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Dirichlet Forms and Symmetrizable Jump Processes

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0. Introduction

The initial idea of this paper comes from the study of large deviations. Let E be a countable set, $Q=(q_{ij})$ be a regular Q-matrix which is symmetrizable with respect to $(\pi_i>0;\ i\in E)$. In our previous paper [6], we have proved

$$(0.1)\qquad I(\mu)=\frac{1}{2}\sum_{i,j}[\sqrt{\mu_i q_{ij}}-\sqrt{\mu_j q_{ji}}]^2$$

for all probability measure μ satisfying $\sum_i \mu_i q_i<\infty$. But we do not know whether the above expression holds or not if $\sum_i \mu_i q_i=\infty$. On the other hand, from Donsker and Varadhan [9] and Stroock [15], it is known that

$$(0.2)\qquad I(\mu)=D(\phi,\phi),\ \ \phi_i=\mu_i/\pi_i,\ \ i\in E$$

where D is the Dirichlet form determined by the process $(P_{ij}(t))$. This leads us to study the Dirichlet forms for jump processes.

Next, from a recent paper: Carlen, Kusuoka and Stroock [3] (who will refer you to Bakry and Emery [1], Davies [8] and Varopoulos [16, 17]), it becomes clear that the Dirichlet forms are powerful tools in the study of uniform decay estimates on the symmetric semigroups. In order to use the general theory to our special case, we also need some concrete understanding about the Dirichlet forms for jump processes.

Because the theory of Dirichlet forms has been a good deal of research (see Fukushima [10] and Silverstein [14]), one may think that our special case is well understood. However, some new information has appeared. For example, we prove in the sections 1 and 3 that our basic Dirichlet form is the maximum in the case the Q-matrix being conservative (Definition (1.6)). This is not the general case since the maximum exten-

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sion—Krein extension is so large that it is sometimes even not sub-Markovian (see [10]). It is more surprising that the Krein extension is sometimes not large enough since there are some Dirichlet forms with larger domain than the Krein's.

In practice, the most important case is that there exists only one Dirichlet form for the given symmetric Q-matrix on $L^2(\pi)$. For this, we present some uniqueness criteria in section 4. The proofs given there are quite technical and depending heavily on our previous works.

We study in this paper the Dirichlet forms for jump processes with general state space which is not necessarily countable. Once the picture of Dirichlet forms for jump processes becomes clear, it is not difficult to answer the question mentioned at the beginning. For example, we prove in section 5 that

$$D(\phi,\phi)=\frac{1}{2}\sum_{i,j}[\sqrt{\mu_i q_{ij}}-\sqrt{\mu_j q_{ji}}]^2,\quad \phi_j=\frac{\mu_i}{\pi_i}$$

for all μ. This seems the first completely explicit result on the rate function of large deviations for a kind of Markov processes.

1. Basic Dirichlet Form for Symmetrizable Jump Processes

Let $(E,\mathscr{E})$ be a separable measurable space with the property that all the singletons $\{x\}$ $(x\in E)$ belong to $\mathscr{E}$. Let $P(t,x,dy)$ $(t\geqslant 0, x\in E)$ be a sub-Markovian transition function on $(E,\mathscr{E})$. It is called a jump process if

(1.1) $$\lim_{t\downarrow 0}P(t,x,A)=P(0,x,A)=I_A(x)$$

for all $x\in E$ and $A\in\mathscr{E}$. Then it is known that the limits

(1.2) $$\lim_{t\downarrow 0}\frac{1-P(t,x,\{x\})}{t}\equiv q(x)\leqslant+\infty$$

exist for all $x\in E$ and the limits

(1.3) $$\lim_{t\downarrow 0}\frac{P(t,x,A)}{t}\equiv q(x,A)$$

exist for all $x\notin A\in\mathscr{R}$, where

(1.4) $$\mathscr{R}=\{A\in\mathscr{E}:\ \lim_{t\downarrow 0}\sup_{x\in A}[1-P(t,x,\{x\})]=0\}$$

Furthermore, if we define $q(x,A)=q(x,A-\{x\})$, then we have

(i) For every $A\in\mathscr{R}$, $q(\cdot)$ and $q(\cdot,A)$ are $\mathscr{E}$-measurable;

(1.5)

(ii) For every $x\in E, q(x,\cdot)$ is a finite measure on $\mathscr{R}$ and $q(x,A)\leqslant q(x)$,

$x \in E$, $A \in \mathscr{R}$.

In the case that $q(x) < +\infty$, $q(x,\cdot)$ can be extended to $\mathscr{E}$ as a finite measure. This leads us to the following

Definition 1.6 A family of functions $q(x) - q(x,dy)$ is called a q-pair if the properties in (1.4) replacing $\mathscr{R}$ by $\mathscr{E}$ hold and $q(x,\{x\}) = 0$ for all $x \in E$. We call x stable if $q(x) < \infty$, and we call $q(x) - q(x,dy)$ totally stable if each x is stable. Similarly we can define instantaneous point and instantaneous q-pair. Finally, $x \in E$ is called a conservative point if $q(x, E) = q(x)$, and $q(x) - q(x,dy)$ is called conservative if each x is conservative.

Definition 1.7 Given a q-pair $q(x) - q(x,dy)$, a jump process $P(t,x,dy)$ $(t \geq 0,\ x \in E)$ is called a q-process if

$$\lim_{t \downarrow 0} [P(t,x,A) - I_A(x)]/t = q(x,A) - q(x) I_A(x)$$

for all $x \in E$ and $A \in \mathscr{R}$, where $\mathscr{R}$ is defined by (1.4). The process is called symmetrizable if there exists a σ-finite measure π on $(E, \mathscr{E})$ such that $\int_A \pi(dx) P(t,x,B) = \int_B \pi(dx) P(t,x,A)$

for all $t \leqslant 0$, and $A, B \in \mathscr{E}$. Then we have $\int_A q(x,B) \pi(dx) = \int_B q(x,A)\, \pi(dx)$ for all $A, B \in \mathscr{E}$. In this case, the q-pair $q(x) - q(x,dy)$ is called symmetrizable, and the measure π is called a symmetrizing measure for the q-process or the q-pair.

In the case that E is countable, one uses traditionally Q-matrix $Q = (q_{ij})$ and Q-process instead of the terms q-pair and q-process respectively.

For a given q-pair, the symmetrizing measures may not be unique. From now on we will fix the q-pair $q(x)-q(x,dy)$ and a symmetrizing measure π. Also, we will consider the totally stable case only in this paper.

Lemma 1.8 Let $P(t,x,dy)$ be a symmetrizable q-process, define $P_t f(x) = \int P(t,x,dy) f(y)$ for all bounded $\mathscr{E}$-measurable functions. Then $\{P_t\}$ can be extended to $L^2(\pi)$ as a strongly continuous, contractive, self-adjoint and sub-Markovian semigroup.

Proof. What we need to check is the strongly continuous. By the jump condition (1.1), we have

$$\|P_t f - f\|_2^2 = \int \pi(dx) \left(\int P(t,x,dy) f(y) - f(x) \right)^2$$
$$= \int \pi(dx) [f(x)(1 - P(t,x,\{x\}) - \int_{E-\{x\}} P(t,x,dy) f(y)]^2$$

$$\leq 2\int \pi(dx) f(x)^2 (1-P(t,x,\{x\}))^2$$
$$+2\int \pi(dx)\left(\int_{E-\{x\}} P(t,x,dy) f(y)\right)^2$$
$$\leq 2\int \pi(dx) f(x)^2 (1-P(t,x,\{x\}))$$
$$+2\int \pi(dx) f(x)^2 \int_{E-\{x\}} P(t,x,dy) f(y)^2$$
$$=4\int \pi(dx) f(x)^2 (1-P(t,x,\{x\})) \to 0, \text{ as } t \downarrow 0.$$

In the last step but one we have used the symmetry of $P(t,x,dy)$. Q. E. D.

