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STATIONARY DISTRIBUTIONS OF INFINITE PARTICLE SYSTEMS WITH NONCOMPACT STATE SPACE*

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Abstract

This paper deals with the problem of existence and uniqueness of the stationary distributions (abbr., s. d.'s) for the processes constructed in [4]. The main results are stated in § 1. For the reader's convenience we first restate the existence theorems (Theorem 1 and 2) of the processes given in [4]. Then two existence theorems (Theorem 3 and 4) and a uniqueness theorem (Theorem 5) for the s. d.'s of the processes are presented. The last result (Theorem 6), as an application of the previous ones, is about the Schlögl model which comes from nonequilibrium statisticali physics. The details of the proofs of Theorem 3—6 are given in § 2—4.

§ 1. Statements of the Main Results

Throughout the paper, we suppose that S is a countable set. For each $u\in S$, let $(E_u, \rho_u, \mathscr{E}_u)$ be a complete separable metric space, where $\mathscr{E}_u$ is the σ-algebra generated by the metric ρ_u. Denote by $(E, \mathscr{E})$ the usual topological product space of (E_u, ρ_u) $(u\in S)$. Choose an arbitrary reference point $\theta=(\theta_u: u\in S)$, and suppose that we are given a positive summable sequence (k_u) on S.

For $x=(x_u: u\in S)$, $y=(y_u: u\in S)\in E$ and $\alpha\subset S$. Define

$$p_\alpha(x, y)=\sum_{u\in\alpha}\rho_u(x_u, \theta_u)k_u.$$

For simplicity, we also use the notations:

$$p_u(x)=p_u(x_u, \theta_u), \quad p_\alpha(x)=p_\alpha(x, \theta).$$

Set $E^\alpha=\{x\in E: p_{S\setminus\alpha}(x,\theta)=0\}$. The σ-algebra $\mathscr{E}^\alpha$ is induced on E^α by the σ-algebra $\mathscr{E}$. Denote by x^α the projection of x on E^α:

$$p_\alpha(x^\alpha, x)+p_{S\setminus\alpha}(x^\alpha, \theta)=0$$

Let us recall some notions. We say that $q(x)-q(x, \cdot)$ is a q-pair on a measurable spafe $(X, \mathscr{B})$ if $q(\cdot)$ and $q(\cdot, B)$ are $\mathscr{B}$-measurable for each $B\in\mathscr{B}$, $q(x, \cdot)$ is a nonnegative measure on $\mathscr{B}$ for each $x\in X$ and

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$$q(x, B) \leqslant q(x) < \infty, \quad x \in X, \ B \in \mathscr{B}.$$

A q-pair $q(x) - q(x, \cdot)$ is called regular if it is conservative:

$$q(x, X) = q(x), \quad x \in X$$

and there exists only one transition probability function $P(t, x, B)$ $(t \geqslant 0, \ x \in X, B \in \mathscr{B})$ such that

$$\frac{d}{dt} P(t, x, B)\,|_{t=0} = q(x, B) - I_B(x) q(x), \quad x \in X, \ B \in \mathscr{B},$$

$$\lim_{t \to 0} P(t, x, \{x\}) = 1, \quad x \in X.$$

The function $P(t, x, \cdot)$ is often called a q-process or a jump process. For a given q-pair, the general uniqueness criteria for q-processes were obtained in [6], and some more practical sufficient conditions for uniqueness of q-processes are given in [3]. Thus, we may and will consider only the regular q-pair subsequently.

Since the q-pair $q(x) - q(x, \cdot)$ corresponds naturally an operator:

$$\Omega f(x) = \int q(x, dy) f(y) - q(x) f(x). \quad x \in X, \ f \in b\mathscr{B}$$

where $b\mathscr{B}$ = the set of all bounded $\mathscr{B}$-measurable functions, we also use Ω instead of q-pair.

We now return to our main context.

Suppose that there is fixed a sequence $\{\wedge_n\}_1^\infty$ of finite subsets of S such that $\wedge_n \uparrow S$. For each $n \geqslant 1$, there is also fixed a regular q-pair $q_n(x) - q_n(x, \cdot)$ on $(E^{\wedge_n}, \mathscr{E}^{\wedge_n})$. The problem we are interested in is to find a limit process of those q-processes $P_n(t, x, \cdot)$ determined by the q-pairs $q_n(x) - q_n(x, \cdot)$ $(n \geqslant 1)$. To this end, let p be an $\mathscr{E}$-measurable function (may be valued $+\infty$) satisfying:

1° $0 \leqslant p(x) < +\infty$ for each $x \in E^{\wedge_n}$ and $n \geqslant 1$;

2° for each $0 \leqslant d < +\infty$ and $n \geqslant 1$, the set $\{x \in E: p(x^{\wedge_n}) > d\}$ is an open set in E;

3° for each $x \in E$, $p(x^{\wedge_n}) \uparrow p(x)$ as $n \uparrow \infty$.

Put $E_0 = \{x \in E; p(x) < \infty\}$, the σ-algebra $\mathscr{E}_0$ is again induced on E_0 by $\mathscr{E}$.

