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STABILITY OF CIRCULATION DECOMPOSITIONS AND SELF-ORGANIZATION PHENOMENA*

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Dedicated to Professor Lee Kwok-ping (Li Guoping) On the Occasion of his 50th Year of Educational and Scientific Work

The circulation decomposition for a Markov chain with discrete parameteir was introduced in 1979 by Qian Min-ping[1]. However, from physical and mathematical point of view, the analogue for a Markov chain with continuous parameter may be more interesting. It is certainly true that we have a circulation decomposition for such chain at each time t. Theorefore, the point we are really interesting is whether the circulation decompositions change when the time varies, and if so, does there exist any stable property?

Look at the following example. Let $E=\{1,2,3\}$ be a state space, and let

$$Q=\begin{pmatrix}-1 & 1 & 0\\ 0 & -1 & 1\\ 1 & 0 & -1\end{pmatrix}$$

be a Q - matrix. The corresponding Q - process is $P(t)=(P_{ij}(t):i,j\in E)$:

$$p_{11}(t)=p_{22}(t)=p_{33}(t)=1/3+(2/3)e^{-3t/2}\cos(\sqrt{3}t/2);$$

$$p_{12}(t)=p_{23}(t)=p_{31}(t)=1/3+(2/3)e^{-3t/2}\cos\left(\frac{\sqrt{3}}{2}t-\frac{2}{3}\pi\right);$$

$$p_{13}(t)=p_{32}(t)=p_{21}(t)=1/3+(2/3)e^{-3t/2}\cos\left(\frac{\sqrt{3}}{2}t+\frac{2}{3}\pi\right);$$

Its stationary distribution is $\pi_1=\pi_2=\pi_3=1/3$. Since

$$\pi_1q_{12}-\pi_2q_{21}=\pi_2q_{23}-\pi_3q_{32}=\pi_2q_{31}-\pi_1q_{13}=1/3,$$

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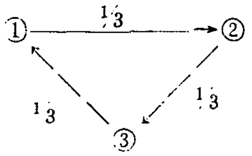
$Q=(q_{ij})$ has a circulation as follows:



For each t, we have

$$\pi_1 p_{12}(t)-\pi_2 p_{21}(t)=\pi_2 p_{23}(t)-\pi_3 p_{32}(t)=\pi_3 p_{31}(t)-\pi_1 p_{13}(t)$$
$$\equiv r(t)=(2\sqrt{3}/9)e^{-3t/2}\sin(\sqrt{3}\,t/2)$$

when $r(t_0)\neq 0$ we also have the same circulation at time t_0 as above. The flow of the circulation is $r(t)$, but the direction of the circulation may not be the same. Notice that $r(2\pi/\sqrt{3})=0$, hence there is no circulation at time $t=2\pi/\sqrt{3}$.

In view of the above example the circulation decompositions may not be the same for different times. It can be happened that there exist infinite circulations for some $t=t_1$, but there is no circulation for another $t=t_2$. However, we will prove, under some hypotheses, that the circulation decompositions are almost stable (Theorem(24)). This result posseses obviously some practical meaning. In particular, it shows that the self-organization phenomenon exists widely, for which we will discuss in §3.

In this paper, we will also study the reversibility for q-processes, the relation between the reversibility for a q-pair and an entropy production, and the distribution of detailed balance points and non-detailed balance points on $[0,\infty]$.

§ 1. Reversibility and Entropy Production

Let $X=(X_t:t\geqslant 0)$ be a Markov process with initial distribution μ and Markovian transition function $p(t,x,A)$ $(t\geqslant 0,\ x\in E,\ A\in\mathscr{E})$ [2; Definition (2.2)]. Denote the absolute distribution of the process at time t by

(1) $$p(t,A)=\int\mu(dx)p(t,x,A),\quad t\geqslant 0,\ A\in\mathscr{E}$$

By the proof of [2; Theorem 2.5], we get

(2) **Proposition** $X=(X_t)$ is reversible iff

(3) $$\int_A p(s,dx)\ p(t,x,B)=\int_B p(s,dx)\ p(t,x,A),\ \forall s,t\geqslant 0,\ \forall A,B\in\mathscr{E}$$

(4) **Proposition** For any sub-Markovian transition function, eq. (3) implies that $p(t,x,A)$ is reversible with respect to μ [2; Definition(3.3)]. If $p(t,x,A)$ is Markovian, then eq. (3) is equivalent to

the reversibility for $p(t,x,A)$.