We will also denote the above extension by $\{P_t\}$ itself. By virtue of the lemma, a q-process determines uniquely a Dirichlet form on $L^2(\pi)$ (cf. [2] and [10]). Since the symmetrizable q-processes are not unique in general for a given symmetrizable q-pair, the corresponding Dirichlet forms are not unique. In this sense, our previous work on the constructions of symmetrizable q-processes is just the constructions of Dirichlet forms (cf. [5]). We prove in this paper that among the Dirichlet forms there is only one which is completely explicit. This is the reason why we call it the basic Dirichlet form.

First of all, it should be no surprise to write down the symmetric form on $L^2(\pi)$:

$$D^*(f,g)=\frac{1}{2}\int \pi(dx) q(x,dy)(f(y)-f(x))(g(y)-g(x)) + \int \pi(dx) d(x) f(x) g(x) \tag{1.9}$$

$$\mathscr{D}(D^*)=\{f\in L^2(\pi):\ D^*(f,f)<\infty\}$$

where $d(x)=q(x)-q(x,E)\geq 0$ $(x\in E)$. For simplicity, we set

$$\pi_q(dx,dy)=\pi(dx)q(x,dy), \qquad \pi_d(dx)=\pi(dx)d(x).$$

Lemma1.10 The measures π_q and π_d are σ-finite. Moreover, π_q is symmetric:

$$\int \pi_q(dx,dy) f(x,y)=\int \pi_q(dx,dy) f(y,x)$$

for all $f\in(\mathscr{E}\times\mathscr{E})^+$ and all $\mathscr{E}\times\mathscr{E}$-measurable and π_q-integrable functions.

Proof. Since π is σ-finite and $q(x,E)+d(x)=q(x)<+\infty$ for all $x\in E$, the first assertion follows immediately. The second one follows from the symmetry of the q-pair and the monotone class theorem. Q.E.D.

Lemma 1.11 D^* defined by (1.9) is a Dirichlet form.

Proof. The proof of this lemma is standard (cf.[10; example 1.2.4]). For example, to show the denseness, let $f\in L^2(\pi)$. Without loss of generality, we may and will assume that $|f|<\infty$ everywhere. Choose $\{B_m\}_1^\infty\subset\mathscr{E}$ such that

$$\pi(B_m)<\infty \text{ and } K_m=\sup\{q(x)\vee|f(x)|;\ x\in B_m\}<\infty.$$

Then $f_m=fI_{B_m}\to f$ in $L^2(\pi)$ and by Lemma (10), we have

$$\begin{aligned}D^*(f_m,f_m)&=\frac{1}{2}\int\pi_q(dx,dy)(f_m(y)-f_m(x))^2+\int\pi_d(dx)f_m(x)^2\\&\leqslant 2\int\pi_q(dx,dy)f_m(x)^2+\int\pi_d(dx)f_m(x)^2\\&\leqslant 2\int_{B_m}\pi(dx)(q(x,E)+d(x))f(x)^2\\&=2\int_{B_m}\pi(dx)q(x)f(x)^2\leqslant 2K_m^3\pi(B_m)<\infty.\end{aligned}$$

Therefore, $\mathscr{D}(D^*)\ni f_m\to f$ in $L^2(\pi)$. Q.E.D.

Since the one-to-one corresponding between the Dirichlet forms and the semi-groups, the Dirichlet form D^* determines uniquely a strongly continuous, contractive, self-adjoint and sub-Markovian semi-group $\{P_t^*\}$. A remaining question is whether it is a q-process or not. The answer is affirmative. To show this, we need more work.

A basic fact in the study of q-processes is that the $P(t,x,dy)$ and its Laplace transform

$$P(\lambda,x,dy)=\int_0^\infty e^{-\lambda t}P(t,x,dy)\,dt,\ \lambda>0$$

are determined by each other. Now, the condition

$$\lim_{t\downarrow 0}[P(t,x,A)-I_A(x)]/t=q(x,A)-q(x)I_A(x), x\in E,\ A\in\mathscr{R}$$

is equivalent to

$$\lim_{\lambda\uparrow\infty}\lambda[\lambda P(\lambda,x,A)-I_A(x)]=q(x,A,)-q(x)I_A(x),\ x\in E,\ A\in\mathscr{R}.$$

The set $\mathscr{R}$ appeared above is not very convenient, but it is not difficult to prove that each of the above two conditions is equivalent to

(1.12) $$\lim_{\lambda\uparrow\infty}\lambda[\lambda P(\lambda,x,A)-I_A(x)]=q(x,A)-q(x)I_A(x),\ x\in E,\ A\in\mathscr{E}\cap E_n,\ n\geqslant 1.$$

where $E_n=\{x\in E;\ q(x)\leqslant n\}$, $n\geqslant 1$. For details, see [5].

Definition 1.13 The following equations

(B); $$P(\lambda,x,A)=\int\frac{q(x,dy)}{\lambda+q(x)}P(\lambda,y,A)+\frac{1}{\lambda+q(x)}I_A(x)$$

(F): $P(\lambda,x,A)=\int P(\lambda,x,dy)\int\frac{q(y,dz)}{\lambda+q(z)}+\frac{1}{\lambda+q(x)}I_A(x)$, $\lambda>0$, $x\in E$, $A\in\mathscr{E}$

are called the backward and the forward Kolomogorov equations respectively.

Remark 1.14 *It is known (cf. [5]) that if the q-pair is conservative, then the equation (B) holds. If the equation (B) (or (F)) holds, then (1.12) holds for all $A\in\mathscr{E}$ and $x\in E$. Moreover, in the symmetrizable case, the two equations are π-equivalent. That is, for example, if (B) holds, then there exists a set $N\in\mathscr{E}$, such that $\pi(N)=0$ and (F) holds for all $\lambda>0$, $A\in\mathscr{E}$ and $x\notin N$.*

Now, we can return to our main context. Note that when we are concerned with L^2-theory, we only consider the equivalent classes. Hence we should replace "$x\in E$" in (1.1) and (1.12) with "$x\notin N$" for a π-null set N. As well known, in general, it is not easy to construct a version of resolvent having kernel from a given resolvent. Usually we need some topology on the state space $(E,\mathscr{E})$. For example, locally compactness. In which case we will have at least one such kernel, but may not be unique. About this problem, some deep results are due to Fukushima [20] and [21] (also see [14; sections 1-4 and 19-20]). But we would not go to this direction and leave it as an assumption; there exists a resolvent having kernel $P^*(\lambda,x,dy)$ such that

$$(1.14)\qquad \int P^*(\lambda,x,dy)f(y)=P^*_\lambda f(x),\ \pi\text{-a.e.},\ f\in L^\infty(\pi)$$

Also, we may assume that $q(x,\cdot)\ll\pi$, $x\in E$. For Markov chains, these assumptions are fulfilled since we can restrict ourselves to the case that $\pi_i>0$, $i\in E$. Thus, in the following we will use these assumptions and use the version of P^*_λ given by the left hand side of (1.14) without further restate.

