UNIQUENESS CRITERION FOR q-PROCESSES

CHEN MUFA (陈木法) AND ZHENG XIAOGU (郑小谷)
(*Beijing Normal University*)

Received December 24, 1980.

ABSTRACT

The uniqueness problem for the totally stable q-processes in the case of the state spaces being countable was solved by C. T. Hou. In this paper, we shall extend Hou's theorem to arbitrary abstract state spaces.

The construction of q-processes has been studied for about half a century. When the state spaces were countable, the existence and uniqueness problem was being successively improved by W.Feller[14,15], J. L. Doob[16], G. E. H. Reuter[2,3,4] and others. Finally, C. T. Hou fully solved the case of all states being stable. When the state spaces were abstract, the problem was studied by D. H. Hu[6,7,11], but his definition of q-processes did not fully include Q-processes of countable state spaces. In this paper, a new definition of q-processes is presented, and it can not be extended further. We have obtained a uniqueness criterion for q-processes, which includes Hou's uniqueness theorem.

I NOTATION

Let (E, ε) be an abstract measurable space. And all singletons $\{x\}\,(x \in E)$ belong to ε. $r\mathscr{E}$ $(r\mathscr{E}_+, b\mathscr{E}, b\mathscr{E}_+, \mathscr{E}_+)$ denotes the set of all finite measurable (nonnegative finite measurable, bounded measurable, nonnegative bounded measurable, nonnegative measurable) functions; $\mathscr{L}(\mathscr{L}_+, \bar{\mathscr{L}}_+)$ denotes the set of all finite σ-additive set functions (finite measures, σ-finite measures).

Definition 1.1. The functions $q(x)-q(x, A)(x \in E, A \in \mathscr{E})$ are called a q-pair, if $q(\cdot) \in \mathscr{E}_+$, $q(\cdot, A) \in r\mathscr{E}_+$ for A; $q(x, \cdot) \in \mathscr{L}_+$ for x, $q(x, \{x\})=0$ and $q(x, E) \leqslant q(x)$ for every $x \in E$. We call x stable if

$$q(x)<+\infty, \tag{1.1}$$

and we call $q(x)-q(x, A)$ totally stable if (1.1) holds for every $x \in E$. x is called a conservative point if

$$q(x, E)=q(x), \tag{1.2}$$

and $q(x)-q(x, A)$ conservative if (1.2) holds for every $x \in E$.

Definition 1.2. Given a q-pair $q(x)-q(x, A)$, a Markov process $P(t, x, A)$ $(t \geqslant 0, x \in E, A \in \mathscr{E})$ [6; Definition 1.1] is called a q-process if

$$\lim_{t\to 0+} [P(t,x,A)-\delta(x,A)]\cdot t^{-1} = q(x,A)-\delta(x,A)q(x)$$
$$(x\in E, A\in \mathscr{R}), \tag{1.3}$$

where

$$\mathscr{R} = \{A\in\mathscr{E}: \limsup_{t\to 0+ x\in A} [1-P(t,x,\{x\})] = 0\}. \tag{1.4}$$

$$\delta(x,A)=1 \text{ if } x\in A; \delta(x,A)=0 \text{ if } x\notin A.$$

Theorem 1.3. *Suppose $(E,\mathscr{E})$ satisfies $\{(x,x): x\in E\}\in\mathscr{E}\times\mathscr{E}$ (for example, E is a Hausdorff space). Then every totally stable Markov process is a q-process with respect to some q-pair $q(x)-q(x,A)$, where "totally stable" means that*

$$\bar{q}(x) = \lim_{t\to 0+} [1-P(t,x,\{x\})]\cdot t^{-1} < \infty, \ \forall x\in E.$$

Proof. By [13; Theorem 4.1], the above limits always exist, and

$$\hat{q}(x,A) \triangleq \lim_{t\to 0+} P(t,x,A)t^{-1} < \infty, \forall A\in\mathscr{R}, E\ni x\notin A.$$

In [13] Kendall proved that if we define $q(x,A)=\hat{q}(x,A\backslash\{x\})$, then

i) $\mathscr{R}$ is a ring;

ii) $\bar{q}(\cdot)$ and $\bar{q}(\cdot,A)\in\mathscr{E}_+$ for every $A\in\mathscr{R}$;

iii) $\bar{q}(x,\cdot)$ is a measure on $\mathscr{R}$ for every $x\in E$;

iv) if $\sup_{x\in A}\bar{q}(x)<\infty$, then $A\in\mathscr{R}$.

Therefore there exists $A_n\in R$ such that $A_n\uparrow E$, and hence $q(x,\cdot)$ can be extended to a unique measure on $\mathscr{E}$ for every $x\in E$, and we may also write $\bar{q}(x,\cdot)$. Clearly, the extended $\bar{q}(x)-\bar{q}(x,A)$ is also a $\bar{q}$-pair, and for every $\forall x\in E, \forall A\in\mathscr{R}$, we have

$$\lim_{t\to 0+} [P(t,x,A)-\delta(x,A)]\cdot t^{-1} = \bar{q}(x,A)-\bar{q}(x)\delta(x,A).$$

Lemma 1.4. *Suppose that $q(x)-q(x,A)$ is totally stable. Then, for every fixed $A\in\mathscr{E}$, $\inf_{x\in A} P^{\min}(\lambda,x,E)>0$ holds for some $\lambda_0>0$ if and only if it holds for all $\lambda>0$, where $P^{\min}(\lambda,x,A)(\lambda>0, x\in E, A\in\mathscr{E})$ is the minimal nonnegative solution of the following equation:*

$$\text{(B)}\ P(\lambda,x,A) = \int \frac{q(x,dy)}{\lambda+q(x)} P(\lambda,y,A) + \frac{\delta(x,A)}{\lambda+q(x)}, \ (\lambda>0, x\in E, A\in\mathscr{E}).$$

