \delta_{jj_0},\quad j\in E.$$

Then

$$e^{-2\sigma t}\ \|f-\pi(f)\|^2 \geqslant \|P(t)f-\pi(f)\|^2 \geqslant \pi_{i_0}|P_{i_0j_0}(t)-\pi_{j_0}|^2.$$

Since i_0 and j_0 are arbitrary, we obtain

$$\mathrm{gap}(D) = \sigma \leqslant \hat{\alpha}.$$ □

For birth-death processes, we will prove in Section 5 that (3.19) is indeed an equality.

4. Reversible Case, an Approximation Theorem

In view of Theorem (3.12), in some cases, we can reduce the non-symmetric case to a symmetric one. Hence the symmetric case is more important and often easier to handle.

Throughout this section, we assume (3.3).

For a bounded q-pair, some nice results were obtained by Lawler and Sokal (1988). The purpose of this section is to reduce the unbounded case to a bounded one. To do this, take compact sets $E_n \uparrow E\,(n\geqslant 0)$. Assume that

$$\pi(E_n^c)>0,\quad n\geqslant 0. \tag{4.1}$$

Regard $\Delta_n = E_n^c$ as a singular point and set

$$\hat{E}_{n+1}=E_n\cup\{\Delta_n\},\quad \hat{\mathscr{E}}_{n+1}=\sigma(\mathscr{E}\cap(E_n\cup\{\Delta_n\})),$$

$$\hat{q}_{n+1}(x,A)=q(x,A\cap E_n)+I_A(\Delta_n)q(x,E_n^c),\quad x\in E_n,\ A\in\hat{\mathscr{E}}_{n+1},$$

$$\hat{q}_{n+1}(\Delta_n,A)=\int_{A\cap E_n}\pi(dx)q(x,E_n^c)/\,\pi(E_n^c),\quad A\in\hat{\mathscr{E}}_{n+1},$$

$$\hat{q}_{n+1}(x)=\hat{q}_{n+1}(x,\hat{E}_{n+1}),\quad x\in\hat{E}_{n+1},\ n\geqslant 0.$$

It is easy to see that $\hat{q}_{n+1}(x)-\hat{q}_{n+1}(x,dy)$ is a bounded conservative q-pair, and hence is regular. Finally, let

$$\hat{\pi}_{n+1}(A)=\pi(A\cap E_n)+\pi(E_n^c)\,I_A(\Delta_n),\quad A\in\hat{\mathscr{E}}_{n+1}.$$

From the reversibility of $q(x)-q(x,dy)$ with respect to π, we obtain

$$\int_A\pi(dx)\,q(x,B)=\int_B\pi(dx)q(x,A),\quad A,B\in\mathscr{E}.$$

For all $A,B\in\hat{\mathscr{E}}_{n+1}$, we have

$$\begin{aligned}
&\int_A\hat{\pi}_{n+1}(dx)\hat{q}_{n+1}(x,B)\\
&=\int_{A\cap E_n}\pi(dx)[q(x,B\cap E_n)+I_B(\Delta_n)q(x,E_n^c)]\\
&\quad+I_A(\Delta_n)\pi(E_n^c)\hat{q}_{n+1}(\Delta_n,B)\\
&=\int_{A\cap E_n}\pi(dx)\,q(x,B\cap E_n)+I_B(\Delta_n)\int_{A\cap E_n}\pi(dx)q(x,E_n^c)\\
&\quad+I_A(\Delta_n)\int_{B\cap E_n}\pi(dx)q(x,E_n^c).
\end{aligned}$$

This is symmetric with respect to A and B. Therefore $\hat{q}_{n+1}(x)-\hat{q}_{n+1}(x,dy)$ is reversible with respect to $\hat{\pi}_{n+1}$.