Lemma 1.15 *The Dirichlet form D^* determines a q-process $P^*(\lambda,x,dy)$.*

Proof. By the construction of resolvent $\{P^*_\lambda\}$ from the Dirichlet form D^*, we have

$$(1.16)\qquad D^*(P^*_\lambda f,g)+\lambda(P^*_\lambda f,g)=(f,g),\ f\in L^2(\pi),\ g\in\mathscr{D}(D^*).$$

We now need to prove that $\{P^*_\lambda\}$ is indeed a q-process. From now on, we will often use the following notation;

$$(1.17)\qquad \mathscr{E}\ni E_n\uparrow E;\ \pi(E_n)<\infty,\ \sup\{q(x);\ x\in E_n\}\leqslant n.$$

Now we fix $f=I_A$ and let $g\in L^2(q(x)\pi(dx))$. Then $g\in\mathscr{D}(D^*)$ and it follows that

$$D^*(f,g)=\int\pi_q(dx,dy)(f(y)-f(x))(g(y)-g(x))+\int\pi_d(dx)f(x)g(x)$$

$$=\int\pi(dx)g(x)[q(x)f(x)-\int q(x,dy)f(y)].$$

By (1.16), we have

$$\int\pi(dx)g(x)(\lambda+q(x))P_\lambda^*f(x)=\int\pi(dx)g(x)[\int q(x,dy)P_\lambda^*f(y)+f(x)].$$

By the monotone class theorem, this equality holds for all $g\in L^\infty(\pi)^+$. Combining this fact and [2, Theorem (1.1)], we arrive at: for each $A\in\mathscr{E}\cap E_n$, $n\geqslant 1$,

$$P_\lambda^*I_A\geqslant 0,\quad \lambda P_\lambda^*1\leqslant 1$$

(1.18)
$$P_\lambda^*I_A=\int\frac{q(\cdot,dy)}{\lambda+q(\cdot)}P_\lambda^*I_A(y)+\frac{1}{\lambda+q(\cdot)}I_A$$

hold almost everywhere. The exceptinal set depends on A and λ. Since $(E,\mathscr{E})$ is separable, we can choose the exceptional set depends on λ only. On the other hand, the strongly continuous means that $\{P_\lambda^*\}$ determined by a dense set of $\lambda>0$, hence the exceptional set can be choosen so that it is also independent of $\lambda>0$. Now, denote the π-null set by N. Then (1.18) holds for all $x\notin N$. Letting $\lambda\to\infty$ in the equation in (1.18) we see that (1.1) holds for all $x\notin N$. Then using the equation again, it is easy to check that (1.12) also holds for all $x\notin N$.
This completes our proof. Q.E.D.

Put

(1.19) $\mathscr{K}=\{f\in L^\infty(\pi):\ \exists n \text{ such that } \{x\in E:\ f(x)\neq 0\}\subset E_n\}.$

Indeed, we have proved the following result:

Lemma 1.20 Let $P(\lambda,x,dy)$ be a symmetric q-process on $L^2(\pi)$. Then $P(\lambda,x,dy)$ satisfies (B) if and only if

(1.21) $D_\lambda^*(P_\lambda f,g)=D^*(P_\lambda f,g)+\lambda(P_\lambda f,g)=(f,g),\ f\in L^2(\pi),\ g\in\mathscr{K}.$

Now, we have in mind a symmetric q-process $P^*(\lambda,x,dy)$ determined by the Dirichlet form D^*. On the other hand, it is well-known in the study of q-processes that there exists the minimal q-process for a given q-pair which can be obtained by the following procedure. Let

$$P^{(0)}(\lambda,x,A)=0,\quad \lambda>0,\quad x\in E,\quad A\in\mathscr{E}$$

$$P^{(n+1)}(\lambda,x,A)=\int\frac{q(x,dy)}{\lambda+q(x)}P^{(n)}(\lambda,y,A)+\frac{1}{\lambda+q(x)}I_A(x),\ n\geqslant 1,\ \lambda>0,$$

$$x\in E,\quad A\in\mathscr{E}$$

then $P^{(n)}(\lambda,x,A)\uparrow P^{\min}(\lambda,x,A)$, as $n\uparrow\infty$

and $P^{min}(\lambda,x,dy)$ is a q-process which is the minimal one in the following sense: for any q-process $P(\lambda,x,dy)$, we have

(1.22) $$P(\lambda,x,A)\geqslant P^{min}(\lambda,x,A),\ \lambda>0,\ x\in E,\ A\in\mathscr{E}.$$

It is also known that the given q-pair $q(x)-q(x,dy)$ is symmetrizable with respect to π if and only if so does the minimal q-process. Hence we indeed have two existence results for the symmetric q-processes on $L^2(\pi)$. Since this fact one may guess that the Dirichlet form D^{min} corresponding to the q-process $P^{min}(\lambda,x,dy)$ coincides with D^*. On the other hand, because the domain of D^* is so large, one may guess that D^* is the maximum. However, there usually does not exist the maximal Dirichlet form (cf. [10]). In the following we will show that the answer to the first guess is negative, the answer to the second guess is affirmative in the conservative case and is probably negative in the non-conservative case.

Example 1.23 Take $E=\{0,1,2,\cdots\}$ and $q_{ij}=0$ for all $i\neq j$, $\sum_i 1/q_i<\infty$. Then $P_{ij}^{min}(\lambda)=\delta_{ij}/(\lambda+q_i)$. It is easy to check that

$$P_{ij}(\lambda)=P_{ij}^{min}(\lambda)+\frac{q_i}{(\lambda+q_i)(\lambda+q_j)}\Big/\sum_k\frac{\lambda}{\lambda+q_k}$$

is also a Q-process. Moreover, the later is honest(i.e., $\lambda\sum_j P_{ij}(\lambda)=1, i\in E$) and symmetrizable with respect to the measure $\pi_i=1/q_i$ only. Next

$$L^2(\pi)=\left\{f:\ \sum_i\pi_i f_i^2=\sum_i f_i^2/q_i<\infty\right\}$$

$$D^*(f,f)=\sum_i\pi_i d_i f_i^2=\sum_i = f_i^2$$

$$\mathscr{D}(D^*)=\left\{f:\ \sum_i f_i^2<\infty\right\}.$$

Since
$$D^{min}(f,f)=\lim_{\lambda\uparrow\infty}\lambda(f-\lambda P_\lambda^{min}f,f)$$
$$=\lim_{\lambda\uparrow\infty}\sum_i\frac{\lambda}{\lambda+q_i}f_i^2$$

we have $D^{min}=D^*$ and $\mathscr{D}(D^{min})=\mathscr{D}(D^*)$.

On the other hand,

$$D(f,f)=\lim_{\lambda\uparrow\infty}\lambda(f-\lambda P_\lambda f,f)=\lim_{\lambda\uparrow\infty}\frac{\lambda^2}{2}\sum_{i,j}\pi_i P_{ij}(\lambda)(f_j-f_i)^2$$

$$=\lim_{\lambda\uparrow\infty}\frac{1}{2}\sum_i\sum_{j\neq i}\frac{\lambda^2}{(\lambda+q_i)(\lambda+q_j)}(f_j-f_i)^2\Big/\sum_k\frac{\lambda}{\lambda+q_k}$$

Let $f\in\mathscr{D}(D^*)$, then

$$D(f,f)\leqslant\lim_{\lambda\uparrow\infty}\sum_i\frac{\lambda}{\lambda+q_i}\sum_{j\neq i}\frac{\lambda}{\lambda+q_j}(f_j^2+f_i^2)\Big/\sum_k\frac{\lambda}{\lambda+q_k}$$
$$=2\lim_{\lambda\uparrow\infty}\sum_i\frac{\lambda}{\lambda+q_i}\sum_{j\neq i}\frac{\lambda}{\lambda+q_j}f_i^2\Big/\sum_k\frac{\lambda}{\lambda+q_k}$$
$$\leqslant 2\lim_{\lambda\uparrow\infty}\sum_i\frac{\lambda}{\lambda+q_i}f_i^2=2\sum_i f_i^2.$$

Thus, $\mathscr{D}(D^*)\subset\mathscr{D}(D)$. But $1\in\mathscr{D}(D)$, $1\notin\mathscr{D}(D^*)$ and so $\mathscr{D}(D^*)\neq\mathscr{D}(D)$.