Proof. It is enough to prove the necessity. Since $P^{\min}(\lambda,x,A)$ satisfies the resolvent equation [7; Theorem 5.1. (iii)], if $\lambda<\lambda_0$, then we have $P^{\min}(\lambda,x,E)\downarrow\ (\lambda\uparrow)$ for fixed x. Hence

$$\inf_{x\in A}\lambda P^{\min}(\lambda,x,E) \geqslant \lambda\lambda_0^{-1}\cdot \inf_{x\in A} P^{\min}(\lambda_0,x,E)>0.$$

On the other hand, since

$$z_\lambda(x)-z_\mu(x)+(\lambda-\mu)\int p^{\min}(\lambda,x,dy)z_\mu(y)=0 \tag{1.5}$$
$$(\lambda,\mu>0, x\in E),$$

where $z_\lambda(x) \triangleq 1 - \lambda P^{\min}(\lambda, x, E)(\lambda > 0, x \in E)$, then we have

$$\lambda P^{\min}(\lambda, x, E) \uparrow (\lambda \uparrow), \tag{1.6}$$

and hence $\inf\limits_{x \in A} \lambda P^{\min}(\lambda, x, E) > 0$ for $\lambda \geqslant \lambda_0$.

Definition 1.5. A q-pair $q(x) - q(x, A)$ is given. A measurable partition $\{E_n\}_{n \geqslant 1}$ is called a q-partition if

$$\sup_{x \in E_n} q(x) < +\infty, (\forall n \geqslant 1), \tag{1.7}$$

$$\inf_{x \in E_n} \lambda P^{\min}(\lambda, x, E) > 0, (\forall n \geqslant 1, \forall \lambda > 0). \tag{1.8}$$

Remarks. Fixed an arbitrary $\lambda > 0$. Put

$$E_{m,n}(\lambda) = \{x \in E: n - 1 \leqslant q(x) < n, (m+1)^{-1} < \lambda P^{\min}(\lambda, x, E) \leqslant m^{-1}\} (m, n \geqslant 1).$$

From Lemma 1.4, we know that $\{E_{m,n}\}$ is a q-partition. From now on, we always assume that we have taken and fixed a q-partition.

Theorem 1.6. *Let $q(x) - q(x, A)$ be a total stable q-pair. Again let $\{E_n\}_{n \geqslant 1}$ be a q-partition and $P(t, x, A)$ be a Markov process, then*

i) *$P(t, x, A)$ is a q-process if and only if*

$$\lim_{t \to 0+} [P(t, x, A) - \delta(x, A)] t^{-1} = q(x, A) - \delta(x, A) q(x)$$

$$(\forall x \in E, A \in \mathscr{E} \cap E_n, n \geqslant 1). \tag{1.9}$$

ii) *If $q(x) - q(x, A)$ is conservative, then $P(t, x, A)$ is a q-process iff*

$$\lim_{t \to 0+} [P(t, x, A) - \delta(x, A)] t^{-1} = q(x, A) - \delta(x, A) q(x),$$

$$(\forall x \in E, A \in \mathscr{E}). \tag{1.10}$$

Proof. Assertion i) follows from the proof of Theorem 1.3, and assertion ii) follows from [13] or Theorem 2 of § 6.1 in [10].

As usual, the Laplace transform $P(\lambda, x, A)$ of a Markov process $P(t, x, A)$ is also called a Markov process. It is not difficult to prove the following

Theorem 1.7. *A Markov process $P(\lambda, x, A)$ is a q-process if and only if*

$$\lim_{\lambda \to \infty} \lambda [\lambda P(\lambda, x, A) - \delta(x, A)] = q(x, A) - \delta(x, A) q(x),$$

$$(\forall x \in E, A \in \mathscr{E} \cap E_n, n \geqslant 1). \tag{1.11}$$

II. Some Lemmas

Hereafter, we always suppose $q(x) - q(x, A)$ is totally stable.

Definition 2.1. We say that a Markov process $P(\lambda, x, A)$ satisfies $(\mathrm{B_n})$ or $(\mathrm{F_n})$ if we have respectively:

$$(\mathrm{B_n}) \int [(\lambda + q(x)) \delta(x, dy) - q(x, dy)] P(\lambda, y, A) = \delta(x, A),$$

$$(\lambda > 0, x \in E, A \in \mathscr{E} \cap E_n), \tag{2.1}$$

and

$$(\mathrm{F}_n)\ \int P(\lambda, x, dy)[(\lambda + q(y))\delta(y, A) - q(y, A)] = \delta(x, A),$$
$$(\lambda > 0, x \in E, A \in \mathscr{E} \cap E_n). \tag{2.2}$$

Proposition 2.2. *In order to make* (*B*) *hold, it is necessary and sufficient that* (B_n) *holds for all* n; *and in order to make* (F_n) *hold for all* n, *it is necessary and sufficient that there must be*

$$(\mathrm{F})\ P(\lambda, x, A) = \int P(\lambda, x, dy) \int_A q(y, dz)(\lambda + q(z))^{-1}$$
$$+ \delta(x, A)(\lambda + q(x))^{-1}, \quad (\lambda > 0, x \in E, A \in \mathscr{E}). \tag{2.3}$$

Proof. Here we only prove: (2.2) holds for all n which is equivalent to (2.3) to be held. The first is equivalent to

$$\int_A P(\lambda, x, dy)(\lambda + q(y)) = \int P(\lambda, x, dy) q(y, A) + \delta(x, A),$$
$$(\lambda > 0, x \in E, A \in \mathscr{E}). \tag{2.4}$$

Regarding both sides as measures in A and taking integrals for $(\lambda + q(x))^{-1}\ \delta(x, A)$, we obtain (2.3). Conversely, if (2.3) holds, then regarding both sides as measures in A and taking integrals for $(\lambda + q(x)) \cdot \delta(x, A)$, we obtain (2.4).