Next, for $f\in\mathscr{K}_L$, we have

$$\begin{aligned}
&\int\pi_q(dx,dy)\,(f(y)-f(x))^2\\
&=\int_{E_n}\pi(dx)\int_{E_n}q(x,dy)\,(f(y)-f(x))^2\\
&\quad+2\int_{E_n}\pi(dx)\int_{E_n^c}q(x,dy)\,(c-f(x))^2
\end{aligned}$$

(if f = a constant c off E_n)

$$= \int_{E_n} \pi_{n+1}(dx) \int_{E_n} \hat{q}_{n+1}(x, dy)(f(y)-f(x))^2$$

$$+ 2\int_{E_n} \hat{\pi}_{n+1}(dx)\, \hat{q}_{n+1}(x, \Delta_n)(c-f(x))^2$$

$$= \int_{\hat{E}_{n+1}} \hat{\pi}_{n+1}(dx) \int_{\hat{E}_{n+1}} \hat{q}_{n+1}(x, dy)(f(y)-f(x))^2 .$$

By Theorem (3.11), we have

$$\begin{aligned} \text{gap}(D) &= \inf\{D^*(f) : \pi(f)=0,\ \|f\|=1,\ f = \text{constant off } E_n \text{ for some } n \geqslant 0\} \\ &= \lim_{n\to\infty} \downarrow \inf\{D^*(f) : \pi(f)=0,\ \|f\|=1,\ f=\text{const. off } E_n\} \\ &= \lim_{n\to\infty} \downarrow \inf\left\{D^*(f):\ \hat{\pi}_{n+1}(f)=0, \int f^2 d\hat{\pi}_{n+1}=1\right\} \\ &= \lim_{n\to\infty} \downarrow \text{gap}(\hat{D}_{n+1}), \end{aligned}$$

where $\lim\limits_{n\to\infty} \downarrow h_n = h$ meas that $h_n \downarrow h$ as $n \to \infty$.

Thus, we have proved the following approximation result :

(4.2) **Theorem**. *Let $q(x) - q(x, dy)$ be a regular q-pair which is reversible with respect to π. Take compacts $E_n \uparrow E$ and assume that $\pi(E_n^c) > 0$ for all $n \geqslant 0$. Define $\hat{q}_{n+1}(x) - \hat{q}_{n+1}(x, dy)$ on $\hat{E}_{n+1}$ as above. If $\mathscr{K}_L$ is a core of D^*, then*

$$\text{gap}(D) = \lim_{n\to\infty} \downarrow \text{gap}(\hat{D}_{n+1}).$$

In particular, we have

(4.3) **Corollary**. *Let $E = \{0, 1, 2, \cdots\}$, $Q = \{q_{ij}\}$ be an irreducible regular Q-matrix which is reversible with respect to (π_i). Take*

$$(4.4) \qquad \hat{Q}_{n+1} = \begin{pmatrix} -q_0 & q_{01} & \cdots & q_{0,n} & \sum_{j>n} q_{0j} \\ q_{10} & -q_1 & \cdots & q_{1,n} & \sum_{j>n} q_{1j} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ q_{n0} & q_{n1} & \cdots & -q_n & \sum_{j>n} q_{nj} \\ \hat{q}_{n+1,0} & \hat{q}_{n+1,1} & \cdots & \hat{q}_{n+1,n} & -\hat{q}_{n+1} \end{pmatrix},$$

where

$$\hat{q}_{n+1,j} = \pi_j \sum_{k>n} q_{jk} \Big/ \sum_{k>n} \pi_k ,\ j = 0, 1, \cdots, n ,$$

$$\hat{q}_{n+1} = \sum_{j=0}^{n} \hat{q}_{n+1,j} .$$

Then

$$\text{gap}(D) = \lim_{n\to\infty} \downarrow \text{gap}(\hat{D}_{n+1}).$$

5. Spectral Gap for Markov Chains

Again, we need only to consider the reversible case.

