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Derived equivalences from cohomological approximations and mutations of Φ-Yoneda algebras

Wei Hu
School of Mathematical Sciences,
Laboratory of Mathematics and Complex Systems,
Beijing Normal University, 100875 Beijing,
People's Republic of China

Steffen Koenig
Institut für Algebra and Zahlentheorie, Universität Stuttgart,
Pfaffenwaldring 57, 70569 Stuttgart, Germany

Changchang Xi
School of Mathematical Sciences, Capital Normal University,
100048 Beijing, People's Republic of China
(xicc@bnu.edu.cn)

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A new construction of derived equivalences is given, which relates different endomorphism rings and, more generally, cohomological endomorphism rings, including higher extensions, of objects in triangulated categories. These objects need to be connected by certain universal maps that are cohomological approximations and that exist in very general circumstances. The construction turns out to be applicable to a wide variety of situations, covering finite-dimensional algebras as well as certain infinite-dimensional algebras, Frobenius categories and n-Calabi–Yau categories.

1. Introduction

Derived equivalences have become increasingly important in representation theory, Lie theory and geometry. Examples range from mirror symmetry over non-commutative geometry to the Kazhdan–Lusztig conjecture and to Broué's conjecture for blocks of finite groups. In all of these situations, and in many others, derived equivalences that involve finite or infinite-dimensional algebras are used. Derived equivalences between algebras, or rings, exist if and only if there exist suitable tilting complexes, as explained quite satisfactorily by Rickard's Morita theory for derived categories of rings [20]. Derived equivalences have been shown to preserve many significant algebraic and geometric invariants, and often to provide unexpected and useful new connections.

A crucial question in this context has, however, not yet received enough answers: *how to construct derived equivalences between rings in a general setup.*

A good (but certainly not unique) answer to this question should be general, flexible and systematic and apply to a multitude of algebraic and geometric situations.

One well-developed approach is based on the theory of tilting modules, building upon results by Happel [8]. Other answers use ring theoretic constructions, such as trivial extensions [21].

The aim of this paper is to provide a rather different approach. The input of the technology developed here is a triple of objects (X, M, Y) in a triangulated category. These objects are required to be related by certain universal maps (cohomological approximations, a new concept introduced here, continuing the approximation theory of Auslander *et al.* [1]) and some cohomological orthogonality conditions in non-zero degrees only. The output is a derived equivalence between cohomological endomorphism rings of $X \oplus M$ and of $M \oplus Y$.

The flexibility of the construction lies in the following features: we enhance endomorphism rings by higher extensions to produce cohomological endomorphism rings, broadening the classical concept of Yoneda extension algebras. Here, we can choose a set of cohomological degrees to define the cohomological endomorphism ring. Choosing degree zero only gives endomorphism rings in the usual sense, and then no orthogonality assumption is needed. Choosing all integers, or a suitable subset thereof (satisfying an associativity constraint), amplifies the concept of Yoneda extension algebras $\bigoplus_j \operatorname{Ext}^j(S, S)$. There is also some flexibility in the choice of M.

A special case of such a triple is given by any Auslander–Reiten triangle $X \to M \to Y$ in a derived module category; this already indicates generality of the construction. Our assumptions are actually much more general and not limited to objects in derived categories of algebras.

A particular feature of the derived equivalences constructed by this method is that they also provide a very general *mutation procedure*, turning one ring into another in a systematic way. Tilting theory has arisen as a far-reaching extension of reflection functors for quivers. Under some assumptions, but not in general, it provides mutation procedures between two given quivers or algebras, both of which are endomorphism rings of tilting modules; in the case of quivers, one may reflect at sink or source vertices. Mutations similar in style have also come up in various geometric situations. The theory of cluster categories, or more generally of Calabi–Yau categories, has extended reflections to a mutation procedure that works for representations of quivers at all vertices. Such mutations fit into the present framework as well. There is, though, a new feature introduced by our approach: reflection does not work in general in derived categories (of quivers or algebras). Therefore, cluster theory passes to the cluster category, a 'quotient' of a derived category modulo the action of some functor; endomorphism rings are taken there. In contrast to this, the approach herein always produces equivalences on the level of derived categories, not just of quotient categories; throughout, we consider derived equivalences between (cohomological) endomorphism rings or quotients thereof. In the case of quivers, this possibility of passing to quotient algebras allows mutation at an arbitrary vertex.

More generality and flexibility is added by extending the concept of 'higher extensions', that is, of shifted morphisms; it is possible to replace the shift functor by any other auto-equivalence of the ambient triangulated category. There is even a version using two such functors.

The main result of this paper provides a construction of derived equivalences in a setup that is very general in several respects. In the following explanation we start with a special case and then add generality step by step, finally arriving at the main result.

The setup is always a triangulated category $\mathcal{T}$, which is an R-category for some commutative Artinian ring R, with identity; so, morphism sets in $\mathcal{T}$ are R-modules.

- Initially, we choose any object M in $\mathcal{T}$ and a triangle $X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \to X[1]$, where α and β are $\mathrm{add}(M)$-approximations, that is universal maps from X to objects in $\mathrm{add}(M)$ or from $\mathrm{add}(M)$ to Y, respectively; in particular, M_1 is in $\mathrm{add}(M)$. For instance, Auslander–Reiten triangles (over algebras) provide such situations. If the triangle is induced by an exact sequence in an abelian category, then the theorem implies a derived equivalence between the two endomorphism rings $\mathrm{End}_{\mathcal{T}}(X \oplus M_1)$ and $\mathrm{End}_{\mathcal{T}}(M_1 \oplus Y)$. This can be seen as a mutation procedure relating the two endomorphism rings. The derived equivalence has already been established in [11].

- In the second step, recasting an idea of [10], endomorphism rings are replaced by *cohomological endomorphism rings* in the following sense: higher extensions between modules S and T are shifted morphisms in the derived category, $\mathrm{Ext}^j(S,T) \simeq \mathrm{Hom}(S, T[j])$. Using Yoneda multiplication of extensions, this defines an algebra structure on the cohomological endomorphism ring, or generalized Yoneda algebra, $\bigoplus_{j\in\mathbb{Z}} \mathrm{Hom}(S, S[j])$. When S is a complex, or any object in a triangulated category $\mathcal{T}$, negative degrees j may occur. The main theorem provides derived equivalences between such generalized Yoneda algebras. The construction works, however, not only for these Yoneda algebras, but also for 'perforated' ones in the following sense: choose a subset $\Phi \subset \mathbb{Z}$. Then, under some associativity constraint requiring Φ to be 'admissible' (see § 2.3), the space $\bigoplus_{j\in\Phi} \mathrm{Hom}(S, S[j])$ is an associative algebra, that in general is neither a subalgebra nor a quotient algebra of the Yoneda algebra $\bigoplus_{j\in\mathbb{Z}} \mathrm{Hom}(S, S[j])$. This algebra is called a *Φ-Yoneda algebra* or a *Φ-perforated Yoneda algebra*. We shall use the notation $\mathbb{E}^{\Phi}_{\mathcal{T}}(Z)$ for the algebra $\bigoplus_{j\in\Phi} \mathrm{Hom}(Z, Z[j])$, where Z is any object in $\mathcal{T}$.

  The assumptions of the first step get modified by using cohomological approximations, in the degrees specified by Φ, instead of approximations in degree zero only. Auslander–Reiten triangles still satisfy these properties. Adding higher extensions requires the addition of an orthogonality assumption without which the result would be wrong: assume

  $$\mathrm{Hom}(M, X[j]) = 0 = \mathrm{Hom}(Y, M[j]) \quad \text{for all } j \in \Phi,\ j \neq 0.$$

  For the sake of exposition, also assume for a moment that the above triangle $X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \to X[1]$ is in a derived module category and it is induced from an exact sequence with corresponding properties. Then there are derived equivalences between Φ-Yoneda algebras

  $$\mathcal{D}^{\mathrm{b}}(\mathbb{E}^{\Phi}_{\mathcal{T}}(X \oplus M)) \simeq \mathcal{D}^{\mathrm{b}}(\mathbb{E}^{\Phi}_{\mathcal{T}}(M \oplus Y)).$$

- This result needs to be modified if the triangle is no longer induced by an exact sequence. Then some annihilators have to be factored out of the degree-zero parts of the cohomological endomorphism rings, and the derived equivalences connect the quotient algebras $\mathbb{E}_{\mathcal{T}}^{\Phi}(X \oplus M)/I$ and $\mathbb{E}_{\mathcal{T}}^{\Phi}(M \oplus Y)/J$. Here, the ideals I and J can be described as follows: let $\Gamma_0 = \mathrm{End}_{\mathcal{T}}(M \oplus Y)$ and let e be the idempotent element in Γ_0 corresponding to the direct summand M. Then J is the submodule of the left Γ_0-module $\Gamma_0 e \Gamma_0$, which is maximal with respect to $eJ = 0$. Let $\Lambda_0 = \mathrm{End}_{\mathcal{T}}(X \oplus M)$, and f be the idempotent in Λ_0 corresponding to the direct summand M. Then I is the submodule of the right Λ_0-module $\Lambda_0 f \Lambda_0$, which is maximal with respect to $If = 0$.

  Another, equivalent, description of I and J is that I consists of all elements $(x_i)_{i\in\Phi} \in \mathbb{E}_{\mathcal{T}}^{\Phi}(X \oplus M)$ such that $x_i = 0$ for $0 \neq i \in \Phi$ and x_0 factorizes through $\mathrm{add}(M)$ and $x_0\tilde{\alpha} = 0$, and J consists of all elements $(y_i)_{i\in\Phi} \in \mathbb{E}_{\mathcal{T}}^{\Phi}(M \oplus Y)$ such that $y_i = 0$ for $0 \neq i \in \Phi$ and y_0 factorizes through $\mathrm{add}(M)$ and $\bar{\beta} y_0 = 0$, where $\tilde{\alpha}$ is the diagonal morphism $\mathrm{diag}(\alpha, 1)\colon X \oplus M \to M_1 \oplus M$, and $\bar{\beta}$ is the skew-diagonal morphism $\mathrm{skewdiag}(1, \beta)\colon M_1 \oplus M \to M \oplus Y$.

- The fourth level of generalization allows replacement of the shift functor by any auto-equivalence of the triangulated category $\mathcal{T}$, thus providing a new and versatile meaning of ‘higher extensions’ in terms of morphisms with one variable shifted by powers of the auto-equivalence. The additional datum F gets mentioned, when necessary, in the notation as an additional superscript, as in $\mathbb{E}_{\mathcal{T}}^{F,\Phi}(Z)$.

In this general form, the main theorem is as follows.

THEOREM 1.1. *Let Φ be an admissible subset of $\mathbb{Z}$, and let $\mathcal{T}$ be a triangulated R-category and M an object in $\mathcal{T}$. Assume that F is a triangle functor from $\mathcal{T}$ to itself, which is an auto-equivalence, that is, provided with a quasi-inverse. Suppose that*

$$X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \xrightarrow{w} X[1]$$

is a triangle in $\mathcal{T}$ such that the following hold:

(i) *The morphism α is a left $(\mathrm{add}(M), F, \Phi)$-approximation of X and β is a right $(\mathrm{add}(M), F, -\Phi)$-approximation of Y;*

(ii) $\mathrm{Hom}_{\mathcal{T}}(M, F^iX) = 0 = \mathrm{Hom}_{\mathcal{T}}(F^{-i}Y, M)$ *for all $0 \neq i \in \Phi$.*

Then $\mathbb{E}_{\mathcal{T}}^{F,\Phi}(X \oplus M)/I$ and $\mathbb{E}_{\mathcal{T}}^{F,\Phi}(M \oplus Y)/J$ are derived equivalent, where I and J are the above ideals of the Φ-Yoneda algebras $\mathbb{E}_{\mathcal{T}}^{F,\Phi}(X \oplus M)$ and $\mathbb{E}_{\mathcal{T}}^{F,\Phi}(M \oplus Y)$, contained in $\mathrm{End}_{\mathcal{T}}(X \oplus M)$ *and* $\mathrm{End}_{\mathcal{T}}(M \oplus Y)$*, respectively.*

A fifth level of generalization, using two functors F and G, will be discussed in the appendix. A further generalization of some results in this paper to n-angulated categories will be considered in [4].

The second level of generality, where F is the shift functor and both I and J are zero, is already widely applicable. This case happens frequently for the derived category $\mathcal{D}^{\mathrm{b}}(A)$ of an R-algebra A.

COROLLARY 1.2. *Let Φ be an admissible subset of $\mathbb{N}$, and let A be an R-algebra and M an A-module. If $0 \to X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \to 0$ is an exact sequence in A-mod such that α is a left $(\mathrm{add}(M), \Phi)$-approximation of X and β is a right $(\mathrm{add}(M), -\Phi)$-approximation of Y in $\mathcal{D}^{\mathrm{b}}(A)$, and that $\mathrm{Ext}^i_A(M, X) = 0 = \mathrm{Ext}^i_A(Y, M)$ for all $0 \neq i \in \Phi$, then the Φ-Yoneda algebras $\mathbb{E}^{\Phi}_A(X \oplus M)$ and $\mathbb{E}^{\Phi}_A(M \oplus Y)$ are derived equivalent.*

These results partly generalize some results of [11].

The setup here, and the main result, covers, combines and extends several classical concepts.

Auslander algebras (endomorphism rings of direct sums of 'all' modules of an algebra of finite representation type) are the ingredients of the celebrated Auslander correspondence, characterizing finite representation type via homological dimensions. Auslander algebras of derived equivalent algebras are, in general, not derived equivalent; positive results in this direction (for self-injective algebras of finite representation type) have previously been obtained in [10]. In the current approach, new results can be obtained by appropriate choices of $X \oplus M$.

Another intensively studied class of algebras is that of Yoneda algebras, that is, algebras of self-extensions of a semisimple module, or, more generally, of any module. The constructions in corollary 1.2 and in [10] would appear to provide the first general class of derived equivalences for Yoneda algebras. Perforated Yoneda algebras were first defined in [10], under the name Φ-Auslander–Yoneda algebras. The approach developed there was based on the existence of particular kinds of derived equivalences for algebras, which were then used to construct derived equivalences for perforated Yoneda algebras.

The main novelty of the present approach is the systematic use of cohomological data, such as cohomological approximations and perforated Yoneda algebras. This effectively relates our approach to a wide variety of concepts, such as Auslander–Reiten sequences and triangles, dominant dimension, Calabi–Yau categories and Frobenius categories.

The paper is organized as follows. In § 2, we first fix notation and then recall definitions and basic results on derived equivalences as well as on admissible sets and perforated Yoneda algebras. Also, we extend the notion of $\mathcal{D}$-approximation to what we call cohomological $\mathcal{D}$-approximation with respect to (F, Φ), where F is a functor and Φ is a subset of $\mathbb{N}$. In § 3, the main result, theorem 1.1, is proven and various easier to access situations are described, for which the assumptions of theorem 1.1 are satisfied. Section 4 explains how theorem 1.1 applies to a variety of situations: derived categories of Artin algebras, Frobenius categories and Calabi–Yau categories. Also, the connection to the concept of dominant dimension is explained. In § 5, two examples are given to illustrate the results and to show the necessity of some assumptions in theorem 1.1. In the appendix, a more general formulation of theorem 1.1 is stated, which involves two functors, in order to add more flexibility with a view to potential future applications.

2. Preliminaries

In this section, we shall recall basic definitions and facts that will be needed in the proofs later on.

2.1. Conventions

Throughout this paper, R is a fixed commutative Artinian ring with identity. Given an R-algebra A, by an A-module we mean a unitary left A-module; the category of all (respectively, finitely generated) A-modules is denoted by A-Mod (respectively, A-mod), the full subcategory of A-Mod consisting of all (respectively, finitely generated) projective modules is denoted by A-Proj (respectively, A-proj). There is a similar notation for right A-modules. The stable module category A-$\underline{\mathrm{mod}}$ of A is, by definition, the quotient category of A-mod modulo the ideal generated by homomorphisms factorizing through projective modules in A-proj. An equivalence between the stable module categories of two algebras is called a *stable equivalence*.

An R-algebra A is called an *Artin R-algebra* if A is finitely generated as an R-module. For an Artin R-algebra A, we denote by D the usual duality on A-mod, and by ν_A the Nakayama functor $D\operatorname{Hom}_A(\cdot, {}_AA)\colon A\text{-proj} \to A\text{-inj}$. For an A-module M, we denote the first syzygy of M by $\Omega_A(M)$, and call Ω_A the *Heller loop operator* of A. The transpose of M, which is an A^{op}-module, is denoted by $\mathrm{Tr}(M)$.

Let $\mathcal{C}$ be an additive R-category, that is, $\mathcal{C}$ is an additive category in which the set of morphisms between two objects in $\mathcal{C}$ is an R-module, and the composition of morphisms in $\mathcal{C}$ is R-bilinear. For an object X in $\mathcal{C}$, we denote by $\mathrm{add}(X)$ the full subcategory of $\mathcal{C}$ consisting of all direct summands of finite direct sums of copies of X. An object X in $\mathcal{C}$ is called an *additive generator* for $\mathcal{C}$ if $\mathcal{C} = \mathrm{add}(X)$. For two morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ in $\mathcal{C}$, we write fg for their composite. Thus, for an A-module X, we always have a natural A-$\mathrm{End}_A(X)$-bimodule structure on X. We shall not consider thus any bi-structure of categories for two functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$. However, we write GF for the composite instead of FG.

Given an object $M \in \mathcal{C}$, we say that a morphism $f\colon X \to Y$ in $\mathcal{C}$ factorizes through $\mathrm{add}(M)$ if there are morphisms $f_1\colon X \to M'$ and $f_2\colon M' \to Y$ in $\mathcal{C}$ with $M' \in \mathrm{add}(M)$ such that $f = f_1f_2$. Given a morphism $g\colon U \to V$ in $\mathcal{C}$, we say that a morphism $\alpha\colon W \to V$ (respectively, $\beta\colon U \to W$) factorizes through g if there exists a morphism $\alpha'\colon W \to U$ (respectively, $\beta'\colon V \to W$) such that $\alpha = \alpha' g$ (respectively, $\beta = g\beta'$).

If $f\colon X \to Y$ is a map between two sets X and Y, we denote the image of f by $\mathrm{Im}(f)$. Moreover, if f is a homomorphism between two abelian groups, we denote the kernel and cokernel of f by $\mathrm{Ker}(f)$ and $\mathrm{Coker}(f)$, respectively.

Recall that a functor $F\colon \mathcal{C} \to \mathcal{D}$ is an *equivalence* if there is a functor $G\colon \mathcal{D} \to \mathcal{C}$ such that $GF \simeq \mathrm{id}_{\mathcal{C}}$ and $FG \simeq \mathrm{id}_{\mathcal{D}}$. The functor G is called a *quasi-inverse* of F. In this case, we write F^{-1} for G. If $\mathcal{C} = \mathcal{D}$, then an equivalence F is called an *auto-equivalence*. An auto-equivalence F is called an *auto-isomorphism* if F has a quasi-inverse G such that $FG = GF = \mathrm{id}_{\mathcal{C}}$. If F is a functor from $\mathcal{C}$ to $\mathcal{C}$, then we write $F^0 = \mathrm{id}_{\mathcal{C}}$, and $F^{-i} = (F^{-1})^i$ for $i > 0$ if F^{-1} exists, and $F^{-i} = 0$ otherwise.

Let $\mathcal{T}$ be a triangulated R-category with a shift functor [1]. For two objects X and Y in $\mathcal{T}$, we sometimes write $\mathrm{Ext}^i_{\mathcal{T}}(X, Y)$ for $\mathrm{Hom}_{\mathcal{T}}(X, Y[i])$. Let Φ be a subset of $\mathbb{Z}$. An object M (or a full subcategory $\mathcal{M}$) of $\mathcal{T}$ is called *Φ-self-orthogonal* provided that $\mathrm{Ext}^i_{\mathcal{T}}(M, M) = 0$ (or $\mathrm{Ext}^i_{\mathcal{T}}(\mathcal{M}, \mathcal{M}) = 0$) for all $0 \neq i \in \Phi$, where $\mathrm{Ext}^i_{\mathcal{T}}(\mathcal{M}, \mathcal{M}) = 0$ means that $\mathrm{Ext}^i_{\mathcal{T}}(X, Y) = 0$ for all $X, Y \in \mathcal{M}$. In the case when $\Phi = \mathbb{Z}$, we say that M is *self-orthogonal*. For $\Phi = \{0, 1, \ldots, n\}$, we say that M is n-self-orthogonal, which is sometimes, perhaps less suggestively, referred to as n-rigid.