By [7; Theorem 3.1] and [8; Theorem 5.2], we obtain

Proposition 2.3. $P^{\min}(\lambda, x, A)$ *satisfies* (B) *and* (F).

Let

$$\mathscr{U}_{-\lambda} = \{f \in b\mathscr{E}_+ : (\lambda + q(\cdot))f(\cdot) = \int q(\cdot, dy) f(y)\},$$
$$\mathscr{V}_\lambda = \{\varphi \in \mathscr{L}_+ : \varphi(\cdot) = \int \varphi(dx) \int q(x, dy)(\lambda + q(y))^{-1}\},$$

$$\mathscr{V}'_\lambda = \{\varphi \in \mathscr{L}_+ : \int \varphi(dx)[(\lambda + q(x))\delta(x, A) - q(x, A)] = 0, \forall A \in \mathscr{E} \cap E_n, \forall n \geqslant 1\},$$
$$R(\lambda, \mu, x, A) = \delta(x, A) + (\lambda - \mu) P^{\min}(\mu, x, A), (\lambda, \mu > 0, x \in E, A \in \mathscr{E}).$$

Definition 2.4. $f_\lambda \in b\mathscr{E}_+ (\lambda > 0)$ are called a (functional) coordinated family if

$$f_\lambda(\cdot) = \int R(\mu, \lambda, \cdot, dy) f_\mu(y), \quad (\lambda, \mu > 0).$$

$\varphi_\lambda \in \mathscr{L}_+ (\lambda > 0)$ are called a (measures) coordinated family if

$$\varphi_\lambda(\cdot) = \int \varphi_\mu(dx) R(\mu, \lambda, x, \cdot), \quad (\lambda, \mu > 0).$$

By [7; Proposition 6.2], and using the proofs of Proposition 2.2 and [7; Proposition 6.5], we can prove

Proposition 2.5.

i) *For any fixed* $\lambda_0 > 0$ *and* $f_{\lambda_0} \in \mathscr{U}_{\lambda_0}$, *put* $f_\mu(\cdot) \triangleq \int R(\lambda_0, \mu, \cdot, dy) f_{\lambda_0}(y) (\mu > 0)$, *then* $f_\mu \in \mathscr{U}_\mu$, *and* $f_\lambda (\lambda > 0)$ *is a coordinated family.*

ii) $\mathscr{V}_\lambda = \mathscr{V}'_\lambda$. *For any fixed* $\lambda_0 > 0$ *and* $\varphi_{\lambda_0} \in \mathscr{V}_{\lambda_0}$, *put* $\varphi_\mu(\cdot) \triangleq \int \varphi_{\lambda_0}(dx) R(\lambda_0, \mu, x, \cdot)(\mu > 0)$, *then* $\varphi_\mu \in \mathscr{V}_\mu$ *and* $\varphi_\lambda (\lambda > 0)$ *is a coordinated family.*

iii) *The dimensions of* $\mathscr{U}_\lambda$ *and* $\mathscr{V}_\lambda$ *are all independent of* $\lambda > 0$, *and denoted by* $\dim \mathscr{U}$ *and* $\dim \mathscr{V}$ *respectively.*

Lemma 2.6. *In order that* $\eta_\lambda(\lambda > 0)$ *is a coordinated family of measures it is necessary and sufficient that there exists a* $\omega \in \bar{\mathscr{L}}_+$ *such that*

$$\int \omega(dx)P^{\min}(\lambda, x, \cdot) \in \mathscr{L}_+, \tag{2.5}$$

and there exists a coordinated family $\bar{\eta}_\lambda \in \mathscr{V}_\lambda(\lambda > 0)$ *such that*

$$\eta_\lambda(\cdot) = \int \omega(dx)P^{\min}(\lambda, x, \cdot) + \bar{\eta}_\lambda(\cdot). \tag{2.6}$$

Then ω *and* $\bar{\eta}_\lambda(\lambda > 0)$ *are unique.*

Proof. It is enough to prove the sufficiency. If η_λ is a coordinated family, then

$$\eta_\nu(A) + (\nu - \lambda)\int \eta_\nu(dx)P^{\min}(\lambda, x, A) = \eta_\lambda(A) \geqslant 0,$$

so that

$$\nu\eta_\nu(A) \geqslant (1 - \lambda^{-1}\nu)\int \eta_\nu(dx)\lambda[\lambda P^{\min}(\lambda, x, A) - \delta(x, A)].$$

From [13; Theorem 4.1], we know that $P^{\min}(t, x, \{x\}) \geqslant \exp[-q(x)t]$, and hence $\lambda[1 - \lambda P^{\min}(\lambda, x, \{x\})] \leqslant q(x)(\lambda > 0, x \in E)$. Fixing ν and letting $\lambda \to \infty$, we obtain

$$\nu\eta_\nu(A) \geqslant \varliminf_{\lambda\to\infty}(1 - \lambda^{-1}\nu)\int_{E\setminus A}\eta_\nu(dx)\lambda^2P^{\min}(\lambda, x, A) + \varliminf_{\lambda\to\infty}(1 - \lambda^{-1}\nu)$$
$$\cdot\int_A \eta_\nu(dx)\lambda[\lambda P^{\min}(\lambda, x, A) - 1] \geqslant \int\eta_\nu(dx)[q(x, A) - \delta(x, A)q(x)],$$

for every $A \in \mathscr{E} \cap En(n \geqslant 1)$.