Let $(P_{ij}(t))$ be an irreducible reversible Markov chain with stationary distribution (π_i) and regular Q-matrix $Q=(q_{ij})$. Suppose that $q_{i,i+1}>0$ $(i\in E)$. For the lower bound of the spectral gap, we have

(5.1) **Theorem.** *If*

$$\sum_{j>i} \pi_j \leqslant c\,\pi_i q_{i,i+1}, \ i\in E,$$

$$\sum_{j>i} \pi_j q_{j,j+1} \leqslant b\pi_i q_{i,i+1}, \ i\in E,$$

then

$$\mathrm{gap}(D)\geqslant 1/[c(\sqrt{b+1}+\sqrt{b})^2]>1/[2c(1+2b)].$$

(5.2) **Theorem.** *If*

$$\sum_{j\geqslant 1} \pi_j \leqslant c\,\pi_i, \ i\in E,$$

$$\pi_{i+1}\leqslant b\pi_i q_{i,i+1}, \ i\in E,$$

then

$$\mathrm{gap}(D)\geqslant 1/[2bc(1+2c)]>1/(4bc^2).$$

The first theorem was proved by Liggett (1989) under two more assumptions: $\mathscr{K}_L$ is a core of the generator Ω and $\sum_i \pi_i q_i<\infty$. The second one was proved by Sullivan (1984) under two more assumptions: $\inf_{i\geqslant 1} q_{i,i+1}>0$ and $\sup_i q_i<\infty$. By Theorem (3.12), it is easy to check that Liggett's proof still works for the above theorems. We omit the details here.

The above results are incomparable. For example, consider the birth-death process: $q_{i,i+1}=\alpha$, $q_{i,i-1}=\beta$, $\alpha<\beta$. If $\alpha\geqslant 1$, then (5.1) is better than (5.2). If $\alpha<1$, (5.2) can be better than (5.1).

Actually, Theorems (5.1) and (5.2) are based on comparing the original process with the birth-death process:

$$\widetilde{q}_{ij}=q_{ij},\ j=i\pm 1;\ \widetilde{q}_i=\sum_{j\neq i}\widetilde{q}_{ij};\quad \widetilde{\pi}_i=\pi_i.$$

The main part of the proofs for (5.1) and (5.2) is to show that the lower bounds hold for the birth-death process, and then to apply Theorem (3.5) to deduce our assertions. This induces us to study more carefully the spectral gap for birth-death processes.

Let $Q=(q_{ij})$ be a birth-death Q-matrix:

$$q_{i,i+1}=b_i>0, \quad i\geqslant 0,$$

$$q_{i,i-1}=a_i>0, \quad i\geqslant 1,$$

$$q_i=-q_{ii}=a_i+b_i, \quad i\geqslant 0.$$

Set

$$\mu_0=1,\ \mu_i=\frac{b_0\cdots b_{i-1}}{a_1\cdots a_i},\ i\geqslant 1,$$

$$\rho=1+\sum_{i=1}^{\infty}\mu_i.$$

Then

$$\pi_i = \mu_i / \rho, \quad i \geqslant 0.$$

The next result is an improvement over Theorem (3.18) in the existing circumstances.

(5.3) Theorem. *For every positive recurrent birth-death process, the exponential L^2-convergence is equivalent to the exponential ergodicity. In other words,*

$$\text{gap}(D) = \hat{\alpha}.$$

Proof. If $\hat{\alpha} = 0$, then by (3.19), gap$(D) = 0$. Thus, we may and will assume that $\hat{\alpha} > 0$. Set

$$H_0(x) = 1,$$

$$-xH_0(x) = -b_0 H_0(x) + b_0 H_1(x),$$

$$-xH_n(x) = a_n H_{n-1}(x) - (a_n + b_n) H_n(x) + b_n H_{n+1}(x), \quad n \geqslant 1, x \in \mathbb{R}.$$