One of our main tools in the study is the KRW-distance:

$$R_\alpha(P, Q) = \inf_\mu \int_{E^\alpha \times E^\alpha} p_\alpha(x, y) \mu(dx, dy),$$

where $\alpha \in \mathscr{S}_0$ = the set of all finite subsets of S, and the greatest lower bound is computed over all the measures μ on $\mathscr{E}^\alpha \times \mathscr{E}^\alpha$ satisfying

$$\mu(A \times E^\alpha) = P(A),$$

$$\mu(E^\alpha \times A) = Q(A), \quad A \in \mathscr{E}^\alpha.$$

The probability measure having the above marginality is called a couple measure of P and Q.

To get the upper estimate for KRW-distance of two q-processes $P_i(t, x_i, \cdot)$ with state space $(E_i, \mathscr{E}_i)$ and q-pair $q_i(x_i) - q_i(x_i, \cdot)$ (resp. Ω_i), $i = 1, 2$, the goal is coupling. The most interesting coupling is certainly the Markov coupling which is also a q-process $P(t; x_1, x_2; dy_1, dy_2)$ with conservative q-pair $q(x_1, x_2) - q(x_1, x_2; dy_1, dy_2)$ (resp. Ω). Then, it is easy to show that the operator Ω must satisfy the marginalities:

$$\Omega f(\cdot, x_2)=\Omega_1 f, \quad x_2\in E_2, f\in b\mathscr{E}_1,$$
$$\Omega f(x_1, \cdot)=\Omega_2 f, \quad x_1\in E_1, f\in b\mathscr{E}_2.$$

We then call Ω a coupling operator of Ω_1 and Ω_2. For the given Ω_i ($i=1, 2$), there are infinite choices of coupling operator Ω. The main result in (3) claims that any coupling operator Ω determines a unique q-process iff so do the marginal operators Ω_1 and Ω_2.

Now we can state our first existence result for the limit process.

Theorem 1. Suppose that the following conditions hold:

1) there exists a constant $c\in\mathbb{R}$ such that

$$\int q_n(x, dy)(p(y)-p(x))\leqslant c(1+p(x)), \quad x\in E_0, n\geqslant 1; \tag{1}$$

2) for each $1\leqslant n\leqslant m$ there exists a coupling operator $\Omega_{n,m}$ of Ω_n and Ω_m such that

$$\Omega_{n,m}p_w(x_1, x_2)\leqslant \sum_{u\in\wedge_n} c_{uw}p_u(x_1, x_2)+c_w(n, m)(1+p(x_1)+p(x_2)), \tag{2}$$
$$w\in\wedge_n, \quad x_1, x_2\in E_0,$$

where the non-diagonal elements of $(c_{u,w}: u, w\in S)$ and the elements of $c_w(n, m)$ ($w\in\wedge_n$, $m\geqslant n\geqslant 1$) are nonnegative and satisfying

$$c_w(t, n, m)=\sum_{k=0}^{\infty}\frac{t^{k+1}}{(k+1)!}[(B_n^*)^k c.(n, m)](w)\to 0, \quad m\geqslant n\to, \infty t\geqslant 0. \tag{3}$$

where $B_n=(c_{u,v}: u, v\in\wedge_n)$ and B_n^* is the transpose of B_n.

Then there exists a Markov process with transition probability function $P(t, x, \cdot)$ on state space $(E_0, \mathscr{E}_0)$ such that for each finite subset α of S,

$$\lim_{n\to 0} R_\alpha(P_n(t, x, \cdot), P(t, x, \cdot))=0, \quad x\in E_0, t\geqslant 0. \tag{4}$$

Moreover, the convergence is uniformly in $x\in E_0^N=\{x\in E_0: p(x)\leqslant N\}$. Finally, for fixed t, $P(t, x, \cdot)$ is continuous in the following sense: if $x, x_n\in E_0$, $n\geqslant 1$, $\sup_n p(x_n)<\infty$ and $\lim_{n\to\infty}\rho_u(x_n, x)=0$ for every $u\in S$, then

$$\lim_{n\to 0} R_\alpha(P(t, x_n, \cdot), P(t, x, \cdot))=0 \tag{5}$$

for every $\alpha\in\mathscr{S}_0$ = the set of all finite subsets of S.

Remark 1. If

$$\lim_{m>n\to\infty} c_u(n, m)=0, \quad u\in S,$$
$$\sup_{m>n,\, u\in\wedge_n} c_u(n, m)+\sup_u\sum_v|c_{uv}|<\infty$$

then the condition (3) holds.

The condition (3) means that the interactions are decreasing when the distance between the components increases. The next theorem relaxes the restriction for the special p defined by

$$p(x)=\sum_{u\in S}\rho_u(x)k_u, \quad x\in E, \tag{6}$$

Denote by $\mathscr{L}$ the set of all Lipschitz continuous functions with respect to the above p. For $f\in\mathscr{L}$, denote by $L(f)$ the Lipschitz constant of f.

Theorem 2. Let p be the function given by (6). Suppose that the following two conditions hold.