Set

$$\omega_\nu(A) = \nu\eta_\nu(A) - \int \eta_\nu(dx)[q(x, A) - \delta(x, A)q(x)]$$
$$(A \in \mathscr{E} \cap En, n \geqslant 1), \tag{2.7}$$

then

$$0 \leqslant \omega_\nu(A) < +\infty(\forall A \in \mathscr{E} \cap E_n, n \geqslant 1). \tag{2.8}$$

ω_ν can be extended to a unique measure $\omega_\nu \in \bar{\mathscr{L}}_+$. We will prove that

$$\int \omega_\nu(dx)P^{(n)}(\nu, x, A) \leqslant \eta_\nu(A), (\forall A \in \mathscr{E}, n \geqslant 0), \tag{2.9}$$

where $P^{(0)}(\nu, x, A) = 0, P^{(n+1)}(\lambda, x, A) = \int P^{(n)}(\lambda, x, dy).\int_A q(y, dz)(\lambda + q(z))^{-1} + \delta(x, A)(\lambda + q(x))^{-1}(n \geqslant 0)$. When $n = 0$, (2.9) is trivial. Now suppose that it holds for n. From (2.7) and (2.8), $0 \leqslant \int \eta_\nu(dx)q(x, A) < \infty$ for $A \in \mathscr{E} \cap E_m$, hence

$$0 \leqslant \int \eta_\nu(dy)\int_A q(y, dz)(\nu + q(z))^{-1} < \infty,$$

so

$$\begin{aligned}&\int \omega_\nu(dx)P^{(n+1)}(\nu, x, A)\\&= \int \omega_\nu(dx)\left[\int P^{(n)}(\nu, x, dy)\int_A \frac{q(y, dz)}{\nu+q(z)} + \frac{\delta(x, A)}{\nu+q(x)}\right]\\&\leqslant \int \eta_\nu(dy)\int_A \frac{q(y, dz)}{\nu+q(z)} + \int_A \frac{\omega_\nu(dx)}{\nu+q(x)} = \eta_\nu(A) \quad A\in\mathscr{E}\cap E_m, m\geqslant 1.\end{aligned}$$

This shows that (2.9) holds. In (2.9), letting $n\to\infty$, from [8; Theorem 8.2 and Lemma 4.1] and the monotone convergence theorem, we obtain

$$\eta_\nu(A) \geqslant \int \omega_\nu(dx)P^{\min}(\nu, x, A), (A\in\mathscr{E}). \tag{2.10}$$

Now set

$$\bar\eta_\nu(A) = \eta_\nu(A) - \int \omega_\nu(dx)P^{\min}(\nu, x, A), (A\in\mathscr{E}), \tag{2.11}$$

then from (2.7) and the fact that $P^{\min}(\lambda, x, A)$ satisfies (B), we obtain

$$\begin{aligned}&\int \bar\eta_\nu(dx)[(\nu+q(x))\delta(x, A) - q(x, A)]\\&= \omega_\nu(A) - \int \omega_\nu(dx)\int P^{\min}(\nu, x, dy)[(\nu+q(y))\delta(y, A) - q(y, A)]\\&= \omega_\nu(A) - \omega_\nu(A) = 0,\end{aligned}$$

for every $A\in\mathscr{E}\cap E_n$ and $n\geqslant 1$. This means $\bar\eta_\nu\in\mathscr{V}_\nu$. Now we want to prove that ω_ν and $\bar\eta_\nu$ are determined uniquely by η_ν. If ω_ν' and $\bar\eta_\nu'$ also satisfy (2.11), then

$$\begin{aligned}&\int \omega_\nu(dx)P^{\min}(\nu, x, A) + \bar\eta_\nu(A)\\&\quad= \int \omega_\nu'(dx)P^{\min}(\nu, x, A) + \bar\eta_\nu'(A) \quad (\forall A\in\mathscr{E}).\end{aligned}$$

Regarding both-hand sides as measures in A and taking integrals for

$$[(\nu+q(x))\delta(x, A) - q(x, A)] \quad (A\in\mathscr{E}\cap E_n),$$

from $P^{\min}(\lambda, x, A)$ satisfying (F_n) and $\bar\eta_\nu, \bar\eta_\nu'\in\mathscr{V}_\nu$, we obtain

$$\omega_\nu(A) = \omega_\nu'(A), \quad (\forall A\in\mathscr{E}).$$

Thus, we have already proved that the decomposition is unique for any fixed $\nu>0$. However

$$\begin{aligned}\eta_\lambda(A) &= \int \eta_\nu(dx)R(\nu, \lambda, x, A)\\&= \int \omega_\nu(dx)\int P^{\min}(\nu, x, dy)R(\nu, \lambda, y, A) + \int \bar\eta_\nu(dx)R(\nu, \lambda, x, A)\\&= \int \omega_\nu(dx)P^{\min}(\lambda, x, A) + \int \bar\eta_\lambda(dx)R(\nu, \lambda, x, A).\end{aligned} \tag{2.12}$$

Clearly, $\int \omega_\nu(dx)P^{\min}(\lambda, x, \cdot)\in\mathscr{L}_+$, and from ii) of Proposition 2.5, we know that

$$\int \bar\eta_\nu(dx)R(\nu, \lambda, x, \cdot)\in\mathscr{V}_\lambda(\lambda>0).$$

Therefore from the uniqueness of ω_λ and $\bar{\eta}_\lambda$, we know that ω_λ is independent of λ, and therefore it can be denoted by ω. Furthermore $\bar{\eta}_\lambda(\lambda>0)$ is a coordinated family.