Then $H_n(0) = 1$, $n \geqslant 0$. Recall the Karlin and Mcgregor's representation theorem :

$$P_{ij}(t) = \mu_j \int_0^\infty e^{-xt} H_i(x) H_j(x) d\psi(x), \tag{5.4}$$

where ψ is a (unique) non-decreasing function which is left continuous and

$$\psi(x) = 0 \quad \text{for} \quad x \leqslant 0, \ \psi(x) \to 1 \text{ as } x \to \infty.$$

Also,

$$\mu_j \int_0^\infty H_i(x) H_j(x) \, d\psi(x) = \delta_{ij}.$$

Write

$$\hat{f}(x) = \sum_i \pi_i H_i(x) f_i, \ f \in \mathscr{K}.$$

From (5.4), it follows that

$$(f, P(t) f) = \rho \int_0^\infty e^{-xt} \hat{f}(x)^2 d\psi(x), \ f \in \mathscr{K}. \tag{5.5}$$

In particular,

$$(f, f) = \rho \int_0^\infty \hat{f}(x)^2 d\psi(x), \quad f \in \mathscr{K}. \tag{5.6}$$

This gives us an isometric imbedding from $L^2(\pi)$ to $L^2([0, \infty], \rho d\psi)$. Thus, (5.5) and (5.6) hold for $f \in L^2(\pi)$. Moreover, by (5.5), we see that

$$D(f, f) = \rho \int_0^\infty x \hat{f}(x)^2 d\psi(x).$$

From the exponential ergodicity, by Van Doorn (1985), Theorem 2.1, Theorem 3.1 and Lemma 3.2, the first two points of the spectrum of ψ are

$$x_1 = 0, \ \hat{\alpha} = x_2 > x_1$$

(x is called a point of the spectrum of ψ if $\Delta\psi(x) = \psi(x+) - \psi(x-) > 0$). Notice that $\Delta\psi(0) > 0$, and so $\hat{f}(0) = \sum_i \pi_i H_i(0) f_i = \pi(f) = 0$. On the other hand, from

$x_2 = \hat{\alpha}$ and

$$P_{00}(t) - \pi_0 = \int_0^\infty e^{-xt}\, d\psi'(x),$$

where $\psi' = \psi - \Delta\psi(0)$, it follows that

$$\psi(x_2-) = \psi(0+).$$

Hence

$$\begin{aligned}
\mathrm{gap}(D) &= \inf\{D(f,f): \pi(f)=0, \|f\| = 1\} \\
&= \inf\left\{\rho\int_0^\infty x\hat{f}(x)^2 d\psi(x) : \pi(f)=0, \|f\| = 1\right\} \\
&= \inf\left\{\rho\int_0^\infty x\hat{f}(x)^2 d\psi(x) : \hat{f}(0)=0, \|f\| = 1\right\} \\
&\geqslant (x_2-\varepsilon)\inf\left\{\rho\int_0^\infty \hat{f}(x)^2 d\psi(x) : \hat{f}(0)=0, \|f\| = 1\right\} \\
&= (x_2-\varepsilon)\ (\text{by } (5.6)) \\
&= \hat{\alpha} - \varepsilon,
\end{aligned}$$

for all small $\varepsilon > 0$. Therefore $\mathrm{gap}(D) \geqslant \hat{\alpha}$. □

Now, we can combine Theorem (5.3) with the previous results (cf. Van Doorn (1981)) to give some examples.

(5.7) **Examples.**

(i) $b_i = b,\ i \geqslant 0$,

$a_i = ai,\ i = 0, 1, \cdots, s-1$,

$= sa,\ i = s, s+1, \cdots,\quad a/sb \equiv \rho < 1$.

