APPLICATIONS OF MALLIAVIN CALCULUS TO STOCHASTIS DIFFERENTIAL EQUATIONS WITH TIME-DEPENDENT COEFFICIENTS*†

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Abstract

In this paper, we apply Malliavin calculus to discuss when the solutions of stochastic differential equations (SDE's) with time-dependent coefficients have smooth density. Under Hörmander's condition,we conclude that the solutions of the SDE's have smooth density. As a consequence, we get the hypoellipticity for inhomogeneous differential operators.

1. Introduction

Given a probability space $(\Omega, \mathcal{F}, \boldsymbol{F}, P)$, where $\boldsymbol{F} = \{\mathcal{F}_t\}_{t\geq 0}$ is a family of σ-algebras satisfying the usual conditions. Consider SDE:

$$\begin{cases} dX(t) = \sum_{k=1}^{d} V_k(t, X(t))d\theta_k(t) + \tilde{V}_0(t, X(t))dt, \\ X(0) = x, \end{cases} \tag{1.1}$$

where $\{\theta_1(t), \cdots, \theta_d(t)\}$ is the d-dimensional Brownian motion corresponding to $\boldsymbol{F}$, and

$$V_k(t,x) = \sum_{i=1}^{N} V_k^i(t,x)\frac{\partial}{\partial x_i}, \quad k = 0, 1, \cdots, d$$

$$\tilde{V}_0(t,x) = V_0(t,x) + \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{N} V_k^j(t,x)\frac{\partial V_k(t,x)}{\partial x_j} =: \sum_{i=1}^{N}\tilde{V}_0^i(t,x)\frac{\partial}{\partial x_i}$$

In this paper, we always assume that $\tilde{V}_0$ and V_k $(k = 1, \cdots, d)$ belong to $\tilde{S}([0,T]\times R^N)$ for some $T > 0$, where $\tilde{S}([0,T]\times R^N)$ consists of the functions $f \in C([0,T]\times R^N \to R^N$ for which there exist positive constants $C_n(T)$ and $r_n(T)$ $(n \geq 0)$ with $r_0(T) = 0$ such that

$$\max_{\{\alpha\}=n}\sup_{0\leq t\leq T}\left|\frac{\partial^\alpha f(t,y)}{\partial y^\alpha}\right|_{R^N} \leq c_n(T)(1+|y|_{R^N})^{r_n(T)}, \quad \forall y \in R^N$$

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where $\alpha = (\alpha_1, \cdots, \alpha_n) \in (0, 1, \cdots, n)^N$ and $\{\alpha\} = \alpha_1 + \cdots + \alpha_n$.

Under this hypothesis, it is well known that there exists uniquely a strong solution $X(t,x)$ (also denoted by $X(t)$, $X_t(x)$ or X_t) satisfying SDE(1.1). Set

$$H = \{h \in C([0,T] \to R^N) : h(0) = 0,\ h \text{ is absolute continuous with respect to Lebesgue measure on } R^+ \text{ and } \int_0^t |h^{(1)}(t)|^2\, dt < \infty\}.$$

For any $h \in H$, the Malliavin derivative $DX(t)(h)$ of $X(t)$ satisfies (see [3] or [6]):

$$\begin{aligned} DX(t)(h) = &\sum_{k=1}^{d} \int_0^t V_k^{(1)}(s, X(s)) DX(s)(h)\, d\theta_k(s) \\ &+ \int_0^t \bar{V}_0^{(1)}(s, X(s)) DX(s)(h)\, ds \\ &+ \sum_{k=1}^{d} \int_0^t V_k(s, X(s)) h_k^{(1)}(s)\, ds, \end{aligned} \tag{1.2}$$

where

$$V_k^{(1)}(s, X(s)) = \left(\frac{\partial V_k^i(s, X(s))}{\partial x_j} \right)_{1 \le i,j \le N}, \quad k = 0, 1, \cdots, d. \tag{1.3}$$

Similarly, we can define $\bar{V}_0^{(1)}(s, X(s))$. The Malliavin covariance matrix of $X(t,x)$ is denoted by

$$A(t,x) = (\langle DX^i(t,x), DX^j(t,x) \rangle_H)_{1 \le i,j \le N}. \tag{1.4}$$

To prove that $X(t)$ has smooth density by using Malliavin calcullus, it needs to check two conditions. One is that $X(t)$ has Malliavin derivatives with any order, and the other is that

$$\det{}^{-1} A(t,x) \in \cap_{p \ge 1} L^p(\Omega). \tag{1.5}$$

For homogeneous case (i.e., in SDE (1.1), $V_k(t,x) = V_k(x)$ for all $t \in [0,T]$, $x \in R^N$ and $k = 0, 1, \cdots, d$), however, (1.5) has been proved by many authors if the cofficients $V_0, \cdots, V_d$ satisfy the so-called general Hörmander's condition at each point $x \in R^N$. (see [1] or [7])

For inhomogeneous case (i.e., in SDE (1.1), $V_k(t,y)$ depends on the variable t), Kusuoka and Stroock (see [6]) proved that (1.5) holds under a special condition. Thus, they concluded that the solution $X(t)$ of SDE(1.1) has smooth density. However, the condition given by them is actually the uniform non-degenerate one. It is natural to concern when (1.5) holds if the coefficients in SDE(1.1) are degenerate. To consider this problem, we need to assume that $V_1(t,x), \cdots, V_d(t,x)$ have nice continuity with respect to variable t because of some techniques.

For simplicity, let $\beta \ge 0$, $l \ge 1$ and set $S_{\beta l}([0,T] \times R^N) = \{f \in \tilde{S}([0,T] \times R^N)$: there exists $\delta > 0$ for each $\alpha = (\alpha_1, \cdots, \alpha_N)$ with $\{\alpha\} \le l$, such that the function $\frac{\partial^{\{\alpha\}} f(t,y)}{\partial y^\alpha}$ at the point x is β-Hölder continuous with respect to $t \in [0, \delta]\}$

One of our main results in this paper is as follows:

Theorem 1.1. Assume that there exists an integer $L \ge 1$ such that:

The vector space spanned by the vector fields $V_i(0,x)$, $1 \le i \le d$; $1 \le i_1, \cdots, i_L \le d$ at each point x is R^N where

$$[V_i(0,x), V_j(0,x)] = \sum_{k=1}^{N} \left(V_i^k(0,x) \frac{\partial V_j(0,x)}{\partial x_k} - V_j^k(0,x) \frac{\partial V_i(0,x)}{\partial x_k} \right) \frac{\partial}{\partial x_k}. \tag{1.6}$$

If $V_0, \cdots, V_d \in S_{\beta l}$ $([0,T] \times R^N)$ for some $\beta \in (\frac{1}{2}, 1]$ and sufficient large constant $l = l(L)$ (for example, $l(L) = 6^{L+1}$), then the solution $X(t,x)$ of SDE(1.1) has smooth density.

The condition (1.6) given in Theorem 1.1 is actually the restricted Hörmander's condition. To show that the solution $X(t,x)$ of SDE(1.1) still has smooth density under the general Hörmander's condition, it needs more hypotheses on the coefficients.

Theorem 1.2. Assume $\tilde{V}_0, V_1, \cdots, V_d \in \tilde{S}([0,T] \times R^N)$. If there exists a constant $L \geq 1$, such that the vector space spanned by the vector fields

$$V_i(0,x),\ 1 \leq i \leq d;\quad [V_i(0,x),\ V_j(0,x)],\ 0 \leq i,j \leq d;\ \cdots;$$

$$[V_{i_1}(0,x),\ \cdots [V_{i_{L-1}}(0,x), V_{i_L}(0,x)] \cdots],\ 0 \leq i_1, \cdots, i_L \leq d \tag{1.7}$$

at each point x is R^N. Moreover, for any $k = 0, 1, \cdots, d$, and $\alpha = (\alpha_1, \cdots, \alpha_N)$ with $\{\alpha\} \leq 2L$

$$\left(\left|\frac{\partial^{\{\alpha\}} V_k(t,y)}{\partial y^\alpha} - \frac{\partial^{\{\alpha\}} V_k(0,y)}{\partial y^\alpha}\right|\right)\Bigg|_{y=x} = o(t^{2L}),$$

as $t \to 0$. Then the solution $X(t,x)$ of SDE(1.1) has smooth density.

The proof of Theorem 1.2 is given in Section 3, and the proof of Theorem 1.1 is given in Section 4, which can not be deduced directly from the homogeneous case. To complete the proofs of the Theorems we make some preparations in the next section.

After proving Theorem 1.1 and Theorem 1.2, in Section 5, we discuss the uniqueness of the solutions for some kind of heat equations. Furthermore, we estimate their fundamental solutions. As in the homogeneous case (see [7, §7]), in Section 6, we give a condition which is weaker than Hörmander's one under which the solution $X(t,x)$ of SDE (1.1) still has smooth density. Having done these, we discuss the hypoellipticity properties for inhomogeneous second order differential operators in Section 7. This problem was also studied by Chaleyat-Maurel and Michel (see [2]) using pure analytical method. Moreover, they also got the Hörmander's theorem for inhomogeneous case. Although their restriction on the coefficients is weaker than ours, there is still difference between the other conditions (see Corollary 7.1).

2. Preliminary

Set

$$\mathcal{A} = \cup_{l=1}^{\infty} \{0, 1, \cdots, d\}^l$$

. For $\alpha = (\alpha_1, \cdots, \alpha_l) \in \{0, 1, \cdots, d\}^l$, we let $|\alpha| = 1$ and

$$\|\alpha\| = |\alpha| + \#\{j :\ \alpha_j = 0,\ j = 1, \cdots, l\}.$$

For

$$V(s,x) = \sum_{i=1}^N V^i(s,x) \frac{\partial}{\partial x_i};\quad W(t,x) = \sum_{i=1}^N W^i(t,x) \frac{\partial}{\partial x_i},$$

we let

$$[V(s), W(t)](x) = \sum_{i=1}^N \left(V^i(s,x) \frac{\partial W(t,x)}{\partial x_j} - W^j(t,x) \frac{\partial V(s,x)}{\partial x_j}\right).$$

Throughout this paper we set $\theta_0(t) = t$ for all $t \in [0,T]$.

It is well known that the SDE(1.1) has uniquely a strong solution $X(t,x)$ and $X(t,x)$ is a flow of diffeomorphisms on R^N (see [5] or [4]). This is saying that $X(t,x)$ is smooth with respect to $x \in R^N$ and $\det(J(t,x)) \neq 0$ for any $(t,x) \in [0,T] \times R^N$, where

$$J(t,x) = X^{(1)}(t,x) = \left(\frac{\partial X^i(t,x)}{\partial x_j}\right)_{1\leq i,j\leq N}.$$

In addition, $J(t,x)$ solves the SDE

$$J(t,x) = I + \sum_{k=0}^{d} \int_0^t V_k^{(1)}(s,X(s))J(s,x) \circ d\theta_k(s), \tag{2.1}$$

where I is $N \times N$ unit matrix, $V_k^{(1)}(s,X(s))$ is determined by (1.3) for each $k = 1,\cdots,d$.