Proof The first assertion is obvious. The second assertion folows from

$$\int_A p(s,dx)p(t,x,B) = \int_A\left(\int\mu(dy)p(s,y,dx)\right)p(t,x,B)$$
$$= \int_A \mu(dx)p(s,x,E)p(t,x,B) = \int_A \mu(dx)p(t,x,B)$$

(5) **Definition** $p(t,x,A)$ is called strong reversible (with respect to μ) if eq. (3) holds.

If $p(t,x,A)$ is a (totally stable) q-process, then, by the proof of [2; proposition (3.1)], it follows that

(6) $\int_A p(s,dx)q(x,B) = \int_B p(s,dx)q(x,A), \quad \forall s\geqslant 0 \ \forall A, \ B\in\mathscr{E}$

we then call the q-pair $q(x)$-$q(x,A)$ strong reversible (with respect to μ).

Set

$$J^{\mu\ p}(t,A,B) = \int_A p(t,dx)q(x,B) - \int_B p(t,dx)q(x,A),$$

$$K^{\mu\ p}(t,A,B) = \log\left[\int_A p(t,dx)q(x,B) \Big/ \int_B p(t,dx)q(x,A)\right],$$

$$G(\mu,p) = \bigvee_{t\geqslant 0} \ \bigvee_{A,B\in} J^{\mu\ p}(t,A,B)K^{\mu\ p}(t,A,B).$$

we call $G(\mu,p)$ an entropy production for $p(t,x,A)$ with respect to μ.

The following conclusions are obvious.

(7) **Proposition**

(i) $G(\mu,p)\geqslant 0$;

(ii) $G(\mu,p)=0$ iff the q-pair $q(x)-q(x,A)$ is strong reversible with respect to μ.

(8) **Remark** For discrete state space, the entropy production is usually defined by

$$\sum_{i,j\in E}\left(p_i(t)q_{ij} - p_j(t)q_{ji}\right)\log\left(p_i(t)q_{ij}/p_j(t)q_{ji}\right),$$

This is not available for the general state space, so we use the new definition of the entropy production, but they coinide in the stron reversible case.

(9) **Theorem** $G(\mu,p^{\min})=0$ iff $p^{\min}(t,x,A)$ is strong reversible with respect to μ.

Proof Use the proof of [2;Theorem(5.3)], (6)and proposition(7).

(10)**Theorem** Let $p(t,x,A)$be a conservative q-process satisfying the forward Kolmogorov equation

(F) $\qquad p\ \ ,\ \ ,A)\left(\equiv\int_0^\infty e^{-\lambda t}p(t,x,A)dt\right)$

$$=\int p(\lambda,x,dy)\int_A\frac{q(y,dz)}{\lambda+q(z)}+\frac{\delta(x,A)}{\lambda+q(x)},$$

and let $G(\mu,p)=0$. Then the q-process is unique in μ-equivalent sense,*) and $p(t,x,A)$ is stationary with stationary distribution μ.

Proof Denote the Laplace transform of $p(t,A)$ by $p(\lambda,A)$. We have

$$\begin{aligned}p(\lambda,A)&=\int_0^\infty e^{-\lambda t}\int p(t,x,A)\mu(dx)dt\\&=\int p(\lambda,x,A)\mu(dx)\\&=\int\Big[\int p(\lambda,x,dy)\int_A\frac{q(y,dz)}{\lambda+q(z)}+\frac{\delta(x,A)}{\lambda+q(x)}\Big]\mu(dx)\\&=\int_A\frac{\mu(dx)}{\lambda+q(x)}+\iint\mu(dx)p(\lambda,x,dy)\int_A\frac{q(y,dz)}{\lambda+q(z)}\\&=\int_A\frac{\mu(dx)}{\lambda+q(x)}+\int p(\lambda,dy)\int_A\frac{q(y,dz)}{\lambda+q(z)}\\&=\int_A\frac{\mu(dx)}{\lambda+q(x)}+\int p(\lambda,dy)q\Big(I_A/(\lambda+q(\cdot))\Big)(y),\end{aligned}$$