Replacing the shift functor by a triangle auto-equivalence F, one may also define the notion of (F, Φ)-self-orthogonality, but we refrain from introducing this notion here.

2.2. Derived equivalences

Let $\mathcal{C}$ be an additive R-category.

By a *complex* $X^\bullet$ over $\mathcal{C}$ we mean a sequence of morphisms d_X^i between objects X^i in

$$\mathcal{C}\colon\ \cdots \to X^i \xrightarrow{d_X^i} X^{i+1} \xrightarrow{d_X^{i+1}} X^{i+2} \to \cdots,$$

such that $d_X^i d_X^{i+1} = 0$ for all $i \in \mathbb{Z}$; we write $X^\bullet = (X^i, d_X^i)$. For a complex $X^\bullet$, the *brutal truncation* $\sigma_{<i}X^\bullet$ of $X^\bullet$ is a quotient complex of $X^\bullet$ such that $(\sigma_{<i}X^\bullet)^k$ is X^k for all $k < i$ and zero otherwise. We define $\sigma_{\geqslant i}X^\bullet$ similarly. For a fixed $n \in \mathbb{Z}$, we denote by $X^\bullet[n]$ the complex obtained from $X^\bullet$ by shifting degree by n, that is, $(X^\bullet[n])^0 = X^n$.

The category of all complexes over $\mathcal{C}$ with chain maps is denoted by $\mathcal{C}(\mathcal{C})$. The homotopy category of complexes over $\mathcal{C}$ is denoted by $\mathcal{K}(\mathcal{C})$. When $\mathcal{C}$ is an abelian category, the derived category of complexes over $\mathcal{C}$ is denoted by $\mathcal{D}(\mathcal{C})$. The full subcategories of $\mathcal{K}(\mathcal{C})$ and $\mathcal{D}(\mathcal{C})$ consisting of bounded complexes over $\mathcal{C}$ are denoted by $\mathcal{K}^{\mathrm{b}}(\mathcal{C})$ and $\mathcal{D}^{\mathrm{b}}(\mathcal{C})$, respectively. As usual, for an algebra A, we simply write $\mathcal{C}(A)$ for $\mathcal{C}(A\text{-mod})$, $\mathcal{K}(A)$ for $\mathcal{K}(A\text{-mod})$ and $\mathcal{K}^{\mathrm{b}}(A)$ for $\mathcal{K}^{\mathrm{b}}(A\text{-mod})$. Similarly, we write $\mathcal{D}(A)$ and $\mathcal{D}^{\mathrm{b}}(A)$ for $\mathcal{D}(A\text{-mod})$ and $\mathcal{D}^{\mathrm{b}}(A\text{-mod})$, respectively.

For an R-algebra A, the categories $\mathcal{K}(A)$ and $\mathcal{D}(A)$ are triangulated R-categories. For basic results on triangulated categories, we refer the reader to [8, 18, 24, 25].

The following result, due to Rickard [20, theorem 6.4] by a direct approach, and to Keller by working in the more general setup of differential graded algebras, is fundamental in the investigation of derived equivalences.

THEOREM 2.1 (Rickard [20]). *Let Λ and Γ be two rings. The following conditions are equivalent.*

(a) $\mathcal{K}^-(\Lambda\text{-}Proj)$ *and* $\mathcal{K}^-(\Gamma\text{-}Proj)$ *are equivalent as triangulated categories.*

(b) $\mathcal{D}^{\mathrm{b}}(\Lambda\text{-}Mod)$ *and* $\mathcal{D}^{\mathrm{b}}(\Gamma\text{-}Mod)$ *are equivalent as triangulated categories.*

(c) $\mathcal{K}^{\mathrm{b}}(\Lambda\text{-}Proj)$ *and* $\mathcal{K}^{\mathrm{b}}(\Gamma\text{-}Proj)$ *are equivalent as triangulated categories.*

(d) $\mathcal{K}^{\mathrm{b}}(\Lambda\text{-}proj)$ *and* $\mathcal{K}^{\mathrm{b}}(\Gamma\text{-}proj)$ *are equivalent as triangulated categories.*

(e) Γ *is isomorphic to* $\mathrm{End}_{\mathcal{K}^{\mathrm{b}}(\Lambda\text{-}proj)}(T^\bullet)$*, where* $T^\bullet$ *is a complex in* $\mathcal{K}^{\mathrm{b}}(\Lambda\text{-}proj)$ *satisfying the following:*

  (1) $T^\bullet$ *is self-orthogonal, that is,* $\mathrm{Hom}_{\mathcal{K}^{\mathrm{b}}(\Lambda\text{-}proj)}(T^\bullet, T^\bullet[i]) = 0$ *for all* $i \neq 0$*;*

  (2) $\mathrm{add}(T^\bullet)$ *generates* $\mathcal{K}^{\mathrm{b}}(\Lambda\text{-}proj)$ *as a triangulated category.*

Two rings Λ and Γ are called *derived equivalent* if the above conditions (a)–(e) are satisfied. A complex $T^\bullet$ in $\mathcal{K}^{\mathrm{b}}(\Lambda\text{-proj})$ as above is called a *tilting complex* over Λ.

For Artin algebras, the above equivalent conditions can be reformulated in terms of finitely generated modules: two Artin R-algebras A and B are said to be *derived* *equivalent* if their derived categories $\mathcal{D}^{\rm b}(A)$ and $\mathcal{D}^{\rm b}(B)$ are equivalent as triangulated categories. In this case, there is a tilting complex $T^\bullet$ in $\mathcal{K}^{\rm b}(A\text{-proj})$ such that $B \simeq \mathrm{End}_{\mathcal{K}^{\rm b}(A)}(T^\bullet)$.

2.3. Admissible subsets and Φ-Yoneda algebras

Let $\mathbb{N} = \{0, 1, 2, \dots\}$ be the set of natural numbers, and let $\mathbb{Z}$ be the set of all integers. For a natural number n or infinity, let $\mathbb{N}_n := \{i \in \mathbb{N} \mid 0 \leqslant i < n+1\}$.

Recall from [10] that a subset Φ of $\mathbb{Z}$ containing 0 is called an *admissible subset* of $\mathbb{Z}$ if the following condition is satisfied:

> *if i, j and k are in Φ such that $i+j+k \in \Phi$, then $i+j \in \Phi$ if and only if $j+k \in \Phi$.*

Clearly, if Φ is an admissible subset of $\mathbb{Z}$, then so is $-\Phi := \{-x \mid x \in \Phi\}$. Any subset $\{0, i, j\}$ of $\mathbb{N}$ is an admissible subset of $\mathbb{Z}$. Moreover, for any subset Φ of $\mathbb{N}$ containing zero and for any positive integer $m \geqslant 3$, the set $\{x^m \mid x \in \Phi\}$ is admissible in $\mathbb{Z}$ (for more examples, see [10]). Nevertheless, not every subset of $\mathbb{N}$ containing zero is admissible; for instance, $\{0, 1, 2, 4\}$ is not admissible. In fact, this is the 'smallest' non-admissible subset of $\mathbb{N}$.

Admissible sets were used to define Φ-Yoneda algebras in [10], under the name of 'Φ-Auslander–Yoneda algebras'. The formulation there works more generally for monoid graded algebras. For our purpose in this paper, we restrict our exposition to the case of an object in a triangulated category.

Let Φ be an admissible subset of $\mathbb{Z}$, and let $\mathcal{T}$ be a triangulated R-category with a shift functor [1]. Suppose that F is a triangle functor from $\mathcal{T}$ to $\mathcal{T}$. Recall that we set $F^i = 0$ for $i < 0$ if F^{-1} does not exist.

Let $\mathbb{E}^{F,\Phi}_{\mathcal{T}}(\cdot,\cdot)$ be the bi-functor

$$\bigoplus_{i\in\Phi} \mathrm{Hom}_{\mathcal{T}}(\cdot, F^i\cdot)\colon \mathcal{T}\times\mathcal{T} \to R\text{-Mod}, \tag{2.1}$$

$$(X, Y) \mapsto \mathbb{E}^{F,\Phi}_{\mathcal{T}}(X, Y) := \bigoplus_{i\in\Phi} \mathrm{Hom}_{\mathcal{T}}(X, F^iY), \tag{2.2}$$

$$X \xrightarrow{f} X' \mapsto \bigoplus_{i\in\Phi} \mathrm{Hom}_{\mathcal{T}}(f, F^iY), \qquad Y \xrightarrow{g} Y' \mapsto \bigoplus_{i\in\Phi} \mathrm{Hom}_{\mathcal{T}}(X, F^ig). \tag{2.3}$$

Suppose that X, Y and Z are objects in $\mathcal{T}$. Let $f = (f_i)_{i\in\Phi} \in \mathbb{E}^{F,\Phi}_{\mathcal{T}}(X, Y)$ and $g = (g_i)_{i\in\Phi} \in \mathbb{E}^{F,\Phi}_{\mathcal{T}}(Y, Z)$. We define a composition as follows:

$$\mathbb{E}^{F,\Phi}_{\mathcal{T}}(X, Y) \times \mathbb{E}^{F,\Phi}_{\mathcal{T}}(Y, Z) \to \mathbb{E}^{F,\Phi}_{\mathcal{T}}(X, Z),$$
$$(f, g) \mapsto fg := \Bigg(\sum_{\substack{u,v\in\Phi\\ u+v=i}} f_u * g_v \Bigg)_{i\in\Phi},$$

where $f_u * g_v := f_u F(g_v)\chi(u, v)_Z$ with $\chi(u, v)$ being a natural transformation from F^uF^v to F^{u+v} defined as follows.

If F^{-1} does not exist, then $\chi(u, v) = 0$ if u or v is negative, and $\chi(u, v) = \mathrm{id}_{F^{u+v}}$ otherwise.

If F^{-1} exists, then (F, F^{-1}) is an adjoint pair. Let $\epsilon\colon \mathrm{id}_{\mathcal{T}} \to F^{-1}F$ be the unit and let $\eta\colon FF^{-1} \to \mathrm{id}_{\mathcal{T}}$ be the counit. The natural transformation $\chi(u,v)$ is defined to be $\mathrm{id}_{F^{u+v}}$ if $uv \geqslant 0$. If $u > 0$ and $v < 0$, then $\chi(u,v)$ is defined as the composite of a sequence of natural transformations

$$F^uF^v \xrightarrow{F^{u-1}\eta_{F^{v+1}}} F^{u-1}F^{v+1} \xrightarrow{F^{u-2}\eta_{F^{v+2}}} F^{u-2}F^{v+2} \to \cdots \to F^{u-i}F^{v+i} \to \cdots \to F^{u+v}.$$

If $u < 0$ and $v > 0$, then $\chi(u,v)$ is defined as the composite of the following natural transformations

$$F^uF^v \xrightarrow{F^{u+1}\epsilon^{-1}_{F^{v-1}}} F^{u+1}F^{v-1} \xrightarrow{F^{u+2}\epsilon^{-1}_{F^{v-2}}} F^{u+2}F^{v-2} \to \cdots \to F^{u+i}F^{v-i} \to \cdots \to F^{u+v}.$$

In this setting, the above-defined composition is associative. To prove this, one needs to check that the multiplication $f_u * g_v$ is associative. This follows if the following diagram is commutative:

$$\begin{array}{ccc} F^iF^jF^k & \xrightarrow{F^i\chi(j,k)} & F^iF^{j+k} \\ \downarrow{\scriptstyle \chi(i,j)_{F^k}} & & \downarrow{\scriptstyle \chi(i,j+k)} \\ F^{i+j}F^k & \xrightarrow{\chi(i+j,k)} & F^{i+j+k} \end{array}$$

for all integers $i, j, k \in \Phi$. However, using the fact that

$$F(\epsilon)\eta_F = \mathrm{id}_F \quad \text{and} \quad \eta_{F^{-1}}F^{-1}(\epsilon) = \mathrm{id}_{F^{-1}},$$

one can get the above commutative diagram by drawing a big commutative diagram with the above two sequences of natural transformations. Here, we omit the details. Note that if F is an auto-isomorphism, that is, $FF^{-1} = \mathrm{id}_{\mathcal{T}} = F^{-1}F$, then $\chi(u,v)$ is an identity for all $u, v \in \mathbb{Z}$, and therefore will not appear in the definition of the multiplication.

Thus, $\mathbb{E}^{F,\Phi}_{\mathcal{T}}(X,X)$ is an R-algebra. It is called the *Φ-Yoneda algebra* or, when Φ is fixed, the *perforated Yoneda algebra* of X with respect to F. Then $\mathbb{E}^{F,\Phi}_{\mathcal{T}}(X,Y)$ is a left $\mathbb{E}^{F,\Phi}_{\mathcal{T}}(X,X)$-module. When $\Phi = \mathbb{N}$, the algebra $\mathbb{E}^{F,\Phi}_{\mathcal{T}}(X,X)$ is the orbit algebra of X under F (see [2]).

For convenience, we write $\mathbb{E}^{F,\Phi}_{\mathcal{T}}(X)$ for $\mathbb{E}^{F,\Phi}_{\mathcal{T}}(X,X)$. In the case $\mathcal{T} = \mathcal{D}^{\mathrm{b}}(A)$, where A is a ring with identity, we write $\mathbb{E}^{F,\Phi}_{A}(X,Y)$ for $\mathbb{E}^{F,\Phi}_{\mathcal{D}^{\mathrm{b}}(A)}(X,Y)$ and $\mathbb{E}^{F,\Phi}_{A}(X)$ for $\mathbb{E}^{F,\Phi}_{\mathcal{D}^{\mathrm{b}}(A)}(X)$.

When F coincides with the shift functor, we omit the upper index F, and call $\mathbb{E}^{\Phi}_{\mathcal{T}}(X)$ the Φ-Yoneda algebra of X, without referring to the shift functor. This is the algebra introduced in [10] and therein called an Auslander–Yoneda algebra.

The following lemma is essentially taken from [10, lemma 3.5], where a variation of it appears. The proof given there carries over to the present situation.

Lemma 2.2. *Let $\mathcal{T}$ be a triangulated R-category with a triangle endo-functor F, and let U be an object in $\mathcal{T}$. Suppose that U_1, U_2 and U_3 are in* $\mathrm{add}(U)$*, and that Φ is an admissible subset of $\mathbb{Z}$. Then, we have the following.*

(i) *There exists a natural isomorphism*

$$\mu\colon \mathbb{E}_{\mathcal{T}}^{F,\Phi}(U_1,U_2) \to \operatorname{Hom}_{\mathbb{E}_{\mathcal{T}}^{F,\Phi}(U)}(\mathbb{E}_{\mathcal{T}}^{F,\Phi}(U,U_1),\mathbb{E}_{\mathcal{T}}^{F,\Phi}(U,U_2))$$

that sends $x \in \mathbb{E}_{\mathcal{T}}^{F,\Phi}(U_1,U_2)$ *to the morphism* $a \mapsto ax$ *for* $a \in \mathbb{E}_{\mathcal{T}}^{F,\Phi}(U,U_1)$*. Moreover, if* $x \in \mathbb{E}_{\mathcal{T}}^{F,\Phi}(U_1,U_2)$ *and* $y \in \mathbb{E}_{\mathcal{T}}^{F,\Phi}(U_2,U_3)$*, then* $\mu(xy)=\mu(x)\mu(y)$*.*

(ii) *The functor* $\mathbb{E}_{\mathcal{T}}^{F,\Phi}(U,\cdot)\colon \operatorname{add}(U) \to \mathbb{E}_{\mathcal{T}}^{F,\Phi}(U)$*-proj is faithful.*

(iii) *If* $\operatorname{Hom}_{\mathcal{T}}(U_1,F^iU_2)=0$ *for all* $i \in \Phi \setminus \{0\}$*, then the functor* $\mathbb{E}_{\mathcal{T}}^{F,\Phi}(U,\cdot)$ *induces an isomorphism of* R*-modules:*

$$\mathbb{E}_{\mathcal{T}}^{F,\Phi}(U,\cdot)\colon \operatorname{Hom}_{\mathcal{T}}(U_1,U_2) \to \operatorname{Hom}_{\mathbb{E}_{\mathcal{T}}^{F,\Phi}(U)}(\mathbb{E}_{\mathcal{T}}^{F,\Phi}(U,U_1),\mathbb{E}_{\mathcal{T}}^{F,\Phi}(U,U_2)).$$

The properties described in lemma 2.2 will be used frequently in the proofs below.

The class of Φ-Yoneda algebras with respect to a functor includes a large class of algebras, including the following.

(a) The endomorphism algebra of a module, in particular, the Auslander algebras of representation-finite algebras; Here we choose $\Phi=\{0\}$.

(b) The generalized Yoneda algebra of a module if we take $\Phi=\mathbb{N}$: this includes the pre-projective algebras (see [2]) and the Hochschild cohomology rings of given algebras. Choosing $\Phi=2\mathbb{N}$, we get, for instance, the even Hochschild cohomology rings of algebras.

(c) Certain trivial extensions: for an Artin algebra A and an A-module M, we choose $\Phi=\{0,i\}$ for $i \geqslant 1$ an arbitrary natural number. Then $\mathbb{E}_A^{\Phi}(M)$ is the trivial extension of $\operatorname{End}_A(M)$ by the bimodule $\operatorname{Ext}_A^i(M,M)$. Such rings appear naturally in the (bounded) derived category $\mathcal{D}^{\mathrm{b}}(\mathbb{X})$ of coherent sheaves of a smooth projective variety $\mathbb{X}$ over $\mathbb{C}$. Indeed, if X is a d-spherical object in $\mathcal{D}^{\mathrm{b}}(\mathbb{X})$, then its cohomological ring $\operatorname{End}^{\bullet}_{\mathcal{D}^{\mathrm{b}}(\mathbb{X})}(X)$ is $\mathbb{E}^{\{0,d\}}_{\mathcal{D}^{\mathrm{b}}(\mathbb{X})}(X)$; this is a graded ring isomorphic to $\mathbb{C}[t]/(t^2)$ with t of degree d. For further information on spherical objects, we refer the reader to [22, § 3c].

In general, if $\Phi=\{0,a_1,\dots,a_n\} \subseteq \mathbb{N}$ such that $a_i > 2a_{i-1}$ for $i=2,\dots,n$, then $\mathbb{E}_A^{\Phi}(X)$ is the trivial extension of $\operatorname{End}_A(X)$ by the bimodule

$$\bigoplus_{0 \neq i \in \Phi} \operatorname{Ext}_A^i(X,X).$$

Note that $\Phi=\{0\}\cup\{2n+1 \mid n \in \mathbb{N}\}$ is admissible. In this case, we also get a trivial extension.

(d) The polynomial ring $R[t]$: if we take $\Phi=m\mathbb{N}$ for $m \geqslant 1$, then the perforated Yoneda algebra $\mathbb{E}^{\Phi}_{R[x]/(x^2)}(R)$ is isomorphic to $R[t^m]$ with t a variable. If $\Phi=\{0,1,\dots,n\}$, then

$$\mathbb{E}^{\Phi}_{R[x]/(x^2)}(R) \simeq R[t]/(t^n).$$

2.4. $\mathcal{D}$-split sequences and cohomological $\mathcal{D}$-approximations

$\mathcal{D}$-split sequences have been defined in [11] in the context of constructing derived equivalences between certain endomorphism algebras. Let us recall the definition and a result in [11].

Let $\mathcal{C}$ be an additive category and $\mathcal{D}$ be a full subcategory of $\mathcal{C}$. A sequence

$$X \xrightarrow{f} M \xrightarrow{g} Y$$

in $\mathcal{C}$ is called a *$\mathcal{D}$-split sequence* if

(i) $M \in \mathcal{D}$,

(ii) f is a left $\mathcal{D}$-approximation of X and g is a right $\mathcal{D}$-approximation of Y and

(iii) f is a kernel of g and g is a cokernel of f.

Typical examples of $\mathcal{D}$-split sequences are Auslander–Reiten sequences. Every $\mathcal{D}$-split sequence provides a derived equivalence (see [11, theorem 1.1]). Here are some details, for later reference.

Theorem 2.3 (Hu and Xi [11]). *Let $\mathcal{C}$ be an additive category and M be an object in $\mathcal{C}$. Suppose that*

$$X \to M' \to Y$$

is an $\operatorname{add}(M)$*-split sequence in $\mathcal{C}$. Then the endomorphism ring* $\operatorname{End}_{\mathcal{C}}(M \oplus X)$ *of $M \oplus X$ is derived equivalent to the endomorphism ring* $\operatorname{End}_{\mathcal{C}}(M \oplus Y)$ *of $M \oplus Y$ via a tilting module of projective dimension at most* 1.