Lemma 2.7. *If $\eta_\lambda(\lambda>0)$ is a coordinated family of measures, then*

$$\eta_\lambda \downarrow 0, \quad (\lambda \uparrow \infty). \tag{2.13}$$

Furthermore, if $\eta_\lambda \in \mathscr{Y}_\lambda(\lambda>0)$, then

$$\begin{aligned}
&\lambda\eta_\lambda(A) - \nu\eta_\nu(A) \\
&= (\lambda-\nu)\int \eta_\lambda(dx)[\delta(x, A) - \nu P^{\min}(\nu, x, A)] \\
&= (\lambda-\nu)\int \eta_\nu(dx)[\delta(x, A) - \lambda P^{\min}(\lambda, x, A)] \\
&\qquad (\lambda, \nu > 0, A \in \mathscr{E}).
\end{aligned} \tag{2.14}$$

Proof. By the coordination, we have

$$\begin{aligned}
\eta_\lambda(A) &= \eta_\nu(A) + (\nu-\lambda)\int \eta_\nu(dx) P^{\min}(\lambda, x, A) \\
&= \eta_\nu(A) + \nu\int \eta_\nu(dx) P^{\min}(\lambda, x, A) - \int \eta_\nu(dx)\lambda P^{\min}(\lambda, x, A),
\end{aligned}$$

so that (2.13) follows. If $\eta_\lambda \in \mathscr{Y}_\lambda$ also, then from the fact that $P^{\min}(\lambda, x, A)$ satisfies (F_n) $(n \geqslant 1)$, we know that

$$\begin{aligned}
&\lambda\eta_\lambda(A) - \nu\eta_\nu(A) \\
&= \sum_n [\lambda\eta_\lambda(A\cap E_n) - \nu\eta_\nu(A\cap E_n)] \\
&= \sum_n \Big\{\int \eta_\lambda(dx)[q(x, A\cap E_n) - q(x)\delta(x, A\cap E_n)] \\
&\quad - \int \eta_\nu(dx)[q(x, A\cap E_n) - q(x)\delta(x, A\cap E_n)]\Big\} \\
&= \sum_n (\nu-\lambda)\int \eta_\lambda(dx)\int P^{\min}(\nu, x, dy)[q(y, A\cap E_n) - q(y)\delta(y, A\cap E_n)] \\
&= (\nu-\lambda)\sum_n \int \eta_\lambda(dx)[\nu P^{\min}(\nu, x, A\cap E_n) - \delta(x, A\cap E_n)] \\
&= (\lambda-\nu)\int \eta_\lambda(dx)[\delta(x, A) - \nu P^{\min}(\nu, x, A)],
\end{aligned}$$

and the second assertion follows from the symmetry between λ and ν.

Lemma 2.8. *Let $\eta_\lambda(\lambda>0)$ be a coordinated family of measures, and similarly to Lemma 2.6, we have*

$$\eta_\lambda(\cdot) = \int \omega(dx) P^{\min}(\lambda, x, \cdot) + \bar{\eta}_\lambda(\cdot), \quad (\lambda>0), \tag{2.15}$$

then

$$\begin{aligned}
&P(\lambda, x, A) \triangleq P^{\min}(\lambda, x, A) + [1-\lambda P^{\min}(\lambda, x, E)](c+\lambda\eta_\lambda(E))^{-1}\eta_\lambda(A) \\
&\qquad (\lambda>0, \ x\in E, A\in\mathscr{E}; c\geqslant 0 \text{ and } c+\lambda\eta_\lambda(E)>0)
\end{aligned} \tag{2.16}$$

is a Markov process. It is a q-process if and only if one of the following conditions holds:

1) $q(x) - q(x, A)$ *is conservative*;

2) $\omega = 0$;

3) $\omega(E) = +\infty$;

4) $\lim\limits_{\lambda\to\infty} \lambda\bar{\eta}(E) = +\infty$.

Proof. In order to prove that $P(\lambda, x, A)$ defined by (2.16) satisfies the resolvent equation, it suffices to prove that

$$\sigma_\lambda\eta_\lambda(A) - \sigma_\mu\eta_\mu(A) + (\lambda - \mu)\int \sigma_\lambda\eta_\lambda(dx)P(\mu, x, A) = 0,$$

where $\sigma_\lambda = (c + \lambda\eta_\lambda(E))^{-1}$, $(\lambda > 0)$. From the coordination of $\eta_\lambda(\lambda > 0)$, it suffices to prove that

$$\sigma_\lambda - \sigma_\mu + (\lambda - \mu)\sigma_\lambda\sigma_\mu\int \eta_\lambda(dx)[1 - \mu P^{\min}(\mu, x, E)] = 0.$$

But $\sigma_\lambda[1 + \lambda\sigma_\mu\eta_\lambda(E)] = \sigma_\lambda\sigma_\mu[c + \lambda\eta_\lambda(E) + \mu\eta_\mu(E)]$ is symmetric for λ and μ, so that the above equality holds.

Since $P^{\min}(\lambda, x, A)$ is a q-process and from Theorem 1.7, we know that $P(\lambda, x, A)$ defined by (2.16) is a q-process if and only if

$$\lim_{\lambda\to\infty} \lambda^2[1 - \lambda P^{\min}(\lambda, x, E)](c + \lambda\eta_\lambda(E))^{-1}\eta_\lambda(A) = 0$$

$$(x \in E, A \in \mathscr{E} \cap E_n, n \geqslant 1). \tag{2.17}$$

From [7; Theorem 3.1. (6)], we have

$$\lim_{\lambda\to\infty} \lambda[1 - \lambda P^{\min}(\lambda, x, E)] = q(x) - q(x, E), \tag{2.18}$$

$$(\forall x \in E).$$

ν_0 being fixed, there exists a $\exists N = N(\nu_0)$ for every $\forall A \in \mathscr{E} \cap E_n, n \geqslant 1$ such that $\sum\limits_{m=N+1}^{\infty}\int_{E_m} \eta_{\nu_0}(dx)q(x, A) < \varepsilon$, so from Lemma 2.7, we have

$$\int \eta_\nu(dx)q(x, A) \leqslant \sum_{m=1}^{N}\int_{E_m} \eta_\nu(dx)q(x, A) + \varepsilon,$$

for $\nu \geqslant \nu_0$. From this and [10; Appendix, Lemma 11], we know that

$$\nu\eta_\nu(A) = \omega(A) + \int \eta_\nu(dx)[q(x, A) - q(x)\delta(x, A)] \underset{\nu\to\infty}{\longrightarrow} \omega(A). \tag{2.19}$$