where

$$(q\ g)(y)\int g(z)\ q(y,dz).$$

Next, by (7.ii), we have

$$\int_A p(\lambda,dx)q(x,B)=\int_B p(\lambda,dx)q(x,A),\ \forall\lambda>0,\ \forall A,B\in\mathscr{A}$$

From this and the monotone class theorem, we get

$$\int f(x)p(\lambda,dx)(qg)(x)=\int g(x)p(\lambda,dx)(qf)(x),\ \forall f,g\in\mathscr{E}_+$$

hence

$$\begin{aligned}p(\lambda,A)&=\int_A\frac{\mu(dx)}{\lambda+q(x)}+\int_A\frac{q(x,E)}{\lambda+q(x)}p(\lambda,dx)\\&=\int_A\frac{\mu(dx)}{\lambda+q(x)}+p(\lambda,A)-\int_A\frac{\lambda}{\lambda+q(x)}p(\lambda,dx),\end{aligned}$$

i.e.

$$\int_A\frac{\mu(dx)}{\lambda+q(x)}=\int_A\frac{\lambda}{\lambda+q(x)}p(\lambda,dx),$$

and so

$$\mu(A)=\lambda p(\lambda,A),\ \forall\lambda>0,\ \forall A\in\mathscr{E}$$

On the other hand

$$\begin{aligned}&|p(t,A)-p(s,A)|\\&\leqslant\int|p(t,x,A)-p(s,x,A)|\mu(dx)\end{aligned}$$

*) See (2; Definition (6.1)). For the measurability of the set given there, we need a little condition on $(E;\mathscr{E})$.

so $p(\cdot, A)$ is continuous. By the uniqueness theorem for Laplace transform, we get

$$\mu(A)=p(t,A), \qquad \forall t\geqslant 0, \ \forall A\in\mathscr{E}.$$

In particular,

$$\int p(t,x,E)\mu(dx)=p(x,E)=\mu(E)=1$$

therefore

$$p(t,\cdot,E)=1, \qquad \mu\text{—}a.e.,$$

(11) **Theorem** For a given conservative q-pair $q(x)-q(x,A)$, there exists a strong reversible q-process or there exists a q-process with $G(\mu,p)=0$ iff the q-pair is reversible (with respect to μ).

Proof The necessity is obvious. We prove the sufficiency. Since the q-pair is conservative, we can always construct an honest reverssible q-process [2; Theorem (6.3)]. By proposition (4), the reversibility and the strong reversibility are the same, so there exists at least one strong reversible q-process. Finally, $G(\mu,p)=0$ follows from proposition (7.ii).

(12) **Proposition** Let μ be a finite measure on $\mathscr{E}$. Set

$$\nu(dx)=q(x)\mu(dx)I_{[q(\cdot)>0]}(x)+I_{[q(\cdot)=0]}(x)\mu(dx)$$

then

$$\nu[x:q(x)=0]=\mu[x:q(x)=0]$$

$$\int_{[q(\cdot)>0]}\frac{\nu(dx)}{q(x)}=\mu[x:q(x)<0]<\infty$$

Moreover, $q(x)-q(x,A)$ is potential with respect to μ iff

$$\pi(x,A)=\frac{q(x,A)}{q(x)}I_{[q>0]}(x)+\delta(x,A)I_{[q=0]}(x)$$

is potential with respect to ν (Recall that the potentiality is a extension to the reversibility, i.e. the measure μ may not be an probability measure, but shold be a σ-finite non-negative measure).