Now, the question arises of whether theorem 2.3 can be extended to Φ-Yoneda algebras. The second example in § 5 demonstrates that this is no longer true if we just replace the endomorphism algebras in theorem 2.3 by Φ-Yoneda algebras. Nevertheless, we shall show that under certain orthogonality conditions there is still a positive answer. This will be discussed in detail in the next section.

The condition (3) of a $\mathcal{D}$-split sequence is a substitute in this general setup for requiring the short sequence to be exact. Since triangles in triangulated categories are replacements of short exact sequences, we may reformulate the notion of $\mathcal{D}$-split sequences in the following sense for triangulated categories.

Let $\mathcal{T}$ be a triangulated category with a shift functor [1], and let $\mathcal{D}$ be a full additive subcategory of $\mathcal{T}$. A triangle

$$X \xrightarrow{\alpha} M' \xrightarrow{\beta} Y \to X[1]$$

in $\mathcal{T}$ is called a *$\mathcal{D}$-split triangle* if $M' \in \mathcal{D}$, the map α is a left $\mathcal{D}$-approximation of X and the map β is a right $\mathcal{D}$-approximation of Y.

Thus, for an Artin R-algebra A, every $\mathcal{D}$-split sequence in A-mod extends to a $\mathcal{D}$-split triangle in $\mathscr{D}^{\mathrm{b}}(A)$.

Next, we introduce the left and right cohomological $\mathcal{D}$-approximations with respect to (F, Φ), which generalize the notions of left and right $\mathcal{D}$-approximations, respectively.

Suppose that $\mathcal{C}$ is a category with an endo-functor $F \colon \mathcal{C} \to \mathcal{C}$. Let $\mathcal{D}$ be a full subcategory of $\mathcal{C}$, and let Φ be a non-empty subset of $\mathbb{N}$. If F has an inverse, then Φ may be chosen to be a subset of $\mathbb{Z}$. Suppose that X is an object of $\mathcal{C}$. A morphism $f\colon X \to D$ in $\mathcal{C}$ is called a *left cohomological $\mathcal{D}$-approximation* of X with respect to (F, Φ) (or, for short, a left $(\mathcal{D}, F, \Phi)$-approximation of X) if $D \in \mathcal{D}$, and for any morphism $g\colon X \to F^i(D')$ with $D' \in \mathcal{D}$ and $i \in \Phi$ there is a morphism $g'\colon D \to F^i(D')$ such that $g = fg'$. Here $F^0 = \mathrm{id}_{\mathcal{C}}$. Similarly, we have the notion of a right $(\mathcal{D}, F, \Phi)$-approximation of X in $\mathcal{T}$, i.e. a morphism $f\colon D \to X$ with D in $\mathcal{D}$ is called a right $(\mathcal{D}, F, \Phi)$-approximation of X if, for any $i \in \Phi$ and any morphism $g\colon F^iD' \to X$ with D' in $\mathcal{D}$, there is a morphism $g'\colon F^iD' \to D$ such that $g = g'f$.

Note that if $F = \mathrm{id}_{\mathcal{C}}$ and $\Phi = \{0\}$, then we get the original notion of approximations in the sense of Auslander and Smalø. (In ring theory, such approximations are called pre-envelope and pre-cover, respectively). Moreover, if $0 \in \Phi$, then every left $(\mathcal{D}, F, \Phi)$-approximation of X is also a left $\mathcal{D}$-approximation of X and every right $(\mathcal{D}, F, \Phi)$-approximation of X is also a right $\mathcal{D}$-approximation of X.

If $F = [1]$ and $\mathcal{T} = \mathcal{D}^{\mathrm{b}}(A)$ for an Artin algebra A, then $\mathrm{Hom}_{\mathcal{T}}(X, F^iY) \simeq \mathrm{Ext}^i_A(X, Y)$ for all $X, Y \in A$-mod and all $i \geqslant 0$. For this reason, a $(\mathcal{D}, F, \Phi)$-approximation has been called a *cohomological* approximation.

In this paper, we are mainly interested in the case where $\mathcal{C}$ is a triangulated R-category $\mathcal{T}$ with a triangle functor F from $\mathcal{T}$ to itself and $\mathcal{D}$ is a full subcategory of $\mathcal{T}$. Thus, a morphism $f\colon X \to D$ with $D \in \mathcal{D}$ and $X \in \mathcal{T}$ is a left $(\mathcal{D}, F, \Phi)$-approximation of X if and only if the canonical map

$$\mathbb{E}^{F,\Phi}_{\mathcal{T}}(f, D')\colon \mathbb{E}^{F,\Phi}_{\mathcal{T}}(D, D') \to \mathbb{E}^{F,\Phi}_{\mathcal{T}}(X, D'),$$

defined by $(x_i)_{\in\Phi} \mapsto (fx_i)_{i\in\Phi}$, is surjective for all $D' \in \mathcal{D}$. Similarly, a morphism $g\colon D \to X$ with $D \in \mathcal{D}$ and $X \in \mathcal{T}$ is a right $(\mathcal{D}, F, \Phi)$-approximation of X if and only if the canonical map

$$\mathrm{Hom}_{\mathcal{T}}(F^jD', g)\colon\ \mathrm{Hom}_{\mathcal{T}}(F^jD', D) \to \mathrm{Hom}_{\mathcal{T}}(F^jD', X)$$

is surjective for every $D' \in \mathcal{D}$ and $j \in \Phi$. If, moreover, F is a triangle auto-equivalence, then a morphism $g\colon D \to X$ with $D \in \mathcal{D}$ and $X \in \mathcal{T}$ is a right $(\mathcal{D}, F, \Phi)$-approximation of X if and only if the canonical map

$$\mathbb{E}^{F,-\Phi}_{\mathcal{T}}(D', g)\colon \mathbb{E}^{F,-\Phi}_{\mathcal{T}}(D', D) \to \mathbb{E}^{F,-\Phi}_{\mathcal{T}}(D', X)$$

is surjective for all $D' \in \mathcal{D}$. Note that here we need the minus sign for Φ and that F^{-1} exists.

Note that if Φ contains zero and if $\mathrm{Hom}_{\mathcal{T}}(X, F^iD') = 0$ for all $0 \neq i \in \Phi$ and $D' \in \mathcal{D}$, then f is a left $(\mathcal{D}, F, \Phi)$-approximation of X if and only if f is a left $\mathcal{D}$-approximation of X. A dual statement is also true for a right $(\mathcal{D}, F, \Phi)$-approximation of X.

If F coincides with the shift functor [1], then we simply speak of $(\mathcal{D}, \Phi)$-approximations, without mentioning F.

Here is a source of examples of $(\mathcal{D}, \Phi)$-approximations. Suppose that $\mathcal{T} = \mathcal{D}^{\mathrm{b}}(A)$ for A an Artin R-algebra and that Φ is a subset of $\mathbb{Z}$. Let

$$X \xrightarrow{\alpha} M \xrightarrow{\beta} Y \to X[1]$$

be an Auslander–Reiten triangle in $\mathcal{T}$. If neither X nor Y belongs to $\mathrm{add}(M[i])$ for every $0 \neq i \in \Phi$, then α is a left $(\mathrm{add}(M), \Phi)$-approximation of X and β is a right $(\mathrm{add}(M), \Phi)$-approximation of Y.

Finally, we note the difference of a left $(\mathcal{D}, F, \Phi)$-approximation of X from a left $(\bigcup_{i\in\Phi} F^i\mathcal{D})$-approximation of X in the sense of Auslander and Smalø, where $\bigcup_{i\in\Phi} F^i\mathcal{D}$ is the full subcategory of $\mathcal{T}$ with all objects in $F^i\mathcal{D}$ for all $i \in \Phi$. Suppose that $0 \in \Phi$. Then a $(\mathcal{D}, F, \Phi)$-approximation is a $(\bigcup_{i\in\Phi} F^i\mathcal{D})$-approximation, but the converse is not true in general. If $0 \notin \Phi$, then the two concepts are independent. So, roughly speaking, a cohomological $\mathcal{D}$-approximation with respect to (F, Φ) emphasizes not only the factorizations but also that the object belongs to the given subcategory $\mathcal{D}$ (and not to $F^i\mathcal{D}$ for $0 \neq i \in \Phi$).

3. Derived equivalences for Φ-Yoneda algebras

In this section, we shall prove theorem 1.1 and derive some consequences and some simplifications in special cases.

Suppose that $\mathcal{T}$ is a triangulated R-category with a shift functor [1] and M is an object in $\mathcal{T}$. Suppose that F is a triangle auto-equivalence of $\mathcal{T}$, which may be different from the shift functor.

For a subset Φ of $\mathbb{Z}$, we define $-\Phi := \{-x \mid x \in \Phi\}$ and

$$\mathcal{X}^{F,\Phi}_{\mathcal{T}}(M) = \{X \in \mathcal{T} \mid \mathrm{Hom}_{\mathcal{T}}(X, F^iM) = 0 \text{ for all } i \in \Phi \setminus \{0\}\},$$
$$\mathcal{Y}^{F,\Phi}_{\mathcal{T}}(M) = \{Y \in \mathcal{T} \mid \mathrm{Hom}_{\mathcal{T}}(M, F^iY) = 0 \text{ for all } i \in \Phi \setminus \{0\}\}.$$

Let n be a positive integer. For brevity, we write $\mathcal{X}^{F,n}(M)$ for $\mathcal{X}^{F,\{0,1,2,\dots,n\}}_{\mathcal{T}}(M)$ and $\mathcal{X}^{F,\infty}(M)$ for $\mathcal{X}^{F,\mathbb{N}}_{\mathcal{T}}(M)$ if $\mathcal{T}$ is clear in the context. $\mathcal{Y}^{F,n}(M)$ and $\mathcal{Y}^{F,\infty}(M)$ are defined similarly.

As usual, F is omitted in notation when it coincides with the shift functor.

Given a triangle

$$X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \xrightarrow{w} X[1] \quad \text{in } \mathcal{T}$$

with $M_1 \in \mathrm{add}(M)$, we define

$$\tilde{w} = (w, 0)\colon Y \to (X \oplus M)[1], \qquad \bar{w} = (0, w)^{\mathrm{T}}\colon M \oplus Y \to X[1],$$

where $(0, w)^{\mathrm{T}}$ denotes the transpose of the matrix $(0, w)$, and

$$I := \{x = (x_i) \in \mathbb{E}^{F,\Phi}_{\mathcal{T}}(X \oplus M) \mid x_i = 0 \text{ for } 0 \neq i \in \Phi,$$
$$x_0 \text{ factorizes through } \mathrm{add}(M) \text{ and } \tilde{w}[-1]\},$$
$$J := \{y = (y_i) \in \mathbb{E}^{F,\Phi}_{\mathcal{T}}(M \oplus Y) \mid y_i = 0 \text{ for } 0 \neq i \in \Phi,$$
$$y_0 \text{ factorizes through } \mathrm{add}(M) \text{ and } \bar{w}\}.$$

The sets I and J are indeed independent of F and $\Phi \setminus \{0\}$, and contained in $\mathrm{End}_{\mathcal{T}}(X \oplus M)$ and $\mathrm{End}_{\mathcal{T}}(M \oplus Y)$, respectively.

The main result of this paper is the following theorem, which is a reformulation of theorem 1.1.

Theorem 3.1. *Let Φ be an admissible subset of $\mathbb{Z}$, let $\mathcal{T}$ be a triangulated R-category with a triangle auto-equivalence F and let M be an object in $\mathcal{T}$. Suppose that*

$$X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \xrightarrow{w} X[1]$$

is a triangle in $\mathcal{T}$ such that the morphism α is a left $(\mathrm{add}(M), F, \Phi)$-approximation of X, the morphism β is a right $(\mathrm{add}(M), F, -\Phi)$-approximation of Y and $X \in \mathcal{Y}^{F,\Phi}(M)$ and $Y \in \mathcal{X}^{F,\Phi}(M)$. Then the algebras

$$\mathbb{E}_{\mathcal{T}}^{F,\Phi}(X \oplus M)/I \quad \text{and} \quad \mathbb{E}_{\mathcal{T}}^{F,\Phi}(M \oplus Y)/J$$

are derived equivalent.

Proof. Let $V = X \oplus M$ and $W = M \oplus Y$. Set

$$\bar{\alpha} := (\alpha, 0)\colon X \to M_1 \oplus M, \qquad \tilde{\alpha} := \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} : X \oplus M \to M_1 \oplus M,$$

$$\bar{\beta} := \begin{pmatrix} 0 & \beta \\ 1 & 0 \end{pmatrix} : M_1 \oplus M \to M \oplus Y, \qquad \tilde{\beta} := \begin{pmatrix} \beta \\ 0 \end{pmatrix} : M_1 \oplus M \to Y,$$

$$\bar{w} := \begin{pmatrix} 0 \\ w \end{pmatrix} : M \oplus Y \to X[1], \qquad \tilde{w} := (w, 0)\colon Y \to (X \oplus M)[1].$$

Then there are two triangles in $\mathcal{T}$:

$$X \xrightarrow{\bar{\alpha}} M_1 \oplus M \xrightarrow{\bar{\beta}} W \xrightarrow{\bar{w}} X[1],$$

$$Y[-1] \xrightarrow{-\tilde{w}[-1]} V \xrightarrow{\tilde{\alpha}} M_1 \oplus M \xrightarrow{\tilde{\beta}} Y.$$

Since F is a triangle functor, there exists a natural isomorphism $\delta\colon F[1] \to [1]F$. That is, for any object X in $\mathcal{T}$, there is an isomorphism $\delta_X\colon F(X[1]) \to (FX)[1]$ that is natural in X. The isomorphism $F^i(X[j]) \to (F^iX)[j]$ is denoted by $\delta(F, i, X, j)$.

First, we have the following lemma.

Lemma 3.2.

(i) *For any morphism $y_i\colon V \to F^iV$ with $i \in \Phi$, there is a morphism $t_i\colon Y[-1] \to (F^iY)[-1]$ such that $(\tilde{w}[-1])y_i = t_i\delta(F, i, Y, -1)^{-1}(F^i(\tilde{w}[-1]))$.*

(ii) *For any morphism $x_i\colon W \to F^iW$ with $i \in \Phi$, there is a morphism $t_i\colon X[1] \to (F^iX)[1]$ such that $x_i(F^i\bar{w})\delta(F, i, X, 1) = \bar{w}t_i$.*

Proof. (i) Note that $\tilde{\alpha}$ is a left $(\mathrm{add}(M), F, \Phi)$-approximation of V. Thus, given $y_i\colon V \to F^iV$, there is a morphism $z_i\colon M_1 \oplus M \to F^i(M_1 \oplus M)$ such that $\tilde{\alpha}z_i = y_i(F^i\tilde{\alpha})$. Since F is a triangle functor, the second triangle implies that there is a triangle (see [8, p. 4])

$$(F^iY)[-1] \xrightarrow{\delta(F,i,Y,-1)^{-1}(-F^i(\tilde{w}[-1]))} F^iV \xrightarrow{F^i\tilde{\alpha}} F^i(M_1 \oplus M) \xrightarrow{F^i\tilde{\beta}} F^iY.$$

Thus, there is a morphism $t_i\colon Y[-1] \to (F^iY)[-1]$ such that

$$(\tilde{w}[-1])y_i = t_i\delta(F, i, Y, -1)^{-1}(F^i(\tilde{w}[-1])).$$

(ii) The proof is similar to that of (i), using the following triangle:

$$F^iX \xrightarrow{F^i\bar{\alpha}} F^i(M_1 \oplus M) \xrightarrow{F^i\bar{\beta}} F^iW \xrightarrow{(F^i\bar{w})\delta(F,i,X,1)} (F^iX)[1].$$

□

Now we prove that the quotient rings in theorem 3.1 are well defined.

LEMMA 3.3. *The I and J appearing in theorem 3.1 are ideals of $\mathbb{E}_{\mathcal{T}}^{F,\Phi}(V)$ and $\mathbb{E}_{\mathcal{T}}^{F,\Phi}(W)$, respectively.*

Proof. We shall only prove that I is an ideal in $\mathbb{E}_{\mathcal{T}}^{F,\Phi}(V)$. The proof for J can be carried out analogously.

The set I is closed under addition in $\mathbb{E}_{\mathcal{T}}^{F,\Phi}(V)$. To show that I is a two-sided ideal in $\mathbb{E}_{\mathcal{T}}^{F,\Phi}(V)$, we pick an $x = (x_i)_{i\in\Phi} \in I$ and a $y = (y_i)_{i\in\Phi} \in \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V)$ and calculate the products xy and yx in $\mathbb{E}_{\mathcal{T}}^{F,\Phi}(V)$. We write $x_0 = uv$ for $u\colon V \to M'$ and $v\colon M' \to V$, where M' is an object in $\mathrm{add}(M)$, and $x_0 = s(\tilde{w}[-1])$ for a morphism $s\colon V \to Y[-1]$. Note that $xy = (x_0 y_i)_{i\in\Phi}$ and $yx = (y_i F^i x_0)_{i\in\Phi}$, since $x_i = 0$ for $0 \neq i \in \Phi$.

We first show that I is a right ideal.

(i) Let $i = 0$. The map $x_0 y_0$ factorizes through an object in $\mathrm{add}(M)$. Since x_0 factorizes through $\tilde{w}[-1]$, it follows from lemma 3.2(i) that $x_0 y_0$ also factorizes through $\tilde{w}[-1]$.

(ii) Let $0 \neq i \in \Phi$. In this case, $\mathrm{Hom}_{\mathcal{T}}(M, F^i X) = 0$ by the assumption $X \in \mathcal{Y}^{F,\Phi}(M)$. Let p_X and p_M be the projections of V onto X and M, respectively. Then the composite

$$v y_i F^i p_X\colon M' \xrightarrow{v} V \xrightarrow{y_i} F^i V \xrightarrow{F^i p_X} F^i X$$

belongs to $\mathrm{Hom}_{\mathcal{T}}(M', F^i X) = 0$. Thus,

$$x_0 y_i F^i p_X = uv y_i F^i p_X = 0.$$

By lemma 3.2(i), there is a morphism $t_i\colon Y[-1] \to F^i Y[-1]$ such that

$$(\tilde{w}[-1]) y_i = t_i \delta(F, i, Y, -1)^{-1} F^i(\tilde{w}[-1]).$$

Hence,

$$\begin{aligned} x_0 y_i (F^i p_M) &= s(\tilde{w}[-1]) y_i (F^i p_M) = s t_i \delta(F, i, Y, -1)^{-1} F^i(\tilde{w}[-1])(F^i p_M) \\ &= s t_i \delta(F, i, Y, -1)^{-1} F^i(\tilde{w}[-1] p_M) \\ &= s t_i \delta(F, i, Y, -1)^{-1} F^i\left((w[-1], 0)\begin{pmatrix} 0 \\ 1_M \end{pmatrix}\right) = 0. \end{aligned}$$

Altogether, $x_0 y_i = x_0 y_i (F^i p_X, F^i p_M) = 0$ for $0 \neq i \in \Phi$.

Hence, $xy \in I$, and I is a right ideal in $\mathbb{E}_{\mathcal{T}}^{F,\Phi}(V)$.

Next, we show that I is a left ideal, that is, we check $(y_i F^i x_0)_{i\in\Phi} \in I$.

(iii) The map $y_0 x_0$ factorizes through an object in $\mathrm{add}(M)$ and through $\tilde{w}[-1]$.

(iv) Let $0 \neq i \in \Phi$. Note that $\tilde{\alpha}\colon V \to M_1 \oplus M$ is a left $(\mathrm{add}(M), F, \Phi)$-approximation of V. Thus, there is a morphism

$$h_i\colon M_1 \oplus M \to F^i(M')$$

such that $y_i(F^i u) = \tilde{\alpha} h_i$. By assumption, $\mathrm{Hom}_{\mathcal{T}}(M, F^i X) = 0$. This implies that $h_i(F^i v)(F^i p_X) = 0$, and therefore $y_i(F^i x_0)(F^i p_X) = 0$. Since $(F^i \tilde{w}[-1])(F^i p_M) = 0$, we get $y_i(F^i x_0)(F^i p_M) = 0$. Thus, $y_i F^i x_0 = 0$ for $0 \neq i \in \Phi$.