From (1.6), the monotone convergence theorem and $\lim\limits_{\lambda\to\infty} \lambda P^{\min}(\lambda, x, A) = \delta(x, A)$, we have

$$\lim_{\lambda\to\infty}\int \omega(dx)\lambda P^{\min}(\lambda, x, E) = \omega(E).$$

From this, Lemma 2.7 and (2.15), we obtain that

$$\lim_{\lambda\to\infty}\lambda\eta_\lambda(E)=\lim_{\lambda\to\infty}\lambda\bar{\eta}_\lambda(E)+\omega(E). \tag{2.20}$$

If $\lim\limits_{\lambda\to\infty}\lambda\bar{\eta}_\lambda(E)+\omega(E)=0$, then $\omega=\bar{\eta}_\lambda=0$, hence $\eta_\lambda=0(\lambda>0)$, i. e. $P(\lambda,x,A)\equiv P^{\min}(\lambda,x,A)$ and at the moment the Lemma is trivial. Now we assume that

$$\lim_{\lambda\to\infty}\lambda\bar{\eta}_\lambda(E)+\omega(E)>0.$$

From (2.18)—(2.20), we obtain that

$$\lim_{\lambda\to\infty}\lambda^2[1-\lambda P^{\min}(\lambda,x,E)](c+\lambda\eta_\lambda(E))^{-1}\eta_\lambda(A)=[q(x)$$
$$-q(x,E)]\omega(A)[c+\omega(E)+\lim_{\lambda\to\infty}\lambda\bar{\eta}_\lambda(E)]^{-1}(x\in E, A\in\mathscr{E}\cap E_n n\geqslant 1). \tag{2.21}$$

Therefore, from (2.17), $P(\lambda,x,A)$ is a q-process if and only if one of the conditions i)—iv) holds.

Lemma 2.9. *There exists a $\lambda>0$ such that*

$$c(\lambda)\triangleq\inf_{x\in E}\lambda P^{\min}(\lambda,x,E)=0, \tag{2.22}$$

if and only if there exists a $\omega\in\mathscr{L}_+$ such that $\omega(E_n)<+\infty(\forall n\geqslant 1)$, $\omega(E)=+\infty$ and $\int\omega(dx)P^{\min}(\lambda,x,E)<\infty(\forall\lambda>0)$.

Proof. The sufficiency follows from

$$\int\omega(dx)P^{\min}(\lambda,x,E)\geqslant\lambda^{-1}c(\lambda)\omega(E).$$

We return to prove the necessity. Assume that (2.22) holds. Then, there is an infinite subset $\hat{N}$ of the set N of positive integers such that

$$\sum_{n\in\hat{N}}\inf_{x\in E_n}\lambda P^{\min}(\lambda,x,E)<\infty.$$

Furthermore, we can take $x_n\in E_n(n\in\hat{N})$ such that $\sum\limits_{n\in\hat{N}}\lambda P^{\min}(\lambda,x_n,E)<\infty$. Take an arbitrary $x_n\in E_n(n\in N\backslash\hat{N})$, and put

$$\omega(\{x_n\})=\begin{cases}1, & \text{if } n\in\hat{N}\\ 0, & \text{if } n\in N\backslash\hat{N}\end{cases}$$

$$\omega(A)\triangleq\sum_{n\in N}\omega(A\cap\{x_n\}),\quad A\in\mathscr{E},$$

then $\omega(E_n)=\omega(\{x_n\})\leqslant 1$, $\omega(E)=\sum\limits_n\omega(\{x_n\})=\infty$ and

$$\int\omega(dx)\lambda P^{\min}(\lambda,x,E)=\sum_{n\in\hat{N}}\lambda P^{\min}(\lambda,x_n,E)<\infty.$$

The proof is completed.

By using the typical methods in [12], it is easy to prove

Lemma 2.10. *Let $U(\cdot,A)$, $T(\cdot,A)$, $P(\cdot,A)$, $Q(\cdot,A)\in\mathscr{E}_+$ for each $A\in\mathscr{E}$;*

$U(x, \cdot), T(x, \cdot), P(x, \cdot) \in \mathscr{L}_+$ *for each* $x \in E$.

i) *If for each* A, $P(\cdot, A)$ *is the minimal nonnegative solution of*

$$f(\cdot) = \int U(\cdot, dy)f(y) + T(\cdot, A),$$

then $\int P(\cdot, dy)g(y)$ *is the minimal nonnegative solution of*

$$f(\cdot) = \int U(\cdot, dy)f(y) + \int T(\cdot, dy)g(y)$$

for every $g \in b\mathscr{E}_+$.

ii) *If for each* $x \in E, Q(x, \cdot)$ *is the minimal nonnegative solution of*

$$\varphi(\cdot) = \int \varphi(dy)U(y, \cdot) + T(x, \cdot),$$

then $\int U(dx)Q(x, \cdot)$ *is the minimal nonnegative solution of*

$$\varphi(\cdot) = \int \varphi(dy)U(y, \cdot) + \int U(dy)T(y, \cdot)$$

for every $\forall U \in \mathscr{L}_+$.