1) there exist $c_1\in\mathbb{R}$ and a nonnegative matrix $(b(u, v): u, v\in S)$ such that

$$\int q_n(x, dy)(\rho_v(y)-\rho_v(x))\leqslant\beta_v+c_1\rho_v(x)+\sum_{u\in\wedge_n}\rho_u(x)b(u,v),\quad v\in\wedge_n,\ x\in E_0, n\geqslant 1, \tag{7}$$

where

$$\beta_u\geqslant 0,\ u\in S;\ \sum_u\beta_u k_u<\infty$$

and

$$\sum_v b(u, v)k_v\leqslant Mk_u,\quad u\in S$$

for some $M>0$,

2) for every $1\leqslant n\leqslant m$ there exists a coupling operator $\Omega_{n,m}$ of Ω_n and Ω_m such that

$$\Omega_{n,m}p_w(x_1, x_2)\leqslant\sum_{u\in\wedge_n}c_{uw}p_u(x_1, x_2)+\sum_{u\in\wedge_m\setminus\wedge_n}p_u(x_2)g_{uw}+p_w(x_2)c_w(n, m),$$

$$w\in\wedge_n,\ x_1, x_2\in E_0 \tag{8}$$

where (c_{uv}), (g_{uv}) and $(c_w(n, m))$ are all nonnegative, they satisfy (5) and

$$c_w(n, n)=0,\ w\in\wedge_n,\ n\geqslant 1;\quad \sup_u\sum_v g_{uv}<\infty. \tag{9}$$

Then, tehre exists a Markov process with transition probability function $P(t, x, \cdot)$ on state space $(E_0, \mathscr{E}_0)$ such that

$$R_{\wedge_n}(P_n(t, x, \cdot), P(t, x, \cdot))\to 0,\quad n\to\infty,\ x\in E_0. \tag{10}$$

Moreover, the convergence is uniformly in t in finite intervals. Finally, the semigroup $\{P(t)\}_{t\geqslant 0}$ on $\mathscr{L}$ induced by $P(t, x, \cdot)$ has properties: $P(0)=I$; $P(t)$ is contractive in the uniform norm; there is a constant $c_3\in R$ such that

$$|P(t)f(x)-P(t)f(y)|\leqslant e^{c_3t}L(f)p(x, y). \tag{11}$$

Remark 2. If $p(x)=\sum_u\rho_u(x)k_u$, then the condition (7) implies (1).

We now turn to consider the stationary distributions for the processes constructed above.

A function h: $0\leqslant h\leqslant+\infty$ is called a compact function if $[x: h(x)\leqslant d]$ is compact set for every $d\in[0, \infty)$.

Theorem 3. Under the assumptions of Theorem 1 with $p(x)=\sum_u\rho_u(x)k_u$, if for each $u\in S$ there is a compact function $h_u<+\infty$ such that

$$\rho_u(x)=h_u(x_u),\ x_u\in E_u;\ \sup_u h_u(\theta_u)k_u<\infty \tag{12}$$

and there are constants $K\in[0, \infty)$ and $\eta\in(0, \infty)$ such that

$$\int q_n(x, dy)(h_n(y)-h_n(x))\leqslant K-\eta h_n(x), \tag{13}$$

where $h_n(x)=\sum_{u\in S}h_u(x_u)k_u$. Then

1) for each $n\geqslant 1$, the process $P_n(t, x, \cdot)$ has at least a stationary distribution π_n of $P_n(t, x, \cdot)$ satisfies

$$\int\pi_n(dx)h_u(x)\leqslant K/\eta, \tag{14}$$

2) the process $P(t, x, \cdot)$ constructed in Theorem 1 has at least a stationary distribution π, which can be obtained as a weak limit of a subsequence of the π_n's and satisfies

$$\int \pi(dx) h(x) \leqslant K/\eta, \tag{15}$$

where

$$h(x) = \sum_u \rho_u(x_u, \theta_u) k_u,$$

Theorem 4. Under the assumptions of Theorem 2, if

$$c_1 + M < 0, \tag{16}$$

then the conclusions of Theorem 3 hold for the process $P_n(t, x, \cdot)$ and the process $P(t, x, \cdot)$ constructed in Theorem 2.

Remark 3. If ρ_u is a compact function for each $u \in S$, then the conditions (7) and (16) imply the conditions (12) and (13) in the case that $h_u = \rho_u$, $u \in S$.

Remark 4. After the author completed the Theorems 1, 2, 3 and the subsequent Theorem 5, Li-Ping Huang has proved Theorem 4 under the following stronger conditions:

$$\sum_v b(u, v) = \sum_v b(v, u), \quad u \in S, \tag{17}$$

$$\sup_u \sum_v b(u, v) + c_1 < 0.$$

Here our Theorem 4 is an improvement to Huang's result.

Theorem 5. Under the assumptions of Theorem 1 (resp. Theorem 2) with $p(x) = \sum \rho_u(x) k_u$, if the coefficients (c_{uw}) given in (2) (resp. (8)) also satisfy

$$\sum_{u \in S} c_{uw} \leqslant -\eta < 0 \tag{18}$$

$$\sum_{u \in S} |c_{uw}| \leqslant K < \infty. \tag{19}$$

Then

1) the process $P(t, x, \cdot)$ constructed in Theorem 1 (resp. Theorem 2) has at most one s. d. π satisfying

$$\int_{E_0} \pi(dx) p(x) < \infty \tag{20}$$

If π is such a distribution, then

$$R_\alpha(P(t, x, \cdot), \pi) \leqslant K(\alpha, t) e^{-\eta t}, \quad x \in E_0, \ \alpha \in \mathscr{S}_0, \tag{21}$$

where $K(\alpha, x)$ is a constant independent of t;

2) for fixed an $n \geqslant 1$, if the coefficients $c_w(n, n)$ given in (2) (resp. (8)) vanish, then the process $P_n(t, x, \cdot)$ has at most one s. d. π_n satisfying

$$\int_{E^{\Lambda_n}} p_{\Lambda_n}(x) \pi_n(dx) < \infty. \tag{22}$$

If π_n is such a distribution, then

$$R_{\Lambda_n}(P_n(t, x, \cdot), \pi_n) \leqslant K_n(x) e^{-\eta t}, \quad x \in E^{\Lambda_n}, \tag{23}$$

where $K_n(x)$ is a constant independent of t.