There exists a $\bar{\rho} < 1$ such that

$$0 < \mathrm{gap}(D) < b(1 - 1/\sqrt{\rho})^2 \quad \text{if } \rho < \bar{\rho},$$
$$\mathrm{gap}(D) = b(1 - 1/\sqrt{\rho})^2 \quad \text{if } \rho \geqslant \bar{\rho}.$$

If $s = 1$ and $b < a$, then

$$\mathrm{gap}(D) = (\sqrt{a} - \sqrt{b})^2.$$

(ii) $b_i = b/(i+1),\ i \geqslant 0$;

$a_i = a,\ i \geqslant 1$.

$\mathrm{gap}(D) = a - (\sqrt{b^2 + 4ab} - b)/2$.

(iii) $b_i = \alpha + \lambda_1,\ i \geqslant 0,\ \alpha > 0$;

$a_i = \lambda_2 i,\ i \geqslant 1,\ 0 \leqslant \lambda_1 < \lambda_2$.

$\mathrm{gap}(D) = \lambda_2$ if $\lambda_1 = 0$,

$= \lambda_2 - \lambda_1$ if $\lambda_1 > 0$.

Because of Theorem (5.3), we can also rely on some sufficient conditions for the exponential ergodicity to estimate the lower bound of $\mathrm{gap}(D)$. Note that in many cases, it is not possible to compute the spectral function ψ. We would like to know some practical methods to estimate $\mathrm{gap}(D)$. Our next result is such a kind of approach without using ψ. The idea is based on Corollary (4.3). In the present case, our approximation Q-matrix (4.4) becomes

$$\hat{Q}_{n+1} = \begin{pmatrix} -b_0 & b_0 & 0 & & & \mathbf{0} \\ a_1 & -(a_1+b_1) & b_1 & & & \\ & \ddots & \ddots & \ddots & & \\ \mathbf{0} & & a_n & -(a_n+b_n) & b_n \\ & & 0 & \hat{a}_{n+1} & -\hat{a}_{n+1} \end{pmatrix},$$

where $\hat{a}_{n+1}=\pi_n b_n / \hat{\pi}_{n+1}$, $\hat{\pi}_{n+1}=\sum_{j>n} \pi_j$, $n\geqslant 0$. For each fixed n, define

$$\begin{aligned} s_0(x) &= b_0+x,\ x\in\mathbb{R}, \\ s_1(x) &= a_1+b_1+x-a_1 b_0 / s_0(x) && \text{if } s_0(x)\neq 0, \\ &= 1 && \text{if } s_0(x)=0, \\ s_i(x) &= a_i+b_i+x-a_i b_{i-1} / s_{i-1}(x) && \text{if } s_{i-1}\neq 0,\ s_{i-2}(x)\neq 0, \\ &= a_i+b_i+x && \text{if } s_{i-2}(x)=0, \\ &= 1 && \text{if } s_{i-1}(x)=0, \\ & && 2\leqslant i\leqslant n,\ x\in\mathbb{R}, \end{aligned}$$

and

$$\begin{aligned} \hat{s}_{n+1}(x) &= x+\frac{\pi_n b_n}{\hat{\pi}_{n+1}}\left(1-\frac{b_n}{s_n(x)}\right) \\ & \qquad \text{if } s_n(x)\neq 0,\ s_{n-1}(x)\neq 0, \\ &= x+\frac{\pi_n b_n}{\hat{\pi}_{n+1}} \qquad \text{if } s_{n-1}(x)=0, \\ &= 1 \qquad \text{if } s_n(x)=0,\ x\in\mathbb{R}. \end{aligned}$$

(5.8) **Theorem**. *For the above* $\hat{Q}_{n+1}$,

(5.9) $$\operatorname{gap}(\hat{D}_{n+1})>\alpha>0,$$

if and only if there is precisely one term of $\{s_0(-\alpha), \cdots, s_n(-\alpha), \hat{s}_{n+1}(-\alpha)\}$ *which is less or equal to zero. Moreover, if the condition holds for all* n, *then*