Remark 2.1. $V_k^{(1)}(s,X(s))J(s,x)$ may not be a semimartingale, hence, the last term in the right hand side of (2.1) has no meaning in terms of the usual Stratonovich integral. Here and what follows we consider $V_k^{(1)}(s,y)$ as a differentiable function with respect to the variable s. Thus, $V_k^{(1)}(s,X(s))J(s,x)$ is a $\boldsymbol{F}$-semimartingale, the right hand side of (2.1) is well defined.

By Itô's formula and (1.2), it is easy to check that

$$J^{-1}(t,x)DX(t,x)(h) = \sum_{k=1}^{d} \int_0^t J^{-1}(s,x)V_k(s,X(s))h_k'(s)\,ds.$$

The inner product of $DX^i(t,x)$ and $DX^j(t,x)$ in H has expression:

$$\begin{aligned}&\langle DX^i(t,x), DX^j(t,x)\rangle_H\\ &= \sum_{k=1}^{d} \int_0^t [J(t,x)J^{-1}(s,x)V_k(s,X(s))]^i[J(t,x)J^{-1}(s,x)V_k(s,X(s))]^j\,ds.\end{aligned}$$

Set

$$\hat{A}(t,x) = \sum_{k=1}^{d} \int_0^t [J^{-1}(s,x)V_k(s,X(s))]^{\otimes^2}\,ds. \tag{2.2}$$

Then the Malliavin covariance matrix $A(t,x)$ defined by (1.4) can be expressed by

$$A(t,x) = J(t,x)\hat{A}(t,x)J(t,x)^*. \tag{2.3}$$

By Itô's formula, we easily get

$$J^{-1}(s,x)V_k(s,X_s) = V_k(s,x) + \sum_{h=0}^{d} \int_0^s J^{-1}(u,x)[V_h(u),V_k(s)](X_u) \circ d\theta_h(u). \tag{2.4}$$

In order to explain the meaning of the last term in the right hand side of (2.4) and get the general exptession of $J^{-1}(s,x)V_k(s,X_s)$, we need the following lemma.

Lemma 2.1. For any fixed h and k, let

$$\xi(s) = \int_0^s [V_h(u),V_k(s)](x) \circ d\theta_h(u),$$

$$\eta(s) = \int_0^s J^{-1}(u,x)[V_h(u), V_k(s)](X_u) \circ d\theta_h(u).$$

Then, both $\{\xi(s)\}_{s\geq 0}$ and $\{\eta(s)\}_{s\geq 0}$ can be considered as a progressive process.

The proof is very easy, hence is omitted.

Expanding further the term $J^{-1}(u,x)[V_h(u), V_k(s)](X_u)$ in the right hand side of (2.4) as we did in Lemma 2.1,

$$\varsigma(u) = \int_0^u J^{-1}(v,x)[V_l(v)[V_h(u), V_k(s)]](X_v) \circ d\theta_l(v)$$

is a progressive process. Hence, the definition of $\int_0^t \varsigma(u) \circ d\theta_i(u)$ is reasonable for $i = 0, 1, \cdots, d$. Thus, $J^{-1}(s,x)V_k(s,X_s)$ can be expresses by

$$J^{-1}(t,x)V_k(t,X_t) = \sum_{\substack{|\alpha|\leq L \\ \alpha_1=k}} \bar{S}^{(\alpha)}(t,x) + \sum_{\substack{|\alpha|=L+1 \\ \alpha_1=k}} R^{(\alpha)}(t,x), \tag{2.5}$$

where $\bar{S}^{(\alpha)}$ and $R^{(\alpha)}$ are defined as follows. Let $\alpha = (\alpha_1, \cdots, \alpha_m) \in \mathcal{A}$. If $m = 1$, then

$$\bar{S}^{(\alpha)}(t,x) = V_{\alpha_1}(t,x); \; R^{(\alpha)}(t,x) = 0.$$

If $m \geq 2$, then

$$\begin{aligned} \bar{S}^{(\alpha)}(t_1,x) = \int_0^{t_1} \cdots \int_0^{t_{m-1}} &[V_{\alpha_m}(t_m), [\cdots [V_{\alpha_2}(t_2), V_{\alpha_1}(t_1)] \cdots](x) \\ &\circ d\theta_{\alpha_m}(t_m) \circ \cdots \circ d\theta_{\alpha_2}(t_2), \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} R^{(\alpha)}(t_1,x) = \int_0^{t_1} \cdots \int_0^{t_{m-1}} &J^{-1}(t_m,x)[V_{\alpha_m}(t_m), [\cdots [V_{\alpha_2}(t_2), V_{\alpha_1}(t_1)] \cdots](X_{t_m}) \\ &\circ d\theta_{\alpha_m}(t_m) \circ \cdots \circ d\theta_{\alpha_2}(t_2). \end{aligned} \tag{2.7}$$

As in Remark 2.1, we can explain the meaning of the right hand side of (2.6) and (2.7). Actually, we can express them in terms of Itô's integral. In other words, $\bar{S}^{(\alpha)}(t,x)$ and $R^{(\alpha)}(t,x)$ can be expressed completely in terms of Itô's integral . For $m = 1$, let $S^{(\alpha)}(t,x) = \bar{S}^{(\alpha)}(t,x)$. For $m \geq 2$, let

$$\begin{aligned} S^{(\alpha)}(t_1,x) = \int_0^{t_1} \cdots \int_0^{t_{m-1}} &[V_{\alpha_m}(t_m), [\cdots [V_{\alpha_2}(t_2), V_{\alpha_1}(t_1)] \cdots]x \\ &d\theta_{\alpha_m}(t_m) \cdots d\theta_{\alpha_2}(t_2). \end{aligned} \tag{2.8}$$

If

$$r^{(\alpha)}(t,x) = \bar{S}^{(\alpha)}(t,x) - S^{(\alpha)}(t,x), \quad \forall \alpha \in \mathcal{A},$$

then we see that $r^{(\alpha)}(t,x)$ can be expressed as a finite sum of the following type

$$\int_0^t \int_0^{t_2} \cdots \int_0^{t_{m-1}} (h_{r_1}^{i_1}(t_1,x) \cdots h_{r_m}^{i}(t_m,x))_{1\leq i\leq N} d\theta_{\beta_m}(t_m) \cdots d\theta_{\beta_2}(t_2), \tag{2.9}$$

where $h^i_{r_k}(t,x)$ is β-Hölder continuous with respect to t and slowly increasing with respect to $x \in R^N$ for each i and r_k, and there must be at least one i $(2 \le i \le m)$, such that $\beta_i = 0$. Set

$$Z_L^{(k)}(t,x) = \sum_{\substack{|\alpha|\le L\\ \alpha_1=k\\ \|\alpha\|\ge L+1}} \bar{S}^{(\alpha)}(t,x) + \sum_{\substack{|\alpha|=L+1\\ \alpha_1=k}} R^{(\alpha)}(t,x),$$

then (2.5) can be written as

$$J^{-1}(t,x)V_k(t,X(t)) = \sum_{\substack{|\alpha|\le L\\ \alpha_1=k}} \bar{S}^{(\alpha)}(t,x) + Z_L^{(k)}(t,x). \tag{2.10}$$

Since we will use [7, Lemma A.7] time after time, we restate it as follows:

Lemma 2.2. Let $\beta(t) = (\beta_k(t))_{1\le k\le d}$ be a d-dimensional continuous progressive measurable process. Set

$$\xi(t) = \sum_{k=1}^{d}\int_0^t \beta_k(t)\,d\theta_k(t);\quad V(t) = \int_0^t |\beta(t)|^2\,dt,\ \forall t \ge 0.$$

Then, for any $r \in [0,\frac{1}{2})$, there is a $c_r < \infty$ and a $\lambda_r > 0$, such that

$$P\left(\sup_{0\le s<t\le T}\frac{|\xi(t)-\xi(s)|}{|V(t)-V(s)|^r} \ge k_1;\ V(T)\le k_2\right) \le c_r \exp\left(\frac{-\lambda_r k_1^2}{k_2^{1-2r}}\right)$$

for all positive k_1 and k_2.

3. Proof of Theorem 1.2

The proof is splitted into two steps. The first step is to show that $D^kX(t,x)$ (in Malliavin's sense) is well defined for all $k \ge 1$. It is not difficult to complete the step since $\bar{V}_0, V_1, \cdots, V_d$ belong to $\tilde{S}([0,T]\times R^N)$ (cf. [6]). The second step is to show that (1.7) and (1.8) imply (1.5). This is what we are working in this section.

Actually, because of (2.3) and [4, Lemma 5.2.1], it suffices to show

$$(\det \hat{A}(t,x))^{-1} \in \cap_{P\ge 1} L^P(\omega), \tag{3.1}$$

where $\hat{A}(t,x)$ is defined by (2.2). Since

$$\det\hat{A}(t,x) \ge \left[\inf_{\eta\in S^{N-1}}\int_0^t\sum_{k=1}^{d}(J^{-1}(s,x)V_k(s,X_s),\eta)^2\,ds\right]^N,$$

(3.1) follows provided

$$P\left(\inf_{\eta\in S^{N-1}}\int_0^t\sum_{k=1}^{d}(J^{-1}(s,x)V_k(s,X_s),\eta)^2\,ds\right) \le c_p n^{-p},\quad \forall p,n\ge 1. \tag{3.2}$$

For this, it suffices to show that

$$P\left(\inf_{\eta\in S^{N-1}}\int_0^{t/n}\sum_{k=1}^{d}(J^{-1}(s,x)V_k(s,X_s),\eta)^2\,ds < n^{-(l+1-\epsilon)}\right) \le c_p n^{-p} \tag{3.3}$$

holds for a large integer $l = l(L) \geq 1$ and all $p, n \geq 1$, where $\varepsilon \in (0,1)$ is a fixed constant, c_p is independent of n. Without loss of generality, we assume $t = 1$. By (2.10), we have

$$\begin{aligned}&\inf_{\eta\in S^{N-1}}\int_0^{\frac{1}{n}}\sum_{k=1}^{d}(J^{-1}(s,x)V_k(s,X_s),\eta)^2\,ds\\ \geq&\frac{1}{2}\inf_{\eta\in S^{N-1}}\int_0^{\frac{1}{n}}\sum_{k=1}^{d}\Big|\sum_{\substack{\|\alpha\|\leq l\\ \alpha_1=k}}S^{(\alpha)}(s,x),\eta\Big|^2 ds-\int_0^{\frac{1}{n}}\sum_{k=1}^{d}|Z_1^{(k)}(s,x)|^2\,ds.\end{aligned}\tag{3.4}$$