Hence, $yx \in I$, and I is a left ideal in $\mathbb{E}_{\mathcal{T}}^{F,\Phi} V$. Thus, I is an ideal in $\mathbb{E}_{\mathcal{T}}^{F,\Phi}(V)$. □

We know that $\mathbb{E}_{\mathcal{T}}^{F,\Phi}(V, Z)$ is an $\mathbb{E}_{\mathcal{T}}^{F,\Phi}(V)$-module for any object Z in $\mathcal{T}$. The following lemma shows that the ideal I of $\mathbb{E}_{\mathcal{T}}^{F,\Phi}(V)$ may annihilate some modules of this form.

Lemma 3.4. *Keep the notation as above. Then we have the following.*

(i) $I \cdot \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V, M) = 0$.

(ii)

$$I \cdot \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V, X) = \{(x_i)_{i\in\Phi} \in \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V, X) \mid x_i = 0 \textit{ for } 0 \neq i \in \Phi, \quad x_0 \textit{ factorizes through } \mathrm{add}(M) \textit{ and } w[-1]\}.$$

(iii) *For* $x = (x_i)_{i\in\Phi} \in \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V', X)$ *with* $V' \in \mathrm{add}(V)$, *we have*

$$\mathrm{Im}(\mu(x)) \subseteq I \cdot \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V, X)$$

if and only if $x_i = 0$ *for all* $0 \neq i \in \Phi$ *and* x_0 *factorizes through* $\mathrm{add}(M)$ *and* $w[-1]$, *where* μ *is defined in lemma 2.2(i).*

(iv) *Let* $f\colon M' \to X$ *with* $M' \in \mathrm{add}(M)$. *Then* $\mathrm{Im}(\mathbb{E}_{\mathcal{T}}^{F,\Phi}(V, f)) \subseteq I \cdot \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V, X)$ *if and only if* f *factorizes through* $w[-1]$.

Proof. (i) We denote by $\lambda_M = (0, 1)\colon M \to V$ the canonical inclusion. Let $(x_i)_{i\in\Phi} \in I$ and $(y_i)_{i\in\Phi} \in \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V, M)$. Then $(x_i)(y_i) = (x_0 y_i)_{i\in\Phi}$, since $x_i = 0$ for $0 \neq i \in \Phi$. It follows, since I is an ideal in $\mathbb{E}_{\mathcal{T}}^{F,\Phi}(V)$, that $x(y_i(F^i\lambda_M))_{i\in\Phi} = (x_0 y_i(F^i\lambda_M))_{i\in\Phi} \in I$. By the definition of I, we have $x_0 y_i(F^i\lambda_M) = 0$ for all $0 \neq i \in \Phi$ and $x_0 y_0 \lambda_M$ factorizes through $\tilde{w}[-1]$. Moreover,

$$x_0 y_0 \lambda_M = (x_0 y_0 \lambda_M p_M)\lambda_M = s(\tilde{w}[-1]p_M)\lambda_M = s \cdot 0 \cdot \lambda_M = 0,$$

where s is a morphism from V to $Y[-1]$. Hence, $x_0 y_i(F^i\lambda_M) = 0$ and

$$x_0 y_i = x_0 y_i(F^i\lambda_M)(F^i p_M) = 0 \cdot F^i p_M = 0 \quad \text{for all } i \in \Phi.$$

Thus, (i) follows.

(ii) Let $\lambda_X\colon X \to V$ be the canonical inclusion. As in (i), it follows that, for $(x_i)_{i\in\Phi} \in I$ and $(y_i)_{i\in\Phi} \in \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V, X)$, we have $(x_i)(y_i) = (x_0 y_i)_{i\in\Phi}$, and that $x_0 y_0 \lambda_X$ factorizes through $\tilde{w}[-1]$ and $\mathrm{add}(M)$. Hence,

$$x_0 y_0 = (x_0 y_0 \lambda_X)p_X = s(\tilde{w}[-1])p_X = s(w[-1]),$$

where s is a morphism from V to $Y[-1]$. Conversely, let $x = (x_i)_{i\in\Phi} \in \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V, X)$ and suppose that $x_i = 0$ for all $0 \neq i \in \Phi$ and that x_0 factorizes through $\mathrm{add}(M)$ and $w[-1]$. For $f\colon U \to Z$ in $\mathcal{T}$, we denote by $\underline{f}$ the element of $\mathbb{E}_{\mathcal{T}}^{F,\Phi}(U, Z)$ concentrated only in degree $0 \in \Phi$. Then it is straightforward to check that $x\underline{\lambda_X}$ belongs to I. Thus, $x = x\underline{\lambda_X}\,\underline{p_X} \in I \cdot \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V, X)$.

(iii) Firstly, suppose $V' = V$ and $\mathrm{Im}(\mu(x)) \subseteq I \cdot \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V,X)$. Then x, the image of $\underline{1_V}$ under $\mu(x)$, belongs to $I \cdot \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V,X)$. Thus, by (ii), we know that $x_i = 0$ for all $0 \neq i \in \Phi$ and that x_0 factorizes through $\mathrm{add}(M)$ and $w[-1]$. Conversely, suppose that $x \in I \cdot \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V,X)$. Then, for any $y \in \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V)$, the image of y under $\mu(x)$ is $y \cdot x$. Since $I \cdot \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V,X)$ is an $\mathbb{E}_{\mathcal{T}}^{F,\Phi}(V)$-submodule of $\mathbb{E}_{\mathcal{T}}^{F,\Phi}(V,X)$, we have $yx \in I \cdot \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V,X)$.

Secondly, suppose that V' is a direct sum of n copies of V, and $x \in \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V',X)$. We identify $\mathbb{E}_{\mathcal{T}}^{F,\Phi}(V',X)$ with $\bigoplus_{i=1}^{n} \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V,X)$ and write $x = (a_1, \dots, a_n)^{\mathrm{T}}$, a column matrix with $a_i \in \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V,X)$. Then the image of $\mu(x)$ is the sum of the image of $\mu(a_i)$ for $1 \leqslant i \leqslant n$. Now the conclusion follows from the first case.

Finally, suppose that V' is a direct summand of n copies of V, that is,

$$\bigoplus_{i=1}^{n} V = V' \oplus V''.$$

If $x \in \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V',X)$, we may consider $(x,0)^{\mathrm{T}}$ to be an element in $\mathbb{E}_{\mathcal{T}}^{F,\Phi}(\bigoplus_{i=1}^{n} V, X)$. Then the proof is reduced to the second case.

Part (iv) follows from (iii) because $\mathbb{E}_{\mathcal{T}}^{\Phi}(V,f) = \mu(\underline{f})$. □

Let $\tilde{T}^{\bullet}$ be the complex

$$\tilde{T}^{\bullet} \colon 0 \to \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V,X) \xrightarrow{\mathbb{E}_{\mathcal{T}}^{F,\Phi}(V,\bar{\alpha})} \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V, M_1 \oplus M) \to 0,$$

where the term $\mathbb{E}_{\mathcal{T}}^{F,\Phi}(V,X)$ is in degree zero. Then it is the direct sum of the following two complexes:

$$0 \to \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V,X) \xrightarrow{\mathbb{E}_{\mathcal{T}}^{F,\Phi}(V,\alpha)} \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V, M_1) \to 0,$$
$$0 \to 0 \to \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V, M) \to 0.$$

Let $P = \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V,X)/I \cdot \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V,X)$ and let $p \colon \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V,X) \to P$ be the canonical surjection. Then, by lemma 3.4(i), we may write $\mathbb{E}_{\mathcal{T}}^{F,\Phi}(V,\bar{\alpha}) = pq$ with $q \colon P \to \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V,X)$. The complex

$$T^{\bullet} \colon 0 \to P \to \mathbb{E}^{F,\Phi}(V, M_1 \oplus M) \to 0$$

in $\mathcal{D}^{\mathrm{b}}(\mathbb{E}_{\mathcal{T}}^{F,\Phi}(V)/I)$ is the direct sum of the complexes

$$0 \to P \xrightarrow{q} \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V, M_1) \to 0,$$
$$0 \to 0 \to \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V, M) \to 0.$$

Each term of $T^{\bullet}$ is a finitely generated projective $\mathbb{E}_{\mathcal{T}}^{F,\Phi}(V)/I$-module.

Before proceeding further, we need to introduce some more notation. Set

$$\Lambda := \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V), \qquad \Gamma := \mathbb{E}_{\mathcal{T}}^{F,\Phi}(W), \qquad \bar{\Lambda} := \Lambda/I, \qquad \bar{\Gamma} := \Gamma/J,$$

where I and J are defined just before theorem 3.1.

LEMMA 3.5. *$T^{\bullet}$ is a tilting complex over $\bar{\Lambda}$.*

Proof. It is clear that

$$\mathrm{Hom}_{\mathcal{K}^{\mathrm{b}}(\bar{\Lambda}\text{-proj})}(T^\bullet, T^\bullet[i]) = 0$$

for $i \leqslant -2$ and $i \geqslant 2$. We have to check that

$$\mathrm{Hom}_{\mathcal{K}^{\mathrm{b}}(\bar{\Lambda}\text{-proj})}(T^\bullet, T^\bullet[1]) = 0 \quad \text{and} \quad \mathrm{Hom}_{\mathcal{K}^{\mathrm{b}}(\bar{\Lambda}\text{-proj})}(T^\bullet, T^\bullet[-1]) = 0.$$

In the following, for a morphism $f^\bullet$ between complexes $U^\bullet$ and $V^\bullet$, we write $[f^\bullet]$ for the class of $f^\bullet$ in the homotopy category.

Let $[f^\bullet] \in \mathrm{Hom}_{\mathcal{K}^{\mathrm{b}}(\bar{\Lambda}\text{-proj})}(T^\bullet, T^\bullet[1])$. Consider the following diagram:

$$\begin{array}{ccccccccc}
 & & & & \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V,X) & & & & \\
 & & & & \big\downarrow{\scriptstyle p} & & & & \\
 & & 0 & \longrightarrow & P & \xrightarrow{\ q\ } & \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V,M_1\oplus M) & \longrightarrow & 0\\
 & & \big\downarrow & & \big\downarrow{\scriptstyle f^0} & & \big\downarrow & & \\
0 & \longrightarrow & P & \xrightarrow{\ q\ } & \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V,M_1\oplus M) & \longrightarrow & 0 & &
\end{array}$$

Since both X and $M_1 \oplus M$ are in $\mathrm{add}(V)$, lemma 2.2(i) provides an isomorphism $\mu\colon \mathbb{E}_{\mathcal{T}}^{F,\Phi}(X, M_1 \oplus M) \simeq \mathrm{Hom}_\Lambda(\mathbb{E}_{\mathcal{T}}^{F,\Phi}(V,X), \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V, M_1 \oplus M))$ and an element $u = (u_i)_{i\in\Phi} \in \mathbb{E}_{\mathcal{T}}^{F,\Phi}(X, M_1 \oplus M)$ such that $pf^0 = \mu(u)$. By assumption, $\bar{\alpha}$ is a left $(\mathrm{add}(M), F, \Phi)$-approximation of X. This yields a morphism

$$u'_i\colon M_1 \oplus M \to F^i(M_1 \oplus M)$$

for each $i \in \Phi$ such that $u_i = \bar{\alpha}u'_i$. Clearly, $u' := (u'_i)_{i\in\Phi} \in \mathbb{E}_{\mathcal{T}}^{F,\Phi}(M_1 \oplus M, M_1 \oplus M)$ and $\mu(u') \in \mathrm{Hom}_\Lambda(\mathbb{E}_{\mathcal{T}}^{F,\Phi}(V, M_1 \oplus M), \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V, M_1 \oplus M))$. Now, we have to check that the following diagram is commutative:

$$\begin{array}{ccc}
\mathbb{E}_{\mathcal{T}}^{F,\Phi}(V,X) & \xrightarrow{\ \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V,\bar{\alpha})\ } & \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V,M_1\oplus M)\\
{\scriptstyle \mu(u)}\big\downarrow & & \big\downarrow{\scriptstyle \mu(u')}\\
\mathbb{E}_{\mathcal{T}}^{F,\Phi}(V,M_1\oplus M) & = \!=\!= & \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V,M_1\oplus M)
\end{array}$$

In fact, if $a = (a_j)_{j\in\Phi} \in \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V,X)$, then it is sent to $b := (a_j F^j(\bar{\alpha}))_{j\in\Phi}$ by $\mathbb{E}_{\mathcal{T}}^{F,\Phi}(V,\bar{\alpha})$ and further sent to $bu' = (a_j(F^j\bar{\alpha}))_{j\in\Phi}u'$ by $\mu(u')$. An easy calculation shows that $bu' = au$, the image of a under $\mu(u)$. Thus, the diagram is commutative and

$$pf^0 = \mu(u) = \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V,\bar{\alpha})\mu(u') = pq\mu(u').$$

This means that $f^0 = q\mu(u')$ (since p is surjective) and that $[f^\bullet] = 0$ in $\mathcal{K}^{\mathrm{b}}(\bar{\Lambda}\text{-proj})$. Therefore, $\mathrm{Hom}_{\mathcal{K}^{\mathrm{b}}(\bar{\Lambda}\text{-proj})}(T^\bullet, T^\bullet[1]) = 0$.

Let $[f^\bullet] \in \mathrm{Hom}_{\mathcal{K}^{\mathrm{b}}(\bar{\Lambda}\text{-proj})}(T^\bullet, T^\bullet[-1])$. Consider the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P & \xrightarrow{\ q\ } & \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V,M_1\oplus M) & \longrightarrow & 0 & & \\
 & & \big\downarrow & & \big\downarrow{\scriptstyle f^1} & & \big\downarrow & & \\
 & & 0 & \longrightarrow & P & \xrightarrow{\ q\ } & \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V,M_1\oplus M) & \longrightarrow & 0
\end{array}$$

Since p is surjective and $\mathbb{E}_{\mathcal{T}}^{F,\Phi}(V, M_1 \oplus M)$ is projective in Λ-mod, the map f^1 can be lifted along p, say $f^1 = gp$ with $g\colon \mathbb{E}_{\mathcal{T}}^{\Phi}(V, M_1 \oplus M) \to \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V, X)$. By assumption, we have $X \in \mathcal{Y}^{F,\Phi}(M)$ and, by lemma 2.2(iii), there is a homomorphism $u\colon M_1 \oplus M \to X$ such that $g = \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V, u)$. Thus,

$$\mathbb{E}_{\mathcal{T}}^{F,\Phi}(V, u\bar{\alpha}) = \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V, u)\mathbb{E}_{\mathcal{T}}^{F,\Phi}(V, \bar{\alpha}) = gpq = f^1 q = 0.$$

Lemma 2.2(ii) implies $u\bar{\alpha} = 0 = u\alpha$. Therefore, u factorizes through $-w[-1]$. By lemma 3.4(iv), the image of $g(= \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V, u))$ is contained in $I \cdot \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V, X)$. It follows that $f^1 = gp = 0$ and $[f^\bullet] = 0$. Hence,

$$\mathrm{Hom}_{\mathcal{K}^{\mathrm{b}}(\bar{\Lambda}\text{-proj})}(T^\bullet, T^\bullet[-1]) = 0.$$

It is easy to see that the subcategory $\mathrm{add}(T^\bullet)$ generates $\mathcal{K}^{\mathrm{b}}(\bar{\Lambda}\text{-proj})$ as a triangulated category. Thus, $T^\bullet$ is a tilting complex over $\bar{\Lambda}$. □

Remark 3.6. To get a tilting complex from $\tilde{T}^\bullet$, one may consider the ideal I_0 of $\mathbb{E}_{\mathcal{T}}^{\Phi}(V)$ consisting of all endomorphisms $V \to V$ which are of the form fg with $f\colon V \to M'$ and $g\colon M' \to V$ such that $M' \in \mathrm{add}(M)$ and $g\tilde{\alpha} = 0$. Then it is easy to show that the quotient complex of $\tilde{T}^\bullet$ modulo $I_0\tilde{T}^\bullet$ is a two-term tilting complex over $\mathbb{E}_{\mathcal{T}}^{\Phi}(V)/I_0$. We shall not use this complex, because its endomorphism algebra cannot be described in a nice way. Note that the ideal I_0 of $\mathbb{E}_{\mathcal{T}}^{\Phi}(V)$ is properly contained in I in general.

Lemma 3.7. *The two rings $\bar{\Gamma}$ and $\mathrm{End}_{\mathcal{K}^{\mathrm{b}}(\bar{\Lambda}\text{-proj})}(T^\bullet)$ are isomorphic.*

Proof. Since $\bar{\Lambda}$ is a quotient algebra of Λ, the category $\bar{\Lambda}$-mod can be viewed as a full subcategory of Λ-mod, and it follows that $\mathcal{K}^{\mathrm{b}}(\bar{\Lambda})$ can be viewed as a full subcategory of $\mathcal{K}^{\mathrm{b}}(\Lambda)$. Thus, we have an isomorphism

$$\mathrm{End}_{\mathcal{K}^{\mathrm{b}}(\bar{\Lambda}\text{-proj})}(T^\bullet) \simeq \mathrm{End}_{\mathcal{K}^{\mathrm{b}}(\Lambda)}(T^\bullet).$$

To prove the lemma, we shall construct an isomorphism from $\mathrm{End}_{\mathcal{K}^{\mathrm{b}}(\Lambda)}(T^\bullet)$ to $\bar{\Gamma}$.