(5.10) $$\operatorname{gap}(D)\geqslant\alpha>0.$$

Proof. Denote by $\widetilde{Q}_{n+1}$ the symmetrized matrix of $\hat{Q}_{n+1}$:

$$\widetilde{Q}_{n+1}=\begin{pmatrix} -b_0 & \sqrt{\pi_0}\, b_0/\sqrt{\pi_1} & 0 & & \mathbf{0} \\ \sqrt{\pi_1}\, a_1/\sqrt{\pi_2} & -(a_1+b_1) & \sqrt{\pi_1}\, b_1/\sqrt{\pi_2} & & \\ & \ddots & \ddots & \ddots & \\ \mathbf{0} & \sqrt{\pi_n}\, a_n/\sqrt{\pi_{n-1}} & -(a_n+b_n) & \sqrt{\pi_n}\, b_n/\sqrt{\hat{\pi}_{n+1}} \\ & 0 & \sqrt{\hat{\pi}_{n+1}}\, \hat{a}_{n+1}/\sqrt{\pi_n} & -\hat{a}_{n+1} \end{pmatrix}$$

$$
= \begin{pmatrix}
-b_0 & \sqrt{a_1 b_0} & 0 & & \mathbf{0} \\
\sqrt{a_1 b_0} & -(a_1+b_1) & \sqrt{a_1 b_1} & & \\
 & \ddots & \ddots & \ddots & \\
\mathbf{0} & \sqrt{a_n b_{n-1}} & -(a_n+b_n) & \sqrt{\pi_n b_n}\,/\sqrt{\hat{\pi}_{n+1}} \\
 & 0 & \sqrt{\pi_n}\; b_n\,/\sqrt{\hat{\pi}_{n+1}} & -\pi_n b_n\,/\sqrt{\hat{\pi}_{n+1}}
\end{pmatrix}.
$$

Then $\hat{Q}_{n+1}$ and $\widetilde{Q}_{n+1}$ have the same eigenvalues which are denoted by

$$0 = \lambda_{n+1,0} > \lambda_{n+1,1} > \cdots > \lambda_{n+1,n+1} .$$

These eigenvalues must be distinct since the matrices are tridiagonal. From the matrix theory, it is known that $-\operatorname{gap}(\hat{D}_{n+1}) = \lambda_{n+1,1} < -\alpha$ if and only if there is precisely one non-positive term among $\{s_0(-\alpha), \cdots, s_n(-\alpha), \hat{s}_{n+1}(-\alpha)\}$. This proves the first assertion. The second follows from the first one plus Corollary (4.3). □

To show that Theorem (5.8) is feasible, let us consider

(5.11) **Example**. Take $b_i = b > 0$, $i \geqslant 0$; $a_i = i$, $a > 0$, $i \geqslant 1$.

For a special case $b=1$ and $a=2$. the bound obtained by Theorem (5.1) is 0.3348. But we have seen in Example (5.7) that $\operatorname{gap}(D) = 2$. Now, we use Theorem (5.8) to show that

$$\operatorname{gap}(D) \geqslant \hat{\alpha} = a > 0 .$$

To do this, assume that

$$b/a \neq 1, 2, \cdots$$

for simplicity. The exceptional cases can be discussed in a similar way. Now,

$$\pi_i = \left(\frac{b}{a}\right)^i \frac{1}{i!}\,/\,\rho, \quad \rho = \exp[b/a] .$$

By induction, it is easy to prove that

$$s_0(-a) = b - a ,$$

$$s_i(-a) = b(b-(i+1)a)\,/\,(b-ia), \quad 1 \leqslant i \leqslant n ,$$

and so

$$\hat{s}_{n+1}(-a) = -a + \frac{ab\pi_n}{\hat{\pi}_{n+1}((n+1)a-b)} .$$

Since

$$\frac{\hat{\pi}_{n+1}}{\pi_n}((n+1)a-b) = \sum_{j>n}\left(\frac{b}{a}\right)^{j-n}\frac{n!}{j!}[(n+1)a-b]$$

$$= \sum_{j=1}^{\infty}\left(\frac{b}{a}\right)^{j} j![(n+1)a-b]\,/\binom{j+n}{n} < b$$

for large n, we have

$$\hat{s}_{n+1}(-a) > 0 \quad \text{for large } n .$$

Clearly, among $\{s_0(-a), \cdots, s_n(-a)\}$ there is precisely one negative term. Hence from Theorem (5.8) we may deduce our assertion.