As an application of the above results, we now discuss Schlögl model which comes from nonequilibrium statistical physics (see [8], [9] or [10]).

Take $E_u=\{0, 1, 2, \cdots\}=Z_+$, and ρ_u the Euclidean distance. For this model, the generator is (formally):

$$\Omega f(x)=\sum_u\left[\lambda_1 a_u\binom{x_u}{2}+\lambda_4 b_u\right](f(x+e_u)-f(x))$$
$$+\sum_u\left[\lambda_2\binom{x_u}{3}+\lambda_3 x_u\right](f(x-e_u)-f(x))$$
$$+\sum_{u,v}\lambda_5 x_u p(u, v)(f(x-e_u+e_v)-f(x)),$$

where e_u is the element in E whose value corresponding to u is one and other values are zero. the constants $\lambda_1,\cdots,\lambda_5$ and a_u, $b_u(u\in S)$ are positive, $(p(u, v))$ is a simple random walk on $S=Z^d$.

In the following, we will often allow S to be any countable set, and $(p(u, v)$ to be a general transition probability matrix on S. We also take $p(x)=\sum_u x_u k_u$.

Theorem 6. For the existence of a process corresponding to Schlögl model), it suffices that $\sup(a_u\vee b_u)<\infty$. Then the process has at least one s. d. Moreover, every $\pi\in\mathscr{P}$(= the set of all s. d.'s of the process) satisfies

$$\int\pi(dx)x_v=\gamma_v, \quad v\in S,$$

where

$$0\leqslant\gamma_v, \ v\in S; \quad \sum_v\gamma_v k_v=\|\gamma\|<\infty.$$

Therefore,

$$\int\pi(dx)p(x)\leqslant\|\gamma\|<\infty.$$

Furthermore, if we set

$$D=\sup\left\{\int\pi(dx)p(x):\ \pi\in\mathscr{P}\right\}<\infty,$$
$$d=\inf\left\{\int\pi(dx)p(x):\ \pi\in\mathscr{P}\right\}>0,$$

then there exist π_* and $\pi^*\in\mathscr{P}$ such that

$$\int\pi^*(dx)p(x)=D,$$
$$\int\pi_*(dx)p(x)=d.$$

Indeed, $\lim\limits_{t\to\infty}P(t, \theta, \cdot)=\pi_0\in\mathscr{P}$ is such a π_*. Finally

$$\mathscr{G}=\{(\lambda_1, \cdots, \lambda_5)\in R_+^5:\ \exists\pi\in\mathscr{P}\text{ such }\pi\neq\pi_0\}\neq R_+^5.$$

In view of the last assertion, what we need for the further study is to show that $\mathscr{G}\neq\phi$ which would give us the existence of phase transition for the Schlögl model.

§ 2. Proofs of Theorem 3 and Theorem 4

Our proofs base on a result due to Dobrushin [7; Theorem 1].

Theorem 7. Let $(E, \rho, \mathscr{E})$ be a complete separable metric space, $P(x, dy)$ be a transition probability function on $(E, \mathscr{E})$. Suppose that

1) $x \mapsto \int P(x, dy) f(y)$ is continuous function for each bounded Lipschitz continuous function f;

2) there exist a compact function $h < \infty$ on E and constants $c \in [0, 1)$, $C \in [0, \infty)$ such that

$$\int P(x, dy) h(y) \leqslant C + ch(x), \quad x \in E.$$

Then, for each $x_0 \in E$, there is a stationary distribution π_{x_0} satisfying

$$\int \pi_{x_0}(dx) h(x) \leqslant \max\{C + ch(x_0), \ C(1-c)^{-1}\}.$$

The following result, due to Basis [1; Proposition 1], is simple but very useful:

Lemma. Let $(E, \mathscr{E})$ be an arbitrary measurable space, $P(x, dy)$ be a trsnsition probability function on $(E, \mathscr{E})$. If there exist a finite nonnegative function g on E and constants $C \in [0, \infty)$, $c \in [0, 1)$, such that

$$\int P(x, dy) g(y) \leqslant C + cg(x), \quad x \in E.$$

Then

$$\int \pi(dx) g(x) \leqslant C(1-c)^{-1}$$

for every stationary distribution π of $P(x, dy)$.

Proofs of Theorem 3 and Theorem 4.