Let $[f^\bullet] \in \mathrm{End}_{\mathcal{K}^{\mathrm{b}}(\Lambda)}(T^\bullet)$. Since $p\colon \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V, X) \to P$ is an epimorphism and $\mathbb{E}_{\mathcal{T}}^{F,\Phi}(V, X)$ is a projective Λ-module, there is a Λ-module homomorphism

$$u^0\colon \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V, X) \to \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V, X)$$

such that $u^0 p = pf^0$. Let $u^1 := f^1$ and $u^i = 0$ for all $i \neq 0, 1$. Then it follows from

$$u^0\mathbb{E}_{\mathcal{T}}^{F,\Phi}(V, \bar{\alpha}) = u^0 pq = pf^0 q = pqf^1 = \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V, \bar{\alpha})u^1$$

that $u^\bullet = (u^i)_{i\in\mathbb{Z}}$ is an endomorphism in $\mathrm{End}_{\mathcal{C}^b(\Lambda)}(\tilde{T}^\bullet)$. By lemma 2.2(i), we can assume that $u^0 = \mu(x)$ and $u^1 = \mu(y)$ with $x = (x_i)_{i\in\Phi} \in \mathbb{E}_{\mathcal{T}}^{F,\Phi}(X)$ and $y = (y_i)_{i\in\Phi} \in \mathbb{E}_{\mathcal{T}}^{F,\Phi}(M_1 \oplus M)$. Now, it follows from

$$\mathbb{E}_{\mathcal{T}}^{F,\Phi}(V, \bar{\alpha})u^1 = u^0\mathbb{E}_{\mathcal{T}}^{F,\Phi}(V, \bar{\alpha})$$

that

$$(\bar{\alpha}y_i)_{i\in\Phi} = (x_i F^i\bar{\alpha})_{i\in\Phi}, \quad \text{that is, } \bar{\alpha}y_i = x_i F^i\bar{\alpha} \text{ for } i \in \Phi.$$

For each $i \in \Phi$, we can form the following commutative diagram in $\mathcal{T}$:

$$\begin{array}{ccccccc} X & \xrightarrow{\bar{\alpha}} & M_1 \oplus M & \xrightarrow{\bar{\beta}} & W & \xrightarrow{\bar{w}} & X[1] \\ \downarrow{\scriptstyle x_i} & & \downarrow{\scriptstyle y_i} & & \downarrow{\scriptstyle h_i} & & \downarrow{\scriptstyle x_i[1]} \\ F^iX & \xrightarrow{F^i\bar{\alpha}} & F^i(M_1 \oplus M) & \xrightarrow{F^i\bar{\beta}} & F^iW & \xrightarrow{(F^i\bar{w})\delta(F,i,X,1)} & (F^iX)[1] \end{array} \quad (3.1)$$

for some morphism $h_i \in \mathrm{Hom}_{\mathcal{T}}(W, F^iW)$. Thus, for each $[f^\bullet] \in \mathrm{End}_{\mathcal{K}^{\mathrm{b}}(\Lambda)}(T^\bullet)$, we get an element $h := (h_i)_{i\in\Phi} \in \Gamma$ which is $\mathbb{E}_{\mathcal{T}}^{F,\Phi}(W)$ by definition. This leads us to defining the following correspondence:

$$\Theta\colon \mathrm{End}_{\mathcal{K}^{\mathrm{b}}(\Lambda)}(T^\bullet) \to \bar{\Gamma} = \Gamma/J, \qquad [f^\bullet] \mapsto h + J.$$

Claim 3.8. *Θ is well defined.*

Proof of claim 3.8. Suppose that $[f^\bullet] \in \mathrm{End}_{\mathcal{K}^{\mathrm{b}}(\Lambda)}(T^\bullet)$ is null-homotopic, that is, there is a map

$$r\colon \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V, M_1 \oplus M) \to P$$

such that $f^0 = qr$ and $f^1 = rq$. Since p is surjective and $\mathbb{E}_{\mathcal{T}}^{F,\Phi}(V, M_1 \oplus M)$ is projective in Λ-mod, there is a map

$$s\colon \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V, M_1 \oplus M) \to \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V, X)$$

such that $sp = r$. Hence, $(u^0 - pqs)p = u^0p - pqsp = u^0p - pqr = u^0p - pf^0 = 0$ and $u^1 = rq = spq$. By the assumption $X \in \mathcal{Y}^{F,\Phi}(M)$, lemma 2.2(iii) yields a map $t\colon M_1 \oplus M \to X$ such that $s = \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V,t) = \mu(\underline{t})$. Therefore,

$$\mu(x - \underline{\bar{\alpha}t})p = (u^0 - \mathbb{E}_{\mathcal{T}}^{\Phi}(V,\bar{\alpha})\mathbb{E}_{\mathcal{T}}^{\Phi}(V,t))p = (u^0 - pqs)p = 0$$

and $\mu(y - \underline{t\bar{\alpha}}) = u^1 - spq = 0$. Consequently, $\mathrm{Im}(\mu(x - \underline{\bar{\alpha}t})) \subseteq I \cdot \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V,X)$ and $y - \underline{t\bar{\alpha}} = 0$. Thus, $y_i = 0$ for all $0 \neq i \in \Phi$ and $y_0 = t\bar{\alpha}$. By lemma 3.4(iii), we have $x_i = 0$ for all $0 \neq i \in \Phi$ and $x_0 - \bar{\alpha}t = ab$ for some morphisms $a\colon X \to M'$ and $b\colon M' \to X$ with $M' \in \mathrm{add}(M)$. Since $\bar{\alpha}$ is a left $\mathrm{add}(M)$-approximation of X, there is a morphism $c\colon M_1 \oplus M \to M'$ such that $a = \bar{\alpha}c$. It follows that

$$x_0 = ab + \bar{\alpha}t = \bar{\alpha}cb + \bar{\alpha}t = \bar{\alpha}(cb + t).$$

Now we consider the commutative diagram (3.1). Suppose that $0 \neq i \in \Phi$. Then we have shown that $x_i = y_i = 0$. Hence, $\bar{\beta}h_i = y_iF^i\bar{\beta} = 0$. This implies that h_i factorizes through $\bar{w}$, and, consequently, that $h_i|_M = 0$, since $\bar{w}|_M = 0$. It follows from $h_i(F^i\bar{w})\delta(F,i,X,1) = \bar{w}(x_i[1]) = 0$ that $h_i\colon W \to F^iW$ factorizes through $F^i(M_1 \oplus M)$. Since $Y \in \mathcal{X}^{F,\Phi}(M)$, we get $h_i|_Y = 0$. Altogether, we have shown that $h_i = 0$ for all $0 \neq i \in \Phi$. Now consider the diagram (3.1) in the case when $i = 0$. First, we have $\bar{\beta}h_0 = y_0\bar{\beta} = t\bar{\alpha}\bar{\beta} = 0$, which means that h_0 factorizes through $\bar{w}$. Second, since

$$h_0\bar{w} = \bar{w}(x_0[1]) = \bar{w}(\bar{\alpha}[1])(cb + t)[1] = 0,$$

the morphism h_0 factorizes through $M_1 \oplus M$ which is in $\mathrm{add}(M)$. Thus, $h \in J$ and $h + J$ is zero in $\bar{\Gamma}$. This shows that Θ is well defined. □

Claim 3.9. *Θ is injective.*

Proof of claim 3.9. Suppose that $\Theta([f^\bullet]) = h + J = 0 + J$. Then $h \in J$, i.e. $h_i = 0$ for all $0 \neq i \in \Phi$, and h_0 factorizes through both $\bar{w}$ and $\mathrm{add}(M)$. Suppose that $h_0 = \bar{w}s$ for a morphism $s\colon X[1] \to W$. For each $0 \neq i \in \Phi$, since $y_i F^i \bar{\beta} = \bar{\beta} h_i = 0$, the morphism $y_i\colon M_1 \oplus M \to F^i(M_1 \oplus M)$ factorizes through $F^i X$, and consequently $y_i = 0$ for all $0 \neq i \in \Phi$, since $X \in \mathcal{Y}^{F,\Phi}(M)$. For each $0 \neq i \in \Phi$, it follows from $\bar{w}(x_i[1]) = h_i(F^i\bar{w})\delta(F,i,X,1) = 0$ that $x_i[1]$ factorizes through $(M_1 \oplus M)[1]$ or, equivalently, the morphism $x_i\colon X \to F^i X$ factorizes through $M_1 \oplus M$. Hence, $x_i = 0$ for all $0 \neq i \in \Phi$, since $X \in \mathcal{Y}^{F,\Phi}(M)$. Now we consider the case when $i = 0$. First, we have $y_0\bar{\beta} = \bar{\beta}h_0 = \bar{\beta}\bar{w}s = 0$, which implies $y_0 = t\bar{\alpha}$ for a morphism $t\colon M_1 \oplus M \to X$. Second, $(x_0 - \bar{\alpha}t)\bar{\alpha} = \bar{\alpha}y_0 - \bar{\alpha}t\bar{\alpha} = \bar{\alpha}y_0 - \bar{\alpha}y_0 = 0$. It follows that $(x_0 - \bar{\alpha}t)\alpha = 0$, and therefore $x_0 - \bar{\alpha}t$ factorizes through $-w[-1]$. Since $h_0\colon W \to W$ factorizes through $\mathrm{add}(M)$ and since $\bar{\beta}\colon M_1 \oplus M \to W$ is a right $\mathrm{add}(M)$-approximation of W, we see that h_0 factorizes through $\bar{\beta}$, say $h_0 = r\bar{\beta}$ for some $r\colon W \to M_1 \oplus M$. Thus, $\bar{w}(x_0[1]) = h_0\bar{w} = r\bar{\beta}\bar{w} = 0$, or equivalently, $(-\bar{w}[-1])x_0 = 0$. It follows that x_0 factorizes through $M_1 \oplus M$. Since $\bar{\alpha}t$ also factorizes through $M_1 \oplus M$, we see that $x_0 - \bar{\alpha}t$ factorizes through $\mathrm{add}(M)$. Thus, we have shown that $x_0 - \bar{\alpha}t$ factorizes through both $\mathrm{add}(M)$ and $-w[-1]$. Now, by lemma 3.4(iii), we have

$$\mathrm{Im}(\mu(x) - \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V, \bar{\alpha}t)) = \mathrm{Im}(\mu(x - \underline{\bar{\alpha}t})) \subseteq I \cdot \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V, X).$$

Hence,

$$p(f^0 - q\mathbb{E}_{\mathcal{T}}^{F,\Phi}(V,t)p) = u^0 p - pq\mathbb{E}_{\mathcal{T}}^{F,\Phi}(V,t)p = (\mu(x) - \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V,\bar{\alpha}t))p = 0.$$

This implies that $f^0 = q(\mathbb{E}_{\mathcal{T}}^{\Phi}(V,t)p)$ since p is surjective. Moreover, one can check that

$$f^1 = u^1 = \mu(y) = \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V,t)\mathbb{E}_{\mathcal{T}}^{F,\Phi}(V,\bar{\alpha}) = (\mathbb{E}_{\mathcal{T}}^{F,\Phi}(V,t)p)q.$$

Hence, $f^\bullet$ is null-homotopic, and consequently Θ is injective. □

Claim 3.10. *Θ is surjective.*

Proof of claim 3.10. Let $h = (h_i)_{i \in \Phi} \in \Gamma$ with $h_i\colon W \to F^i W$ for $i \in \Phi$. Since $\bar{\beta}$ is a right $(\mathrm{add}(M), F, -\Phi)$-approximation of W, we have a morphism $y_i\colon M_1 \oplus M \to M_1 \oplus M$ such that $\bar{\beta}h_i = y_i F^i\bar{\beta}$ for $i \in \Phi$. This means that there is a commutative diagram

$$\begin{array}{ccccccc}
X & \xrightarrow{\bar{\alpha}} & M_1 \oplus M & \xrightarrow{\bar{\beta}} & W & \xrightarrow{\bar{w}} & X[1] \\
\downarrow{\scriptstyle x_i} & & \downarrow{\scriptstyle y_i} & & \downarrow{\scriptstyle h_i} & & \downarrow{\scriptstyle x_i[1]} \\
F^i X & \xrightarrow{F^i\bar{\alpha}} & F^i(M_1 \oplus M) & \xrightarrow{F^i\bar{\beta}} & F^i W & \xrightarrow{(F^i\bar{w})\delta(F,i,X,1)} & F^i X[1]
\end{array}$$

Now, we define $x := (x_i)_{i\in\Phi} \in \mathbb{E}_{\mathcal{T}}^{F,\Phi}(X)$, $y := (y_i)_{i\in\Phi} \in \mathbb{E}_{\mathcal{T}}^{F,\Phi}(M_1 \oplus M)$; $u^0 := \mu(x)$, $u^1 := \mu(y)$ and $u^j := 0$ for $j \neq 0, 1$. Then $u^\bullet := (u^i)_{i\in\mathbb{Z}}$ belongs to $\mathrm{End}_{\mathcal{C}^b(\Lambda)}(\tilde{T}^\bullet)$. Since

$$u^0\colon \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V,X) \to \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V,X)$$

takes elements in $I \cdot \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V,X)$ to elements in $I \cdot \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V,X)$, the image of $I \cdot \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V,X)$ under the map u^0p is zero, and consequently there exists a unique map $f^0\colon P \to P$ such that $pf^0 = u^0p$. Now we have

$$p(f^0q - qu^1) = pf^0q - pqu^1 = u^0pq - pqu^1 = u^0\mathbb{E}_{\mathcal{T}}^{F,\Phi}(V,\bar{\alpha}) - \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V,\bar{\alpha})u^1 = 0.$$

Hence, $f^0q = qu^1$ since p is surjective. Defining $f^1 = u^1$ and $f^j = 0$ for all $j \neq 0, 1$, we see that $f^\bullet = (f^i)_{i\in\mathbb{Z}}$ is an endomorphism in $\mathrm{End}_{\mathcal{C}^b(\Lambda)}(T^\bullet)$ and $\Theta([f^\bullet]) = h + J$. Thus, Θ is surjective. □

Claim 3.11. *Θ is an R-algebra homomorphism.*

Proof of claim 3.11. The map Θ is R-linear, so it preserves addition. For multiplication, we take $[f^\bullet]$ and $[g^\bullet]$ in $\mathrm{End}_{\mathcal{K}^b(\Lambda)}(T^\bullet)$. Let $[u^\bullet]$ and $[v^\bullet]$ be in $\mathrm{End}_{\mathcal{K}^b(\Lambda)}(\tilde{T}^\bullet)$ such that $u^0p = pf^0$, $u^1 = f^1$, $v^0p = pg^0$ and $v^1 = g^1$. Suppose that

$$(u^0, u^1) = (\mu(x), \mu(y)) \quad \text{and} \quad (v^0, v^1) = (\mu(x'), \mu(y'))$$

with $x, x' \in \mathbb{E}_{\mathcal{T}}^{F,\Phi}(X)$ and $y, y' \in \mathbb{E}_{\mathcal{T}}^{\Phi}(M_1 \oplus M)$. Let $h := (h_i)_{i\in\Phi}$ and $h' := (h'_i)_{i\in\Phi}$ be in Γ, making the diagram (3.1) commutative, that is,

$$\begin{aligned}
\bar{\beta} * h_i &= \bar{\beta}h_i = y_iF^i\bar{\beta} = y_i * \bar{\beta},\\
\bar{\beta} * h'_i &= \bar{\beta}h'_i = y'_iF^i\bar{\beta} = y'_i * \bar{\beta},\\
\bar{w}(x_i[1]) &= h_i(F^i\bar{w})\delta(F,i,X,1) = (h_i * \bar{w})\delta(F,i,X,1),\\
\bar{w}(x'_i[1]) &= h'_i(F^i\bar{w})\delta(F,i,X,1) = (h'_i * \bar{w})\delta(F,i,X,1),
\end{aligned}$$

for all $i \in \Phi$. Then, by definition, we have $\Theta([f^\bullet]) = h + J$, $\Theta([g^\bullet]) = h' + J$ and

$$\Theta([f^\bullet])\Theta([g^\bullet]) = \Bigg(\sum_{\substack{i,j\in\Phi\\ i+j=k}} h_i * h'_j\Bigg)_{k\in\Phi} + J.$$

Now we calculate $\Theta([f^\bullet g^\bullet])$. Let $s^\bullet := u^\bullet v^\bullet$. Then $s^0p = pf^0g^0 = p(f^\bullet g^\bullet)^0$, $s^1 = f^1g^1 = (f^\bullet g^\bullet)^1$ and $(s^0, s^1) = (\mu(xx'), \mu(yy'))$, where

$$(xx')_k = \sum_{\substack{i,j\in\Phi\\ i+j=k}} x_i * x'_j \quad \text{and} \quad (yy')_k = \sum_{\substack{i,j\in\Phi\\ i+j=k}} y_i * y'_j.$$

For each $k \in \Phi$, we have

$$\begin{aligned}
(yy')_kF^k\bar{\beta} = (yy')_k * \bar{\beta} &= \Bigg(\sum_{\substack{i,j\in\Phi\\ i+j=k}} y_i * y'_j\Bigg) * \bar{\beta}\\
&= \bar{\beta} * \Bigg(\sum_{\substack{i,j\in\Phi\\ i+j=k}} h_i * h'_j\Bigg) = \bar{\beta}\Bigg(\sum_{\substack{i,j\in\Phi\\ i+j=k}} h_i * h'_j\Bigg).
\end{aligned}$$

Similarly, for each $k \in \Phi$, we have

$$\Bigg(\sum_{\substack{i,j\in\Phi\\ i+j=k}} h_i * h'_j\Bigg)(F^k\bar{w})\delta(F,k,X,1) = \bar{w}((xx')_k[1]).$$

This means that

$$\Theta([f^\bullet][g^\bullet]) = \Theta([f^\bullet g^\bullet]) = \Big(\sum_{\substack{i,j\in\Phi \\ i+j=k}} h_i * h'_j \Big)_{k\in\Phi} + J = \Theta([f^\bullet])\Theta([g^\bullet]).$$

Thus, Θ is a ring homomorphism, and the proof of theorem 3.1 is finished. □

□

Before proceeding, we comment on the conditions in theorem 3.1.

Remark 3.12. (a) Let

$$X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \xrightarrow{w} X[1]$$

be a triangle in $\mathcal{T}$ with $M_1 \in \operatorname{add}(M)$, $X \in \mathcal{Y}^{F,\Phi}(M)$ and $Y \in \mathcal{X}^{F,\Phi}(M)$. If α is a left $(\operatorname{add}(M), F, \Phi)$-approximation of X, then $\operatorname{Hom}_{\mathcal{T}}(X, F^iM) \simeq \operatorname{Hom}_{\mathcal{T}}(M_1, F^iM)$ for $0 \neq i \in \Phi$. Similarly, if β is a right $(\operatorname{add}(M), F, -\Phi)$-approximation of Y, then $\operatorname{Hom}_{\mathcal{T}}(M, F^iY) = \operatorname{Hom}_{\mathcal{T}}(M, F^iM_1)$ for $0 \neq i \in \Phi$. In particular, if M is an (F,Φ)-self-orthogonal object of $\mathcal{T}$, that is, $\operatorname{Hom}_{\mathcal{T}}(M, F^iM) = 0$ for every $0 \neq i \in \Phi$, and if α is a left $(\operatorname{add}(M), F, \Phi)$-approximation of X and β is a right $(\operatorname{add}(M), F, -\Phi)$-approximation of Y, then $X \in \mathcal{X}^{F,\Phi}(M)$ and $Y \in \mathcal{Y}^{F,\Phi}(M)$.

(b) Under the conditions of theorem 3.1, there are isomorphisms $\operatorname{Hom}_{\mathcal{T}}(X, F^iX) \simeq \operatorname{Hom}_{\mathcal{T}}(Y, F^iY)$ for every $0 \neq i \in \Phi$. In fact, this follows from the following general statement.

Let $\mathcal{T}$ be a triangulated category with a shift functor [1]. Suppose that F is a triangle functor from $\mathcal{T}$ to itself and that $\mathcal{D}$ is a full subcategory of $\mathcal{T}$. Let i be a positive integer. Suppose that

$$X_j \xrightarrow{\alpha_j} D_j \xrightarrow{\beta_j} Y_j \to X_j[1]$$

is a triangle in $\mathcal{T}$, such that α_j is a left $(\mathcal{D}, F, \{i\})$-approximation of X_j, and that

$$\operatorname{Hom}_{\mathcal{T}}(D', F^i(\beta_j))\colon \operatorname{Hom}_{\mathcal{T}}(D', F^iD_j) \to \operatorname{Hom}_{\mathcal{T}}(D', F^iY_j)$$

is surjective for every $D' \in \mathcal{D}$ and $j = 1, 2$. If

$$\operatorname{Hom}_{\mathcal{T}}(\mathcal{D}, F^iX_j) = 0 = \operatorname{Hom}_{\mathcal{T}}(Y_j, F^i\mathcal{D})$$

for $1 \leqslant j \leqslant 2$, then $\operatorname{Hom}_{\mathcal{T}}(X_1, F^iX_2) \simeq \operatorname{Hom}_{\mathcal{T}}(Y_1, F^iY_2)$.

Proof. From the given two triangles the following exact commutative diagram can be formed:

$$\begin{array}{ccccccc}
 & & & & \operatorname{Hom}_{\mathcal{T}}(D_1, F^iX_2) & \longrightarrow & \operatorname{Hom}_{\mathcal{T}}(D_1, F^iD_2) \\
 & & & & \downarrow & & \downarrow{\scriptstyle (\alpha_1, F^iD_2)} \\
 & & & & \operatorname{Hom}_{\mathcal{T}}(X_1, F^iX_2) & \longrightarrow & \operatorname{Hom}_{\mathcal{T}}(X_1, F^iD_2) \\
 & & & & \downarrow & & \downarrow{\scriptstyle 0} \\
\operatorname{Hom}_{\mathcal{T}}(Y_1, F^iD_2) & \longrightarrow & \operatorname{Hom}_{\mathcal{T}}(Y_1, F^iY_2) & \longrightarrow & \operatorname{Hom}_{\mathcal{T}}(Y_1, F^iX_2[1]) & \longrightarrow & \operatorname{Hom}_{\mathcal{T}}(Y_1, F^iD_2[1]) \\
\downarrow & & \downarrow & & \downarrow & (*) & \downarrow \\
\operatorname{Hom}_{\mathcal{T}}(D_1, F^iD_2) & \xrightarrow{(D_1, F^i(\beta_2))} & \operatorname{Hom}_{\mathcal{T}}(D_1, F^iY_2) & \xrightarrow{0} & \operatorname{Hom}_{\mathcal{T}}(D_1, F^iX_2[1]) & \longrightarrow & \operatorname{Hom}_{\mathcal{T}}(D_1, F^iD_2[1])
\end{array}$$

Since

$$\mathrm{Hom}_{\mathcal{T}}(Y_1, F^i D_2) = \mathrm{Hom}_{\mathcal{T}}(D_1, F^i X_2) = 0$$

by assumption, and since $\mathrm{Hom}_{\mathcal{T}}(\alpha_1, F^i D_2)$ and $\mathrm{Hom}_{\mathcal{T}}(D_1, F^i \beta_2)$ are surjective by the property of approximation, the conclusion follows from the commutative square $(*)$. □

(c) Let $X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \xrightarrow{w} X[1]$ be an $\mathrm{add}(M)$-split triangle in $\mathcal{T}$. Define $V := X \oplus M$, $\Lambda_0 := \mathrm{End}_{\mathcal{T}}(V)$, $W := M \oplus Y$ and $\Gamma_0 := \mathrm{End}_{\mathcal{T}}(W)$. Let I and J be as defined in theorem 3.1. Then the ideals I and J in theorem 1.1 have the following characterization.