This example shows that the Dirichlet form D^* is not the maximum in general.

Theorem 1.25 D^* is the maximal Dirichlet form among the symmetric q-processes which satisfy the Kolmogorov equation (B). In other words, if $P(\lambda,x,dy)$ is a symmetric q-process satisfying (B), denote by D its Dirichlet form, then we have

(1.26) $$D(f,f)\geqslant D^*(f,f)$$

for all $f\in\mathscr{D}(D)\subset\mathscr{D}(D^*)$. In particular, $D^{\min}(f,f)\geqslant D^*(f,f)$ for all $f\in\mathscr{D}(D^{\min})\subset\mathscr{D}(D^*)$.

Proof. Since $P^{\min}(\lambda,x,dy)$ is the minimal solution to (B), it certainly satisfies (B). Combining the above lemmas we now need only to prove (1.26).

Let $P(\lambda,x,dy)$ be a symmetric q-process and satisfy (B). By [5; Theorem (1.2.25)], it follows that

$$\lim_{\lambda\uparrow\infty}\lambda[\lambda P(\lambda,x,A)-I_A(x)]=q(x,A)-q(x)I_A(x),\quad x\in E,\quad A\in\mathscr{E}.$$

Moreover, by [5; Theorem (7.4.14)], we have

$$\lim_{\lambda\uparrow\infty}\lambda[1-\lambda P(\lambda,x,E)]=d(x),\quad x\in E.$$

On the other hand,

$$\lambda^2P(\lambda,x,A-\{x\})\leqslant\lambda(1-\lambda P(\lambda,x,\{x\}))$$
$$\leqslant\lambda(x)/(\lambda+q(x))\leqslant q(x),\quad x\in E.$$

By the dominated convergence theorem, we get

$$\lambda^2\int_{E_n}\pi(dx)P(\lambda,x,A-\{x\})\longrightarrow\int_{E_n}\pi(dx)q(x,A)\leqslant n\pi(E_n)<\infty$$

for all n. Using the above facts and [5; Theorem (1.2.24)], we obtain

$$D(f,f)=\lim_{\lambda\uparrow\infty}\lambda(f-\lambda P_\lambda f,f)$$
$$=\lim_{\lambda\uparrow\infty}\left[\frac{\lambda^2}{2}\int\pi(dx)\int_{E-\{x\}}P(\lambda,x,dy)(f(y)-f(x))^2\right.$$
$$\left.+\lambda\int\pi(dx)[1-\lambda P(\lambda,x,E)]f(x)^2\right]$$
$$\geqslant\varliminf_{\lambda\uparrow\infty}\left[\frac{\lambda^2}{2}\int_{E_n}\pi(dx)\int_{E-\{x\}}P(\lambda,x,dy)(f(y)-f(x))^2\right.$$

$$+\lambda\int\pi(dx)[1-\lambda P(\lambda,x,E)]f(x)^2\Big]$$

$$\geqslant\int_{E_n}\pi(dx)\int q(x,dy)(f(y)-f(x))^2.$$

The conclusion is now followed by letting $n\to\infty$. Q.E.D.

Now, we have answered the questions mentioned above. In section 3, we will give a more complete picture for the Dirichlet forms.

2. Constructions of the Dirichlet Forms

As we already mentioned in the previous section that the construction of symmetrizable q-processes is the same as the construction of Dirichlet forms. What has already been done is the case that the non-conservative points are finite and the exist boundary consists of finite points. The work is based on the Feller boundary and the Martin boundary theory, and it would be too much to rewrite the Dirichlet forms according to our constructions. Here we only consider a simple case.

Definition 2.1 Let $\dim(\mathscr{U}_\lambda)$ be the independent solutions to the following equation

$$(2.2)\qquad \begin{cases}(\lambda+q(x))u(x)=\int q(x,dy)u(y) & \lambda>0\\ 0\leqslant u\leqslant 1\end{cases}$$

It is known that $\dim(\mathscr{U}_\lambda)$ is independent of $\lambda>0$. The maximal solution $\bar{z}(\lambda,x)$ to (2.2) can be obtained in the following way: let $x^{(0)}(\lambda,x)=1$, $\lambda>0$, $x\in E$

$$x^{(n+1)}(\lambda,x)=\int\frac{q(x,dy)}{\lambda+q(x)}x^{(n)}(\lambda,y),\quad n\geqslant1,\quad \lambda>0,\quad x\in E$$

then
$$x^{(n)}(\lambda,x)\downarrow\bar{z}(\lambda,x)\quad\text{as}\quad n\uparrow\infty.$$

If the q-pair is conservative, then $\bar{z}(\lambda,x)$ is nothing but $z(\lambda,x)=1-\lambda P^{\min}(\lambda,x,E)$.

Theorem 2.3 Let $q(x)-q(x,dy)$ be a symmetric q-pair on $L^2(\pi)$ and $\dim(\mathscr{U}_\lambda)\leqslant1$. If $\dim(\mathscr{U}_\lambda)=0$ or $\dim(\mathscr{U}_\lambda)=1$ but $\int\pi(dx)z(\lambda,x)=\infty$ (which is also independent of $\lambda>0$), then there is precisely one Dirichlet form, $D^{\min}=D^*$. If $\dim(\mathscr{U}_\lambda)=1$ and $\int\pi(dx)z(\lambda,x)<\infty$, then there are infinite Dirichlet forms which have the same representation.

$$(2.4)\qquad D^c(f,f)=D^{\min}(f,f)+\lim_{\lambda\uparrow\infty}\frac{(\lambda\int\pi(dx)z(\lambda,x)f(x))^2}{c+\lambda\int\pi(dx)z(\lambda,x)}$$

with $c\geqslant0$. Moreover, if $c_2\geqslant c_1\geqslant0$, then

$$(2.5)\qquad D^{\min}(f,f)\geqslant D^{c_1}(f,f)\geqslant D^{c_2}(f,f)\geqslant D^*(f,f)$$

for all $f\in\mathscr{D}(D^{\min})\subset\mathscr{D}(D^{c_1})\subset\mathscr{D}(D^{c_2})\subset\mathscr{D}(D^*)$.

Proof. From our previous work (cf. [5]), it is known that in the first case, there is only one symmetric q-process on $L^2(\pi)$, and that in the second case, all of the q-processes have the representation:

$$P_c(\lambda,x,A)=P^{\min}(\lambda,x,A)+\frac{z(\lambda,x)\int_A\pi(dx)z(\lambda,x)}{c+\lambda\int\pi(dx)z(\lambda,x)},$$

$$\lambda>0,\ x\in E,\ A\in\mathscr{E}.$$

From this, the theorem is a straightforward consequence. Q.E.D.

Remark 2.6 As we will see later ((3.10)), for each $f\in\mathscr{D}(D^{\min})$, the inequalities in (2.5) are indeed equalities. That is

$$\lim_{\lambda\uparrow\infty}\frac{\lambda(\int\pi(dx)z(\lambda,x)f(x))^2}{\int\pi(dx)z(\lambda,x)}=0 \tag{2.7}$$

for all $f\in\mathscr{D}(D^{\min})$. Thus,

$$\mathscr{D}(D^{\min})=\{f\in\mathscr{D}(D^*):\ (2.7)\text{ holds}\}.$$

Example (2.8) Take $E=\{0,1,\dots\}$, $q_{i,i+1}=b_i>0$, $q_{i,i-1}=a_i>0$, $q_i=-q_{ii}=(a_i+b_i)$ with $a_0=0$. Then $Q=(q_{ij})$ is a birth-death Q-matrix and so $\dim(\mathscr{U}_\lambda)\leqslant 1$. The Q-matrix is symmetrizable with respect to

$$\pi_0=1,\ \pi_i=\frac{b_0b_1\cdots b_{i-1}}{a_1a_2\cdots a_i}\qquad (i>1)$$

and hence Theorem (2.3) is applicable to this case. However, for this special case, if $z(\lambda,\cdot)\neq 0$, then $\sum_i\pi_iz(\lambda,i)=\infty$ if and only if $\sum_i\pi_i=\infty$.