Next, we prove that

$$P\Big(\int_0^{\frac{1}{n}}\sum_{k=1}^{d}|Z_1^{(k)}(s,x)|^2\,ds\geq n^{-(l+1-\varepsilon)}\Big)\leq c_p n^{-p},\tag{3.5}$$

for all $l \geq 2$ and $n \geq 1$. To do this, we need a lemma:

Lemma 3.1. For $\alpha = (\alpha_1, \cdots, \alpha_l)$, set

$$f_\alpha(t_1,\cdots,t_l,x)=[V_{\alpha_l}(t_l),[\cdots[V_{\alpha_2}(t_2),V_{\alpha_1}(t_1)]\cdots](x),$$

Then

(i) There exist positive constants c and u such that

$$\begin{aligned}&P\Big(\int_0^1\Big|\int_0^{t_l}\cdots\int_0^{t_{l-1}}f_\alpha(\frac{t_1}{n},\cdots,\frac{t_l}{n},x)P\\ &\circ d\theta_{\alpha_1}(t_1)\circ\cdots\circ d\theta_{\alpha_2}(t_2)\Big|^2 dt_l\geq n\Big)\leq ce^{-n^u};\end{aligned}$$

(ii) There exists a positive constant c_p such that

$$\begin{aligned}&P\Big(\int_0^1\Big|\int_0^{t_l}\cdots\int_0^{t_{l-1}}J^{-1}(\frac{t_l}{n},x)f(\frac{t_1}{n},\cdots,\frac{t_l}{n},X(\frac{t_l}{n}))\\ &\circ d\theta_{\alpha_1}(t_1)\circ\cdots\circ d\theta_{\alpha_2}(t_2)\Big|^2 dt_l\geq n\Big)\\ \leq&c_p n^{-p},\quad\forall n\geq 1,\ \forall p\geq 1.\end{aligned}$$

Proof. To prove (i), it needs only to show that

$$\begin{aligned}&P\left(\max_{0\leq t_l\leq 1}\Big|\int_0^{t_l}\cdots\int_0^{t_{l-1}}f_\alpha(\frac{t_1}{n},\cdots,\frac{t_l}{n},x)\circ d\theta_{\alpha_1}(t_1)\circ\cdots\circ d\theta_{\alpha_2}(t_2)\Big|^2\geq n\right)\\ \leq&ce^{-n^u}.\end{aligned}\tag{3.6}$$

In fact, we can prove (3.6) by using Lemma 2.2 repeatedly. The details are omitted here.

To prove (ii), we also assume $N = 1$. Since

$$\max_{0\leq t\leq 1}E\|J^{-1}(t,x)\|^p<\infty;\quad\max_{0\leq t\leq 1}E|X(t,x)|^p<\infty,\quad\forall p\geq 1,$$

and $V_0(t,x),\cdots,V_d(t,x)$ are slowly increasing with respect to $x\in R^N$, it is easy to see that

$$\max_{0\le t_1,\cdots,t_1\le 1} E|J^{-1}(t_1,x)f_\alpha(t_1,\cdots,t_1,X(t_1,x))|^p<\infty,\quad \forall p\ge 1.$$

If $l=2$ and $\alpha_2=0$, the proof is easy. If $l=2$ and $\alpha_2>0$, $\alpha=(\alpha_1,\alpha_2)$, then the Burkholder's inequality implies that

$$\begin{aligned}
&P\left(\int_0^1\left|\int_0^{t_1} J^{-1}(\frac{t_2}{n},x)f(\frac{t_1}{n},\frac{t_2}{n},X(\frac{t_2}{n}))\circ d\theta_{\alpha_2}(t_2)\right|^2 dt_1\ge n\right)\\
\le& P\left(\int_0^1\left|\int_0^{t_1} J^{-1}(\frac{t_2}{n},x)f(\frac{t_1}{n},\frac{t_2}{n},X(\frac{t_2}{n}))d\theta_{\alpha_2}(t_2)\right|^2 dt_1\ge n\right)\\
&+P\left(\int_0^1\left|\int_0^{t_1} J^{-1}(t_2,x)f_{(\alpha_1,\alpha_2,\alpha_2)}(t_1,t_2,t_2,X_{t_2}))\,dt_2\right|^2 dt_1\ge n\right)\\
\le& c_pn^{-p}+n^{-p}\max_{0\le t_1\le 1}E\left|\int_0^{t_1} J^{-1}(\frac{t_2}{n},x)f_\alpha(\frac{t_1}{n},\frac{t_2}{n},X(\frac{t_2}{n}))\,d\theta_{\alpha_2}(t_2)\right|^{2p}\\
\le& c_pn^{-p}+c_pn^{-p}\max_{0\le t_1\le 1}E\left(\int_0^{t_1}\left|J^{-1}(\frac{t_2}{n},x)f(\frac{t_1}{n},\frac{t_2}{n},X(\frac{t_2}{n}))\right|^2 dt_2\right)^p\\
\le& c_pn^{-p}.
\end{aligned}$$

For general $l\ge 3$, (ii) can be also proved by induction. ∎

Set $\alpha=(\alpha_1,\cdots,\alpha_{|\alpha|})$, $|\alpha|\le l$ and $\|\alpha\|\ge l+1$. Then

$$\begin{aligned}
&P\left(\int_0^{\frac1n}|\bar S^{(\alpha)}(t,x)|^2\,dt\ge n^{-(l+1-\epsilon)}\right)\\
\le& P\left(\int_0^1\left|\int_0^{t_1}\cdots\int_0^{t_{|\alpha|-1}} f_\alpha(\frac{t_1}{n},\cdots,\frac{t_{|\alpha|}}{n},x)\circ d\theta_{\alpha_{|\alpha|}}(t_\alpha)\circ\cdots\circ d\theta_{\alpha_2}(t_2)\right|^2 dt_1\ge n^{\epsilon}\right)
\end{aligned}$$

where $l+1-\|\alpha\|\le 0$. By Lemma 3.1 (i), we know that

$$P\left(\int_0^{\frac1n}|\bar S^{(\alpha)}(t,x)|^2\,dt\ge n^{-(l+1-\epsilon)}\right)\le ce^{-n^{\epsilon}}.$$

Now, (3.5) follows by using the expression of $Z_l^{(k)}(t,x)$. Thus, by (3.4), it is easy to see that (3.3) follows from

$$P\Big(\inf_{\eta\in S^{N-1}}\int_0^{1/n}\sum_{k=1}^d\Big(\sum_{\substack{\|\alpha\|\le l\\ \alpha_1=k}}(S^{(\alpha)}(t,x),\eta)^2\,dt\Big)\le n^{-(l+1-\epsilon)}\Big)\le c_pn^{-p}.\tag{3.7}$$

To prove this, we need the following lemma.

Lemma 3.2. For $\alpha=(\alpha_1,\cdots,\alpha_m)$ and $m=2,\cdots,L$ let

$$\begin{aligned}
&Q_n^\alpha(j,t_1,x,\eta)\\
=&\int_0^{t_1}\cdots\int_0^{t_{m-1}}(f_\alpha(\frac{jt_1}{n},\cdots,\frac{jt_m}{n},x),\eta)\circ d\theta_{\alpha_m}(t_m)\circ\cdots\circ d\theta_{\alpha_2}(t_2),\quad j=0,1,
\end{aligned}$$

where $f_\alpha(t_1,\cdots,t_m,x)$ was defined in Lemma 3.1. If $V_0,\cdots,V_d$ satisfy the hypothesis (1.8), then there exist positive constants c and u for each $q\geq 1$, such that

$$P\left(\max_{0\leq t_1\leq 1}|Q_n^\alpha(1,t_1,x,\eta)-Q_n^\alpha(0,t_1,x,\eta)|\geq n^{-q}\right)\leq c\exp(-n^u),\quad \forall n\geq 1.$$

Proof. Without loss of generality, we assume $N=1$ and $\eta=1$. If $m=2$ and $\alpha_2=0$, then the conclusion follows from (1.8) immediately. If $m=2$ and $\alpha_2>0$, then

$$\begin{aligned}&Q_n^\alpha(1,t_1,x,1)-Q_n^\alpha(0,t_1,x,1)\\&=\int_0^{t_1}\left[f_\alpha(\frac{t_1}{n},\frac{t_2}{n},x)-f_\alpha(0,\frac{t_2}{n},x)\right]d\theta_{\alpha_2}(t_2)\\&\quad+\int_0^{t_1}\left[f_\alpha(0,\frac{t_2}{n},x)-f_\alpha(0,0,x)\right]d\theta_{\alpha_2}(t_2)\\&=:I_1(t_1)+I_2(t_1).\end{aligned}$$

In fact, we can assume

$$f_\alpha(t_1,t_2,x)=g_\alpha(t_1,x)q_\alpha(t_2,x).$$

By(1.8), for $q\leq L$, there exists an integer $N(q)$ such that

$$\left|g_\alpha(\frac{t_1}{n},x)-g_\alpha(0,x)\right|\leq Mn^{-(q+1)},\quad n\geq N(q).$$

Meanwhile, $q_\alpha(\frac{t_2}{n},x)$ on $t_2\in[0,1]$ is uniformly bounded for large enough n. Thus, we get that from Lemma 2.2

$$P\left(\max_{0\leq t\leq 1}|I_1(t)|\geq n^{-q}\right)\leq c\exp(-n^u),$$

where the positive constants c and u are independent of n. For $t_2\in[0,1]$ and large n, we have

$$|f_\alpha(0,\frac{t_2}{n},x)-f_\alpha(0,0,x)|\leq Mn^{-1-q}.$$

Then, Lemma 2.2 leads immediately that

$$P\left(\max_{0\leq t_1\leq 1}|I_2(t_1)|\geq n^q\right)\leq \acute{c}\exp(-n^u).$$

So far, we get the conclusion for $m=2$. For general case, it can be obtained also by induction. ∎

Now, we apply [7,Appendix] and Lemma 3.2 to prove (3.7) (choose $l=L$). Note $Q_n^\alpha(1,t,x,\eta)$ has the same distribution as $n^{\frac{\|\alpha\|-1}{2}}(\bar{S}^{(\alpha)}(\frac{t}{n},x),\eta)$. Then

$$\begin{aligned}&P\left(\inf_{\eta\in S^{N-1}}\int_0^{\frac{1}{n}}\sum_{k=1}^{d}\Big(\sum_{\substack{\|\alpha\|\leq L\\ \alpha_1=k}}(\bar{S}^{(\alpha)}(t,x),\eta)^2\Big)dt\leq n^{-(L+1-\epsilon)}\right)\\&\leq ce^{-n^u}+P\left(\inf_{\eta\in S^{N-1}}\int_0^1\sum_{k=1}^{0}\Big(\sum_{\substack{\|\alpha\|\leq L\\ \alpha_1=k}}n^{\frac{L-\|\alpha\|}{2}}Q_n^\alpha(0,t_1,x,\eta)^2\Big)dt_1\leq n^{-(1-\epsilon)}\right).\end{aligned}\tag{3.8}$$