Proof The first assersion is obvious. The second one follows from the following fact:

$$\int_A \nu(dx)\pi(x,B)$$

$$=\int_{A[q>0]}q(x)\mu(dx)\frac{q(x,B)}{q(x)}+\int_{A[q=0]}\pi(x,B)\mu(dx)$$

$$=\int_A \mu(dx)q(x,B)+\mu\Big[A\cap B\cap[q=0]\Big].$$

§ 2. Stability of Circulation Decompositions

(13) **Definition** $R=(r_{ij})_{\leqslant i,j\leqslant n}$ is called a circulation matrix, if there exist some distinct states $i_1, i_2, \cdots, i_k$ ($k\geqslant 3$) and a constant

$a>0$ such that

$$r_{ij}=\begin{cases} a, & i=i_l,\ j=i_{l+1};\ l=1,2,\cdots,k, \quad i_{k+1}=i_1) \\ 0, & \text{otherwise} \end{cases}$$

then, $(i_1,\cdots,i_k,i_1)$ is called the cycle of R with the direction $i_1\to i_2\to\cdots\to i_k\to i_1$. and a is called the flow of R.

(14) **Definition** we call

$$c(t,A,B)=\left[\int_A \mu(dx)p(t,x,B)-\int_B \mu(dx)p(t,x,A)\right]^+$$

a flow at time t from A to B.

(15) **Definition** Let $p(t,x,A)$ be an honest Markov process with stationary distribution μ. If

$$\int_A \mu(dx)p(t,x,B)=\int_B \mu(dx)p(t,x,A),\ \forall A,B\in\mathscr{E},$$

we say that t is a detailed balance point. We abbreviate it by DBP, and we use the abbreviation NDBP for a point which is not a DBP.

(16) **Proposition***) Let $p(t,x,A)$ be honest Markov process with stationary distribution μ. Then

(i) either there is only one DBP (i.e. $t=0$) or there are infinite ones;

(ii) either there is no NDBP, or the set of all NDBP is an open set with positive Lebesgue measure.

(iii) the set of all DBP is not dense in $[0,\infty)$ whenever $p(t,x,A)$ is irreversible.

(iv) if the q-pair $q(x)$-$q(x,A)$ is irreversible with respect to μ, then, for each q-process $p(t,x,A)$, there exists a $\delta>0$ such that each $t\in(0,\delta)$ is a NDBP for $p(t,x,A)$.

Proof (i) Since $p(0,x,A)=\delta(x,A)$, we know that $t=0$ is a DBP. If $t>0$ is also a DBP, then so are $nt(n\geqslant 1)$ by Kolmogorov-Chapman equation.

(ii) If $p(t,x,A)$ is reversible, there is no NDBP. Otherwise, there $t_0>0$ and $A,B\in\mathscr{E}$ such that $C(t_0,A,B)>0$. On the other hand, for the fixed A and B, $C(t,A,B)$ is continuous in t, hence $C(\cdot,A,B)>0$ in a neighbourhood of t_0. This prove our assertions.

(iii) If the assertion was not true, then $p(t,x,A)$ would be reversible since $t\to p(t,x,A)$ is continuous.

(iv) If there is sequence $\{t_n\}$ of DBP such that $t_n\searrow 0$, then the proof of [2; proposition (3.1)] could show that the q-pair would be reversible with respect to μ.

(17) **Hypotheses** From now on, we will suppose that the given

*) We can say more in special case, see(20).

q-pair $q(x)$-$q(x,A)$ is conservative and bounded, the unique q-process $p(t,x,A)$ is stationary with stationary distribution μ.

We will fix $\{A_j\}^\infty$, to be a measurable partition of $\mathscr{E}$.

Endowing the supremum norm on $b\mathscr{E}$ (the set of all bounded $\mathscr{E}$-measurable functions), we get a normed space which is also denoted by $b\mathscr{E}$. Define

$$Qf(x)=\int\Big(q(x,dy)-\delta(x,dy)q(y)\Big)f(y),$$

$$\hat{Q}=Q/\|q\|+I$$

where I is the identity operator on $b\mathscr{E}$. Then we have

(18) **Lemma.** Q and $\hat{Q}$ are bounded linear operators on $b\mathscr{E}$ to itself,

$$\|Q\|\leqslant 2\|q\|,\qquad \|\hat{Q}\|\leqslant 3$$

$\hat{Q}$ is also a positive operator. If $0\leqslant f_m\nearrow f$, f_m, $f\in b\mathscr{E}$, then

$$Q^n f_m(x)\longrightarrow Q^n f(x)\qquad (m\to\infty)$$

for each $x\in E$ and each $n\geqslant 1$.