3. Extension and Regularity

Having the previous results in mind, one can believe that the Dirichlet form $D^{\min}$ is the minimum.

Theorem 3.1 Let $P(\lambda,x,dy)$ be any symmetric q-process on $L^2(\pi)$ with Dirichlet form D, then we have

$$\mathscr{K}\subset\mathscr{D}(D^{\min})\subset\mathscr{D}(D)\quad\text{and} \tag{3.2}$$

$$D^{\min}(f,f)\geqslant D(f,f),\ f\in\mathscr{D}(D^{\min})^+. \tag{3.3}$$

Proof. By the monotone class theorem and the minimal property (1.22), it follows that

$$P_tf\geqslant P_t^{\min}f,\ t\geqslant 0,\ f\in L^2(\pi)^+.$$

and $$(P_tf,g)\geqslant(P_t^{\min}f,g),\ t\geqslant 0,\ f,g\in L^2(\pi)^+.$$

Hence $$D(f,f)=\lim_{t\downarrow 0}(f-P_tf,f)/t$$

$$\leqslant\lim_{t\downarrow 0}(f-P_t^{\min}f,f)/t=D^{\min}(f,f),\ f\in\mathscr{D}(D^{\min})^+.$$

Thus, we have proved (3.3) and $\mathscr{D}(D^{\min})^+\subset\mathscr{D}(D)^+$.

Now (3.2) follows immediately because of the basic property for Dirichlet form:

$$f\in\mathscr{D}(D)\Longleftrightarrow f^{*}\in\mathscr{D}(D).\quad \text{Q.E.D.}$$

Remark 3.4 At the end of this section, we will show that we usually can not use $\mathscr{D}(D^{\min})$ instead of $\mathscr{D}(D^{\min})^{+}$ in (3.2). In other words, one can not say that any Dirichlet form is an extension of the minimum in the sense of [10; §2.3] (or see (3.7) below).

Now, we are going to prove that the Dirichlet form $D^{\min}$ is just the Friedricks extension of Ω_0, which is defined by

$$\Omega_0 f(x)=\int q(x,dy)f(y)-q(x)f(x),\quad x\in E,\ f\in\mathscr{K}.$$

Since Ω_0 is a well-defined, linear, symmetric and non-positive definite ($(\Omega_0 f,f)\leqslant 0$, $f\in\mathscr{K}$) operator on $L^2(\pi)$, with the domain $\mathscr{K}$ being dense in $L^2(\pi)$, the usual procedure will give us the smallest closed extension $\bar{D}^0$ of $D^0(f,g)=-(\Omega_0 f,g)$, $\mathscr{D}(D^0)=\mathscr{K}$. Denote by $\bar{\Omega}_0$ the generator corresponding to $\bar{D}^0$. But we still need to show that $\bar{D}^0$ is a Dirichlet form. For this, it suffices to note that

$$\bar{D}^0(f,f)=D^*(f,f),\quad f\in\mathscr{D}(\bar{D}^0).$$

Definition 3.5 We say that a Dirichlet form D is regular if $\mathscr{K}$ is dense in $\mathscr{D}(D)$ with D_1 norm.

Theorem 3.6 $\bar{D}^0=D^{\min}$ and is regular.

Proof. We have just proved that $\bar{D}^0$ is a Dirichlet form and hence the same proofs given in section 1 will give us a symmetric q-process on $L^2(\pi)$. So by Theorem (3.1), we have $\mathscr{D}(D^{\min})\subset\mathscr{D}(\bar{D}^0)$. On the other hand, $\bar{D}^0=D^{\min}$ on $\mathscr{K}$, the minimum property of $\bar{D}^0$ gives us $\mathscr{D}(\bar{D}^0)\subset\mathscr{D}(D^{\min})$. Therefore, we have $\mathscr{D}(\bar{D}^0)=\mathscr{D}(D^{\min})$ and $\bar{D}^0=D^{\min}$. Q.E.D.

For Markov chains, this theorem was proved in [18].

Clearly, we obtain

Corollary 3.7 If a Dirichlet form D is an extension of $D^{\min}$:

$$D^{\min}(f,f)\geqslant D(f,f),\quad f\in\mathscr{D}(D^{\min})\subset\mathscr{D}(D),$$

then it is regular if and only if $D=D^{\min}$. In particular, D^* is regular if and only if $D^*=D^{\min}$. That is, there exists only one symmetric q-process which satisfies the equation (B).

Corllary 3.8 If $\int\pi(dx)q(x)<\infty$ then there is only one symmetric q-process on $L^2(\pi)$ satisfying (B).

Proof. This result is proved in [5; Theorem (7.2.7)], but we would like to give a new proof here. Since $\mathscr{D}(D^*)\cap L^\infty(\pi)$ is dense in $\mathscr{D}(D^*)$ with D_1^* norm, we need only to prove that $\mathscr{K}$ is dense in $\mathscr{D}(D^*)$ with D_1^* norm. Given $f\in\mathscr{D}(D^*)\cap L^\infty(\pi)$, set $f_n=fI_{E_n}$, then by the dominated convergence theorem, we have

$$\frac{1}{2}\int\pi_q(dx,dy)(f(y)-f_n(y)-f(x)+f_n(x))^2$$

$$\leqslant\int\pi_q(dx,dy)[(f(y)-f_n(y))^2+(f(x)-f_n(x))^2]$$

$$\leqslant 2\int\pi_q(dx,dy)(f(x)-f_n(x))\to 0,\ \text{as } n\to\infty,$$

and $$\int\pi_d(dx)(f(x)-f_n(x))^2\to 0,\ \text{as } n\to\infty.\ \text{Q.E.D.}$$

Definition 3.9 The q-processes satisfying the backward Kolmogorov equation (B) are called B_q-processes.

We are now at the position to complete our picture for the symmetric B_q-processes.

Theorem 3.10 A symmetric q-process on $L^2(\pi)$ is an extension of $P_\lambda^{\min}$ in the sense of (3.7) if and only if it is a B_q-process. Furthermore, for any symmetric B_q-process on $L^2(\pi)$ with Dirichlet form D, we have

(3.11) $$\mathscr{K}\subset\mathscr{D}(D^{\min})\subset\mathscr{D}(D)\subset\mathscr{D}(D^*)$$ and

(3.12) $$D^{\min}(f,f)=D(f,f)=D^*(f,f) \text{ on } \mathscr{D}(D^{\min}),$$

Proof. By Theorem (1.25) and Theorem (3.1), it suffices to prove the first assertion. But this is a straightforward consequence of Lemma (1.20) and Theorem (3.6). Q.E.D.

After finishing the discussions for $D^{\min}$, one may ask the relationship between D^* and the Krein extension. It is known that the Krein extension is so large that it is sometimes even not sub-Markovian (cf. [10; Theorem 2.3.2]). But we would like to point out here that the Krein extension is sometimes not large enough since there are some Dirichlet forms with larger domain than the Krein's. To show this, let us return to consider the example (1.23). In this case, it is easy to check that $\mathscr{N}_a=\{0\}$ (see [10; (2.3.5)] for the notation), hence $D^{\min}=D^*=D^K$ (the Krein extension), and so $\mathscr{D}(D^K)\subsetneqq\mathscr{D}(D)$. The mean reason is that the symmetric Q-process $(P_{ij}(\lambda))$ given in (1.23) does not satisfy the equation (B). More precisely, the non-conservative part of the Martin boundary is not considered by the Krein extension.