(a) Using the first assumption of Theorem 3 and [4; Lmema 1] we get

$$\int P_n(t, x, dy) h_n(y) \leqslant e^{Kt} - 1 + e^{-\eta t} h_n(x), \quad x \in E^{\Lambda_n}, \ n \geqslant 1. \tag{24}$$

Hence by the above lemma, for each $\pi_n \in \mathscr{P}_n$ = the set of all s. d.'s of $P_n(t, x, \cdot)$ and $t > 0$, we have

$$\int \pi_n(dx) h_n(x) \leqslant (e^{Kt} - 1)/(1 - e^{-\eta t}). \tag{25}$$

(b) Fix $t > 0$. Denote by $\mathscr{P}_n(t)$ the set of all s.d.'s of $P(x, dy) = P(t, x, dy)$. Because h_n is compact on E and the function $x \to \int P(x, dy) f(y)$ is continuous for every bounded function f on E^{Λ_n} that is Lipschitz continuons with respect to p_{Λ_n}, we see that Theorem 7 is available, and so $\mathscr{P}_n(t) \neq \phi$.

(c) Using the methods in [1], we can now claim that $\mathscr{P}_n(t)$ is compact in weakly convergence topology, $\mathscr{P}_n \neq \phi$ and (14) holds.

(d) By (12), (14), we have

$$\int \pi_n(dx) h_u(x) \leqslant \max\{K/\eta, \ \sup_w h_w(\theta_w) k_w\}/k_u, \quad u \in S.$$

Hence, by [7; Lemma 2], one sees that $\{\pi_n\}$ $(\pi_n \in \mathscr{P}_n)$ is relatively compact in the finite dimentional weakly convergence topology. We now choose a subsequence $\{\pi_{n_k}\}_{k=1}^{\infty}$ if necessary such that π_{n_k} converges to π as $k \to \infty$. Using (14) again, it follows that

$$\int \pi_n(dx) h_m(x) \leqslant K/\eta, \quad m < n$$

and so, by [7; Lemma 4], we have

$$\int \pi(dx) h_m(x) \leqslant K/\eta.$$

This gives us

$$\int \pi(dx) p(x) \leqslant \int \pi(dx) h(x) \leqslant K/\eta < \infty. \tag{26}$$

To conclude the proofs, we now need only to check that π is an s.d. of $P(t, x, \cdot)$. For this it suffices to show that

$$\int_{E_0} \pi(dx) f(x) = \int_{E_0} \pi(dx) \int_{E_0} P(t, x, dy) f(y) \tag{27}$$

for any $\alpha \in \mathscr{S}_0$ and any bounded function f on E_0, which depends on E^α and is Lipschitz continuous with respect to p_α.

In the following we fix such an f.

First, consider the process given in Theorem 1. Let "$A \overset{\varepsilon}{=} B$" denote that $|A-B| \leqslant \varepsilon$. Then for every $\varepsilon > 0$ and large enough N we have

$$\begin{aligned}
\int_{E_0} \pi(dx) P(t) f(x) &= \lim_{m\to\infty} \int_{E_0} \pi(dx) P_m(t) f(x^{\wedge m}) \\
&= \lim_{m\to\infty} \lim_{k\to\infty} \int_{E_0} \pi_{n_k}(dx) P_m(t) f(x^{\wedge m}) \overset{(\varepsilon)}{=} \lim_{m\to\infty} \lim_{k\to\infty} \int_{E_0^{(N)}} \pi_{n_k}(dx) P_m(t) f(x^{\wedge m}) \\
&= \lim_{k\to\infty} \int_{E_0^{(N)}} \pi_{n_k}(dx) P_{n_k}(t) f(x^{\wedge n_k}) = \lim_{k\to\infty} \int_{E_0^{(N)}} \pi_{n_k}(dx) P_{n_k}(t) f(x) \\
&\overset{(\varepsilon)}{=} \lim_{k\to\infty} \int_{E_0} \pi_{n_k}(dx) P_{n_k}(t) f(x) = \lim_{k\to\infty} \int_{E_0} \pi_{n_k}(dx) f(x) = \int \pi(dx) f(x).
\end{aligned}$$

The first equality comes from (26) and [4; (13)]; the second and the last equalities comes from the convergence of $\pi_{n_k} \to \pi$; the ε-equalities due to (12), (14) and [4; (14)]; finally, the sixth and eighth equlities due to the property that π_{n_k} is a stationary distribution of $P_{n_k}(t, x, \cdot)$.

The proof of Theorem 3 is now finished.

Next, consider the process given by Theorem 2.

Without loss of the generality, we may and will assume that $\pi_n \to \pi$. Clearly, we have

$$\begin{aligned}
&\left| \int \pi_n(dx) P_n(t) f(x) - \int \pi(dx) P(t) f(x) \right| \\
&\leqslant \left| \int \pi_n(dx) P_n(t) f(x^{\wedge m}) - \int \pi_n(dx) P_n(t) f(x) \right| \\
&\quad + \left| \int \pi_n(dx) P_n(t) f(x^{\wedge m}) - \int \pi(dx) P_n(t) f(x^{\wedge m}) \right| \\
&\quad + \left| \int \pi(dx) P_n(t) f(x^{\wedge m}) - \int \pi(dx) P(t) f(x^{\wedge m}) \right| \\
&\quad + \left| \int \pi(dx) P(t) f(x^{\wedge m}) - \int \pi(dx) P(t) f(x) \right| = \mathrm{I}_n^m + \mathrm{II}_n^m + \mathrm{III}_n^m + \mathrm{IV}^m
\end{aligned}$$