Lemma 2. 11.

i) *If* $f \in b\mathscr{E}_+$ and

$$\int [(\lambda + q(\cdot))\delta(\cdot, dy) - q(\cdot, dy)]f(y) = g(\cdot) \geqslant 0,$$

then $f(\cdot) \geqslant \int P^{\min}(\lambda, \cdot, dy)g(y)$. *Moreover, if dim* $\mathscr{U} = 0$ *also, then*

$$f(\cdot) = \int P^{\min}(\lambda, \cdot, dy)g(y).$$

(ii) *If* $\mu \in \mathscr{L}_+$ *and*

$$\int \mu(dx)[(\lambda + q(x))\delta(x, A) - q(x, A)] = U(A) \geqslant 0 (A \in \mathscr{E} \cap E_n, n \geqslant 1), \quad (2.23)$$

then

$$\mu(\cdot) \geqslant \int U(dx)P^{\min}(\lambda, x, \cdot). \quad (2.24)$$

If $\dim \mathscr{V} = 0$ *in addition, then*

$$\mu(\cdot) = \int U(dx)P^{\min}(\lambda, x, \cdot). \quad (2.25)$$

Proof. We only need to prove ii). By the proof of Proposition 2.2, we know that μ is a solution of

$$\varphi(\cdot) = \int \varphi(dx)\int_\cdot q(x, dy)[\lambda + q(y)]^{-1} + \int_\cdot U(dx)[\lambda + q(x)]^{-1}. \quad (2.26)$$

However, $P^{\min}(\lambda, x, A)$ is the minimal nonnegative solution of (F), and from Lemma

2.10. ii), $\int U(dx)P^{\min}(\lambda, x, \cdot)$ is the minimal nonnegative solution of (2.26), so (2.24) holds. Clearly,

$$\mu(\cdot) - \int U(dx)P^{\min}(\lambda, x, \cdot) \in \mathscr{V}_\lambda \quad (\lambda > 0).$$

hence (2.25) holds when dim $\mathscr{V} = 0$.

Lemma 2.12. *We have dim* $\mathscr{U} = 0$, *if there exists a* $\lambda > 0$ *such that*

$$\inf_{x \in E} P^{\min}(\lambda, x, E) > 0.$$

Proof. The assertion follows from [12; Theorem 5].

Lemma 2.13.

i) *If* $f \in r\mathscr{E}$ *and*

$$\int P^{\min}(\lambda, \cdot, dy)f(y) = 0, \tag{2.27}$$

then there exists $f = 0$.

ii) *If* $\mu \in \mathscr{L}$ *and*

$$\int \mu(dx)P^{\min}(\lambda, x, A) = 0 \quad (A \in \mathscr{E} \cap E_n, n \geqslant 1), \tag{2.28}$$

then there exists $\mu = 0$.

Proof. i) Since $P^{\min}(\lambda, x, A)$ satisfies (B) and (2.27), we have

$$f(\cdot) = \int \delta(\cdot, dy)f(y)$$
$$= [\lambda + q(\cdot)] \int P^{\min}(\lambda, \cdot, dy)f(y) - \int q(\cdot, dy) \int P^{\min}(\lambda, y, dz)f(z) = 0.$$

ii) By (2.28), we have

$$\int \mu(dx)P^{\min}(\lambda, \cdot, x) = \sum_n \int \mu(dx)P^{\min}(\lambda, x, \cdot \cap E_n) = 0.$$

Since $P^{\min}(\lambda, x, A)$ satisfies (F), we have

$$\mu(A) = \sum_n \int_{E_n \cap A} \mu(dx) = \sum_n \left\{ \int \mu(dx) \int P^{\min}(\lambda, x, dy)[(\lambda + q(y)) \cdot \delta(y, A \cap E_n) - q(y, A \cap E_n)] \right\} = 0,$$
$$(A \in \mathscr{E}).$$

Lemma 2.14. *Set* $G(, A) \in b\mathscr{E}_+$ *for* $A \in \mathscr{E}$, $G(x, \cdot) \in \mathscr{L}_+$ *for* $x \in E$, *and*

$$0 \leqslant G(\cdot, \cdot) \leqslant 1.$$

Suppose $g \in b\mathscr{E}_+$ *and* f^* *is the minimal nonnegative solution of*

$$f(\cdot) = \int G(\cdot, dy)f(y) + g(\cdot).$$

Put

$$D=\{x: g(x)>0\}, \tag{2.29}$$

then

$$\sup_{x\in E} f^*(x)=\sup_{y\in D} f^*(y). \tag{2.30}$$

Proof. Let $f^{(1)}(x)=g(x)$, $f^{(n+1)}(x)=\int G(x,dy)f^{(n)}(y)+g(x)(n\geqslant 1)$. Since (2.29), we have $f^{(1)}(x)\leqslant\sup_{y\in D} f^{(1)}(y)\leqslant\sup_{y\in D} f^*(y)(\forall x\in E)$. If $f^{(n)}(x)\leqslant\sup_{y\in D} f^*(y)$ $(\forall x\in E)$ for some $n\geqslant 1$, then for every $x\in E\backslash D$, we have

$$\begin{aligned} f^{(n+1)}(x) &= \int G(x,dy)f^{(n)}(y)+g(x) \\ &= \int G(x,dy)f^{(n)}(y)\leqslant \sup_{y\in D} f^*(y)G(x,E)\leqslant\sup_{y\in D} f^*(y). \end{aligned}$$

Letting $n\to\infty$, we obtain $f^*(x)\leqslant\sup_{y\in D} f^*(y)(\forall x\in E)$. Taking supremum for x over E on the left-hand side, we obtain (2.30).

Lemma 2.15. *If* $\dim\mathscr{U}=0$, *then it follows*

$$\inf_{x\in E}\lambda P^{\min}(\lambda,x,E)=\inf_{x\in\dot{E}}\lambda P^{\min}(\lambda,x,E), \tag{2.31}$$

where $\dot{E}=\{x\in E: q(x,E)<q(x)\}$.