Can we find the maximum sub-Markovian extension?

Recall the proof of Theorem (1.25). It is not difficult to show that for any symmetric q-process $P(\lambda,x,dy)$ with Dirichlet form D, we have

$$D(f,f)=\lim_{\lambda\uparrow\infty}\lambda(f-\lambda P_\lambda f,f)$$

$$\geqslant \frac{1}{2}\int \pi_q(dx,dy)(f(y)-f(x))^2 = \tilde{D}(f,f).$$

One might expect $\tilde{D}$ as the maximum extension. However, if the given q-pair is not conservative, $\tilde{D}$ is not a Dirichlet form.

4. Uniqueness

It is usually the most interesting case that the given symmetric q-pair $q(x)-q(x,dy)$ on $L^2(\pi)$ determines a unique Dirichlet form. Then $D^{\min} = D^*$ has the nice property—regular and has an explicit expression.

We have seen that if $\int \pi(dx)q(x)<\infty$, then the Dirichlet form is unique ((3.8)). We have also seen a uniqueness criterion ((3.7)) for the symmetric q-process satisfying the equation (B). It is a simple matter to restate the criterion as follows.

Corollary 4.1 Given a symmetric q-pair $q(x)-q(x,dy)$ on $L^2(\pi)$. The Dirichlet form corresponding the symmetric B_q-processes is unique if and only if

(i) $\lim\limits_{\lambda\uparrow\infty}\lambda^2\int\pi(dx)\int P^{\min}(\lambda,x,dy)\frac{1}{2}(f(y)-f(x))^2 = \int\pi_q(dx,dy)(f(y)-f(x))^2$

(ii) $\lim\limits_{\lambda\uparrow\infty}\lambda\int\pi(dx)[1-\lambda P^{\min}(\lambda,x,E)]f(x)^2 = \int\pi_d(dx)f(x)^2$

hold for all $f\in\mathscr{D}(D^*)\cap L^\infty(\pi)$.

Even though the above conditions are reasonable in the sense that they depend only on the q-pair and the minimal q-process. But it is not very easy to check in practice.

On the other hand, as we have known from the last section, if we consider only the extension of $D^{\min}$, then a part of Dirichlet forms will be lost. In the paper [11], Martingale approach is used to study the uniqueness. Note that for this approach, the equation (F) is needed, and hence we will have the same situation (see Remark (1.14)). From these points of view, it seems not easy to get a uniqueness criterion in our context. For these reasons and the later use we first restate here two results from our previous work.

Of course, the uniqueness criteria [7] for general q-processes will give us the sufficient conditions for the symmetric case. Here, we prefer to show some more practical sufficient conditions.

Theorem 4.2 The following conditions are all sufficient for the uniqueness of q-processes.

(i) The q-pair $q(x)-q(x,dy)$ is bounded, i.e.,

$q(x)\leqslant C<\infty$ for all $x\in E$,

(ii) The q-pair $q(x)-q(x,dy)$ is conservative, there exists a function $\phi\geqslant q$ and a constant $C\in\mathbb{R}$ such that

$$\int q(x,dy)\phi(y)\leqslant(C+q(x))\phi(x),\quad x\in E, \tag{4.3}$$

(iii) The q-pair $q(x)-q(x,dy)$ is conservative, there exists a function $\phi\geqslant 0$, a constant $C\in\mathbb{R}$ and a sequence $\{B_n\}_1^\infty\subset\mathscr{E}$ such that

$$B_n\uparrow E,\quad \sup_{x\in B_n}q(x)<\infty,\quad \varliminf_{n\to\infty}\inf_{x\notin B_n}\phi(x)=\infty$$

and (4.3) holds.

For the proofs of (ii) and (iii), see [4]. The following result is proved in [5; Theorem (7.6.2)].

Theorem 4.4 Let π be a probability measure. Then there exists precisely one symmetric q-process (equivalently, Dirichlet form) on $L^2(\pi)$ if and only if the following conditions: (i) $q(x)-q(x,dy)$ is symmetric on $L^2(\pi)$; (ii) $\bar{x}(\lambda,\cdot)=0$, $\pi-$a.e.; (iii) $\int\pi(dx)d(x)<\infty$ all hold.

In the case that π being a probability measure, we often call the process (resp. q-pair) reversible. If $\pi(E)=\infty$, then the condition (i) is still necessary, but not the conditions (ii) and (iii). We conclude this section with an improvement of Theorem (4.4).

Theorem 4.5 There exists only one Dirichlet form on $L^2(\pi)$ if the following three conditions all hold:

(i) $q(x)-q(x,dy)$ is symmetric on $L^2(\pi)$,

(ii) $\mathscr{U}_\lambda\cap L^1(\pi)=\{0\}$,

(iii) $\inf\{P^{\min}(\lambda,x,E):\ x\in H\}>0$ or $\int\pi(dx)d(x)<\infty$,

where $H=\{x\in E:\ d(x)>0\}$ which is the set of all non-conservative points.

Remark 4.6 It is known that the first condition in (iii) and dim $(\mathscr{U}_\lambda)$ are independent of $\lambda>0$. We now prove that the condition (ii) is also independent of $\lambda>0$. To do this, let $u(\lambda)\in\mathscr{U}_\lambda$ and $0<\|u(\lambda_0)\|_1=\|u(\lambda_0)\|_{L^1(\pi)}<\infty$ for some $\lambda_0>0$. Define

$$u(\mu)=u(\lambda_0)+(\lambda_0-\mu)P_\mu^{\min}u(\lambda_0),\quad \mu>0.$$

Then $u(\mu)\in\mathscr{U}_\lambda-\{0\}$ by [5; Lemma (2.5.10)]. Moreover, by the symmetry, we obtain

$$\|P_\mu^{\min}u(\lambda_0)\|_1=\int\pi(dx)\int P^{\min}(\mu,x,dy)\int\frac{q(y,dz)}{\lambda+q(y)}u(\lambda_0,z)$$

$$=\int\pi(dx)P_\mu^{\min}1(x)\int\frac{q(x,dy)}{\lambda+q(x)}u(\lambda_0,y)$$

$$= \int \pi(dx)\frac{u(\lambda_0,x)}{\lambda+q(x)}\int q(x,dy)P_\mu^{\min}1(y)$$

$$\leqslant \mu^{-1}\int \pi(dx)\frac{u(\lambda_0,x)}{\lambda+q(x)}q(x,E) \leqslant \|u(\lambda_0)\|_1/\mu.$$

Hence $\|u(\mu)\|_1 \leqslant \|u(\lambda_0)\|_1 + |\lambda_0-\mu|\|P_\mu^{\min}u(\lambda_0)\|_1/\mu < \infty.$

Proof of Theorem (4.5).