By (10), (11) and the dominated convergence theorem, we have

$$\lim_{n\to\infty} \mathrm{III}_n^m = 0, \quad m \geqslant 1,$$
$$\lim_{m\to\infty} \mathrm{IV}^m = 0.$$

By the last paragraph but one of [4], (11), (26) and the dominated convergence theorem, we also have

$$\lim_{n\to\infty} \mathrm{II}_n^m = 0, \quad m \geqslant 1.$$

Since $\{P_n(t) f(x^{\wedge_m}) : n \geqslant 1\}$ is a uniformly bounded functions which are equicontinuous at each point of $E^{\wedge_m}$, and $\pi_n \to \pi$, we obtain

$$\lim_{n\to\infty} \mathrm{II}_n^m = 0 \quad \wedge_m \supset \alpha.$$

Finally, by [4; Lemma 1] and the condition (7), we obtain the following more precise estimate than those given in [4; Lemma 4]:

$$\int P_n(t, x, dy) \rho_v(y) \leqslant \sum_{u\in\wedge_n} \rho_u(x) e^{c_1 t} \sum_{l=0}^{\infty} \frac{t^l}{l!} b_n^{(l)}(u, v) + \sum_{u\in\wedge_n} \beta_u \int_0^t e^{c_1 s} \sum_{l=0}^{\infty} \frac{s^l}{l!} b_n^{(l)}(u, v) ds \tag{28}$$

and so

$$\begin{aligned}
&\int \pi_n(dx) (p_{\wedge_n \setminus \wedge_m}(x) \wedge N) \\
&= \lim_{t\to\infty} \int \pi_n(dx) \int P_n(t, x, dy) (p_{\wedge_n\setminus\wedge_m}(y) \wedge N) \\
&\leqslant \int \pi_n(dx) \overline{\lim_{t\to\infty}} \int P_n(t, x, dy) (p_{\wedge_n\setminus\wedge_m}(y) \wedge N) \\
&\leqslant \int \pi_n(dx) \overline{\lim_{t\to\infty}} \int P_n(t, x, dy) p_{\wedge_n\setminus\wedge_m}(y) \\
&\leqslant \sum_{u\in\wedge_n} \beta_u \int_0^\infty e^{c_1 s} \sum_{l=0}^{\infty} \frac{s^l}{l!} \sum_{v\in\wedge_n\setminus\wedge_m} b_n^{(l)}(u, v) k_v ds
\end{aligned}$$

Letting $N \uparrow \infty$, we get

$$\int \pi_n(dx) p_{\wedge_n\setminus\wedge_m}(x) \leqslant \sum_{u\in\wedge_n} \beta_u \int_0^\infty e^{c_1 t} \sum_{l=0}^{\infty} \frac{t^l}{l!} \sum_{v\in\wedge_n\setminus\wedge_m} b_n^{(l)}(u, v) k_v dt$$

the right hand side is dominated by

$$\sum_u \beta_u \int_0^\infty e^{c_1 t} \sum_{l=0}^{\infty} \frac{t^l}{l!} \sum_{v\in\wedge_n\setminus\wedge_m} b_n^{(l)}(u, v) k_v dt \leqslant -\sum \beta_u k_u / (c_1 + M) < \infty,$$

hence, we have

$$\lim_{m\to\infty} \overline{\lim_{n\to\infty}} \int \pi_n(dx) p_{\wedge_n\setminus\wedge_m}(x) \leqslant \lim_{m\to\infty} \sum_u \beta_u \int_0^\infty e^{c_1 t} \sum_{l=0}^{\infty} \frac{t^l}{l!} \sum_{v\notin\wedge_m} b^{(l)}(u, v) k_v dt = 0.$$

This shows that

$$\begin{aligned}
\lim_{m\to\infty} \overline{\lim_{n\to\infty}} \mathrm{I}_n^m &= \lim_{m\to\infty} \overline{\lim_{n\to\infty}} \left| \int \pi_n(dx) (f(x^{\wedge_m}) - f(x)) \right| \\
&\leqslant L(f) \lim_{m\to\infty} \overline{\lim_{n\to\infty}} \int \pi_n(dx) p_{\wedge_n\setminus\wedge_m}(x) = 0.
\end{aligned}$$

The proof of Theorem 4 is now completed.

§ 3. Proof of Theorem 5

By the condition (19) and the main estimate [4; (12)] (resp. [4; Lemma 5]), using the notations in Theorem 1, we obtain

$$R_w(P(t, x_1, \cdot), P(t, x_2, \cdot)) \leqslant \overline{\lim_{n\to\infty}} R_w(P_n(t, x_1, \cdot), P_n(t, x_2, \cdot))$$
$$\leqslant \overline{\lim_{n\to\infty}} (e^{B_n^* t} p.(x_1, x_2))(w) = [e^{B^* t} p.(x_1, x_2)](w), \quad w \in S, \ x_1, x_2 \in E_0.$$

Set

$$e^{B_n t} = (c_{uw}^{(n)}(t): u, w \in \wedge_n),$$
$$e^{Bt} = (c_{uw}(t): u, w \in \wedge_n).$$

