Mu-Fa Chen

Eigenvalues, Inequalities, and Ergodic Theory

September 14, 2004

Springer
Berlin Heidelberg New York
Hong Kong London
Milan Paris Tokyo

Preface

First, let us explain the precise meaning of the compressed title. The word "eigenvalues" means the first nontrivial Neumann or Dirichlet eigenvalues, or the principal eigenvalues. The word "inequalities" means the Poincaré inequalities, the logarithmic Sobolev inequalities, the Nash inequalities, and so on. Actually, the first eigenvalues can be described by some Poincaré inequalities, and so the second topic has a wider range than the first one. Next, for a Markov process, corresponding to its operator, each inequality describes a type of ergodicity. Thus, study of the inequalities and their relations provides a way to develop the ergodic theory for Markov processes. Due to these facts, from a probabilistic point of view, the book can also be regarded as a study of "ergodic convergence rates of Markov processes," which could serve as an alternative title of the book. However, this book is aimed at a larger class of readers, not only probabilists.

The importance of these topics should be obvious. On the one hand, the first eigenvalue is the leading term in the spectrum, which plays an important role in almost every branch of mathematics. On the other hand, the ergodic convergence rates constitute a recent research area in the theory of Markov processes. This study has a very wide range of applications. In particular, it provides a tool to describe the phase transitions and the effectiveness of random algorithms, which are now a very fashionable research area.

This book surveys, in a popular way, the main progress made in the field by our group. It consists of ten chapters plus two appendixes. The first chapter is an overview of the second to the eighth ones. Mainly, we study several different inequalities or different types of convergence by using three mathematical tools: a probabilistic tool, the coupling methods (Chapters 2 and 3); a generalized Cheeger's method originating in Riemannian geometry (Chapter 4); and an approach coming from potential theory and harmonic analysis (Chapters 6 and 7). The explicit criteria for different types of convergence and the explicit estimates of the convergence rates (or the optimal constants in the inequalities) in dimension one are given in Chapters 5 and 6; some generalizations are given in Chapter 7. The proofs of a diagram of nine types of ergodicity (Theorem 1.9) are presented in Chapter 8. Very often, we deal with one-dimensional elliptic operators or tridiagonal matrices (which can be infinite) in detail, but we also handle general differential and integral operators. To avoid heavy technical details, some proofs are split among several locations in the text. This also provides different views of the same problem at different levels. The topics of the last two chapters (9 and 10) are different but closely related. Chapter 9 surveys the study of a class of interacting particle systems (from which a large part of the problems studied in this book are motivated), and illustrates some applications. In the last chapter, one can see an interesting application of the first eigenvalue, its eigenfunctions, and an ergodic theorem to stochastic models of economics. Some related open problems are included in each chapter. Moreover, an effort is made to make each chapter, except the first one, more or less self-contained. Thus, once one has read about the program in Chapter 1, one may freely go on to the other chapters. The main exception is Chapter 3, which depends heavily on Chapter 2. As usual, a quick way to get an impression about what is done in the book is to look at the summaries given at the beginning of each chapter.

One should not be disappointed if one cannot find an answer in the book for one's own model. The complete solutions to our problems have only recently been obtained in dimension one. Nevertheless, it is hoped that the three methods studied in the book will be helpful. Each method has its own advantages and disadvantages. In principle, the coupling method can produce sharper estimates than the other two methods, but additional work is required to figure out a suitable coupling and, more seriously, a good distance. The Cheeger and capacitary methods work in a very general setup and are powerful qualitatively, but they leave the estimation of isoperimetric constants to the reader. The last task is usually quite hard in higher-dimensional situations.

This book serves as an introduction to a developing field. We emphasize the ideas through simple examples rather than technical proofs, and most of them are only sketched. It is hoped that the book will be readable by nonspecialists. In the past ten years or more, the author has tried rather hard to make acceptable lectures; the present book is based on these lecture notes: Chen (1994b; 1997a; 1998a; 1999c; 2001a; 2002b; 2002c; 2003b; 2004a; 2004b) [see Chen (2001c)]. Having presented eleven lectures in Japan in 2002, the author understood that it would be worthwhile to publish a short book, and then the job was started.

Since each topic discussed in the book has a long history and contains a great number of publications, it is impossible to collect a complete list of references. We emphasize the recent progress and related references. It is hoped that the bibliography is still rich enough that the reader can discover a large number of contributors in the field and more related references.

Beijing, The People's Republic of China — Mu-Fa Chen, October 2004

Acknowledgments

As mentioned before, this book is based on lecture notes presented over the past ten years or so. Thus, the book should be dedicated, with the author's deep acknowledgment, to the mathematicians and their universities/institutes whose kind invitations, financial support, and warm hospitality made those lectures possible. Without their encouragement and effort, the book would never exist. With the kind permission of his readers, the author is happy to list some of the names below (since 1993), with an apology to those that are missing:

- Z.M. Ma and J.A. Yan, Institute of Applied Mathematics, Chinese Academy of Sciences. D.Y. Chen, G.Q. Zhang, J.D. Chen, and M.P. Qian, Beijing (Peking) University. T.S. Chiang, C.R. Hwang, Y.S. Chow, and S.J. Sheu, Institute of Mathematics, Academy Sinica, Taipei. C.H. Chen, Y.S. Chow, A.C. Hsiung, W.T. Huang, W.Q. Liang, and C.Z. Wei, Institute of Statistical Science, Academy Sinica, Taipei. H. Chen, National Taiwan University. T.F. Lin, Soochow University. Y.J. Lee and W.J. Huang, National University of Kaohsiung. C.L. Wang, National Dong Hwa University.
- D.A. Dawson and S. Feng [McMaster University], Carleton University. G. O'Brien, N. Madras, and J.M. Sun, York University. D. McDonald, University of Ottawa. M. Barlow, E.A. Perkins, and S.J. Luo, University of British Columbia.
- E. Scacciatelli, G. Nappo, and A. Pellegrinotti [University of Roma III], University of Roma I. L. Accardi, University of Roma II. C. Boldrighini, University of Camerino [University of Roma I]. V. Capasso and Y.G. Lu, University of Bari.
- B. Grigelionis, Akademijios, Lithuania.
- L. Stettner and J. Zabczyk, Polish Academy of Sciences.
- W.Th.F. den Hollander, Utrecht University [Universiteit Leiden].
- Louis H.Y. Chen, K.P. Choi, and J.H. Lou, Singapore University.
- R. Durrett, L. Gross, and Z.Q. Chen [University of Washington Seattle], Cornell University. D.L. Burkholder, University of Illinois. C. Heyde, K. Sigman, and Y.Z. Shao, Columbia University.

- D. Elworthy, Warwick University. S. Kurylev, C. Linton, S. Veselov, and H.Z. Zhao, Loughborough University. T.S. Zhang, University of Manchester. G. Grimmett, Cambridge University. Z. Brzezniak and P. Busch, University of Hull. T. Lyons, University of Oxford. A. Truman, N. Jacod, and J.L. Wu, University of Wales Swansea.
- F. Götze and M. Röckner, University of Bielefeld. S. Albeverio and K.-T. Sturm, University of Bonn. J.-D. Deuschel and A. Bovier, Technical University of Berlin.
- K.J. Hochberg, Bar-Ilan University. B. Granovsky, Technion-Israel Institute of Technology.
- B. Yart, Grenoble University [University, Paris V]. S. Fang and B. Schmit, University of Bourgogne. J. Bertoin and Z. Shi, University of Paris VI. L.M. Wu, Blaise Pascal University and Wuhan University.
- R.A. Minlos, E. Pechersky, and E. Zizhina, the Information Transmission Problems, Russian Academy of Sciences.
- A.H. Xia, University of New South Wales [Melbourne University]. C. Heyde, J. Gani, and W. Dai, Australian National University. E. Seneta, University of Sydney. F.C. Klebaner, University of Melbourne. Y.X. Lin, Wollongong University.
- I. Shigekawa, Y. Takahashi, T. Kumagai, N. Yosida, S. Watanabe, and Q.P. Liu, Kyoto University. M. Fukushima, S. Kotani, S. Aida, and N. Ikeda, Osaka University. H. Osada, S. Liang, and K. Sato, Nagoya University. T. Funaki and S. Kusuoka, Tokyo University.
- E. Bolthausen, University of Zurich, P. Embrechts and A.-S. Sznitman, ETH.
- London Mathematical Society for the visit to the United Kingdom during November 4–22, 2003.

Next, the author acknowledges the organizers of the following conferences (since 1993) for their invitations and financial support.

- The Sixth International Vilnuis Conference on Probability and Mathematical Statistics (June 1993, Vilnuis).
- The International Conference on Dirichlet Forms and Stochastic Processes (October 1993, Beijing).
- The 23rd and 25th Conferences on Stochastic Processes and Their Applications (June 1995, Singapore and July 1998, Oregon).
- The Symposium on Probability Towards the Year 2000 (October 1995, New York).
- Stochastic Differential Geometry and Infinite-Dimensional Analysis (April 1996, Hangzhou).
- Workshop on Interacting Particle Systems and Their Applications (June 1996, Haifa).
- IMS Workshop on Applied Probability (June 1999, Hong Kong).

- The Second Sino-French Colloquium in Probability and Applications (April 2001, Wuhan).
- The Conference on Stochastic Analysis on Large Scale Interacting Systems (July 2002, Japan).
- Stochastic Analysis and Statistical Mechanics, Yukawa Institute (July 2002, Kyoto).
- International Congress of Mathematicians (August 2002, Beijing).
- The First Sino-German Conference on Stochastic Analysis—A Satellite Conference of ICM 2002 (September 2002, Beijing).
- Stochastic Analysis in Infinite Dimensional Spaces (November 2002, Kyoto)
- Japanese National Conference on Stochastic Processes and Related Fields (December 2002, Tokyo).

Thanks are given to the editors, managing editors, and production editors, of the Springer Series in Statistics, Probability and Its Applications, especially J. Gani and S. Harding for their effort in publishing the book, and to the copyeditor D. Kramer for the effort in improving the English language.

Thanks are also given to World Scientific Publishing Company for permission to use some material from the author's previous book (1992a, 2004: 2nd edition).

The continued support of the National Natural Science Foundation of China, the Research Fund for Doctoral Program of Higher Education, as well as the Qiu Shi Science and Technology Foundation, and the 973 Project are also acknowledged.

Finally, the author is grateful to the colleagues in our group: F.Y. Wang, Y.H. Zhang, Y.H. Mao, and Y.Z. Wang for their fruitful cooperation. The suggestions and corrections to the earlier drafts of the book by a number of friends, especially J.W. Chen and H.J. Zhang, and a term of students are also appreciated. Moreover, the author would like to acknowledge S.J. Yan, Z.T. Hou, Z.K. Wang, and D.W. Stroock for their teaching and advice.

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