Let $P(\lambda,x,dy)$ be any symmetric q-process on $L^2(\pi)$. By [5; Theorem (7.4.16)], as a q-process, it must have the representation:

$$P(\lambda,x,dy) = P^{\min}(\lambda,x,dy) + B(\lambda,x,dy) + \int_H X(\lambda,x,dy)F(\lambda,y,dz), \quad \lambda>0, \ x\in E$$

where $X(\lambda,x,A) = \int_A P^{\min}(\lambda,x,dy)d(y)$ and B_λ and F_λ having the properties:

(4.7) For every $\lambda>0$ and $x\in E$, $\lambda B(\lambda,x,E)\leqslant 1$, $\lambda F(\lambda,x,A)\leqslant 1$, $F(\lambda,x,E) = 0$ if $x\notin H$;

(4.8) For every $\lambda>0$ and $A\in\mathscr{E}$, $\lambda B(\lambda,\cdot,A)\in\mathscr{U}_\lambda$;

(4.9) The family $\{F_\lambda: \lambda>0\}$ satisfies

$$F_\lambda - F_\mu + (\lambda-\mu)F_\lambda P_\mu^{\min} + (\lambda-\mu)F_\lambda X_\mu F_\mu = 0, \quad \lambda,\mu>0,$$

(4.10)
$$\lim_{\lambda\to\infty}\lambda^2\int_H X(\lambda,\cdot,dy)F(\lambda,y,A)$$

$$\equiv \lim_{\lambda\to\infty}\lambda^2 X_\lambda F_\lambda I_A = 0, \quad A\in\mathscr{E}\cap E_n, \quad n\geqslant 1.$$

Finally, P_λ satisfies the equation (B) if and only if $F_\lambda\equiv 0$.

Suppose that $B_\lambda\neq 0$. Then there exist a $\lambda_0>0$ and an $n\geqslant 1$ such that

$$0<\int \pi(dx)B(\lambda_0,x,E_n)\leqslant \int \pi(dx)P(\lambda_0,x,E_n)$$

$$= \int_{E_n}\pi(dx)P(\lambda_0,x,E)\leqslant \pi(E_n)/\lambda_0<\infty.$$

This is impossible since (4.8) and the condition (ii).

Now, we have $P_\lambda = P_\lambda^{\min} + X_\lambda F_\lambda$

In the case that $\int\pi(dx)d(x)<\infty$, the same proof of Theorem (4.4) (c.f. [5]) will give us the conclusion. Hence, we need only to consider the case that $\{\inf P^{\min}1(x): x\in H\}>0$.

By (4.9), we have $F_\lambda + (\lambda-\mu)F_\lambda P_\mu^{\min} = F_\mu + (\mu-\lambda)F_\lambda X_\mu F_\mu \geqslant 0$, $\mu\geqslant\lambda$.

Hence $\lambda F_\lambda \geqslant (1-\lambda/\mu)F_\lambda[\mu(\mu P_\mu^{\min} - I)]$, $\mu\geqslant\lambda$.

Let $\mu\to\infty$, we get $\lambda F_\lambda I_A \geqslant -F_\lambda \Omega I_A$, $\lambda>0$, $A\in\mathscr{E}$

where $\Omega f(x) = \int q(x,dy)f(y) - q(x)f(x)$, $f\in L^\infty(\pi)$. Thus,

(4.11) $U(\lambda,x,A)\equiv F_\lambda(\lambda I_A + \Omega I_A)(x)$

$$=\int F(\lambda,x,dy)[(\lambda+q(y))I_A(y)-q(y,A)]\geqslant 0.$$

By [5; Lemma (3.3.1)], it follows that $F_\lambda \geqslant U_\lambda P_\lambda^{\min}$, $\lambda>0$. Set

(4.12) $$V_\lambda = F_\lambda - U_\lambda P_\lambda^{\min}\geqslant 0,\quad \lambda>0.$$

Then we obtain, for each $x_0\in E$, that

$$\int V(\lambda,x_0,dy)\Omega I_A(y)=0,\quad \lambda>0,\quad A\in\mathscr{E}\cap E_n,\quad n\geqslant 1.$$

That is

(4.13) $$V(\lambda,x_0,A)=\int V(\lambda,x_0,dy)\frac{q(y,A)}{\lambda+q(y)},\quad \lambda>0,\quad A\in\mathscr{E}$$

by the monotone class theorem. On the other hand, by the symmetry, $\pi(A)=0$ implies that

$$\begin{aligned}0&=\int_A\pi(dx)P(\lambda,x,E)=\int\pi(dx)P(\lambda,x,A)\\&\geqslant\int\pi(dx)\int P^{\min}(\lambda,x,dy)d(y)F(\lambda,y,A)\\&\geqslant\int\pi(dx)P^{\min}(\lambda,x,\{x\})d(x)F(\lambda,x,A)\\&\geqslant\int\pi(dx)P^{\min}(\lambda,x,\{x\})d(x)V(\lambda,x,A),\quad\lambda>0.\end{aligned}$$

This shows that for all x_0 out of a π-null subset of H, say H_0, we have

$$V(\lambda,x_0,\cdot)\ll\pi,\quad\lambda>0.$$

Define $$V(\lambda,x_0,x)=\frac{dV(\lambda,x_0,\cdot)}{d\pi}(x).$$

Then $$\int\pi(dx)V(\lambda,x_0,x)=V(\lambda,x_0,E)\leqslant F(\lambda,x_0,E)\leqslant 1/\lambda,$$

and so by (4.13), we have

$$\begin{aligned}\int_A V(\lambda,x_0,x)\pi(dx)&=V(\lambda,x_0,A)=\int\pi(dy)V(\lambda,x_0,y)\frac{q(y,A)}{\lambda+q(y)}\\&=\int_A\pi(dy)\int\frac{q(y,dz)}{\lambda+q(y)}V(\lambda,x_0,z),\quad\lambda>0,\quad A\in\mathscr{E}\end{aligned}$$

That is $$V(\lambda,x_0,x)=\int\frac{q(x,dy)}{\lambda+q(x)}V(\lambda,x_0,y),\quad\lambda>0,\quad\pi-a.e(x).$$

Combining the above facts with the conditions (ii), we see that

$$V(\lambda,x_0,E)=0,\quad x_0\notin H_0,\quad\lambda>0.$$

Note that

$$\int_A\pi(dx)\int P^{\min}(\lambda,x,dy)d(y)I_{H_0}(y)V(\lambda,y,E)$$

$$\leqslant (P_\lambda^{\min} dI_{H_0}, V_\lambda 1) \leqslant (P_\lambda^{\min} dI_{H_0}, F_\lambda 1) \leqslant (P_\lambda^{\min} dI_{H_0}, 1)/\lambda$$
$$= (I_{H_0}, dP_\lambda^{\min} 1)/\lambda = 0, \quad A \in \mathscr{E}, \quad \lambda > 0.$$

We arrive at

$$P_\lambda \doteq P_\lambda^{\min} + P_\lambda^{\min} dU_\lambda P_\lambda^{\min} \tag{4.14}$$

Now, we start to use the condition

$$C(\lambda) \equiv \inf\{P_\lambda^{\min} 1(x): x \in H\} > 0.$$

Since $\quad 1 \geqslant \lambda F_\lambda 1 = \lambda U_\lambda P_\lambda^{\min} 1 \geqslant C(\lambda) \lambda U_\lambda 1$

we have $\quad \lambda U_\lambda 1 \leqslant C(\lambda)^{-1}, \ \lambda > 0$

and so $\quad P_\mu^{\min} U_\mu \leqslant (\mu C(\mu))^{-1}, \ \mu > 0.$

Also, by [5; Theorem (7.4.1)], we have

$$\int P^{\min}(\lambda, x, dy) d(y) \leqslant 1, \lambda > 0, \ x \in E.$$

We have got everything we need, the next step is copying the proof of our uniqueness theorem for general q-process (cf [5; pages 130—132)). Q.E.D.

Remark (4.15) The condition (ii) is quite general, which also shows that in the study of symmetrizable q-processes, the usual exit boundary $\mathscr{U}_\lambda$ is too large. The natural one should be $\mathscr{U}_\lambda \cap L^1(\pi)$. This boundary theory needs further study in order to get a complete picture of the theory.

The first condition in (iii) is the main condition to control the non-conservative part of the boundary [5,7,12,19]. It is still stronger since our example (1.23) can not be completely covered by this condition. A more natural condition should use the measure π and is still unknown.