From (18), it follows that

$$\sum_{u\in S} c_{uw}(t) \leqslant e^{-\eta t}, \quad w \in S \tag{30}$$

Note that, for $n \geqslant 1$ and $\alpha \subset \wedge_n \in \mathscr{S}_0$,

$$R_\alpha(P_n(t, x_l, \cdot), P_n(t, x, \cdot)) \to 0$$
$$\text{as } p(x_l, x) \to 0.$$

By triangle inequality, it follows that

$$y \to R_\alpha(P_n(t, x, \cdot), P_n(t, y, \cdot))$$

is continuous with respect to p, and so is $\mathscr{E}_0$-measurable. Thus,

$$y \to R_\alpha(P(t, x, \cdot), P(t, y, \cdot)) = \lim_{n\to\infty} R_\alpha(P_n(t, x, \cdot), P_n(t, y, \cdot))$$

is also $\mathscr{E}_0$-measurable. On the other hand, since our state space is complete and separable, R_α is indeed attainable. That is, there is a coupling measure $\widetilde{P}^{t,x,y}(dx_1, dx_2)$ of $P(t, x, \cdot)$ and $P(t, y, \cdot)$ such that

$$R_\alpha(P(t, x, \cdot), P(t, y, \cdot)) = \int P(x_1, x_2) \widetilde{P}^{t,x,y}(dx_1, dx_2).$$

Clearly, $\int \pi(dy) \widetilde{P}^{t,x,y}(dx_1, dx_2)$ is a coupling of $P(t, x, \cdot)$ and $\int \pi(dy) P(t, y, \cdot)$. From this fact and using (29) and (30), we finally get

$$R_\alpha(P(t, x, \cdot), \pi) = R_\alpha\left(P(t, x, \cdot), \int \pi(dy) P(t, y, \cdot)\right)$$
$$\leqslant \int_{E_0} \pi(dy) R_\alpha(P(t, x, \cdot), P(t, y, \cdot))$$
$$\leqslant \sum_{w\in\alpha} \sum_{u\in S} c_{uw}(t) \left[p_u(x) + \int_{E_0} p_u(y) \pi(dy) \right]$$
$$\leqslant \sum_{w\in\alpha} \sum_{u\in S} c_{uw}(t) \left[p(x) + \int_{E_0} P(y) \pi(dy) \right] \leqslant K(\alpha, x) e^{-\eta t}.$$

This proves the first conclusion of Theorem 5. The second one can be proved in the same way.

§ 4. Proof of Theorem 6

(a) The existence of the process corresponding to Schlögl model was proved in

[2]. One can also check that the assumptions of Theorem 1 are satisfied in the case of $(p(u, v))$ having finite range and the assumptions of Theorem 2 are satisfied in the general case. The following remarks may be helpful for the readers: First, the condition (1) (resp. (7)) guarantees the reguraluty of Ω_n, and hence the reguraluty of any coupling operator $\Omega_{n,\omega}$ of Ω_n and Ω_ω (see [3] for the details). Second, we may use the coupling:

$$\begin{aligned}\tilde{\Omega}_{n,\omega} f(x, y) = &\sum_{u\in\Lambda_n}\sum_{k=\pm1}\{(q_u(x_u, x_u+k)-q_u(y_u, x_u+k))^+\\ &\cdot[f(x+ke_u, y)-f(x, y)]+(q_u(y_u, y_u+k)-q_u(x_u, x_u+k))^+\\ &\cdot[f(x, y+ke_u)-f(x, y)]+q_u(x_u, x_u+k)\\ &\wedge q_u(y_u, y_u+k)[f(x+ke_u, y+ke_u)-f(x, y)]\}\\ &+\sum_{u\in\Lambda_m\setminus\Lambda_n}\sum_{k=\pm1} q_u(y_u, y_u+k)[f(x, y+ke_u)-f(x, y)]\\ &+\lambda_5\sum_{u,v\in\Lambda_n} p(u, v)\{(x_u-y_u)^+[f(x-e_u+e_v, y)-f(x, y)]\\ &+(y_u-x_u)^+[f(x, y-e_u+e_v)-f(x, y)]\\ &+x_u\wedge y_u(f(x-e_u+e_v)-f(x, y)]\}\\ &+\lambda_5[\sum_{u\in\Lambda_m\setminus\Lambda_n} y_u\sum_{v\in\Lambda_n} p(u, v)+\sum_{u\in\Lambda_n} y_u\sum_{v\in\Lambda_m\setminus\Lambda_n} p(u, v)]\\ &\cdot[f(x, y-e_u+e_v)-f(x, y)],\quad m\geqslant n\geqslant 1,\end{aligned} \tag{31}$$

where

$$q_u(i, j)=\begin{cases}\lambda_1 a_u\binom{i}{2}+\lambda_4 b_u, & j=i+1, i\geqslant 0,\\ \lambda_2\binom{i}{3}+\lambda_3 i, & j=i-1,\ i\geqslant 1,\\ 0, & \text{otherwise}.\end{cases}$$

(b) We now prove that Theorem 3 (resp. Theorem 4) with $h_u=\rho_u(u\in S)$ works for Schlögl model in the finite range case (resp. in the general case). To this end, take $\eta>0$ and set

$$d=\sup\left\{(\lambda_5 M+\eta-\lambda_3)i+\lambda_1\binom{i}{2}a_u-\lambda_2\binom{i}{3}:\quad i\geqslant 0,\ u\in S\right\},$$
$$K=\lambda_4\sum_u b_u k_u+d\sum_u k_u.$$

Then

$$0\leqslant d<\infty,\ 0<K<\infty$$

and it is easy to check that the condition (13) holds. For Theorem 4, the proof is similar.