As we have just seen above, for estimating the decay parameter $\hat{\alpha}$, the tridiagonal property of birth-death Q-matrices is very helpful. On the same idea, Van Doorn (1985) obtaind the following bounds.

(5.12) **Theorem**. *For a birth-death process with rates a_i and b_i, the decay parameter $\hat{\alpha}$ satisfies*

(i) $\hat{\alpha} \geqslant \inf_{i\geqslant 1} \{a_i + b_{i-1} - \sqrt{a_{i-1} b_{i-1}} - \sqrt{a_i b_i}\}$,

$$\hat{\alpha} \geqslant \frac{1}{2} \inf_{i\geqslant 1} \{a_i + a_{i+1} + b_i + b_{i-1} - \sqrt{(a_{i+1} + b_i - a_i - b_{i-1})^2 + 16 a_i b_i}\},$$

(ii) $$\hat{\alpha} < \left\{1 + \sum_{i=n+1}^{n+k} \left(1 - 2\left(\frac{a_i b_i}{(a_i + b_{i-1})(a_{i+1} + b_i)}\right)^{1/2}\right)\right\}\left\{\sum_{i=n}^{n+k} \frac{1}{a_{i+1} + b_i}\right\}^{-1},$$

$$n, k \geqslant 0,$$

$$\hat{\alpha} < \frac{1}{2} \{a_i + a_{i+1} + b_i + b_{i-1} - \sqrt{(a_{i+1} + b_i - a_i - b_{i-1})^2 + 4 a_i b_i}\}, \quad i \geqslant 1.$$

Moreover, if

$$\lim_{i\to\infty} \inf \{a_i + b_i - \sqrt{a_i b_{i-1}} - \sqrt{a_{i+1} b_i}\} > 0$$

then $\hat{\alpha} > 0$.

Having worked so much on the birth-death processes, now let us return to the general Markov chains. By comparing a given Markov chain with a birth-death process as we explained before, we obtain

(5.13) **Theorem**. *Let $(P_{ij}(t))$ be a Markov chain given at the beginning of this section. Then*

$$\mathrm{gap}(D) \geqslant \inf_{i\geqslant 1} \{q_{i,i-1} + q_{i-1,i} - \sqrt{q_{i-1,i-2}\, q_{i-1,i}} - \sqrt{q_{i,i-1}\, q_{i,i+1}}\},$$

$$\mathrm{gap}(D) \geqslant \frac{1}{2} \inf_{i\geqslant 1} \{q_{i,i-1} + q_{i+1,i} + q_{i-1,i} + q_{i,i+1} - [(q_{i+1,i} + q_{i,i+1} - q_{i,i-1} - q_{i-1,i})^2 + 16 q_{i,i-1}\, q_{i,i+1}]^{1/2}\}.$$

Moreover, if

$$\lim_{i\to\infty} \inf \{q_{i,i-1} + q_{i,i+1} - \sqrt{q_{i,i-1}\, q_{i-1,i}} - \sqrt{q_{i+1,i}\, q_{i,i+1}}\} > 0,$$

then $\mathrm{gap}(D) > 0$.

6. Non-Positive Recurrent Case

For the non-positive recurrent case, a Markov process has no finite measure as its invariant measure. Thus, the vector 1 does not belong to $L^2(\pi)$ and so the largest eigenevalue of Ω on $L^2(\pi)$ is meaningful. Inded, it determines the convergence rate. However, our previous results work well in this situation with a slight modification.