For $\alpha = (\alpha_1, \cdots, \alpha_m)$, we have

$$\begin{aligned} Q_n^\alpha(0,t_1,x) &= \int_0^{t_1} \cdots \int_0^{t_{m-1}} (f_\alpha(0,\cdots,0,x),\eta) \circ d\theta_{\alpha_m}(t_m) \circ \cdots \circ d\theta_{\alpha_2}(t_2) \\ &= V_{(\alpha)}(0,x)\theta^{(\alpha)}(t_1), \end{aligned}$$

where

$$V_{(\alpha)} = \begin{cases} V_{\alpha_1}(0,x), & \alpha = (\alpha_1) \\ [V_{\alpha_m}(0),\ V_{(\alpha')}(0)](x),\ \alpha = (\alpha_1,\cdots,\alpha_m), & m \geq 2 \end{cases}$$

$$\theta^{(\alpha)}(t_1) = \begin{cases} 1, & m = 1 \\ \int_0^{t_1} \cdots \int_0^{t_{m-1}} \circ d\theta_{\alpha_m}(t_m) \circ \cdots \circ d\theta_{\alpha_2}(t_2), & m \geq 2 \end{cases}$$

and $\alpha' = (\alpha_1, \cdots \alpha_{m-1})$. Comparing $V_{(\alpha)}(0,x)$ and $\theta^{(\alpha)}(t_1)$ with $V_{(\alpha)}(x)$ and $\theta^{(\alpha)}(t)$ respectively as in Appendix of [7], and noting the hypothesis (1.7) and [7,Theorem A.6], we get immediately

$$P\Big(\inf_{\eta \in S^{N-1}} \int_0^1 \sum_{k=1}^d \sum_{\substack{\|\alpha\| \leq L \\ \alpha_1 = k}} Q_n^\alpha(0,t,x,\eta))^2\, dt \leq n^{-(1-\epsilon)}\Big) \leq ce^{-n^u}.$$

Therefore,the right hand side of (3.8) $\leq ce^{-n^u}$.

Combining the above discussions, we conclude that Theorem 1.2 holds.

4. Proof of Theorem 1.1

By the previous discussions, we see that (1.5) implies Theorem 1.1. Indeed, we need only to prove (3.7) for some $l = l(L)$. To do this, for $m \geq 2$ and $\alpha = (\alpha_1, \cdots, \alpha_m)$, let

$$\begin{aligned} \bar{S}_n^{(\alpha)}(t,x,\eta) &= Q_n^{(\alpha)}(1,t,x,\eta), \\ S_n^{(\alpha)}(t_1,x,\eta) &= \int_0^{t_1} \cdots \int_0^{t_{m-1}} \Big(f_\alpha\big(\frac{t_1}{n},\cdots,\frac{t_m}{n} bigr),\eta\Big)\, d\theta_{\alpha_m}(t_m) \cdots d\theta_{\alpha_2}(t_2). \end{aligned}$$

For $m = 1$, let

$$\bar{S}_n^{(\alpha)}(t,x,\eta) = S_n^{(\alpha)}(t,x,\eta) = \big(V_{\alpha_1}(\frac{t}{n},x),\eta\big).$$

So, (3.7) is equivalent to that

$$P\Big(\inf_{\eta \in S^{N-1}} \sum_{k=1}^d \int_0^1 \Big(\sum_{\substack{\|\alpha\| \leq 1 \\ \alpha_1 = k}} n^{-\frac{\|\alpha\|-1}{2}} \bar{S}_n^{(\alpha)}(t,x,\eta)\Big)^2 dt \leq n^{\epsilon - l}\Big) \leq ce^{-n^u}. \tag{4.1}$$

Noting Lemma 3.1 (i) and using a similar argument in [7, Appendix], we easily see that, to complete the proof of (4.1), it is sufficient to show that there exist positive constants $c = c(l)$ and $u = u(l)$, such that

$$P\Big(\int_0^1 \sum_{k=1}^d \Big(\sum_{\substack{\|\alpha\| \leq l \\ \alpha_1 = k}} n^{-\frac{\|\alpha\|-1}{2}} \bar{S}_n^{(\alpha)}(t,x,\eta)\Big)^2 dt \leq n^{2\epsilon - 1}\Big) \leq ce^{-n^u}, \tag{4.2}$$

where $\eta \in S^{N-1}$, $n \geq 1$ and $\varepsilon \in (0, \frac{1}{2})$.

To do so, we first give the next lemmas.

Lemma 4.1. Assume that f is a r-Hölder continuous function on interval $[0,1]$ ($r > 0$), i.e. there exists a constant $c \geq 1$ such that

$$|f(t) - f(s)| \leq c|t-s|^r, \quad \forall t, s \in [0,1].$$

If $\max_{0\leq t\leq 1} |f(t)| \geq M$ ($M \leq 1$), then

$$\int_0^1 f^2(t)\,dt \geq \left(\frac{1}{c}\right)^{\frac{1}{r}} \left(\frac{M}{2}\right)^{2+\frac{1}{r}}$$

Proof. Easy!

Lemma 4.2. If $r \in (0, \frac{1}{2})$, then $\exists u$,

$$P\left(\sup_{0\leq s<t\leq 1} \frac{|\bar{S}_n^{(\alpha)}(t,x,\eta) - \bar{S}_n^{(\alpha)}(s,x,\eta)|}{|t-s|^r} \geq n\right) \leq ce^{-n^u}.$$

Proof. Set $\alpha = (\alpha_1, \cdots, \alpha_m)$. If $m = 1$, then $\bar{S}_n^{(\alpha)}(t,x,\eta) = (V_{\alpha_1}(\frac{t}{n}, x), \eta)$. In this case, the desired result is easy to get. If $m = 2$, then

$$\bar{S}_n^{(\alpha)}(t,x,\eta) = \left(\int_0^t f_\alpha(\frac{t}{n}, \frac{s}{n}, x)\,d\theta_{\alpha_2}(s), \eta\right).$$

Without loss of generality, we still assume $N = 1$ and $\eta = 1$. Thus

$$f_\alpha(t,s,x) = g_\alpha(t,x) q_\alpha(s,x),$$

where $g_\alpha(t,x)$ and $q_\alpha(t,x)$ belong to $S_{\beta l}([0,T] \times R^N)$. If $\alpha_2 = 0$, then

$$\bar{S}_n^{(\alpha)}(t,x,\eta) = \left[\int_0^t q_\alpha(\frac{s}{n}, x)\,ds\right] \times g_\alpha\left(\frac{t}{n}\right).$$

Clearly, the estimation holds for this situation. If $\alpha_2 > 0$, then

$$\begin{aligned}
&|\bar{S}_n^{(\alpha)}(t,x,\eta) - \bar{S}_n^{(\alpha)}(s,x,\eta)| \\
=&\left|\int_0^t f_\alpha(\frac{t}{n}, \frac{t_1}{n}, x)\,d\theta_{\alpha_2}(t_1) - \int_0^s f_\alpha(\frac{s}{n}, \frac{t_1}{n}, x)\,d\theta_{\alpha_2}(t_1)\right| \\
\leq &\left|g_\alpha(\frac{t}{n}, x) - g_\alpha(\frac{s}{n}, x)\right| \max_{0\leq t\leq 1} \left|\int_0^t q(\frac{t_1}{n})\,d\theta_{\alpha_2}(t_1)\right| \\
&+ \max_{0\leq s\leq 1} \left|g_\alpha(\frac{s}{n}, x)\right| \left|\int_s^t q_\alpha(\frac{t_1}{n}, x)\,d\theta_{\alpha_2}(t_1)\right| \\
=:&I_1(t,s) + I_2(t,s).
\end{aligned}$$

By Lemma 2.2, we have

$$P\left(\sup_{0\leq s<t\leq 1} \frac{I_1(t,s)}{|t-s|^r} \geq n\right) \leq P\left(\max_{0\leq t\leq 1} \left|\int_0^t q_\alpha(\frac{t_1}{n}, x)\,d\theta_{\alpha_2}(t_1)\right| \geq n^{\frac{1}{2}}\right) \leq ce^{-n^u} \ (n \gg 1),$$

$$P\left(\sup_{0\leq s<t\leq 1} \frac{|I_2(t,s)|}{|t-s|^r} \geq n\right) \leq P\left(\sup_{0\leq s<t\leq 1} \frac{|\int_s^t q_\alpha(\frac{t_1}{n}, x)\,d\theta_{\alpha_2}(t_1)|}{[\int_s^t q_\alpha^2(\frac{t_1}{n}, x)\,dt_1]^2}\right) \geq nM^{2r-1}(x) \leq ce^{-n^u}.$$

Thus, we get the conclusion for the case $m = 2$. By induction, we can get the conclusion for general $m \geq 3$. ∎

By the hypothesis of Theorem 1.1, we know that there exists a constant $c \in (0, \infty)$ such that

$$\inf_{\eta \in S^{N-1}} \sum_{\substack{\|\alpha\| \leq L \\ \alpha \in \tilde{A}}} (V_{(\alpha)}(0, x), \eta)^2 \geq c,$$

where

$$\tilde{A} = \cup_{l=1}^{\infty} \{1, \cdots, d\}^l.$$

For any fixed $\eta \in S^{N-1}$, there always exists a q $(1 \leq q \leq L)$ such that

$$\sum_{\substack{\|\alpha\| = q \\ \alpha \in \tilde{A}}} (V_{(\alpha)}(0, x), \eta)^2 \geq \frac{c}{L}. \tag{4.3}$$

Choose large $l = l(L) \geq L$ and set

$$\varsigma_n^{(k)}(t, x, \eta) = \sum_{\substack{\|\alpha\| \leq l \\ \alpha_1 = k}} n^{\frac{1-\|\alpha\|}{2}} \bar{S}_n^{(\alpha)}(t, x, \eta).$$

Then, an immediate result of Lemma 4.1 and Lemma 4.2 is that

$$\begin{aligned} &P\left(\bigcap_{k=1}^{d} \left\{\int_0^1 (\varsigma_n^{(k)}(t, x, \eta))^2 \, dt \leq n^{-1}\right\}\right) \\ \leq & ce^{-n^u} + P\left(\bigcap_{k=1}^{d} \left\{\max_{0 \leq t \leq 1} |\varsigma_n^{(k)}(t, x, \eta)| \leq n^{-\frac{l-2}{6}}\right\}\right). \end{aligned}$$

Hence to prove (4.2), it needs only to prove

$$P\left(\max_{0 \leq t \leq 1} \sum_{k=1}^{d} |\varsigma_n^{(k)}(t, x, \eta)| \leq n^{-\frac{l-2}{6}}\right) \leq ce^{-n^u}.$$

For this, we need the following lemma.