(i) Let e be the idempotent in Γ_0 corresponding to the direct summand M of W. Then J is the submodule of the left Γ_0-module $\Gamma_0 e \Gamma_0$, which is maximal with respect to $eJ = 0$.

(ii) Let f be the idempotent in Λ_0 corresponding to the direct summand M of V. Then I is the submodule of the right Λ_0-module $\Lambda_0 f \Lambda_0$, which is maximal with respect to $If = 0$.

Proof. By lemma 3.3, the sets I and J are ideals of Λ_0 and Γ_0, respectively.

(i) Let $p_M \colon W \to M$ and $\lambda_M \colon M \to W$ be the canonical projection and injection, respectively. By definition, $e = p_M \lambda_M$. The set $\Gamma_0 e \Gamma_0$ is precisely the set of all endomorphisms of W that factorize through $\mathrm{add}(M)$. The endomorphisms of W factorizing through $\bar{w}$ are those endomorphisms x that satisfy $\bar{\beta} x = 0$, and consequently

$$ex = p_M \lambda_M x = p_M (\bar{\beta}|_M) x = 0.$$

Hence, J is a submodule of ${}_{\Gamma_0}\Gamma_0 e \Gamma_0$ with $eJ = 0$. Suppose that $\bar{J} \subseteq {}_{\Gamma_0}\Gamma_0 e \Gamma_0$ is another submodule containing J with $e\bar{J} = 0$. Then $e\bar{J} = 0$ implies

$$\mathrm{Hom}_{\Gamma_0}(\mathrm{Hom}_{\mathcal{T}}(W, M), \bar{J}) = 0,$$

and consequently $\mathrm{Hom}_{\Gamma_0}(\mathrm{Hom}_{\mathcal{T}}(W, M'), \bar{J}) = 0$ for all $M' \in \mathrm{add}(M)$. For each $x \in \bar{J}$, the image of the morphism $\mathrm{Hom}_{\mathcal{T}}(W, x)$ is contained in $\bar{J}$ since $\bar{J}$ is a left ideal of Γ_0. Thus, the morphism $\mathrm{Hom}_{\mathcal{T}}(W, \bar{\beta} x)$ is a Γ_0-module morphism from $\mathrm{Hom}_{\mathcal{T}}(W, M_1 \oplus M)$ to the image of $\mathrm{Hom}_{\mathcal{T}}(W, x)$. Hence, $\mathrm{Hom}_{\mathcal{T}}(W, \bar{\beta} x) = 0$, and consequently $\bar{\beta} x = 0$. This implies $x \in J$. Thus, (i) is proved.

(ii) The proof is similar to that of (i). □

Note that if $\mathrm{Hom}_{\mathcal{T}}(X, F^i D') = 0$ for all $0 \neq i \in \Phi$ and $D' \in \mathcal{D}$, then f is a left $(\mathcal{D}, F, \Phi)$-approximation of X if and only if f is a left $\mathcal{D}$-approximation of X. A dual statement is also true for a right $(\mathcal{D}, F, \Phi)$-approximation of X. Thus, a special case of theorem 3.1 is the following corollary for $\mathcal{D}$-split triangles (see § 2.4 for definition).

Corollary 3.13. *Let Φ be an admissible subset of $\mathbb{Z}$, $\mathcal{T}$ be a triangulated R-category with a triangle auto-equivalence F and M be an object in $\mathcal{T}$. Suppose that*

$$X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \xrightarrow{w} X[1]$$

is an add(M)*-split triangle in* $\mathcal{T}$ *and that* X *and* Y *both are in* $\mathcal{X}^{F,\Phi}(M)\cap\mathcal{Y}^{F,\Phi}(M)$. *Then* $\mathbb{E}^{F,\Phi}_{\mathcal{T}}(X\oplus M)/I$ *and* $\mathbb{E}^{F,\Phi}_{\mathcal{T}}(M\oplus Y)/J$ *are derived equivalent.*

Another special case of theorem 3.1 is the following corollary, which is useful for constructing explicit examples.

COROLLARY 3.14. *Let* $\mathcal{T}$ *be a triangulated* R*-category with* $[1]$ *the shift functor, and let* M *be an object in* $\mathcal{T}$. *Suppose that* $X\xrightarrow{\alpha}M_1\xrightarrow{\beta}Y\xrightarrow{w}X[1]$ *is a triangle in* $\mathcal{T}$ *such that* $M_1\in\mathrm{add}(M)$, *and suppose that* $X\in\mathcal{Y}^{n+1}(M)$ *and* $Y\in\mathcal{X}^{n+1}(M)$. *Then, for any admissible subset* Φ *of* $\mathbb{N}_n$, *the algebras* $\mathbb{E}^{\Phi}_{\mathcal{T}}(X\oplus M)/I$ *and* $\mathbb{E}^{\Phi}_{\mathcal{T}}(M\oplus Y)/J$ *are derived equivalent.*

Proof. We show that β is a right $(\mathrm{add}(M),-\Phi)$-approximation of Y. Note that, for $i\in\Phi$, we always have $i+1\leqslant n+1$. Hence, $\mathrm{Hom}_{\mathcal{T}}(M,X[i+1])=0$ for $i\in\Phi$. Now apply $\mathrm{Hom}_{\mathcal{T}}(M[-i],\cdot)$ with $i\in\Phi$ to the triangle $X\xrightarrow{\alpha}M_1\xrightarrow{\beta}Y\xrightarrow{w}X[1]$:

$$\cdots\to\mathrm{Hom}_{\mathcal{T}}(M[-i],M_1)\to\mathrm{Hom}_{\mathcal{T}}(M[-i],Y)\to\mathrm{Hom}_{\mathcal{T}}(M[-i],X[1])\to\cdots.$$

Because

$$\mathrm{Hom}_{\mathcal{T}}(M[-i],X[1])=\mathrm{Hom}_{\mathcal{T}}(M,X[i+1])=0,$$

the map β is a right $(\mathrm{add}(M),-\Phi)$-approximation of Y.

Similarly, it follows from $\mathrm{Ext}^{i+1}_{\mathcal{T}}(Y,M)=0$ for $i\in\Phi$ that α is a left $(\mathrm{add}(M),\Phi)$-approximation of X. Now, corollary 3.14 follows from theorem 3.1. □

An interesting case of theorem 3.1 is when $I=0$ and $J=0$. The following proposition is a sufficient condition for $I=0$ and $J=0$.

PROPOSITION 3.15. *Let* $X\xrightarrow{\alpha}M_1\xrightarrow{\beta}Y\xrightarrow{w}X[1]$ *be an* add(M)*-split triangle in* $\mathcal{T}$. *Define* $V:=X\oplus M$, $\Lambda_0:=\mathrm{End}_{\mathcal{T}}(V)$, $W:=M\oplus Y$ *and* $\Gamma_0:=\mathrm{End}_{\mathcal{T}}(W)$. *Let* I' *be the ideal of* Λ_0 *consisting of all* $f\colon V\to V$ *that factorizes through* $\tilde{w}[-1]\colon Y[-1]\to V$, *and let* J' *be the ideal of* Γ_0 *consisting of all* $g\colon W\to W$ *that factorizes through* $\bar{w}\colon W\to X[1]$.

(i) *Suppose that* Λ_0 *is an Artin algebra. If*

$$\mathrm{add}(\mathrm{top}_{\Lambda_0}\mathrm{Hom}_{\mathcal{T}}(V,X))\cap\mathrm{add}(\mathrm{top}({}_{\Lambda_0}D\Lambda_0))=0,$$

then $I'=0$.

(ii) *Suppose that* Γ_0 *is an Artin algebra. If*

$$\mathrm{add}(\mathrm{top}_{\Gamma_0}\mathrm{Hom}_{\mathcal{T}}(W,Y))\cap\mathrm{add}(\mathrm{soc}({}_{\Gamma_0}\Gamma_0))=0,$$

then $J'=0$.

By definition, there are inclusions $I\subseteq I'$ and $J\subseteq J'$. Sometimes it is easy to verify that I' and J' vanish if the algebras Λ_0 and Γ_0 are described by quivers with relations.

Proof of proposition 3.15. We prove (i). The proof of (ii) is similar and we omit it.

We have a triangle

$$Y[-1]\xrightarrow{-\tilde{w}[-1]}V\xrightarrow{\tilde{\alpha}}M_1\oplus M\xrightarrow{\tilde{\beta}}Y;$$

apply $\mathrm{Hom}_{\mathcal{T}}(\cdot, V)$ to this triangle, and get the following exact sequence of right Λ_0-modules:

$$\mathrm{Hom}_{\mathcal{T}}(M_1 \oplus M, V) \to \mathrm{Hom}_{\mathcal{T}}(V, V) \to C \to 0,$$

where C is the cokernel of $\mathrm{Hom}_{\mathcal{T}}(\tilde{\alpha}, V)$. Now, applying $\mathrm{Hom}_{\Lambda_0^{\mathrm{op}}}(\mathrm{Hom}_{\mathcal{T}}(M, V), \cdot)$ to the above exact sequence, we get another exact sequence, which is isomorphic to the following exact sequence:

$$\mathrm{Hom}_{\mathcal{T}}(M_1 \oplus M, M) \xrightarrow{(\tilde{\alpha}, M)} \mathrm{Hom}_{\mathcal{T}}(V, M) \to \mathrm{Hom}_{\Lambda_0^{\mathrm{op}}}(\mathrm{Hom}_{\mathcal{T}}(M, V), C) \to 0.$$

Since $\tilde{\alpha}$ is a left $\mathrm{add}(M)$-approximation of V, the map $\mathrm{Hom}_{\mathcal{T}}(\tilde{\alpha}, M)$ is surjective, and consequently $\mathrm{Hom}_{\Lambda_0^{\mathrm{op}}}(\mathrm{Hom}_{\mathcal{T}}(M, V), C) = 0$. So, the right Λ_0-module C has no composition factors in $\mathrm{top}(\mathrm{Hom}_{\mathcal{T}}(M, V))$, and C has composition factors only in $\mathrm{top}(\mathrm{Hom}_{\mathcal{T}}(X, V))$. This is equivalent to saying that the Λ_0-module $D(C)$ has composition factors only in $\mathrm{soc}(D\,\mathrm{Hom}_{\mathcal{T}}(X, V))$ which is isomorphic to $\mathrm{top}(\mathrm{Hom}_{\mathcal{T}}(V, X))$.

Let $x \colon V \to V$ be an element in $I' \subseteq \Lambda_0$. Then x factorizes through $-\tilde{w}[-1]$ or, equivalently, $x\tilde{\alpha} = 0$. This implies that $(D\,\mathrm{Hom}_{\mathcal{T}}(x, V))(D\,\mathrm{Hom}_{\mathcal{T}}(\tilde{\alpha}, V)) = 0$. Thus, the image of $D\,\mathrm{Hom}_{\mathcal{T}}(x, V)$ is contained in the kernel of $D\,\mathrm{Hom}_{\mathcal{T}}(\tilde{\alpha}, V)$, which is isomorphic to $D(C)$. Therefore, if $D\,\mathrm{Hom}_{\mathcal{T}}(x, V) \neq 0$, then the top of the image of $D\,\mathrm{Hom}_{\mathcal{T}}(x, V)$ is contained in $\mathrm{add}(\mathrm{top}_{\Lambda_0} \mathrm{Hom}_{\mathcal{T}}(V, X)) \cap \mathrm{add}(\mathrm{top}({}_{\Lambda_0} D\Lambda_0)) = 0$; this is a contradiction. Thus, we must have $\mathrm{Hom}_{\mathcal{T}}(x, V) = 0$. Since $\mathrm{Hom}_{\mathcal{T}}(\cdot, V)$ is a duality from $\mathrm{add}(V)$ to Λ_0^{op}-proj, we obtain $x = 0$. Thus, $I' = 0$. □

REMARK 3.16. If we substitute 'add(M)-split' for 'left $(\mathrm{add}(M), \Phi)$-approximation' and 'right $(\mathrm{add}(M), -\Phi)$-approximation' in proposition 3.15, and if we consider $\mathbb{E}_{\mathcal{T}}^{\Phi}(V)$ and $\mathbb{E}_{\mathcal{T}}^{\Phi}(W)$ instead of Λ_0 and Γ_0, then proposition 3.15 is still true. The proof is almost the same.

For the derived category of an abelian category, the following result provides an explicit example for $I = 0 = J$.

PROPOSITION 3.17. *Let $\mathcal{A}$ be an abelian category and let M be an object of $\mathcal{A}$. Suppose that $0 \to X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \to 0$ is an exact sequence in $\mathcal{A}$ with $M_1 \in \mathrm{add}(M)$. Consider the induced triangle $X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \xrightarrow{w} X[1]$ in $\mathcal{D}^{\mathrm{b}}(\mathcal{A})$. Then the ideals I and J defined in theorem 3.1 vanish.*

Proof. Every exact sequence $0 \to X \to M_1 \to Y \to 0$ in $\mathcal{A}$ gives rise to a triangle $X \to M_1 \to Y \to X[1]$ in $\mathcal{D}^{\mathrm{b}}(\mathcal{A})$. Now we show that the exactness of the given sequence in $\mathcal{A}$ implies that the two ideals I and J in theorem 3.1 are equal to zero. Since I is contained in $\mathrm{End}_{\mathcal{D}^{\mathrm{b}}(\mathcal{A})}(X \oplus M)$, it is sufficient to show that if a morphism $x \colon X \oplus M \to X \oplus M$ factorizes through $\mathrm{add}(M)$ and $\tilde{w}[-1]$, then $x = 0$. In fact, let x be such a morphism. Then we see immediately that $x\tilde{\alpha} = 0$ in $\mathcal{D}^{\mathrm{b}}(\mathcal{A})$. Since $\mathcal{A}$ is fully embedded in $\mathcal{D}^{\mathrm{b}}(\mathcal{A})$, we also have $x\tilde{\alpha} = 0$ in $\mathcal{A}$. Consequently, $x = 0$ since $\tilde{\alpha}$ is injective in $\mathcal{A}$. Thus, $I = 0$. Dually, we can show $J = 0$. Hence, proposition 3.17 holds true. □

As an immediate application of the proof of theorem 3.1 together with a result on derived equivalences in [19], we have the following corollary.

Corollary 3.18. *We keep all assumptions of theorem 3.1. If $\bar{\Lambda}$ and $\bar{\Gamma}$ both are left coherent rings (for example, if Φ is finite and $\mathcal{T} = \mathcal{D}^{\mathrm{b}}(A)$ with A a finite-dimensional algebra over a field), then* $\operatorname{fin}\dim(\bar{\Lambda})-1 \leqslant \operatorname{fin}\dim(\bar{\Gamma}) \leqslant \operatorname{fin}\dim(\bar{\Lambda})-1$, *where* $\operatorname{fin}\dim(\bar{\Lambda})$ *stands for the finitistic dimension of* $\bar{\Lambda}$.

Recall that, given a ring S with identity, the *finitistic dimension* of S is defined to be the supremum of the projective dimensions of finitely generated S-modules of finite projective dimension.

Now, let us make a few remarks on theorem 3.1.

Since the map q in the proof of theorem 3.1 is not always injective, the tilting complex $T^{\bullet}$ is not, in general, isomorphic in $\mathcal{D}^{\mathrm{b}}(\mathbb{E}_{\mathcal{T}}^{F,\Phi}(V)/I)$ to a tilting module. Thus, the derived equivalence presented in theorem 3.1 is not given by a tilting module in general (in contrast with the situation of theorem 2.3). In fact, it is easy to see that the derived equivalence in theorem 3.1 is given by a tilting module if the kernel of $\mathbb{E}_{\mathcal{T}}^{F,\Phi}(V,\alpha)$ is $I \cdot \mathbb{E}_{\mathcal{T}}^{F,\Phi}(V,X)$.

Moreover, a small additive category may be embedded into an abelian category of coherent functors (see [17, ch. IV, § 2]). However, this will not, in general, turn a $\mathcal{D}$-split sequence in the additive category into an exact sequence in the abelian category, since otherwise the sequence would split, and it therefore cannot provide a triangle in the derived category of the abelian category. Consequently, theorem 2.3 cannot be obtained from theorem 3.1 by taking $\Phi = \{0\}$ and embedding an additive category into an abelian category.

Finally, we mention that theorem 3.1 generalizes the result [11, proposition 5.1] by choosing $\Phi = \{0\}$. Indeed, under the conditions of [11, proposition 5.1], the ideals I and J in theorem 3.1 vanish. Theorem 3.1 covers various other situations, some of which will be discussed in the next section.

4. Φ-Yoneda algebras in some explicit situations

In this section, we shall describe some natural habitats for theorem 3.1 and relate them to several widely used concepts that fit with or simplify the assumptions of theorem 3.1. Throughout, we choose F to be the shift functor of the triangulated category considered.

We note that Dugas, in independent work [5] that is also motivated by [11], has constructed derived equivalent pairs of symmetric algebras. As explained in [5, § 4, remark 3] his examples appear in our framework too.

4.1. Derived categories of Artin algebras

A first consequence of theorem 3.1 is the following result for $\mathcal{T} = \mathcal{D}^{\mathrm{b}}(A)$ with A an Artin R-algebra.

Theorem 4.1. *Let Φ be an admissible subset of $\mathbb{N}$, let M be an A-module and let $0 \to X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \to 0$ be an exact sequence in A-mod with α a left $(\operatorname{add}(M),\Phi)$-approximation of X and β a right $(\operatorname{add}(M),-\Phi)$-approximation of Y in $\mathcal{D}^{\mathrm{b}}(A)$ such that $X \in \mathcal{Y}^{\Phi}(M)$ and $Y \in \mathcal{X}^{\Phi}(M)$. Then the perforated Yoneda algebras $\mathbb{E}_A^{\Phi}(X \oplus M)$ and $\mathbb{E}_A^{\Phi}(M \oplus Y)$ are derived equivalent.*

Proof. This is a consequence of theorem 3.1 and proposition 3.17 if we take $\mathcal{T} = \mathcal{D}^{\mathrm{b}}(A)$. □

Under the assumptions of theorem 4.1, the higher cohomology group $\mathrm{Ext}^i_A(X,X)$ of X is isomorphic to the higher cohomology groups $\mathrm{Ext}^i_A(Y,Y)$ of Y for each $0 \neq i \in \Phi$. This follows from the comment (b) before corollary 3.13.

If we relax the conditions on the exact sequence, but strengthen the orthogonality conditions in theorem 4.1, then we get the following consequence.

Corollary 4.2. *Suppose that M is an A-module. Let $0 \to X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \to 0$ be an* add(M)*-split sequence in A-mod such that $X, Y \in \mathcal{X}^n(M) \cap \mathcal{Y}^n(M)$ for n a positive number or infinity. Then, for any admissible subset Φ of $\mathbb{N}_n$, the perforated Yoneda algebras $\mathbb{E}^{\Phi}_A(X \oplus M)$ and $\mathbb{E}^{\Phi}_A(M \oplus Y)$ are derived equivalent.*

Note that the orthogonality conditions in corollary 4.2 occur very naturally in Calabi–Yau categories (see § 4.2).

The following result shows that the orthogonality conditions are related to the concepts of *short cycle* and *short chain* in A-mod [1, ch. IX, p. 313]. Recall that a short cycle of length 2 from an indecomposable module X to X is a sequence of non-zero radical homomorphisms $X \xrightarrow{f} M \xrightarrow{g} X$ with M indecomposable; and a short chain is a sequence of non-zero radical homomorphisms $X \xrightarrow{f} M \xrightarrow{g} D\,\mathrm{Tr}(X)$ with X indecomposable.

Corollary 4.3. *Let A be an Artin algebra and let $0 \to X \to M \to Y \to 0$ be an Auslander–Reiten sequence in A-mod. Suppose neither X nor Y lies on a short cycle of length* 2 *or on a short chain. Then the trivial extension of* $\mathrm{End}_A(X \oplus M)$ *by the bimodule* $\mathrm{Ext}^1_A(X,X) \oplus \mathrm{Ext}^1_A(M,M)$ *is derived equivalent to the trivial extension of* $\mathrm{End}_A(M \oplus Y)$ *by the bimodule* $\mathrm{Ext}^1_A(Y,Y) \oplus \mathrm{Ext}^1_A(M,M)$.