Proof. By [12; Theorem 9], $\lambda P^{\min}(\lambda,x,E)$ is the minimal nonnegative solution of

$$f(\cdot)=\int\frac{q(\cdot,dy)}{\lambda+q(\cdot)}f(y)+\frac{\lambda}{\lambda+q(\cdot)},$$

hence $z_\lambda=1-\lambda P^{\min}(\lambda,\cdot,E)$ is the minimal solution of

$$f(\cdot)=\int\frac{q(\cdot,dy)}{\lambda+q(\cdot)}f(y)+\frac{q(\cdot)-q(\cdot,E)}{\lambda+q(\cdot)},\quad 0\leqslant f\leqslant 1.$$

Because $\dim\mathscr{U}=0$, it follows that z_λ is also the minimal nonnegative solution of the above equation. So from Lemma 2.14 we have

$$\sup_{x\in E}[1-\lambda P^{\min}(\lambda,x,E)]=\sup_{x\in\dot{E}}[1-\lambda P^{\min}(\lambda,x,E)],$$

and hence (2.31) follows.

Lemma 2.16. *For every q-process* $P(\lambda,x,A)$ *we have*

$$\begin{aligned} \lambda P(\lambda,x,A) &\geqslant \int P(\lambda,x,dy)[q(y,A)-q(y)\delta(y,A)] \\ &\quad +\delta(x,A)\ (\lambda>0,\ x\in E, A\in\mathscr{E}\cap E_n, n\geqslant 1). \end{aligned} \tag{2.32}$$

Proof. By the resolvent equation, the assertion follows from the proof of Lemma 2.6 before (2.7).

Proposition 2.17. *There exists only one q-process which satisfies* (F) *if and only if the minimal q-process* $P^{\min}(\lambda,x,A)$ *is honest or* $\dim\mathscr{V}=0$.

Proof. Sufficiency is clear. We will prove the necessity. Suppose that the minimal

q-process $P^{\min}(\lambda, x, A)$ is not honest and dim $\mathscr{V} = 0$. Fixing $\lambda_0 > 0$, taking

$$0 \neq \eta_{\lambda_0} \in \mathscr{V}_{\lambda_0}$$

and putting $\bar{\eta}_\lambda(\cdot) = \int \eta_{\lambda_0}(dx) R(\lambda_0, \lambda, x, \cdot)$, $\lambda > 0$, from Proposition 2.5. ii) we know that $\bar{\eta}_\lambda (\lambda > 0)$ is a coordinated family and $\bar{\eta}_\lambda \in \mathscr{V}_\lambda (\lambda > 0)$. Taking $\bar{\eta}_\lambda$ instead of η_λ in (2.16), we obtain a q-process which satisfies (F) and is different to $P^{\min}(\lambda, x, A)$. This leads to a contradiction.

III Proofs of Main Theorems

Theorem 3.1. *Suppose that q-pair $q(x) - q(x, A)$ is totally stable. Then there exists only one q-process if and only if we simultaneously have*

i) $$c(\lambda) \triangleq \inf_{x \in E} \lambda P^{\min}(\lambda, x, E) > 0, \quad (\exists \lambda > 0). \tag{3.1}$$

ii) *there exists only one q-process which satisfies* (F).

Proof. Necessity. If (3.1) does not hold, then taking ω as in Lemma 2.9 and from Lemma 2.8, we know that (2.16) gives an honest q-process. However, $P^{\min}(\lambda, x, A)$ is not honest because (3.1) does not hold, and so q-processes are not unique.

Sufficiency. Having the preparations of the preceding section in mind, it is not difficult to prove the sufficiency. One can find the idea of this proof from [5]. The details are left to reader.

By Lemmas 2.15 and 2.12 and by Proposition 2.17 and [6; § 4], we obtain

Theorem 3.2. *Suppose q-pair $q(x) - q(x, A)$ is totally stable, then there exists only one q-process if and only if the following three conditions all hold:*

i) $\inf_{x \in E} \lambda P^{\min}(\lambda, x, E) > 0 \quad (\lambda > 0)$;

ii) dim $\mathscr{U} = 0$;

iii) $q(x) - q(x, A)$ *is conservative; or although it is not conservative, it is still dim* $\mathscr{V} = 0$.

From this it is easy to prove that

Theorem 3.3. *If q-pair is bounded, then there exists only one q-process.*

We would like to thank Profs. Yan Shijian and Hou Zhenting for their comments.

References

[1] 侯振挺，中国科学，1974，2:115—130.
[2] Reuter, G. E. H., *Acta. Math.*, **97**(1957), 1—46.
[3] ——————, *J. London. Math. Soc.*, **34**(1959), 81—91.
[4] ——————, *ibid.*, **37**(1962), 63—73.
[5] ——————, *Z. Wahr.*, **33**(1976), 309—315.
[6] 胡迪鹤，数学学报，**16**(1966), 2:150—165.
[7] ————，武汉大学学报，1978, 4:1—18; 1979, 1: 15—38.

[8] 陈木法，数学年刊，**1**(1980)，3—4: 437—451.
[9] 郑小谷，科学通报，**13**(1982)，830.
[10] 王梓坤，随机过程论，科学出版社，1965.
[11] 胡迪鹤，数学学报，**23**(1980)，5: 750—757.
[12] 陈木法，北京师范大学学报，1979，3:63—73.
[13] Kendall, D. G., *TAMS*, **78**(1955), 2:529—540.
[14] Feller, W., *ibid.*, **48**(1940), 488—515; **58**(1945), 474.
[15] ——————, *Ann. of Math.*, II. Ser., **65**(1957), 527—570.
[16] Doob, J. L., *TAMS*, **58**(1945), 455—473.