Proof We only prove the last assertion here. When $n=1$, it is clear that

$$Qf_m(x)=\int q(x,dy)f_m(y)-q(x)f_m(x)$$

$$\xrightarrow[m\to\infty]{}\int q(x,dy)f(y)-q(x)f(x)=Qf(x).$$

Assume that the assertion is true for n, since $\|Q^n f_m\|\leqslant 2^n\|q\|^n\cdot\|f_m\|$ $2^n\|q\|^n\cdot\|f\|$,

$$Q^{n+1}f_m(x)=\int q(x,dy)(Q^n f_m(y))-q(x)Q^n f_m(x),$$

we get $Q^{n+1}f_m(x)\xrightarrow[m\to\infty]{}Q^{n+1}f(x)$ ($\forall\lambda\in E$) by dominated convergence theorem.

(19) **Lemma** For fixed A and B, as a function of t,

$$\int_A\mu(dx)p(t,x,B)$$

can be extended to be an analytic function.

Proof The assertion follows from the following facts:

$$\int_A\mu(dx)p(t,x,B)$$
$$=\int_A\mu(dx)e^{tQ}I_B(x)$$
$$=\sum_{n=0}^{\infty}\frac{t^n}{n!}\int_A(Q^nI_B)(x)\mu(dx);$$

$$\left|\int_A(Q^nI_B)(x)\mu(dx)\right|\leqslant(2\|q\|)^n\mu(AB)\leqslant(\|2q\|)^n.$$

(20) **Corollary** Under hypotheses (17), if $p(t,x,A)$ is irreversible with respect to μ, then each point in $(0,\infty)$ is a NDBP except an isolated set.

Proof From [2; Theorem (5.3)] we know that there exists at $t\in(0,\infty)$ and $A_0,B_0\in\mathscr{E}$ such that

$$\int_{B_0}\mu(dx)p(t,x,B_0)\neq\int_{B_0}\mu(dx)p(t,x,A_0).$$

Putting

$$\Lambda=\left\{t:\ \int_A\mu(dx)p(t,x,B)=\int_B\mu(dx)p(t,x,A),\ \forall A,B\in\mathscr{E}\right\},$$

then, by proposition (16), we see that either $\Lambda=\phi$ or Λ is an infinite set. In the second case, notice that

$$\Lambda\subset\left\{t:\ \int_{A_0}\mu(dx)p(t,x,B_0)=\int_{B_0}\mu(dx)p(t,x,A_0)\right\},$$

the assertion follows from Lemma (19) and the following fact: Assume that Λ is an infinite set containing a convergent sequence, f and g are two analytic functions. If f and g coincide on Λ, then $f=g$ everywhere.

(21) **Lemma** Under Hypotheses (17), we have

$$\sum_j Q^n I_{A_j}(x)=0,$$

$$\sum_j\left|\hat{Q}^n I_{A_j}(x)\right|\leqslant 3^n$$

for each $x\in E$ and $n\geqslant1$.

Proof From Lemma (18), we know that $\hat{Q}^n$ and $Q^n\,(n\geqslant1)$ are bounded operators, $\hat{Q}^n(n\geqslant1)$ are positive and $\|\hat{Q}\|\leqslant3$, therefore

$$\sum_j Q^n I_{A_j}(x)=\Big(Q^n\sum_j I_{A_j}\Big)(x)=(Q^n1)(x)$$

$$=Q^{n-1}(q(\cdot,E)-q(\cdot))(x)=0,$$

$$\sum_j\left|\hat{Q}^n I_{A_j}(x)\right|=\sum_j\hat{Q}^n I_{A_j}(x)=(\hat{Q}^n1)(x)$$

$$\leqslant\|\hat{Q}\|^n\leqslant3^n.$$

(22) **Lemma** Under Hypotheses (17), we have

$$\int\mu(dx)Q^nf(x)=0$$

for each $f\in b\mathscr{E}$ and each $n\geqslant1$.