Lemma 4.3. Suppose that $f \in S_{\beta l}([0, T] \times R^N)$ $(\beta > \frac{1}{2})$ and let

$$g_i(s, t, x) = \sum_{j:\text{finite}} P_j^{(i)}(s, x) q_j^{(i)}(t, x), \quad i = 1, \cdots, d,$$

where $q_j^{(i)} \in S_{\beta l}([0, T] \times R^N)$ $(\beta > \frac{1}{2})$ and $p_j^{(i)}$ satisfies

$$P\left(\max_{0 \leq s \leq 1} \max_{i,j} |p_j^{(i)}(s, x)| \geq n\right) \leq ce^{-n^u}, \quad \forall n \geq 1, \tag{4.5}$$

for constants $c, u \in (0, \infty)$ independent of n. Then

$$P\left(\max_{0 \leq t \leq 1} |\xi(t, x)| \leq n^{-1}; \quad \int_0^1 \sum_{k=1}^{d} g_k^2(t, t, x) \, dt \geq n^{-\frac{1}{2}}\right) \leq ce^{-n^u},$$

where

$$\xi(t,x) = f(t,x) + \sum_{k=0}^{d} \int_0^t g_k(s,t,x)\, d\theta_k(s). \tag{4.6}$$

Proof. Without loss of generality, in the proof below, we always assume $\beta = 1$, and $|f(t,x) - f(s,x)| \le |t-s|$, $\forall t,s \in [0,1]$. By the hypothesis (4.5), we easily get

$$\begin{aligned}
&P\left(\max_{0\le t\le 1} |\xi(t,x)| \le n^{-1};\quad \int_0^1 \sum_{k=1}^{d} g_k^2(t,t,x)\,dt \ge n^{-\frac{1}{2}}\right)\\
&\le \sum_{i=1}^{n} P\left(\sum_{k=1}^{d} \int_{\frac{i-1}{n}}^{\frac{i}{n}} g_k^2(t,t,x)\,dt \ge n^{-3/2};\right.\\
&\qquad \left.\max_{\frac{i-1}{n}\le t\le \frac{i}{n}} \sum_{k=0}^{d} \left| \int_0^t g_k(s,t,x)\,d\theta_k(s) - \int_0^{\frac{i-1}{n}} g_k\big(s,\frac{i-1}{n},x\big)\,d\theta_k(s)\right| \le 3n^{-1}\right)\\
&\le ce^{-n^u} + \sum_{i=1}^{n} P\left(\max_{0\le s\le 1} \max_{i,j} |p_j^{(i)}(s)| \le n^{\frac{1}{16}};\right.\\
&\qquad \max_{\frac{i-1}{n}\le t\le \frac{i}{n}} \sum_{k=1}^{d} \left| \int_0^t g_k\big(s,\frac{i-1}{n}\big)\,d\theta_k(s) - \int_0^{\frac{i-1}{n}} g_k\big(s,\frac{i-1}{n}\big)\,d\theta_k(s)\right| \le n^{-(1-\frac{1}{8})};\\
&\qquad \left.\sum_{k=1}^{d} \int_{\frac{i-1}{n}}^{\frac{i}{n}} g_k^2\big(t,\frac{i-1}{n},x\big)\,dt \ge \frac{1}{2} n^{-\frac{3}{2}}\right).
\end{aligned} \tag{4.7}$$

By Lemma 2.2, we easily get

$$\left| \int_{\frac{i-1}{n}}^{\frac{i}{n}} g_k^2(t,t)\,dt - \int_{\frac{i-1}{n}}^{\frac{i}{n}} g_k^2\big(t,\frac{i-1}{n}\big)\,dt \right| \le cn^{-(2-\frac{1}{8})},$$

where $g_k(s,t) = g_k(s,t,x)$. Using the expressions of $g_k(t,t)$ and $g_k(t,\frac{i-1}{n})$, we easily get

$$\max_{0\le t\le 1} |g_k(t,t)| \vee \left| g_k\big(t,\frac{i-1}{n}\big)\right| \le n^{\frac{1}{16}}$$

and

$$\max_{\frac{i-1}{n}\le t\le \frac{i}{n}} \left| g_k(t,t) - g_k\big(t,\frac{i-1}{n}\big)\right| \le cn^{\frac{1}{16}-1}.$$

Thus, the right hand side of (4.7)

$$\begin{aligned}
&\le c\exp(-n^u) + \sum_{i=1}^{n} P\left(\sum_{k=1}^{d} \int_{\frac{i-1}{n}}^{\frac{i}{n}} g_k^2\big(t,\frac{i-1}{n}\big)\,dt \ge \frac{1}{2} n^{-\frac{3}{2}};\right.\\
&\qquad\qquad \left.\max_{\frac{i-1}{n}\le t\le \frac{i}{n}} \left| \sum_{k=1}^{d} \int_{\frac{i-1}{n}}^{t} g_k\big(s,\frac{i-1}{n}\big)\,d\theta_k(s)\right| \le n^{-(1-\frac{1}{8})}\right);\\
&\le c\exp(-n^u) + nP\big(\max_{0\le t\le 1} |\theta_1(t)| \le n^{-\frac{1}{8}}\big) \le c\exp(-n^u).
\end{aligned}$$

So far, the proof of Lemma 4.3 is finished. ∎

We now prove (4.4). In fact, if $q=1$, we get from (4.3)

$$\varsigma_n^{(k)}(t,x,\eta)=\bar{S}_n^{(k)}(t,x,\eta)+\sum_{\substack{2\le\|\alpha\|\le l\\ \alpha_1=k}} n^{\frac{1-\|\alpha\|}{2}}\bar{S}_n^{(\alpha)}(t,x,\eta).$$

If n is large enough, then

$$\sum_{k=1}^{d}|\bar{S}_n^{(k)}(t,x,\eta)|=\sum_{k=1}^{d}|(V_k(\frac{t}{n},x),\eta|\ge c(2L)^{-1}.$$

By the proof of Lemma 3.1 (i), we know that

$$P\left(\max_{0\le t\le 1}\sum_{2\le\|\alpha\|\le l}|\bar{S}_n^{(\alpha)}(t,x,\eta)|\ge n^{\frac{1}{4}}\right)\le ce^{-n^{u}}.$$

Therefore

$$\begin{aligned}
&P\left(\max_{0\le t\le 1}\sum_{k=1}^{d}|\varsigma_n^{(k)}(t,x,\eta)|\le n^{-\frac{l-2}{6}}\right)\\
\le&P\left(\max_{0\le t\le 1}\left|\sum_{2\le\|\alpha\|\le l} n^{\frac{1-\|\alpha\|}{2}}\bar{S}_n^{(\alpha)}(t,x,\eta)\right|\ge\inf_{0\le t\le 1}\sum_{k=1}^{d}\bar{S}_n^{(k)}(t,x,\eta)-n^{-\frac{l-2}{6}}\right)\\
\le&P\left(\max_{0\le t\le 1}\sum_{2\le\|\alpha\|\le l}|\bar{S}_n^{(\alpha)}(t,x,\eta)|\ge n^{\frac{1}{4}}\right)\le ce^{-n^{u}}.
\end{aligned}$$

For the case $q=1$, we have proved (4.4). If $q=2$, then $\varsigma_n^{(k)}(t,x,\eta)$ can be expressed in the form of (4.6), where

$$\begin{aligned}
&f(t,x)=\bar{S}_n^{(k)}(t,x,\eta),\\
&g_h^{(k)}(s,t,x)=\sum_{\substack{2\le\|\alpha\|\le l\\ \alpha_1=k,\alpha_2=h}} n^{\frac{1-\|\alpha\|}{2}}(S_n^{(\alpha)}(s,t,x,\eta)+r_n^{(\alpha)}(s,t,x,\eta)),
\end{aligned}$$

here $S_n^{(\alpha)}(s,t,x,\eta)$ and $r_n^{(\alpha)}(s,t,x,\eta)$ are defined as follows: If $\alpha=(\alpha_1,\cdots,\alpha_m)$, $m\ge 3$ and

$$S^{(\alpha)}(t_1,x,\eta)=\left(\int_0^{t_1}\cdots\int_0^{t_{m-1}} f_\alpha(\frac{t_1}{n},\cdots\frac{t_m}{n},x)\,d\theta_{\alpha_m}(t_m)\cdots d\theta_{\alpha_2}(t_2),\eta\right),$$

then we set

$$S_n^{(\alpha)}(s,t,x,\eta)=\left(\int_0^{s}\int_0^{t_3}\cdots\int_0^{t_{m-1}} f_\alpha(\frac{t}{n},\frac{s}{n},\cdots,\frac{t_m}{n},x)\,d\theta_{\alpha_m}(t_m)\cdots d\theta_{\alpha_3}(t_3),\eta\right).$$

If $m=2$, then we let

$$S_n^{(\alpha)}(s,t,x,\eta)=\left(f_\alpha(\frac{t}{n},\frac{s}{n},x),\eta\right).$$

In the same way, we can define $r_n^{(\alpha)}(s,t,x,\eta)$, but we should first define $r_n^{(\alpha)}(t,x,\eta)$ in a similar way as defining $r^{(\alpha)}(t,x)$ in Section 2. Of course, each term in the expression of $r_n^{(\alpha)}(t,x,\eta)$ also possesses the form as in (2.9).

By the hypothesis of Theorem 1.1, noting the expressions of $S_n^{(\alpha)}(t,x,\eta)$ and $r_n^{(\alpha)}(t,x,\eta)$ and the expansions of (2.8) and (2.9), we easily know that $g_n^{(k)}(s,t,x)$ also satisfies the hypothesis of Lemma 4.3. Using Lemma 3.1 (i) we know that the hypothesis (4.5) is satisfied too. Thus, to prove (4.4), it is sufficient to show the following estimation in terms of Lemma 4.3.

$$P\left(\sum_{k=1}^{d}\sum_{h=1}^{d}\int_0^1 [g_h^{(k)}(t,t,x)]^2\,dt \le n^{-\frac{l-2}{12}}\right) \le ce^{-n^*}, \quad \forall n \ge 1. \tag{4.8}$$

Just as the discussion before, we easily know

$$P\left(\max_{0\le t\le 1}\sum_{k=1}^{d}\sum_{h=1}^{d}|g_h^{(k)}(t,t,x)| \le \frac{c}{nL}\right) \le ce^{-n^*}, \quad \forall n \ge 1.$$

Therefore, if $l \ge 8\times 12$, then (4.8) follows from Lemma 4.2. Consequently, (4.4) holds for $q=2$. For $q\ge 3$, repeating the above steps for $q-1$ times, it is not difficult to see that (4.4) is also true. But the statement is very tedious, and omitted here.