For example, as an analogue of Theorem (2.4), we have

$$\begin{aligned}\sigma_0 &\equiv \inf_{t>0} \frac{1}{t} \inf\{-\log \| P(t) f \| : \| f \| = 1\} \\ &= \inf\{(-\Omega f, f) : f \in \mathscr{D}(\Omega) \text{ and } \| f \| = 1\} \\ &= \inf\{(-\widetilde{\Omega} f, f) : f \in \mathscr{D}(\widetilde{\Omega}) \text{ and } \| f \| = 1\} \\ &= \inf\{D(f) : f \in \mathscr{D}(D) \text{ and } \| f \| = 1\}.\end{aligned}$$

Also, we can often reduce the non-symmetric case to a symmetric one.

For jump processes, we allow our q-pair $q(x)-q(x, dy)$ to be non-conservative:

$$d(x)\equiv q(x)-q(x,E)\geqslant 0,\ x\in E.$$

Any jump process $P(t, x, dy)$ with a q-pair $q(x)-q(x,dy)$ and an excessive measure π (σ-finite),

$$\pi\geqslant\pi P(t)\ ,\ t\geqslant 0$$

will gives us a strongly continuous and contractive semigroup $\{P(t)\}_{t\geqslant 0}$ on $L^2(\pi)$ (cf. Chen (1987), (11)). Of course, $(D^*, \mathscr{D}(D^*))$ given in Section 3 should be replaced by

$$D^*(f)=\int\pi(dx)f(x)[f(x)q(x)-\int q(x, dy)f(y)]\ ,$$

$$\mathscr{D}(D^*)=\{f\in L^2(\pi):\ D^*(f)<\infty\}.$$

In the symmetric case,

$$D^*(f)=\frac{1}{2}\int\pi_q(dx, dy)(f(y)-f(x))^2+\int\pi_d(dx)f(x)^2,$$

where $\pi_q(dx, dy)=\pi(dx)q(x, dy)$ and $\pi_d(dx)=\pi(dx)d(x)$.

From now on, we consider the symmetric case only.

It is interesting that $\sigma_0=-\lambda_\sigma(\pi)$ which was introduced by Stroock (1981). Several equivalent descriptions of $\lambda_\sigma(\pi)$ were discussed in Stroock (1981). For a related problem, see Chen and Stroock (1983) in which a simple estimate ($\sigma_0\leqslant\inf_{i\in E}q_i$) was obtained.

Now, suppose that the jump process satisfying the backward Kolmogorov equations is unique. Then the symmetric jump process corresponds to a regular Dirichlet form:

$$D(f,f)=D^*(f)$$

(see Chen (1989), Theorem (3.10)). Actually, this process is just the minimal one. Then

$$\begin{aligned}\sigma_0&=\inf\{D(f,f):\ \|f\|=1\}\\&=\inf\{D(f,f):\ f\in\mathscr{K}\text{ and }\|f\|=1\}.\end{aligned}$$

In particular, take a compact K such that $\pi(K)>0$ and set $f=I_K/(\pi(K))^{1/2}$; then

$$D(f,f)=[\pi_q(K\times K^c)+\pi_d(K)]\ /\ \pi(K).$$

Therefore,

$$\sigma_0\leqslant\inf_{\pi(K)>0}\frac{\pi_q(K\times K^c)+\pi_d(K)}{\pi(K)}$$

gives us an upper bound.

We can easily given an approximation theorem for σ_0 as an analogue of Theorem (4.2). Finally, for the birth-death process, we again have

$$\sigma_0=\hat{\alpha}\ ,$$

where $\hat{\alpha}$ is the exponentially ergodic rate (i.e., $P_{ij}(t)=O(\exp(-\hat{\alpha}\ t))$ for all i, j). Thus,

Exponential L^2-convergence $\Longleftrightarrow$ Exponential ergodicity.

Finally, Theorem (5.12) remains valid in the present case.

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