Proof Since $p(t,x,A)$ is stationary, we get

$$0=\int\mu(dx)\frac{p(t,x,A)-\delta(x,A)}{t}$$

$$=\int_{A^c}\mu(dx)p(t,x,A)/t-\int_A\mu(dx)[1-p(t,x,\{x\})]/t$$
$$+\int_A\mu(dx)p(t,x,A\setminus\{x\})/t$$

Using the fact that $\forall x\notin B\in\mathscr{E}$,

$$0\leqslant\frac{p(t,x,B)}{t}\leqslant\frac{-p(t,x,\{x\})}{t}\leqslant q(x)\leqslant\|q\|$$

and dominated convergence theorem, we obtain $\forall A\in\mathscr{E}$,

$$0=\int_{A^c}\mu(dx)q(x,A)-\int_A\mu(dx)q(x)+\int_A\mu(dx)q(x,A)$$
$$=\int\mu(dx)q(x,A)-\int\mu(dx)\int_A\delta(x,dy)q(y)$$
$$=\int\mu(dx)QI_A(x).$$

This shows that the assertion holds for $n=1$. Now, by Lemma (18) and monotone class theorem, it follows that

$$\int\mu(dx)Qf(x)=0,\quad\forall f\in b\mathscr{E}$$

On the other hand, by Lemma (18), we have

$$Q^nf\in b\mathscr{E},\quad\forall f\in b\mathscr{E},\quad\forall n\geqslant 1.$$

Therefore, we get

$$\int\mu(dx)Q^{n+1}f(x)=\int\mu(dx)(Q(Q^nf))(x)=0$$

By induction, this proves our lemma.

Define

$$c_{ij}^{(n)}=\left[\int_{A_i}\mu(dx)\,\hat{Q}^nI_{A_j}(x)-\int_{A_j}\mu(dx)\,\hat{Q}^nI_{A_i}(x)\right]^+,$$
$$d_{ij}^{(n)}=\left(\int_{A_i}\mu(dx)\,\hat{Q}^nI_{A_j}(x)\right)\wedge\left(\int_{A_j}\mu(dx)\,\hat{Q}^{(n)}I_{A_i}(x)\right).$$

We see that $d_{ij}^{(n)}\geqslant 0$ by Lemma(18).

(23) **Lemma** For any i,j, and n, we have

(i) $c_{ij}^{(n)}\geqslant 0$;

(ii) $c_{ij}^{(n)}\cdot c_{ji}^{(n)}=0$;

(iii) $\sum_j(c_{ij}^{(n)}-c_{ji}^{(n)})=0$;

(iv) $\sum_{i,j}c_{ij}^{(n)}<\infty$.

Proof The first two assertions are obvious. It follows from Lemma(21) and Lemma(22) that

$$\sum_j(C_{ij}^{(n)}-C_{ji}^{(n)})=\sum_j\Big(\int_{A_i}\mu(dx)\,\hat{Q}^nI_{A_j}(x)$$
$$-\int_{A_j}\mu(dx)\hat{Q}^nI_{A_i}(x)\Big)$$

$$=\int_{A_i}\mu(dx)\sum_j(Q/\|q\|+I)^n I_{A_j}(x)-\int\mu(dx)(Q/\|q\|+I)^n I_{A_i}(x)$$

$$=\int_{A_i}\mu(dx)\sum_j I_{A_j}(x)-\int\mu(dx)I_{A_i}(x)=0.$$

This proves (iii). As for (iv), by Lemma (21), we have

$$\sum_j C_{ij}^{(n)}\leqslant 2\sum_{i,j}\left|\int_{A_i}\mu(dx)\hat{Q}^n I_{A_j}(x)\right|$$

$$\leqslant 2\int\mu(dx)\sum_j|\hat{Q}^n I_{A_j}(x)|\leqslant 2\cdot 3^n<\infty$$

(24) Theorem (Stability theorem of circulations).