Proof. An Auslander–Reiten sequence $0 \to X \to M \to Y \to 0$ is always an add(M)-split sequence. Since Y does not lie on a short cycle, the Auslander–Reiten formula

$$D\underline{\mathrm{Hom}}_A(\mathrm{Tr}\,D(X), M) \simeq \mathrm{Ext}^1_A(M,X) \simeq D\overline{\mathrm{Hom}}_A(X, D\,\mathrm{Tr}(M))$$

(see [1, p. 131]) implies $\mathrm{Ext}^1_A(M,X) = 0$. Moreover, X not lying on a short cycle implies $\mathrm{Ext}^1_A(Y,M) = 0$. Similarly, the Auslander–Reiten formula yields that $\mathrm{Ext}^1_A(X,M) = 0$ (since X does not lie on a short chain) and that $\mathrm{Ext}^1_A(M,Y) = 0$ (since Y does not lie on a short chain). Thus, corollary 4.3 follows from corollary 4.2 when $n = 1$. □

The next corollary is a consequence of corollary 4.2.

Corollary 4.4. *Let A be an Artin algebra and let X be an A-module such that $\mathrm{Ext}^i_A(X,A) = 0$ for all $1 \leqslant i < n+2$ with n a fixed positive integer or infinity. Then, for any admissible subset Φ of $\mathbb{N}_n$, the perforated Yoneda algebras $\mathbb{E}^{\Phi}_A(A \oplus X)$ and $\mathbb{E}^{\Phi}_A(A \oplus \Omega(X))$ are derived equivalent.*

Proof. If $\mathrm{Ext}^i_A(X,A) = 0$ for a fixed $i \geqslant 1$, then

$$0 \to \Omega^i(X) \to P_{i-1} \to \Omega^{i-1}(X) \to 0$$

is an add$({}_AA)$-split sequence in A-mod, where P_i is a projective cover of $\Omega^i(X)$. Using this fact, corollary 4.4 follows immediately from corollary 4.2. □

The condition $\mathrm{Ext}^i_A(X, A) = 0$ on X in corollary 4.4 is related to the context of the *generalized Nakayama conjecture*. This states that if an A-module T satisfies $\mathrm{Ext}^i_A(A \oplus T, A \oplus T) = 0$ for all $i > 0$, then T should be projective. Corollary 4.4 (or [11, theorem 1.1]) describes the shape of the syzygy modules $\Omega^i(X)$: if X is indecomposable and non-projective and satisfies $\mathrm{Ext}^i_A(X, A) = 0$ for all $i > 0$, then, for each $j \geqslant 0$, there is an indecomposable non-projective module L_j such that $\Omega^j(X) \simeq L_j^{m_j}$ for an integer $m_j > 0$.

In corollary 4.4, there are isomorphisms $\mathrm{Ext}^i_A(X, X) \simeq \mathrm{Ext}^i_A(\Omega(X), \Omega(X))$ for all $i \geqslant 1$. Thus, the algebras $\mathbb{E}^{\Phi}_A(A \oplus X)$ and $\mathbb{E}^{\Phi}_A(A \oplus \Omega(X))$ are the extensions of $\mathrm{End}_A(A \oplus X)$ and $\mathrm{End}_A(A \oplus \Omega(X))$ by the same ideal $\mathbb{E}^{\Phi\setminus\{0\}}_A(X, X)$, respectively. The algebras $\mathbb{E}^{\Phi}_A(X \oplus M)$ and $\mathbb{E}^{\Phi}_A(M \oplus Y)$ in corollary 4.2, however, are the extensions of $\mathrm{End}_A(X \oplus M)$ and $\mathrm{End}_A(M \oplus Y)$ by possibly different ideals

$$\mathbb{E}^{\Phi\setminus\{0\}}_A(M) \oplus \mathbb{E}^{\Phi\setminus\{0\}}_A(X) \quad \text{and} \quad \mathbb{E}^{\Phi\setminus\{0\}}_A(M) \oplus \mathbb{E}^{\Phi\setminus\{0\}}_A(Y),$$

respectively.

Recall that a module $M \in A$-mod is called *reflexive* if the evaluation map

$$\alpha_M : M \to M^{**} := \mathrm{Hom}_{A^{\mathrm{op}}}(\mathrm{Hom}_A(M, A), A_A)$$

is an isomorphism of modules.

Corollary 4.5. *Let M be a reflexive A-module. Then, for any subset $0 \in \Phi \subseteq \{0, 1\}$, the perforated Yoneda algebras $\mathbb{E}^{\Phi}_A(D(A_A) \oplus D\,\mathrm{Tr}(M))$ and $\mathbb{E}^{\Phi}_A(D(A_A) \oplus \Omega^{-1}(D\,\mathrm{Tr}(M)))$ are derived equivalent, where Ω^{-1} is the co-syzygy operator.*

Proof. By [1, ch. IV, proposition 3.2], the kernel and cokernel of the evaluation map α_M are $\mathrm{Ext}^1_{A^{\mathrm{op}}}(\mathrm{Tr}(M), A)$ and $\mathrm{Ext}^2_{A^{\mathrm{op}}}(\mathrm{Tr}(M), A)$, respectively. As

$$\mathbb{E}^{\Phi}_A(U) \simeq \mathbb{E}^{\Phi}_{A^{\mathrm{op}}}(D(U))^{\mathrm{op}}$$

for any A-module U, corollary 4.5 follows from corollary 4.4 for right modules. □

A special case of corollary 4.4, or corollary 4.5, is the following result on self-injective algebras, which was obtained in [10, corollary 3.14].

Corollary 4.6. *If A is a self-injective Artin algebra, then, for any admissible subset Φ of $\mathbb{N}$, the perforated Yoneda algebras $\mathbb{E}^{\Phi}_A(A \oplus X)$ and $\mathbb{E}^{\Phi}_A(A \oplus \Omega(X))$ are derived equivalent.*

Another concept related to the generalized Nakayama conjectures and to modules being projective and injective is the *dominant dimension* of an algebra or a module.

Suppose that A is an Artin R-algebra. By definition, the *dominant dimension* of A is greater than or equal to n if, in the minimal injective resolution of ${}_AA$,

$$0 \to A \to I_0 \to I_1 \to \cdots \to I_{n-1} \to I_n \to \cdots,$$

the first n injective A-modules $I_0, \ldots, I_{n-1}$ are projective. In this case we write $\mathrm{dom\,dim}(A) \geqslant n$. Let C_i be the cokernel of the map $I_{i-1} \to I_i$.

For an A-module X, we define $a(X)$ to be the number of non-isomorphic indecomposable direct summands of M. The *self-injective measure* of A is defined to be the number $m(A) := a(A) - a(I_0)$, where I_0 is an injective hull of A. Thus, if A is self-injective, then $m(A) = 0$. If dom dim $(A) \geqslant 1$, then A is self-injective if and only if $m(A) = 0$. So the famous Nakayama conjecture can be reformulated as: if dom dim $(A) = \infty$, then $m(A) = 0$.

Corollary 4.7. *Let A be an Artin algebra and let T be the direct sum of all non-isomorphic indecomposable projective-injective A-modules.*

(i) *If* dom dim $(A) \geqslant n \geqslant 2$, *then* $\mathrm{End}_A(T \oplus C_i)$ *is derived equivalent to A for* $1 \leqslant i < n$.

(ii) *If* dom dim $(A) \geqslant n + 1 < \infty$, *then* $m(A) = a(C_n)$.

Proof. Since the sequence $0 \to C_{i-1} \to I_i \to C_i \to 0$ is an $\mathrm{add}(I_i)$-split sequence (or an $\mathrm{add}(T)$-split sequence), the orthogonality conditions in corollary 4.2 are trivially satisfied. Derived equivalence preserves the number of non-isomorphic simple modules. Therefore, corollary 4.7 now follows from corollary 4.2. Here we also use the observation that $\mathrm{add}(C_i) \cap \mathrm{add}(I_j) = \{0\}$ for all $0 \leqslant i, j \leqslant n$. Alternatively, one can also use theorem 2.3 to prove this corollary. $\square$

Examples of algebras of dominant dimension at least n can be obtained in the following way: let A be a self-injective algebra and X be an A-module. If $\mathrm{Ext}^i_A(X, X) = 0$ for all $1 \leqslant i \leqslant n$, then dom $\dim(\mathrm{End}_A(A \oplus X)) \geqslant n + 2$.

Finally, we remark that the condition $\mathrm{Ext}^i_A(X, A) = 0$ for an A-module X also appears in Auslander-regular algebras.

Let Λ be a k-algebra over a field k. Recall that Λ is called *Auslander-regular* if Λ has finite global dimension and satisfies the Gorenstein condition: if $p < q$ are non-negative integers and M is a finitely generated (left or right) Λ-module, then $\mathrm{Ext}^p_\Lambda(N, \Lambda) = 0$ for every submodule N of $\mathrm{Ext}^q_{\Lambda^{\mathrm{op}}}(M, \Lambda)$. Here, if M is a right Λ-module, then N is a left Λ-module. Let $j(M)$ be the minimal number $r \geqslant 0$ such that $\mathrm{Ext}^r_{\Lambda^{\mathrm{op}}}(M, \Lambda) \neq 0$. Then, for any submodule N of $\mathrm{Ext}^{j(M)}_{\Lambda^{\mathrm{op}}}(M, \Lambda)$, we have $\mathrm{Ext}^i_\Lambda(N, \Lambda) = 0$ for $0 < i < j(M)$. Thus, the following corollary holds.

Corollary 4.8. *Let Λ be an Auslander-regular k-algebra and M be a finitely generated right Λ-module. Then, for any submodule X of $\mathrm{Ext}^{j(M)}_{\Lambda^{\mathrm{op}}}(M, \Lambda)$ and any admissible subset Φ of $\mathbb{N}_{j(M)-2}$, the algebras $\mathbb{E}^\Phi_\Lambda(\Lambda \oplus X)$ and $\mathbb{E}^\Phi_\Lambda(\Lambda \oplus \Omega(X))$ are derived equivalent.*

4.2. Calabi–Yau categories

The theory of Calabi–Yau and cluster categories provides very natural contexts for our construction of derived equivalences.

Let k be a field and let $\mathcal{T}$ be a k-linear triangulated category which is Hom-finite, i.e. the Hom-space $\mathrm{Hom}_{\mathcal{T}}(X, Y)$ is finite dimensional over k for all X and Y in $\mathcal{T}$.

Recall that $\mathcal{T}$ is called $(n + 1)$-*Calabi–Yau* for some non-negative integer n if there is a natural isomorphism between $D\,\mathrm{Hom}_{\mathcal{T}}(X, Y)$ and $\mathrm{Hom}_{\mathcal{T}}(Y, X[n + 1])$ for all X and Y in $\mathcal{T}$, where $D = \mathrm{Hom}_k(\cdot, k)$ is the usual duality. It follows that $\mathcal{X}^n_{\mathcal{T}}(M) = \mathcal{Y}^n_{\mathcal{T}}(M)$ for $M \in \mathcal{T}$. (See [13] for more information on Calabi–Yau categories.)

Note that if $\Phi = \{0, 1, \ldots, n\}$, then $n - i \in \Phi$ for each $i \in \Phi$.

Lemma 4.9. *Let $\Phi = \{0, 1, \dots, n\}$. Suppose that $\mathcal{T}$ is an $(n+1)$-Calabi–Yau triangulated category and that M is an object in $\mathcal{T}$. Let*

$$X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \to X[1]$$

be a triangle in $\mathcal{T}$ with $M_1 \in \operatorname{add}(M)$. Then we have the following.

(i) *The morphism α is a left $(\operatorname{add}(M), \Phi)$-approximation of X if and only if the morphism β is a right $(\operatorname{add}(M), -\Phi)$-approximation of Y.*

(ii) *If α is a left $(\operatorname{add}(M), \Phi)$-approximation of X and if M is n-self-orthogonal, then $X \in \mathcal{X}^n(M) \cap \mathcal{Y}^n(M)$ and $Y \in \mathcal{X}^n(M) \cap \mathcal{Y}^n(M)$.*

Proof. We shall abbreviate $\operatorname{Hom}_{\mathcal{T}}(\cdot,\cdot)$ by $(\cdot,\cdot)$. First we assume that α is a left $(\operatorname{add}(M), \Phi)$-approximation of X. Now, for each $i \in \Phi$, there is a commutative diagram with exact rows

$$\begin{array}{ccccccc}
(M[-i], M_1) & & \xrightarrow{(M[-i],\beta)} & & (M[-i], Y) & & \\
\downarrow \simeq & & & & \downarrow \simeq & & \\
(M, M_1[i]) & \xrightarrow{(M,\beta[i])} & (M, Y[i]) & \longrightarrow & (M, X[i+1]) & \xrightarrow{(M,\alpha[i+1])} & (M, M_1[i+1]) \\
 & & & & \downarrow \simeq & & \downarrow \simeq \\
 & & & & D(X, M[n-i]) & \xrightarrow{D(\alpha, M[n-i])} & D(M_1, M[n-i])
\end{array}$$

Since $n-i$ is in Φ, and since α is a left $(\operatorname{add}(M), \Phi)$-approximation of X, the map $(\alpha, M[n-i])$ is surjective, and consequently $D(\alpha, M[n-i])$ is injective. Hence, $(M, \alpha[i+1])$ is injective, and therefore $(M[-i], \beta)$ is surjective. This shows that β is a right $(\operatorname{add}(M), -\Phi)$-approximation of Y. The other implication in (i) can be proved similarly.

(ii) It follows from (i) and remark 3.12(a) that $X \in \mathcal{X}^{\Phi}_{\mathcal{T}}(M)$ and $Y \in \mathcal{Y}^{\Phi}_{\mathcal{T}}(M)$. Since $\mathcal{T}$ is $(n+1)$-Calabi–Yau, we have $(M, X[i]) \simeq D(X, M[n+1-i]) = 0$ and $(M, Y[i]) \simeq D(Y, M[n+1-i]) = 0$ for all $0 \neq i \in \Phi$. Thus, $X \in \mathcal{Y}^{\Phi}_{\mathcal{T}}(M)$ and $Y \in \mathcal{X}^{\Phi}_{\mathcal{T}}(M)$. □

Corollary 4.10. *Let $\Phi = \{0, 1, \dots, n\}$ and let $\mathcal{T}$ be an $(n+1)$-Calabi–Yau triangulated category. Suppose that M is n-self-orthogonal and $Y \in \mathcal{Y}^n(M)$. Let $X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \xrightarrow{w} X[1]$ be a triangle in $\mathcal{T}$ with β a right $\operatorname{add}(M)$-approximation of Y. Then the algebras $\mathbb{E}^{\Phi}_{\mathcal{T}}(M \oplus X)/I$ and $\mathbb{E}^{\Phi}_{\mathcal{T}}(M \oplus Y)/J$ are derived equivalent, where I and J are defined as in theorem 3.1.*

Proof. Since $Y \in \mathcal{Y}^{\Phi}_{\mathcal{T}}(M)$, for each $0 \neq i \in \Phi$, the map $(M[-i], M_1) \to (M[-i], Y) = 0$ induced by β is surjective. Taking into account that β is a right $\operatorname{add}(M)$-approximation of Y, we see that β is, in fact, a right $(\operatorname{add}(M), -\Phi)$-approximation of Y. By proposition 4.9(i), the map α is a left $(\operatorname{add}(M), \Phi)$-approximation of X. Since M is n-self-orthogonal, the proof can be finished by applying proposition 4.9(ii) and corollary 3.13 to the triangle. □

Corollary 4.10 is related to mutations in a Calabi–Yau category. We now give some definitions from [12].

Let $\mathcal{T}$ be an $(n+1)$-Calabi–Yau category. An object T in $\mathcal{T}$ is called an *n-cluster tilting object* if T is n-self-orthogonal and if any $X \in \mathcal{T}$ with $\mathrm{Ext}^i_{\mathcal{T}}(T, X) = 0$ for $1 \leqslant i \leqslant n$ is in $\mathrm{add}(T)$. The object T is called *basic* if the multiplicity of each indecomposable direct summand of T is 1.

Let T be an n-cluster basic tilting object in an $(n+1)$-Calabi–Yau category $\mathcal{T}$ and let Y be a direct summand of T, i.e. $T = Y \oplus M$. Let $\beta\colon M_1 \to Y$ be a minimal right $\mathrm{add}(M)$-approximation of Y and let

$$X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \to X[1]$$

be a triangle containing β. Note that we allow Y to be decomposable, and that X is indecomposable if and only if Y is indecomposable. The object $X \oplus M$ is called the *left mutation* of T at Y. In the case of tilting modules, X is called a *tilting complement* to M in the literature (see, for example, [9]). It was pointed out in [12] that the left mutation of T at Y is again an n-cluster tilting object (for some special cases, see [3, 7] and [16, p. 314]). In fact, this can be seen in the following way: the proof of corollary 4.10 and comment (b) on the conditions of theorem 3.1 imply that $T' := M \oplus X$ is n-self-orthogonal. Moreover, let $X' \in \mathcal{X}^n(T')$ and consider a triangle $X' \xrightarrow{\alpha'} M' \to Y' \to X'[1]$ with α' a left $\mathrm{add}(M)$-approximation of X'. Then $Y' \in \mathcal{X}^n(T)$ by lemma 4.9 and comment (b). Thus, $Y' \in \mathrm{add}(T)$, $X' \in \mathrm{add}(T')$ and $T' := X \oplus M$ is again an n-cluster tilting object in $\mathcal{T}$. The notion of a right mutation of T at Y is dual.

Usually, $\mathrm{End}_{\mathcal{T}}(X \oplus M)$ and $\mathrm{End}_{\mathcal{T}}(M \oplus Y)$ are not derived equivalent. When they *are* derived equivalent is an interesting question. We now give a sufficient condition.

Corollary 4.11. *Let $\Lambda := \mathrm{End}_{\mathcal{T}}(X \oplus M)$ and $\Gamma := \mathrm{End}_{\mathcal{T}}(M \oplus Y)$. Then we have the following.*

(i) $\mathrm{End}_{\mathcal{T}}(X \oplus M)/I$ *and* $\mathrm{End}_{\mathcal{T}}(M \oplus Y)/J$ *are derived equivalent.*

(ii) *Suppose that Y is indecomposable. Let S_X be the simple Λ-module corresponding to X and let S_Y be the simple Γ-module corresponding to Y. Suppose that S_Y is not a submodule of Γ, and that S_X is not a quotient of $D(\Lambda)$. Then Λ and Γ are derived equivalent.*

Proof. Statement (i) is a direct consequence of corollary 4.10, and (ii) follows from (i) and proposition 3.15. □

Remark 4.12. Consider a 2-Calabi–Yau category, and assume that

$$\mathrm{Ext}^1_{\Gamma}(S_Y, S_Y) = 0.$$

Then we once more obtain the result [14, theorem 5.3] from corollary 4.11(ii).

4.3. Frobenius categories

As is known, triangulated categories are closely related to Frobenius categories. In fact, the only known general method to get triangulated categories is first to construct Frobenius categories and then pass to their stable categories (see [8]). In this section, we shall apply our results to Frobenius abelian categories.

Let $\mathcal{A}$ be a *Frobenius* abelian category, that is, $\mathcal{A}$ is an abelian category with enough projective objects and enough injective objects such that the projective objects coincide with the injective objects. We denote by $\underline{\mathcal{A}}$ the stable category of $\mathcal{A}$ modulo projective objects. It is shown in [8] that $\underline{\mathcal{A}}$ is a triangulated category, in which the shift functor [1] is just the co-syzygy functor Ω^{-1}, and the triangles in $\underline{\mathcal{A}}$ are all induced by short exact sequences in $\mathcal{A}$. For each morphism $f\colon U \to V$ in $\mathcal{A}$, we denote by $\underline{f}$ the image of f under the canonical functor from $\mathcal{A}$ to $\underline{\mathcal{A}}$. Note that the objects of $\underline{\mathcal{A}}$ are the same as those of $\mathcal{A}$.