Remark 4.1 From the proof above, we see that $r^{(\alpha)}(t,x)$ defined by (2.9) does not interfere the estimation (4.4) if the hypothesis (1.6) is satisfied. In fact, in [7, Appendix], Kusuoka and Stroock choose $l = l(L) = L$. Here we need to choose $l = l(L) > L$, and from the procedure of the proof above, we easily see that l may be chosen 6^{L+1} or larger.

5. On the Transition Probability Function

From now on, we consider the differential operator

$$L = \frac{1}{2}\sum_{i,j=1}^{N} a_{i,j}(t,x)\frac{\partial^2}{\partial x_i\partial x_j} + \sum_{i=1}^{N} b_i(t,x)\frac{\partial}{\partial x_i},$$

where

$$a(t,x) = (a_{ij}(t,x))_{1\le i,j\le N} = \sum_{k=1}^{d}(V_k(t,x))^{\otimes^2},$$

$$b(t,x) = V_0(t,x) + \frac{1}{2}\sum_{k=1}^{d}\sum_{j=1}^{N} V_k^j(t,x)\frac{\partial V_k(t,x)}{\partial x_j}.$$

For convenience, in this section and Section 7, we always assume for some $\beta > \frac{1}{2}, V_0, \cdots, V_d \in S_{\beta l}([0,T]\times R^N)$ (Of course, there $\delta = T$ and x appearing in the definition of $S_{\beta l}([0,T]\times R^N)$ is not fixed and various for all points in R^N, or see the definition of $\hat{C}_\beta([0,T]\times R^N)$ in Section 6) and their derivatives for any order are bounded. Just as in [7,§3], we discuss the uniqueness and the existence of solutions of heat equations with time-dependent coefficients, and express the solutions by the transition probability functions (see Theorem 5.1). Furthermore, we give the estimation for the transition probability functions. Certainly, this estimation is very useful to discuss the hypoellipticity for inhomogeneous differential operators.

Now, we suppose that $X_s(t,x)$ solves SDE:

$$X_s(t,x) = x + \sum_{k=1}^{d} \int_s^t V_k(u, X_s(u,x)) \circ d\theta_k(u) + \int_s^t V_0(u, X_s(u,x))\, du. \tag{5.1}$$

Set

$$P_{s,x} = P \circ (X_s(\cdot, x))^{-1}.$$

From the discussion in [8, Chapter 5 and 6], we know that

$$\left(\varphi(X_s(t)) - \varphi(x) - \int_s^t L\varphi(X_s(u))\, du, \mathcal{M}_{s,t}, P_{s,t}\right),$$

is a martingale for any $x \in R^N$ and $\varphi \in C_0^\infty(R^N)$, and $P_{s,x}$ has Feller continuity, where $\mathcal{M}_{s,t} = \sigma\{X(u) : s \le u \le t\}$. For any $c \in \bar{C}_b([0,T] \times R^N) = \{c : c(\cdot, x) \in C([0,T])$ for $x \in R^N$, $c(t,\cdot) \in C_b(R^N)$ for $t \in [0,T]$, and $\sup_{0\le t\le 1, y\in R^N} |\frac{\partial^{\{\alpha\}} c(t,y)}{\partial y^\alpha}| \le M < \infty$, where $\alpha \in (0,1,\cdots,\{\alpha\})^N\}$, set

$$\begin{aligned} {}^c p(s,x;t,\Gamma) &= E^{P_{s,x}}\left[\exp\left(\int_s^t c(u,X_u)\, du\right), X_t \in \Gamma\right] \\ &= E^P\left[\exp\left(\int_s^t c(u, X_s(u,x))\, du\right), X_s(t,x) \in \Gamma\right], \end{aligned}$$

where $t \in [s,T]$, $x \in R^N$, $\Gamma \in \mathcal{B}(R^N)$. For any $\varphi \in C_\uparrow(R^N)$, let

$${}^c p_{s,t}\varphi(x) = \int \varphi(y)\ {}^c p(s,x,t,dy).$$

Obviously

$$\begin{aligned} &\lim_{s\uparrow t} {}^c p_{s,t}\varphi(x) = \varphi(x), \quad \forall \varphi \in C_\uparrow(R^N); \quad \forall x \in R^N \\ &{}^c p_{s,s+h} \cdot {}^c p_{s+h,t}\varphi(x) = {}^c p_{s,t}\varphi(x), \\ &{}^c p_{s,t} : \varphi \in C_\uparrow(R^N) \mapsto {}^c p_{s,t}\varphi \in C_\uparrow(R^N), \\ &{}^c p_{s,t}\varphi(x) - \phi(x) = \int_s^t {}^c p_{s,u}(L+c)\phi(x)\, du. \end{aligned} \tag{5.2}$$

From SDE (5.1), we easily see that $X_s(t,x)$ coincides with the solution of

$$\bar{X}(t,x) = x + \sum_{k=0}^{d} \int_0^t V_k(u+s, \bar{X}(u)) \circ d\bar{\theta}_k(u), \tag{5.3}$$

where $\bar{\theta}_k(u) = \theta_k(u+s) - \theta_k(s)$, $k = 0,1,\cdots,d$. Here, we regard $V_k(s,x)$ as $V_k(0,x)$ appearing in SDE (1.1), then SDE (5.3) is changed into the type of SDE (1.1). To prove the main result of this section, we need a lemma. Set

$$\begin{aligned} {}^c Y_s(t,x) &= \int_s^t c(u, X_s(u,x))\, du, \\ {}^c Z_s(t,x) &= \begin{pmatrix} X_s(t,x) \\ {}^c Y_s(t,x) \end{pmatrix}. \end{aligned}$$

Lemma 5.1. If $\phi \in C_{\uparrow}^{\infty}(R^N)$, then ${}^cP_{s,t}\varphi \in C_{\uparrow}^{\infty}(R^N)$. Moreover, there are polynomials $P_{n,m} : \prod_{\nu}$ H.S. $((R^N)^{\otimes^{\nu}}, R^{N+1}) \to$ H.S. $((R^N)^{\otimes m}, (R^N)^{\otimes n})$, $0 \le m \le n$, such that

$$\begin{aligned}&({}^cP_{s,t}\phi)^{(n)}(x)\\ &= \sum_{m=0}^{n} E^P[\exp({}^cY_s(t,x))P_{n,m}({}^cZ_s^{(1)}(t,x),\cdots,{}^cZ_s^{(n)}(t,x))\varphi^{(m)}(X_s(t,x))].\end{aligned} \tag{5.4}$$

The proof of Lemma 5.1 is quite similar to those of [7, Lemma 3.9], here is omitted. We now state the main result of this section.

Theorem 5.1. For $c \in \bar{C}_b([0,T] \times R^N)$, $\{ {}^cp_{s,t}, s \le t \le T\}$ is a family of linear bounded operators on $C_{\uparrow}^{\infty}(R^N)$. Moreover, for each $\varphi \in C_{\uparrow}^{\infty}(R^N)$, ${}^cp_{s,t}\varphi$ is the unique solution of the following equation on $C'([0,T] \mapsto C_{\uparrow}^{\infty}(R^N))$:

$$\begin{cases} \frac{\partial u}{\partial s} = -(L+c)u, & 0 \le s \le t,\\ u(t,\cdot) = \varphi(\cdot). \end{cases} \tag{5.5}$$

Proof. We need only to check that

$$\lim_{h\to 0} \frac{{}^cp_{s+h,t}\varphi(x) - {}^cp_{s,t}\varphi(x)}{h} = (L+c)\ {}^cp_{s,t}\varphi(x).$$

Without loss of the generality, assume $h < 0$. Then

$${}^cp_{s+h,t}\varphi(x) = {}^cp_{s+h,s} \cdot {}^cp_{s,t}\varphi(x).$$

Hence

$$\begin{aligned}{}^cp_{s+h,t}\varphi(x) - {}^cp_{s,t}\varphi(x) &= ({}^cp_{s+h,s} - I) \cdot {}^cp_{s,t}\varphi(x)\\ &= \int_{s+h}^{s} {}^cp_{s+h,u}(L+c)\ {}^cp_{s,t}\varphi(x)\,du.\end{aligned}$$

Since $\{ {}^cp_{s+h,u}(L+c)\ {}^cp_{s,t}\varphi(x), s+h \le u \le s\}$ is bounded on $C_{\uparrow}^{\infty}(R^N)$ and

$$\lim_{\substack{h\uparrow 0\\ s+h\le u\le s}} {}^cp_{s+h,u}\ {}^cp_{s,t}\varphi(x) = (L+c)\ {}^cp_{s,t}\varphi(x),$$

we have the desired limit as $h \uparrow 0$ and the equality

$$\frac{d\ {}^cp_{s,t}\varphi(x)}{ds} = -(L+c)\ {}^cp_{s,t}\varphi(x).$$

The uniqueness of the solution of the differential equation (5.5) is easy to prove, here is omitted. ∎

From Theorem 5.1, we see that it is sufficient to concern ${}^cp_{s,t}\varphi$ in the study of the solution of (5.5). Now, we give an upper bound for the density function of $X_s(t,x)$ if the transition probability function ${}^cp(s,x,t,dy)$ is regular.

Theorem 5.2. Set $D_1 = \{(s,t) : 0 \le s \le t \le 1\}$, $D_2 = \{(s,t) : t \ge s \ge 1\}$ and $U \subset R^N$ be an open set. If there are $\rho : D_1 \cup D_2 \to (0,\infty)$ and $M_p > 0$ for every $p \ge 1$ such that

$$\sup_{x\in U} \left\| \frac{1}{\Delta_s(t,x)} \right\|_{L^p(\Omega)} \le M_p\rho^{-1}(s,t), \quad \forall (s,t) \in D_1 \cup D_2.$$

Then there exists a function ${}^cp \in C^{1,\infty,1,\infty}([0,T]\times U\times[0,T]\times R^N)$, such that

$$^cp(s,x,t,dy) = {}^cp(s,x,t,y)dy.$$

Moreover, there are positive constants $k_n(t)$,u_n and λ_n for every $n\geq 1$, such that for any $(x,y)\in U\times R^N$, (s,t) with $0\leq s\leq t\leq T$, and $\alpha,\beta\in\mathcal{N}^N$ with $|\alpha+\beta|\leq n$,

$$\begin{aligned}&(1+|y-x|^2)^{\frac{n}{2}}|D_x^\alpha D_y^\beta\ {}^cp(s,x,t,y)|\\ &\leq\begin{cases} k_n(t)(1+|x|^2)^{u_n}\exp(c(t-s)-\lambda_n|y-x|^2/(t-s))\rho^{-\nu_n}(s,t), & (s,t)\in D_1\\ k_n(t)(1+|x|^2)^{u_n}\exp(c(t-s)-\lambda_n|y-x|^2/(t-s))\rho^{-\nu_n}(s\vee 1,t), & 0\leq s\leq t\geq 1,\end{cases}\end{aligned}$$

where

$$\Delta_s(t,x)=\det(A_s(t,x))$$

and $A_s(t,x)$ is the Malliavin covariance matrix of $X_s(t,x)$, i.e.