(c) As we have proved in (b), there exists $c_1\in R$ such that $c_1+M<0$ and (28) holds. Thus, by (10), we have

$$\int P(t, x, dy)y_v\leqslant\sum_u x_u e^{c_1 t}\sum_{l=0}^{\infty}\frac{t^l}{l!}b^{(l)}(u, v)+\sum_u\beta_u\int_0^t e^{c_1 s}\sum_{l=0}^{\infty}\frac{s^l}{l!}b^{(l)}(u, v)ds.$$

Now, the same argument used in the last part of the proof of Theorem 4 shows that

$$\int \pi(dx)x_v \leqslant \sum_u \beta_u \int_0^\infty e^{c_1 t} \sum_{l=0}^\infty \frac{t^l}{l!} b^{(l)}(u, v)dt = \gamma_v$$

for each $\pi \in \mathscr{P}$. Also,

$$\|\gamma\| = \sum \gamma_u k_u \leqslant -\|\beta\|/(c_1+M) < \infty.$$

(d) We have known that

$$\int \pi(dx)p(x) \leqslant \|\gamma\|$$

for each $\pi \in \mathscr{P}$. Now, let $\{\pi_n\} \in \mathscr{P}$ be a sequence so that

$$\int \pi_n(dx)p(x) \to D.$$

We may choose a subsequence $\{\pi_{n_k}\}$ if necessary such that $\pi_{n_k} \to \pi^*$. We now prove that $\pi^* \in \mathscr{P}$. Obviously,

$$D = \varliminf_{k\to\infty} \int \pi_{n_k}(dx)p(x) \geqslant \int \pi^*(dx)p(x).$$

Therefore, for each $f \in \mathscr{E}^\alpha \cap \mathscr{L}_b(\alpha \in \mathscr{S}_0)$, we have

$$\begin{aligned}
&\left|\int \pi^*(dx)f(x) - \int \pi^*(dx)P(t)f(x)\right| \\
&= \lim_{k\to\infty}\left|\int \pi_{n_k}(dx)P(t)f(x) - \int \pi^*(dx)P(t)f(x)\right| \\
&\leqslant \varlimsup_{m\to\infty} \varlimsup_{k\to\infty}\left|\int \pi_{n_k}(dx)P(t)f(x^{\wedge m}) - \int \pi^*(dx)P(t)f(x^{\wedge m})\right| \\
&\quad + \varlimsup_{m\to\infty} \varliminf_{k\to\infty}\left|\int \pi_{n_k}(dx)P(t)f(x^{\wedge m}) - \int \pi_{n_k}(dx)P(t)f(x)\right| \\
&\quad + \varlimsup_{m\to\infty}\left|\int \pi^*(dx)P(t)f(x^{\wedge m}) - \int \pi^*(dx)P(t)f(x)\right| \\
&\leqslant L(P(t)f) \varlimsup_{m\to\infty} \varliminf_{k\to\infty} \int \pi_{n_k}(dx)p_{S\setminus\wedge_m}(x) = 0.
\end{aligned}$$

In the last step we have used the same approach used in the last step of the proof of theorem 4, since we now have (32).

Similarly, one can get a $\pi_* \in \mathscr{P}$ such that

$$\int \pi_*(dx)p(x) = d.$$

(e) Since

$$\int P(t, \theta, dy)p(y) \leqslant \|\gamma\| < \infty, \quad t \geqslant 0,$$

one sees that $\{P(t, \theta, \cdot): t \geqslant 0\}$ is relatively compact in the finite dimensional weakly convergence topology. On the other hand, Schlögl model is monotone, by the standard coupling argument, we get $P(t, \theta, \cdot) \to$ a limit $\pi_0 \in \mathscr{P}$. Using the coupling argument again, it follows that

$$\int \pi_0(dx)f(x) = \int \pi_*(dx)f(x)$$

for each monotone function f, and so

$$\int \pi_0(dx)p(x)=d.$$

(f) For Theorem 5, the following condition is sufficient

$$K'_3+\lambda_5(M-1)<0,$$

where the constant K_3^1 is computed in [2]:

$$K'_3=\sup_{\substack{j_2>j_1\geqslant 0\\ u\in S}}\left\{(\lambda_1 a_u+\lambda_2)(j_1+j_2)-\left(\frac{\lambda_1 a_u}{2}+\frac{\lambda_2}{3}+\lambda_3\right)-\frac{1}{6}\lambda_2(j_1^2+j_1j_2+j_2^2)\right\}.$$

From this we see that $\mathscr{G}\neq \mathbf{R}_+^5$. An important special case is that $(P(u, v))$ is the simplest random walk on Z^d, $a_u\equiv 1$ and $\lambda_5=1$. In which the condition

$$\frac{\lambda_1}{\lambda_2}+1<\frac{2\lambda_3}{\lambda_1}$$

will guarantee the uniqueness of s.d. of the process with suitably chosen $M>1$ and (R_u).

The proof of Theorem 6 is now completed.

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