5. Application to Large Deviations

As usual, we assume that $(E, \mathscr{E})$ is a Polish space in the study of large deviations. Also, we restrict ourselves to the processes which have right continuous path having left limits at every $t \in (0, \infty)$. Hence, we assume that the given q-pair is symmetric on $L^2(\pi)$ and regular. That is, the q-pair is conservative and determines uniquely a q-process.

Now, we can answer completely a question mentioned in [6; Remark (9)].

Theorem (5.1) Let $q(x) - q(x, dy)$ be a regular symmetric q-pair on $L^2(\pi)$. Suppose that there exist a σ-finite measure λ and an $\mathscr{E} \times \mathscr{E}$-measurable function $q(x, y)$ such that

$$q(x, dy) = q(x, y) \lambda(dy), \quad x, y \in E$$

and $\quad \pi(\cdot) = d\pi / d\lambda > 0$, a.s. (λ). Then we have

$$(5.2)\quad I(\mu)=\begin{cases}+\infty, & \text{if } \mu \not\ll \pi,\\ \dfrac{1}{2}\displaystyle\int_{E^0}\int_{E^0}\left[\sqrt{\frac{d\mu}{d\lambda}(x)q(x,y)}-\sqrt{\frac{d\mu}{d\lambda}(y)q(y,x)}\right]^2\lambda(dx)\lambda(dy), & \text{if } \mu\ll\pi.\end{cases}$$

where $E^0=\{x\in E;\ q(x)>0\}$.

Proof. By Theorem (3.10), [6] and [15; Theorem (7.44)], we may assume that $\mu\ll\pi$ and compute:

$$D^*\left[\sqrt{\frac{d\mu}{d\pi}},\sqrt{\frac{d\mu}{d\pi}}\right]=\frac{1}{2}\int J_{\pi q}(dx,dy)\left[\sqrt{\frac{d\mu}{d\pi}}(y)-\sqrt{\frac{d\mu}{d\pi}}(x)\right]^2$$
$$=\frac{1}{2}\int\lambda(dx)\int\lambda(dy)\frac{d\mu}{d\lambda}(x)q(x,y)\left[\sqrt{\frac{d\mu}{d\pi}}(y)-\sqrt{\frac{d\mu}{d\pi}}(x)\right]^2$$
$$=\frac{1}{2}\int_{E^0}\int_{E^0}\left[\sqrt{\frac{d\mu}{d\lambda}(x)q(x,y)}-\sqrt{\frac{d\mu}{d\lambda}(y)q(y,x)}\right]^2\lambda(dx)\lambda(dy).$$

Q.E.D.

Corollary (5.3) Let $Q=(q_{ij})$ be a regular Q-matrix, symmetrizable with respect to $\{\pi_i>0;\ i\in E\}$, then for every probability measure μ, we have

$$(5.4)\qquad I(\mu)=\frac{1}{2}\sum_{i,j}[\sqrt{\mu_i q_{ij}}-\sqrt{\mu_j q_{ji}}]^2.$$

6. Application to Decay Estimates

Let us begin this section with comparing two kinds of decay estimates for Markov chains. The first one was studied by Kingmann [13] a long time ago. He proved that for an irreducible Markov chain $(P_{ij}(t))$, the limit

$$(6.1)\qquad t^{-1}\log P_{ij}(t)\to -v,\ \text{as } t\to\infty$$

always exists and

$$(6.2)\qquad 0\leqslant v\leqslant \inf_i q_i.$$

If $v>0$, we call the chain $(P_{ij}(t))$ exponential decay. Now, let $(P_{ij}(t))$ be symmetric on $L^2(\pi)$, if there exist constant $C<\infty$ and $v>0$ such that

$$(6.3)\qquad \sup_{i,j}P_{ij}(t)/\pi_j\leqslant Ct^{-v},\ t>0$$

we call the chain $(P_{ij}(t))$ uniform decay. This concept was introduced by Varopoulos [16,17] recently.

Since

$$\sup_{i,j}P_{ij}(t+s)/\pi_j=\sup_{i,j}\sum_k P_{ik}(t)/\pi_k P_{jk}(s)$$
$$\leqslant\sup_{i,k}P_{ik}(t)/\pi_k\sup_j\sum_k P_{jk}(s)\leqslant\sup_{i,k}P_{ik}(t)/\pi_k,$$

$\sup_{i,j} P_{ij}(t)/\pi_j$ is decreasing as t increases. Hence, the estimate in (6.3) is asking for the polynomial decay parameter.

It is believable that the exponential decay is stronger than the polynomial decay. But it is not always true since the measure (π_i) appeared in (6.3). For example take $q_i=1$ for all $i\in E$ in (1.23), then the Q-process is unique since the Q-matrix is bounded.

Also $$P_{ij}(t)=\delta_{ij}e^{-t},\quad t\geqslant 0,\quad i,j\in E$$

and so (6.1) holds with $v=1$. However, if we take (π_i) having the property: $\inf_i \pi_i=0$, then (6.3) is false.

Remark (6.4) In the case that $(P_{ij}(t))$ being irreducible and positive recurrent, we have $\lim_{t\to\infty} P_{ij}(t)=\pi_j>0$ independent of $i\in E$. Then the estimates in (6.1) and (6.3) have no meaning. This is just the main case which large deviations deal with.

Let us consider the uniform decay. In order to get (6.3), or equivalently,

$$\sup_{i,j} t^v P_{ij}^{\min}(t)/\pi_j\leqslant C,\quad t>0,$$

we usually proceed as follows. First prove that for each fix $j\in E$, $\{t^v P_{ij}^{\min}(t):\ i\in E\}$ is the minimal solution to

$$x_i(t)=\sum_{k\neq i} q_{ik}\int_0^t e^{-q_i(t-s)}x_k(s)\,ds+\delta_{ij}e^{-q_i t}t^v/\pi_j\quad i\in E.$$

then use the comparison theorem. But this time the above approach is not useful. Hence we need a different method. In the symmetrizable case, the Dirichlet forms are very useful tools.

Set $\|f\|_p=\|f\|_{L^p(\pi)}$ and denote $\|K\|_{p\to q}=\sup\{\|Kf\|_q:\ f\in\mathscr{K}$ with $\|f\|_p=1\}$ for an operator K defined on $\mathscr{K}$.

Theorem (6.5) Let $P(t,x,dy)$ be a symmetric q-process on $L^2(\pi)$ with Dirichlet form D, and $v\in(0,\infty)$, $\delta\in[0,\infty)$ be given. If

(6.6) $$\|f\|_2^{2+4/v}\leqslant A[D(f,f)+\delta\|f\|_2^2]\|f\|_1^{4/v},\quad f\in L^2(\pi)$$

for some $A\in(0,\infty)$, then there is a $B\in(0,\infty)$ which depends only on v and A such that

(6.7) $$\|P_t\|_{1\to\infty}\leqslant Be^{\delta t}/t^{v/2},\quad t>0.$$

Conversely, if (6.7) holds for some B, then (6.6) holds for an A depending only on B and v.

This theorem is restated from [3] except we allow the (P_t) to be sub-Markovian. In which case, the (P_t) is usually transient and often has uniform decay. The most results in [3] are still correct and the proofs given there need only a slight modification.

A lot of interesting results are presented in [2], [3] and [17]. They can apply to our case directly.

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Dirichlet型和可配称跳过程

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摘 要

本文研究了跳过程的Dirichlet型理论，改进了可配称跳过程的唯一性定理。作为本文结果的直接应用，给出了可配称跳过程所导出的半群的一致衰减估计以及这类过程的大偏差速率函数的完全刻画。