$$A_s(t,x) = < DX_s(t,x), DX_s(t,x) >_H .$$

Proof. Observing the proof of [7,Theorem 3.17], we can prove this theorem without any difficult. ∎

Regarding $V_k(s,x)$ as $V_k(0,x)$ in Section 3 for $k=0,1,\cdots,d$, we get the definition of $V_{(\alpha)}(s,x)$ similarly for $\alpha\in\mathcal{A}$. For $s\leq t$, let

$$V_{Ls}(x,\eta)=\sum_{k=1}^{d}\sum_{\substack{\|\alpha\|\leq L-1\\ \alpha_1=k\\ \alpha\in\mathcal{A}}}(V_{(\alpha)}(s,x),\eta)^2,\quad V_L(s,x)=\left(\inf_{\eta\in S^{N-1}}V_{Ls}(x,\eta)\right)\wedge 1$$

and

$$U_L =: \{x\in R^N : V_L(s,x)>0,\quad \forall s\in[0,T]\}.$$

Corollary 5.1. Set $U=\cup_{L=1}^\infty U_L$. For every $p\geq 1$, there exist positive constants $M_p(L)$ and k_L such that

$$\|\Delta_s^{-1}(t,x)\|_{L^p(\Omega)}\leq M_p(L)(V_L(s,x)(t-s))^{k_Lp},\quad \forall x\in U_L$$

where $(s,t)\in D_1\cup D_2$, and $0\leq s\leq t\leq T$. Moreover, there is a function ${}^cp\in C^{1,\infty,1,\infty}([0,T]\times U\times[0,T]\times R^N)$, such that

$$^cp(s,x;t,dy) = {}^cp(s,x;t,y)dy.$$

Meanwhile, there are positive constants λ_n, u_n, $k_n(T,L)$ and $k_L(n)$ for each α,β with $\alpha+\beta=n$, such that

$$\begin{aligned}&(1+|y-x|^2)^{\frac{n}{2}}|D_x^\alpha D_y^{\beta c}p(s,x,t,y)|\\ &\leq R_n(T,L)(1+|x|^2)^{u_n}\exp\left(c(t-s)-\lambda\frac{|y-x|^2}{|t-s|}\right)/(V_L(s,x)(t-s))^{k_L(n)}.\end{aligned}\tag{5.6}$$

Proof. We have

$$\Delta_s^{-1}(t,x)\leq(\det J_s^{-1}(t,x))^2/\hat{\lambda}_s(t,x)^N,$$

where

$$\hat{\lambda}_s(t,x) = \hat{\lambda}_s(t,x,S^{N-1}) = \inf_{\eta\in S^{N-1}}(\eta, \hat{A}_s(t,x)\eta).$$

Moreover

$$\sup_{0\le s\le t\le T}\sup_{x\in R^N}\|(\det J_s^{-1}(t,x))^2\|_{L^p(\Omega)} \le M_p < \infty.$$

Hence, if $l = l(L)$ is large enough (see Section 4), then

$$P\left(\frac{\hat{\lambda}_s(t,x)}{(t-s)^l} \le k^{-1}\right) \le \frac{\hat{c}(L,p)}{(V_L(s,x))^{p/l}}k^{-p}, \quad \forall p\ge 1; \quad \forall k \ge 1.$$

It is not difficult to get that for $\forall x \in U_L$

$$\|\Delta_s^{-1}(t,x)\|_{L^p(\Omega)} \le M_p(L)/(V_L(s,x))^{Np/l}(t-s)^{Np}.$$

By Theorem 5.2 we easilly get the desired conclusion. ∎

6. Regularity in the Presence of Degeneracy on Thin Sets

For homogeneous case, Kusuoka and Stroock (see [7, §7]) gave a condition which is weaker than Hörmander's one, under which the solution of SDE has smooth density too. In this section, we discuss a similar problem for inhomogeneous case.

Given $\psi \in \hat{C}_\beta(0,T\times R^N)$ ={f: $\frac{\partial^{\{\alpha\}}f(t,y)}{\partial y^\alpha}$ is β-Hölder continuous with respect to t and bounded uniformly for $(t,y)\in[0,T]\times R^N$, $\forall\alpha\in(0,1,\cdots,\{\alpha\})^N$} and

$$\alpha = (\alpha_1,\cdots,\alpha_m) \in \{\emptyset\}\cup \mathcal{A}.$$

If $\alpha=\emptyset$,then $|\alpha| = \|\alpha\| = 0$. Set

$$\begin{aligned}
\psi_{(\emptyset)}(t,x) &= \psi(t,x),\\
\psi_{(k)}(t_1,t_2,x) &= \sum_{i=1}^N V_k^i(t_2,x)\psi_{x_i}^{(1)}(t_1,x),\\
\psi_{(\alpha)}(t_1,\cdots,t_{m+1},x) &= \sum_{i=1}^N V_{\alpha_m}^i(t_{m+1},x)\psi_{(\alpha')x_i}^{(1)}(t_1,\cdots,t_m,x),\\
\psi_{(\alpha)}(t,x) &= V_{\alpha_*}\psi_{(\alpha')}(t,x),\\
\sigma_{(L)}^2(\psi)(t,x) &= \sum_{\substack{\|\alpha\|\le L-1\\ \alpha\in\overline{\mathcal{A}}\cup\{\emptyset\}}}(\psi_{(\alpha)}(t,x))^2,
\end{aligned}$$

where the notations $\overline{\mathcal{A}}, \mathcal{A}, \alpha_*, \alpha', \|\alpha\|$ and $|\alpha|$ were given in Section 2 and [7].

From Section 2 and Section 4, we easily obtain the next lemma.

Lemma 6.1. There are constants $c(L,p)\in(0,\infty)$ and $l=l(L)\ge L$ for $L\ge 1$ and $p\ge 1$, such that

$$P\left(\sup_{0\le t\le Tk^{-1}}\psi(t,X_0(t,x))^2/T^{l-1}\le k^{-l}\right) \le c(L,p)(k\sigma_{(L)}(0,x))^{-p}.$$

We now prove the main result of this section.

Theorem 6.1. Suppose h is a non-negative function on R^+, and there exists a positive constant c_0 and a sufficient large constant $r_0 > 0$, such that$(kh(k^{r_0}))^{-1} \le c_0$ for any $k \ge 1$. If

$$\begin{aligned}&\psi(t,x) \ge 0, \quad \forall (t,x) \in [0,T] \times R^N,\\ &\sigma^2_{(L)}(\psi)(0,x) \ge 1, \quad \forall x \in R^N,\\ &V_{L,t}\left(x, \overline{F}^{(\delta)}\right) \ge h \circ \psi(t,x), \quad \forall (t,x) \in [0,T] \times R^N,\end{aligned}$$

where $\delta \in (0,1)$, $\psi \in \hat{C}_\beta([0,T] \times R^N)$ and

$$F^{(\delta)} = \{\eta \in S^{N-1}, |\eta - F| < \delta\},$$

where F is a non-empty closed set in S^{N-1}. Then, there exists a constant $\varepsilon > 0$, such that for all $\rho \in [0, \varepsilon)$

$$P(\hat{\lambda}_0(tk^{-\rho}, x, F)/t^l \le 4k^{-1}) \le c(L,p)k^{-p}, \quad \forall p \ge 1; \quad \forall k \ge 1,$$

where $c(L,p) \in (0,\infty)$, and

$$\begin{aligned}&V_{L,t}(x,F) = (\inf_{\eta \in F} V_{L,t}(x,\eta)) \wedge 1,\\ &\hat{\lambda}_s(t,x,F) = \inf\{(\eta, \hat{A}_{(s)}(t,x)\eta) : \eta \in F\},\\ &\hat{A}_{(s)}(t,x) = \sum_{k=1}^{d} J_0(s,x) \int_s^t (J_0^{-1}(u,x)V_k(u,X_0(u,x)))^{\bullet 2}\, du J_0(u,x)^*,\end{aligned}$$

where the definition of $V_{L,t}(x,\eta)$ was given in Section 5.

Proof. Set

$$\begin{aligned}&\tau(x) = \inf\{t \ge 0 : \psi(t, X_0(t,x))^2 \ge k^{-r}\}, \quad r > 0,\\ &\varsigma(x) = \inf\{t \ge 0 : |X_0(t,x) - x| \ge 1 \text{ or } \|J_0^{-1}(t,x) - I\| \ge \delta/2 + \delta\}.\end{aligned}$$

By Lemma 6.1 and the standard estimation, we easily get that

$$P(\tau(x) \ge k^{-h}) + P(\varsigma(x) \ge k^{-h\rho}) \le c(L,p)k^{-p}, \quad \forall p \ge 1, \quad \forall k \ge 1,$$

where $c(L,p) \in (0,\infty)$. Thus, we need only to show the following

$$P\left(\hat{\lambda}_0(tk^{-\rho}, x, F)/t^l \le 4k^{-1};\ \tau(x) < k^{-h\rho} < \varsigma(x)\right) \le \bar{c}(L,p)k^{-p}, \quad \forall p \ge 1, \quad \forall k \ge 1.$$

However, applying the strong Markov property to $(X(\cdot,x), A_{(0)}(\cdot,x))$ and noting the following facts

$$tk^{-\rho} \ge 4k^{-h\rho}; \quad \hat{\lambda}_s(t,x,\overline{F}^{(\delta)}) \le \frac{1}{4}\hat{\lambda}_0(t,x,F),$$

we easily get the desired result. ∎

The following corollary is an immediate consequence of Theorem 6.1.

Corollary 6.1. Suppose that h satisfies the hypotheses in Theorem 6.1 and there exist $L \geq 1$ and $\psi \in \hat{C}_\beta([0,T] \times R^N)$, such that

$$V_L(s,x) \geq' h \circ \psi(s,x), \quad \forall x \in R^N.$$

If $\sigma^2_{(L)}(\psi)(s,x) \geq 1$, then

$$\|\Delta_s^{-1}(t,x)\|_{L^p(\Omega)} \leq M_p(\rho)(t-s)^\rho, \quad \forall p \geq 1; \quad \forall \rho \geq 1,$$

where $M_p(\rho) \in (0,\infty)$, $t > s$.

Proof. Since

$$\Delta_s(t,x) \geq (\det(J_s(t,x)))^2(\hat{\lambda}_s(t,x))^N,$$
$$\|J_s^{-1}(t,x)\| \leq A_p \exp(B_p(t-s)), \quad \forall p \geq 1.$$

By Theorem 6.1, we get

$$\|\lambda_s^{-1}(t,x)\|_{L^p(\Omega)} \leq \sum_{k=1}^{\infty} k^p P(\hat{\lambda}_s(t,s)) \leq M_p(\rho)(t-s)^{-\rho}.$$

∎

7. Hypoellipticity

Our aim in this section is to show the hypoellipticity for the operator L defined in Section 5 by using probability method (For the definition of hypoellipticity for a differential operator, one may refer [7,§8]. For the hypoellipticity of inhomogeneous differential operator, Chaleyat - Maurel and Michel already studied before by pure analytic method.