Let $q(x)-q(x,A)$ be a bounded conservative q-pair, the q-process $p(t,x,A)$ is stationary with stationary distribution μ, and let $\{A_j\}_1^\infty$ be a measurable partition of $\mathscr{E}$. Define

$$a_{ij}(t)=\int_{A_i}\mu(dx)p(t,x,A_j)$$

then the matrix $A(t)=(a_{ij}(t))$ can be decomposed into two parts:

$$A(t)=D(t)+\sum_{k\in H}R_k(t)$$

where the first part $D(t)$ which is a symmetric matrix is the detailed balance part; the second part $\sum_{k\in H}R_k(t)$ is the circulation part, each $R_k(t)$ is a circulation matrix; H is at most denumerable and may be empty. Moreover, when time varies, each cycle of $R_k(t)$ $(k\in H)$ is not changed (but the direction and the flow of $R_k(t)$ can be changed) maybe except a countable set of $[0,\infty)$. In other words, the circulation decompositions are almost stable.

Proof Because

$$a_{ij}(t)=\int_{A_i}\mu(dx)e^{tQ}I_{A_j}(x)$$

$$=e^{-t\|q\|}\int_{A_i}\mu(dx)e^{t\|q\|\hat{Q}}I_{A_j}(x)$$

$$=e^{-t\|q\|}\sum_{n=0}^{\infty}\frac{(t\|q\|)^n}{n!}\int_{A_i}\mu(dx)\hat{Q}^n I_{A_j}(x),$$

hence

$$A(t)=\exp(-t\|q\|)\sum_{n=0}^{\infty}\frac{(t\|q\|)^n}{n!}[D^{(n)}+C^{(n)}],$$

where

$$D^{(n)}=(d_{ij}^{(n)}),\ n\geqslant 0;\quad c^{(n)}(c_{ij}^{(n)}),\ n\geqslant 1,\quad C^{(0)}=0.$$

These matrices do not depend on t. clearly, $D^{(n)}$ is symmetric, hence $D(t)\equiv\exp(-t\|q\|)\sum_{n=0}^{\infty}\frac{(t\|q\|)^n}{n!}D^{(n)}$ is also symmetric. On the other hand, each matrix $C^{(n)}$ satisfies the conditions in Lemma (23), using the proof in [1; §5.3] we get the following circulation decompositions:

$$C^{(n)}=\sum_{l\in H_n}\widetilde{R}_{n,l},\quad n\geqslant 1$$

where H_n is at most denumerable. Here, we agree with the convention that a summation over an empty set is zero. The flow of $\widetilde{R}_{n,l}$ is denoted by $\widetilde{r}_{n,l}<0$. Clearly $\widetilde{R}_{n,l}\geqslant 0$. Some of the circulation matrices in $\{\widetilde{R}_{n,l}: n\geqslant 1, l\in H_H\}$ may have the same cycle but different directions. we relabel all the cycles by 1, 2, $\cdots$ Thus, among the circulations with kth cycle, there may be two opposite directions. Putting the circulation matrices which have the same direction together, we get two circulation matrices: $\overline{r}_{k1}(t)\overline{R}_{k1}$ and $\overline{r}_{k2}(t)\overline{R}_{k2}$, where $\overline{R}_{k1}$ and $\overline{R}_{k2}$ do not depend on t. We have arrived at:

$$A(t)=D(t)+\sum_{k\in H}(\overline{r}_{k1}(t)\overline{R}_{k1}+\overline{r}_{k2}(t)\overline{R}_{k2}).$$

We are now at the position to discuss the stability of circulation decompositions. All together there are four cases:

(i) Among $\overline{r}_{k1}(t)\overline{R}_{k1}$ and $\overline{r}_{k2}(t)R_{k2}$, someone, for instance $\overline{R}_{k2}$, is disappeared. Then kth cycle never vanishes since $\overline{r}_{k1}(t)>0$ for all $t>0$ and $\overline{R}_{k1}\neq 0$. Indeed, $\overline{r}_{k1}(t)$ can be expressed by

$$\overline{r}_{k1}(t)=\exp(-t\|q\|)\sum_{a}c_a\frac{(t\|q\|)^a}{a!},$$

where $c_a\geqslant 0$, and there exists an a_0 such that $c_{a_0}>0$.

(ii) $\overline{r}_{k1}(t)\equiv\overline{r}_{k2}(t)$. In this case, $\overline{r}_{k1}(t)\overline{R}_{k1}+\overline{r}_{k2}(t)R_{k2}$ is a symmetric matrix, so can be merged into $D(t)$.

(iii) $\overline{r}_{k1}(t)>\overline{r}_{k2}(t)$ for each $t>0$. In this case, we can replace $\overline{r}_{k1}(t)\overline{R}_{k1}+\overline{r}_{k2}(t)\overline{R}_{k2}$ by a symmetric matrix (also merged into $D(t)$) and a new circulation martrix $R_k(t)\equiv(\overline{r}_{k1}(t)-\overline{r}_{k2}(t))\overline{R}_{k1}$. By (i), $R_k(t)$ never vanishes. Similarly, we can discuss the case of $\overline{r}_{k1}(t)<\overline{r}_{k2}(t)$ for each $t>0$.

(iv) The cases (i)-(iii) are disappeared, then for some t, we have $\overline{r}_{k1}(t)=\overline{r}_{k2}(t)$, and for another t, we have $\overline{r}_{k1}(t)\neq\overline{r}_{k2}(t)$.

As discussed in (iii), we can always replace $\overline{r}_{k1}(t)\overline{R}_{k1}+\overline{r}_{k2}(t)\overline{R}_{k2}$ by a symmetric matrix and a circulation matrix $R_k(t)$. If $\overline{r}_{k1}(t)\neq\overline{r}_{k2}(t)$, then the flow of $R_k(t)$ is either $\overline{r}_{k1}(t)-\overline{r}_{k2}(t)$ (in the case of $\overline{r}_{k1}(t)>\overline{r}_{k2}(t)$), or $\overline{r}_{k2}(t)-\overline{r}_{k1}(t)$ (in the case of $\overline{r}_{k1}(t)<\overline{r}_{k2}(t)$); in the two cases, the cycles of $R_k(t)$ are the same but the directions are different. If $\overline{r}_{k1}(t)=\overline{r}_{k2}(t)$, then the circulation $R_k(t)$ vanishes. In other words, kth cycle vanishes only at the time t belonging to

$$\Lambda_k=\{t\geqslant 0:\quad \overline{r}_{k1}(t)=\overline{r}_{k2}(t)\}.$$

If Λ_k is a finite set, then Λ_k must be an isolated set. Therefore, we may assume that Λ_k is an infinite set. Next, by the above construction, we know that both $\overline{r}_{k1}(t)$ and $\overline{r}_{k2}(t)$ can be regarded as analytic functions in t. Hence, Λ_k must be an isolated set. Set

$$\Lambda=\bigcup_{k\in H}\Lambda_k$$

We see that Λ is at most denumerable.

We have now proved our main theorem.

§ 3. Self-Organization Phenomena

Various dynamical patterns have been observed for numerous systems far from thermal equilibrium in physics, chemistry, biology and so on. This is just the self-organization phenomena[3,4]. Many nonequilibrium systems can be studied by Markov processes with continuous parameter [3,4,5]. When the system (resp. the Markov process) arrives stable (resp. stationary Markov process), either the system is detailed balance (resp. reversible Markov process), or it is not detailed balance (resp. irreversible Markov process). Theorem (24) tells us that the patterns (resp. circulations) should be arised in any nonequilibrium system, and the patterns are almost stable even through the flows of circulations now strong, now weak, and the directions of circulations are not the same when time t varies. It seems that the self-organization phenomena exists widely, not only for the systems far from thermal equilibrium, but also for the systems near equilibrium (i.e. linear nonequilibrium). The reason we can not easily observe the patterns is the flows of circulations are too weak.

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