For simplicity, in this section we assume the coefficients $V_0, \cdots, V_d$ are the same as those given in Section 5, and $c \in \tilde{C}_b([0,T] \times R^N)$. Consider

$$L + c = \frac{1}{2}\sum_{k=1}^{d} V_k^2(t,x) + V_0(t,x) + c(t,x).$$

Then, its dual operator is

$$\hat{L} + \hat{c} = \frac{1}{2}\sum_{k=1}^{d} V_k^2(t,x) + \hat{V}(t,x) + \hat{c}(t,x),$$

where

$$\hat{V}(t,x) = -V_0(t,x) + \frac{1}{2}\sum_{k=1}^{d} \operatorname{div}(V_k(t,x))V_k(t,x)$$

and

$$\hat{c}(t,x) = c(t,x) - \operatorname{div}(V_0(t,x)) + \frac{1}{2}\sum_{k=1}^{d} V_k(\operatorname{div}(V_k)) + \frac{1}{2}\sum_{k=1}^{d}(\operatorname{div}V_k)^2.$$

Suppose that $\hat{X}_s(t,x)$ solves SDE:

$$\hat{X}_s(t,x) = x + \sum_{k=1}^{d}\int_s^t V_k(u,\hat{X}_s(u,x)) \circ d\theta_k(u) + \int_0^t \hat{V}(u,\hat{X}_s(u,x))\,du.$$

Regarding $\hat{X}_s(t,x)$ and $\hat{V}(u,\hat{X}_s(u,x))$ as $X_s(t,x)$ and $V_0(u,X_s(u,x))$ as in Section 5, we can define $\hat{\Delta}_s(t,x)$ in a similar way.

For the hypoellipticity of L, we have

Theorem 7.1. If there exists a function ρ: $D_1 \cup D_2 \to (0,\infty)$, such that

$$\|\Delta_s^{-1}(t,x)\|_{L^p(\Omega)} \le M_p \rho^{-1}(s,t), \quad \forall x \in R^N; \quad \forall (s,t) \in D_1 \cup D_2; \quad \forall p \ge 1$$

where $M_p \in (0,\infty)$, D_1 and D_2 are given in Section 5, then $L+c$ is a hypoelliptic operator.

Proof. For $\Gamma \in \mathcal{B}(R^N)$, set

$$^{\hat{c}}\hat{P}(s,x,t,\Gamma) = E^p\left[\exp\left(\int_s^t \hat{c}(u,\hat{X}_s(u,x))du\right),\ \hat{X}_s(t,x) \in \Gamma\right].$$

By Theorem 5.2, we know that there exists a function

$$^{\hat{c}}\hat{p} \in C^{1,\infty,1,\infty}([0,T] \times R^N \times [0,T] \times R^N),$$

such that

$$^{\hat{c}}\hat{P}(s,x,t,dy) = {}^{\hat{c}}\hat{p}(s,x,t,y)dy.$$

Moreover

$$\frac{\partial\, {}^{\hat{c}}\hat{P}_{s,t}\varphi(x)}{\partial s} = -(\hat{L}+\hat{c})\, {}^{\hat{c}}\hat{P}_{s,t}\varphi(x), \quad \forall \varphi \in C_0^\infty(R^N).$$

Let

$$\mathcal{W}(u) = \int {}^{\hat{c}}\hat{P}_{u,t}\varphi(x)\, {}^{\hat{c}}\hat{P}_{s+(t-u),t}\psi(x)\,dx, \quad s \le u \le u \le t; \quad \psi \in C_0^\infty(R^N),$$

then

$$\frac{d\mathcal{W}(u)}{du} = 0.$$

Therefore

$$\mathcal{W}(s) = \mathcal{W}(t).$$

That means

$$^{\hat{c}}\hat{P}_{s,t}\psi(y) = \int \psi(y)\, {}^{\hat{c}}\hat{p}(s,x,t,y)\,dx.$$

For any $\varphi \in C_0^\infty(R^N)$, we have

$$\lim_{t \downarrow s} {}^{\hat{c}}\hat{p}_{s,t}\varphi(x) = \varphi(x), \quad \forall x \in R^N,$$

$$\lim_{t \downarrow s} {}^{\hat{c}}\hat{p}_{s,t}\varphi(y) = \varphi(y), \quad y \in R^N,$$

meanwhile

$$\sup_{s \le u \le T} \|\, {}^{\hat{c}}\hat{p}_{s,u}\varphi\|_{C_b^n(R^N)} \le c_n \|\varphi\|_{C_b^n(R^N)},$$

$$\sup_{s \le u \le T} \|\, {}^{\hat{c}}\hat{p}_{s,u}\varphi\|_{C_b^n(R^N)} \le c_n \|\varphi\|_{C_b^n(R^N)}.$$

Some results in Section 5 are used here. In order to apply [7,Theorem 8.6], we need to prove that

$$\frac{d\, {}^{\hat{c}}\hat{p}(s,x,t,y)}{dt} = (\hat{L}+\hat{c})_y\, {}^{\hat{c}}\hat{p}(s,x,t,y). \tag{7.1}$$

Actually, from (5.2), we get

$$\frac{d\,{}^c\hat{p}_{s,t}\varphi(x)}{dt} = {}^c\hat{p}_{s,t}(L+c)\varphi(x), \quad \forall \varphi \in C_0^\infty(R^N).$$

This means that

$$\int \left[\frac{d\,{}^c\hat{p}(s,x,t,y)}{dt} - (\hat{L}+\hat{c})\,{}^c\hat{p}(s,x,t,y)\right]\varphi(y)\,dy = 0.$$

Hence, (7.1) is true. By the discussions in Section 5, we also know that

$$\max_{|\alpha|+|\beta|\le n}\ \sup_{s\le t\le T}\ \sup_{|y-x|\ge\epsilon} |D_x^\alpha D_y^\beta\,{}^c\hat{p}(s,x,t,y)| < \infty.$$

Replacing $q(t,x,y)$ in [7,Theorem 8.6] by ${}^c\hat{p}(s,x,t,y)$, we see that $L+c$ is a hypoelliptic operator. ∎

Corollary 7.1. Suppose that there exist $\psi \in \hat{C}_\beta([0,T]\times R^N)$ $(\beta > \frac{1}{2})$ and non-negative function h satisfying the hypotheses in Theorem 6.1 such that for some $L \ge 1$,

$$V_L(t,x) \ge h\circ\psi(t,x), \quad \forall x \in R^N; \quad \forall t \in [s,T],$$

where $V_L(t,x)$ is defined in Section 5. Moreover, there exists an open set W in R^N, such that

$$\sigma^2_{(L)}(\psi)(tx) \ge 1, \ \forall x \in W, \ \forall t \in [s,T].$$

Then, for any $c \in \hat{C}_b([0,T]\times R^N)$, ${}^cL = \frac{1}{2}\sum_{k=1}^d V_k^2 + V_0 + c$ is a hypoelliptic operator on W. In particular, if

$$\mathrm{Lie}(V_1(s,x),\cdots V_d(s,x)) = R^N, \quad \forall s \in [0,t], \quad \forall x \in R^N, \tag{7.2}$$

then ${}^cL$ is a hypoelliptic operator, where $V_0,\cdots,V_d$ satisfy the hypotheses given in Section 5.

Proof. For any point $x^0 \in W$, if there exist $\varepsilon > 0$ and a hypoelliptic operator $\hat{L}$ on R^N, such that ${}^cL$ is equal to $\hat{L}$ on $C_0^\infty(B(x^0,\varepsilon))$, then ${}^cL$ possesses the hypoellipticity on W.

For any fixed point $x^0 \in W$, there is a positive constant ε with $\overline{B(x^0,4\varepsilon)} \subset\subset W$. Choose $\eta_1 \in C_0^\infty(B(x^0,2\varepsilon))$ so that $\eta_1 = 1$ on $\overline{B(x^0,\varepsilon)}$ and $\eta_2 \in C_0^\infty(B(x^0,3\varepsilon))$ so that $\eta_2 = 1$ on $\overline{B(x^0,2\varepsilon)}$. Set

$$\begin{aligned}\tilde{V}_k(s,x) &= \eta_2(x)V_k(s,x), \quad 0 \le k \le d,\\ \tilde{V}_{k+i}(s,x) &= (1-\eta_1(x))\frac{\partial}{\partial x_i}, \quad 1 \le i \le N,\\ \tilde{c}(s,x) &= \eta_1(x)c(s,x).\end{aligned}$$

Similarly, we can define $\tilde{V}_L(s,x)$, $\tilde{\sigma}^2_{(L)}(s,x)$ and $\tilde{\Delta}_s(t,x)$. Moreover, on $B(x^0,2\varepsilon)$ we have

$$\tilde{V}_L(s,x) \ge V_L(s,x); \quad \tilde{\sigma}^2_{(L)}(\psi)(s,x) \ge \sigma^2_{(L)}(\psi)(s,x),$$

on $B(x^0,2\varepsilon)^c$,

$$\tilde{V}_L(s,x) \ge \tilde{V}_1(s,x) \ge 1.$$

Without loss of generality, we assume $h \leq 1$. Then

$$\tilde{V}_L(s,x) \geq h \circ \psi(s,x), \quad \forall x \in R^N; \quad s \in [0,T].$$

Meanwhile, on $B(x^0, 2\varepsilon)$

$$\tilde{\sigma}_{(L)}(\psi)(s,x) \geq 1.$$

In view of the discussion in Section 6, we know that there is an $M_p(\rho) \in (0,\infty)$ for every $p \geq 1$ and $\rho > 0$ such that

$$\|\tilde{\Delta}_s^{-1}(t,x)\|_{L^p(\Omega)} \leq M_p(\rho)(t-s)^{-\rho}, \ t > s.$$

By Theorem 7.1, we know that

$$^{\tilde{c}}\tilde{L} = \frac{1}{2}\sum_{k=1}^{d+N} \tilde{V}_k^2 + \tilde{V}_0 + \tilde{c}$$

is hypoelliptic. Note that $^cL =^{\tilde{c}} \tilde{L}$ on $C_0^\infty(B(x^0,\varepsilon))$, ^cL is also hypoelliptic.

In particular, noting that $V_0(s,x), \cdots, V_d(s,x)$ are β-Hölder continuous with respect to s for $\beta > \frac{1}{2}$, we easily know that there is a constant $L \geq 1$ with $V_L(s,x) \geq c > 0$ for any $(s,x) \in [0,T] \times R^N$ if (7.2) is satisfied. Choosing $h \equiv c$, by the above discussion, it is easily to see that ^cL is hypoelliptic. ∎

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