LEMMA 4.13. *Let Φ be an admissible subset of $\mathbb{N}$ and let M, X and Y be objects in $\mathcal{A}$. Then*

(i) *for arbitrary $0 \neq i \in \mathbb{N}$ and $U, U' \in \mathcal{A}$, there exists an isomorphism*

$$\mathrm{Hom}_{\mathcal{D}^{\mathrm{b}}(\mathcal{A})}(U, U'[i]) \simeq \mathrm{Hom}_{\underline{\mathcal{A}}}(U, U'[i]),$$

which is functorial in U and U',

(ii) *a monomorphism $\alpha\colon X \to M_1$ in $\mathcal{A}$ is a left* $(\mathrm{add}(M), \Phi)$*-approximation of X in $\mathcal{D}^{\mathrm{b}}(\mathcal{A})$ if and only if $\underline{\alpha}$ is a left* $(\mathrm{add}(M), \Phi)$*-approximation of X in $\underline{\mathcal{A}}$,*

(iii) *an epimorphism $\beta\colon M_2 \to Y$ in $\mathcal{A}$ is a right* $(\mathrm{add}(M), -\Phi)$*-approximation of Y in $\mathcal{D}^{\mathrm{b}}(\mathcal{A})$ if and only if $\underline{\beta}$ is a right* $(\mathrm{add}(M), -\Phi)$*-approximation of Y in $\underline{\mathcal{A}}$.*

Proof. (i) For $0 \neq i \in \mathbb{N}$, the isomorphisms

$$\mathrm{Hom}_{\mathcal{D}^{\mathrm{b}}(\mathcal{A})}(U, U'[i]) \simeq \mathrm{Ext}^i_{\mathcal{A}}(U, U') \simeq \mathrm{Hom}_{\underline{\mathcal{A}}}(U, \Omega^{-i}U') = \mathrm{Hom}_{\underline{\mathcal{A}}}(U, U'[i]).$$

are functorial in U and U'. Thus, (i) follows.

(ii) First, let $0 \neq i$ be in Φ. By (i), there is a commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{D}^{\mathrm{b}}(\mathcal{A})}(M_1, M[i]) & \xrightarrow{(\alpha, M[i])} & \mathrm{Hom}_{\mathcal{D}^{\mathrm{b}}(\mathcal{A})}(X, M[i]) \\
\downarrow \simeq & & \downarrow \simeq \\
\mathrm{Hom}_{\underline{\mathcal{A}}}(M_1, M[i]) & \xrightarrow{(\underline{\alpha}, M[i])} & \mathrm{Hom}_{\underline{\mathcal{A}}}(X, M[i])
\end{array}$$

Thus, the map $\mathrm{Hom}_{\underline{\mathcal{A}}}(\underline{\alpha}, M[i])$ is surjective if and only if $\mathrm{Hom}_{\mathcal{D}^{\mathrm{b}}(\mathcal{A})}(\alpha, M[i])$ is surjective. Now we consider the case $i = 0$. If every morphism from X to M in $\mathcal{A}$ factorizes through α, then every morphism from X to M in $\underline{\mathcal{A}}$ factorizes through $\underline{\alpha}$. Conversely, assume that every morphism from X to M in $\underline{\mathcal{A}}$ factorizes through $\underline{\alpha}$. Let $f\colon X \to M$ be a morphism in $\mathcal{A}$. Then $\underline{f} = \underline{\alpha h}$ for some $h\colon M_1 \to M$ in $\mathcal{A}$. Thus, $f - \alpha h$ in $\mathcal{A}$ factorizes through a projective object P, say $f - \alpha h = st$ for some $s\colon X \to P$ and $t\colon P \to M$ in $\mathcal{A}$. Since P is also injective and α is a monomorphism, there is some morphism $r\colon M_1 \to P$ such that $s = \alpha r$. Altogether, $f = \alpha h + st = \alpha h + \alpha rt = \alpha(h + rt)$ factorizes through α. Thus, statement (ii) follows. The proof of (iii) is similar to that of (ii). $\square$

PROPOSITION 4.14. *Let Φ be an admissible subset of $\mathbb{N}$. Suppose that $\mathcal{A}$ is a Frobenius abelian category, M is an object in $\mathcal{A}$ and $0 \to X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \to 0$ is a short exact sequence in $\mathcal{A}$ with $M_1 \in \mathrm{add}(M)$ such that the induced triangle*

$$X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \to X[1]$$

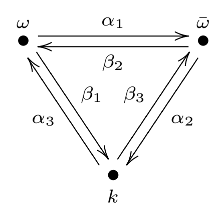
in $\underline{\mathcal{A}}$ satisfies the conditions in theorem 3.1. Then the algebras $\mathbb{E}^{\Phi}_{\mathcal{D}(\mathcal{A})}(M \oplus Y)$ and $\mathbb{E}^{\Phi}_{\mathcal{D}(\mathcal{A})}(X \oplus M)$ are derived equivalent.

Proof. This follows from lemma 4.13 and proposition 3.17. □

5. Examples

First, we present an explicit example which satisfies all conditions in theorem 3.1.

EXAMPLE 5.1. Let k be an algebraically closed field of characteristic 2 and let $A := kA_4$ be the group algebra of the alternating group A_4. Then there are three simple A-modules, which are denoted k, ω and $\bar{\omega}$, respectively. Their projective covers are $P(k)$, $P(\omega)$ and $P(\bar{\omega})$, respectively. It was shown in [6, § V2.4.1, p. 129] that kA_4 is Morita equivalent to the following algebra given by quiver



and relations $\alpha_i\beta_{i+1} - \beta_i\alpha_{i+2} = \alpha_i\alpha_{i+1} = \beta_i\beta_{i-1} = 0$, where the subscripts are considered modulo 3.

As this algebra is symmetric, the Auslander–Reiten translation $D\,\mathrm{Tr}$ is just the second syzygy Ω^2 (see [1, proposition 3.8, p. 127]). Thus, a direct computation shows that the Auslander–Reiten quiver of this algebra has a component of the form shown in figure 1.

Consider the Auslander–Reiten sequence

$$0 \to \Omega^3(\omega) \to \Omega^2(k) \oplus \Omega^2(\bar{\omega}) \to \Omega(\omega) \to 0.$$

Let $X = \Omega^3(\omega)$, $Y = \Omega(\omega)$ and $M = \Omega^2(k) \oplus \Omega^2(\bar{\omega})$. This sequence provides an Auslander–Reiten triangle in the triangulated category A-$\underline{\mathrm{mod}}$:

$$X \to M \to Y \to X[1].$$

We shall check that this triangle satisfies the conditions of theorem 3.1.

We choose $\Phi = \{0, 1\}$ and $F = [1]$. Since this is an Auslander–Reiten triangle in A-$\underline{\mathrm{mod}}$, the map $X \to M$ is a left $(\mathrm{add}(M), \Phi)$-approximation of X, and the map $M \to Y$ is a right $(\mathrm{add}(M), -\Phi)$-approximation of Y (see the example at the end of § 2). It follows from the above Auslander–Reiten quiver of A that

$$\mathrm{Ext}^1_A(M, X) \simeq \underline{\mathrm{Hom}}_A(M, \Omega^{-1}(X)) \simeq \underline{\mathrm{Hom}}_A(\Omega^2(k) \oplus \Omega^2(\bar{\omega}), \Omega^2(\omega)) = 0$$

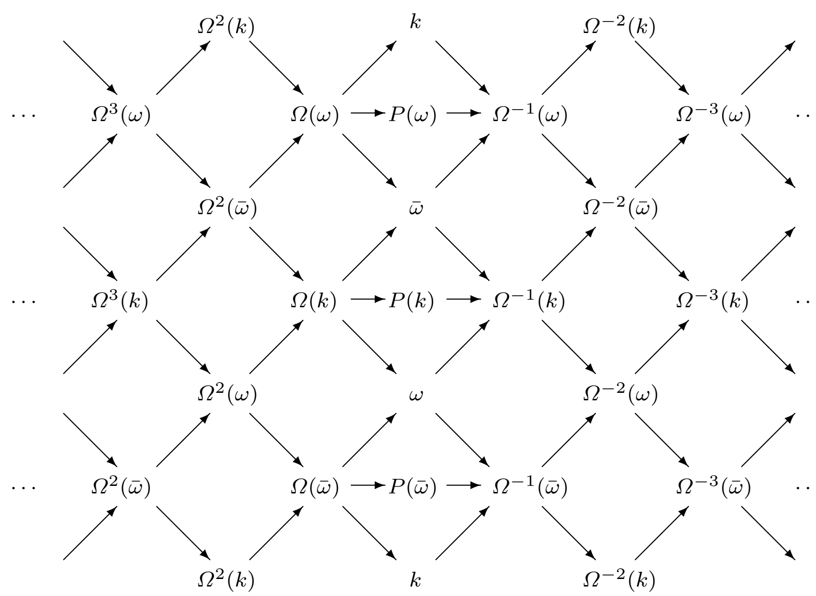


Figure 1. A component of Auslander–Reiten quiver of A in Example 5.1.

and

$$\mathrm{Ext}^1_A(Y,M) \simeq \underline{\mathrm{Hom}}_A(Y,\Omega^{-1}(M)) = \underline{\mathrm{Hom}}_A(\Omega(\omega),\Omega(k)\oplus\Omega(\bar\omega)) = 0.$$

Thus, the above triangle in A-$\underline{\mathrm{mod}}$ satisfies all conditions in theorem 3.1, and therefore, by proposition 4.14, the algebras $\mathbb{E}^{\Phi}_A(M\oplus X)$ and $\mathbb{E}^{\Phi}_A(M\oplus Y)$ are derived equivalent.

Furthermore, we have

$$\mathrm{Ext}^1_A(M,M) \simeq \underline{\mathrm{Hom}}_A(M,\Omega^{-1}M) \simeq \underline{\mathrm{Hom}}_A(\Omega(k)\oplus\Omega(\bar\omega),k\oplus\bar\omega).$$

There are an epimorphism from $\Omega(k)$ to $\bar\omega$ and an epimorphism from $\Omega(\bar\omega)$ to k. The latter cannot factorize through a projective module, we get $\dim_k \mathrm{Ext}^1_A(M,M) = 2$. Moreover, there are an epimorphism from $\Omega(k)$ to ω and an epimorphism from $\Omega(\bar\omega)$ to ω. This implies $\dim_k \mathrm{Ext}^1_A(M,Y) = 2$. Similarly, $\dim_k \mathrm{Ext}^1_A(X,M) = 2$. Note that all the indecomposable modules appearing in the Auslander–Reiten triangle are 1-self-orthogonal. A more precise calculation shows that $\dim_k \mathbb{E}^{\Phi}_A(M\oplus X) = 33$ and $\dim_k \mathbb{E}^{\Phi}_A(M\oplus Y) = 21$.

The following example shows that the Ext-orthogonality conditions in corollary 4.2 and therefore in theorem 3.1 cannot be dropped.

EXAMPLE 5.2. Let A be the algebra (over a field k) given by the quiver with relations shown in figure 2.

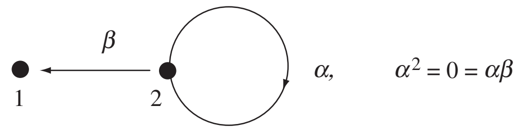


Figure 2. The quiver and relations of A in Example 5.2.

This example is in a class of examples constructed by Small [23]. The algebra A is of finite representation type, its finitistic dimension equals one, while the finitistic dimension of the opposite algebra $A^{\rm op}$ is zero.

We denote by $S(i)$ and $P(i)$ the simple and projective modules corresponding to the vertex i, respectively. Let M_i be the quotient module of $P(2)$ by $S(i)$ and let $M := M_1 \oplus M_2 = D(A_A)$, where D is the usual duality. Then there is an Auslander–Reiten sequence

$$0 \to X := P(2) \to M \to S(2) =: Y \to 0.$$

This is an $\mathrm{add}(M)$-split sequence in A-mod.

If we take $\Phi = \{0, 1\}$, then $\mathbb{E}_A^{\Phi}(X \oplus M) = \mathrm{End}_A(X \oplus M)$. An easy calculation shows that $\mathrm{End}_A(X \oplus M)$ is a quasi-hereditary algebra, and thus has finite global dimension. The algebra $\mathbb{E}_A^{\Phi}(M \oplus Y)$ contains a loop which is given by the short exact sequence induced by the loop α at the vertex 2. Thus, it has infinite global dimension by [15]. It follows that $\mathbb{E}_A^{\Phi}(X \oplus M)$ and $\mathbb{E}_A^{\Phi}(M \oplus Y)$ cannot be derived equivalent since derived equivalences preserve the finiteness of global dimensions. Also, one can see that $\mathrm{Ext}_A^i(X, M) = 0 = \mathrm{Ext}_A^1(M, X)$ and $\mathrm{Ext}_A^i(Y, M) = 0 \neq \mathrm{Ext}_A^1(M, Y)$ for $i \geqslant 1$. This example shows that the orthogonality conditions in corollary 4.2 cannot be omitted. Moreover, it shows that the result in [11, theorem 1.1] cannot be extended from endomorphism algebras to Φ-Yoneda algebras without any additional conditions.

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Appendix A. A two-functors version of theorem 1.1

In theorem 3.1, there is only one functor F involved. When working with the derived category of a hereditary algebra, the stable category of a self-injective algebra or the derived category of coherent sheaves of a projective variety over $\mathbb{C}$, apart from the shift functor there are other prominent functors, for example, the Auslander–Reiten translation $D\,\mathrm{Tr}$. To have available a general statement of construction of derived equivalences, which is similar to theorem 3.1, we define Φ-perforated Yoneda algebras for two functors over a triangulated category and formulate a two-functor version of theorem 3.1. In this appendix, we summarize the ingredients for a generalization of theorem 3.1. The proof of this generalization is analogous to that of theorem 3.1, but more technical and tedious. So we only sketch it here.

Let Φ be a subset of $\mathbb{N} \times \mathbb{N}$ which we consider as a semigroup with ordinary addition. Let $\mathcal{T}$ be a triangulated R-category with shift functor [1] and let X be an object in $\mathcal{T}$.

Suppose that F and G are two triangle functors from $\mathcal{T}$ to itself, such that FG is naturally isomorphic to GF. For X in $\mathcal{T}$, let $\delta(i,j,X)\colon F^jG^iX \to G^iF^jX$ be an isomorphism induced from the natural transformation $FG \sim GF$. Then we define

$$\mathbb{E}_{\mathcal{T}}^{F,G,\Phi}(X) := \bigoplus_{(i,j)\in\Phi} \mathrm{Hom}_{\mathcal{T}}(X, G^iF^jX),$$

with elements of the form $(f_{i,j})_{(i,j)\in\Phi}$, where $f_{i,j}\colon X \to G^iF^jX$. The multiplication on $\mathbb{E}_{\mathcal{T}}^{F,G,\Phi}(X)$ is given by

$$(f_{i,j})_{(i,j)\in\Phi} \cdot (g_{i,j})_{(i,j)\in\Phi} = \Bigg(\sum_{\substack{(p,q),(u,v)\in\Phi \\ (u+p,v+q)=(l,m)\in\Phi}} f_{u,v}(G^uF^vg_{p,q})(G^u\delta(p,v,F^qX)) \Bigg)_{(l,m)\in\Phi}.$$

A general model for the above definition is as follows. Given a bi-graded algebra $\Lambda = \bigoplus_{i,j\in\mathbb{Z}} \Lambda_{i,j}$, we define

$$\Lambda(\Phi) = \bigoplus_{(i,j)\in\Phi} \Lambda_{i,j},$$

and a multiplication by $a_{i,j} \cdot a_{p,q} = a_{i,j}a_{p,q}$ if $(i+p, j+q) \in \Phi$, and zero otherwise. If Φ is admissible, for example, Φ is the Cartesian product of two admissible sets in $\mathbb{Z}$, then $\Lambda(\Phi)$ is an associative algebra. So, we have to check that, given two auto-isomorphism functors F and G on $\mathcal{T}$, the R-module

$$\mathbb{E}_{\mathcal{T}}^{F,G}(X) := \bigoplus_{i,j\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{T}}(X, G^iF^jX)$$

is an associative algebra with respect to the above multiplication. This can be based on the following lemma.

Lemma A.1. *Suppose that F and G are two triangle functors from $\mathcal{T}$ to itself such that FG is naturally isomorphic to GF. For any triangle functor L from $\mathcal{T}$ to itself, there is a natural isomorphism $\delta(i,j,L)\colon F^jG^iL \to G^iF^jL$ for all $i,j \geqslant 0$ such that, for $p,q,r,s \in \mathbb{N}$,*

(i) $\delta(p+q,r,L) = \delta(p,r,G^qL)(G^p\delta(q,r,L))$,

(ii) $\delta(p,r+s,L) = (F^s\delta(p,r,L))\delta(p,s,F^rL)$.

Proof. For functors L_1 and L_2 from $\mathcal{T}$ to itself, we define $L_1\delta(1,1,L_2)\colon L_1FGL_2 \to L_1GFL_2$ to be the induced natural isomorphism from the functor L_1FGL_2 to the functor L_1GFL_2. So, $\delta(1,1,1_{\mathcal{T}})$ is just the given natural isomorphism from FG to GF. Now we shall construct inductively a natural isomorphism $\delta(i,j,L)$ from

F^jG^iL to G^iF^jL for all non-negative integers i and j and functors L from $\mathcal{T}$ to itself.

If $i=0$ or $j=0$, then $F^jG^iL = G^iF^jL$, and we define $\delta(i,j,L)$ to be the identity natural transformation. For each positive integer $j>1$, we assume that $\delta(1,j-1,L)$ is defined. Now we define

$$\delta(1,j,L) := (F\delta(1,j-1,L))\delta(1,1,F^{j-1}L).$$

For each positive integer $i>1$, assume that $\delta(i-1,j,L)$ is defined. We define

$$\delta(i,j,L) := \delta(1,j,G^{i-1}L)(G\delta(i-1,j,L)).$$

(i) It is straightforward to check that (i) holds for $p+q \leqslant 2$. We shall prove (i) by induction on $p+q$. Now assume that $p+q>2$. Then we have

$$\begin{aligned}
&\delta(p+q,r,L)\\
&\quad = \delta(1,r,G^{p+q-1}L)(G\delta(p+q-1,r,L)) && \text{(by definition)}\\
&\quad = \delta(1,r,G^{p+q-1}L)G(\delta(p-1,r,G^qL)(G^{p-1}\delta(q,r,L))) && \text{(by induction)}\\
&\quad = (\delta(1,r,G^{p+q-1}L)(G\delta(p-1,r,G^qL)))(G^p\delta(q,r,L))\\
&\quad = \delta(p,r,G^qL)(G^p\delta(q,r,L)) && \text{(by definition).}
\end{aligned}$$

This proves (i).

(ii) We first prove (ii) for $p=0,1$. If $p=0$, then (ii) is clearly true. Now suppose $p=1$. We shall show (ii) by induction on $r+s$. In fact, if $r+s \leqslant 2$, it is straightforward to check (ii). Now we assume that $r+s>2$. Then we have

$$\begin{aligned}
&\delta(1,r+s,L)\\
&\quad = (F\delta(1,r+s-1,L))\delta(1,1,F^{r+s-1}L) && \text{(by definition)}\\
&\quad = F((F^{s-1}\delta(1,r,L))\delta(1,s-1,F^rL))\delta(1,1,F^{r+s-1}L) && \text{(by induction)}\\
&\quad = (F^s\delta(1,r,L))((F\delta(1,s-1,F^rL))\delta(1,1,F^{r+s-1}L))\\
&\quad = (F^s\delta(1,r,L))\delta(1,s,F^rL) && \text{(by definition).}
\end{aligned}$$

This proves (ii) for $p=1$. Now assume $p>1$. Then

$$\begin{aligned}
&\delta(p,r+s,L)\\
&\quad = \delta(1,r+s,G^{p-1}L)(G\delta(p-1,r+s,L)) && \text{(by definition)}\\
&\quad = (F^s\delta(1,r,G^{p-1}L))\delta(1,s,F^rG^{p-1}L)\\
&\qquad \times G((F^s\delta(p-1,r,L))\delta(p-1,s,F^rL)) && \text{(by induction)}\\
&\quad = (F^s\delta(1,r,G^{p-1}L))(\delta(1,s,F^rG^{p-1}L)(GF^s\delta(p-1,r,L)))\\
&\qquad \times (G\delta(p-1,s,F^rL)).
\end{aligned}$$

Since

$$\delta(1,s,F^rG^{p-1}L)$$

is a natural transformation from $F^sGF^rG^{p-1}L$ to $GF^sF^rG^{p-1}L$, the following diagram of natural transformations is commutative:

$$\begin{array}{ccc} F^sGF^rG^{p-1}L & \xrightarrow{\delta(1,s,F^rG^{p-1}L)} & GF^sF^rG^{p-1}L \\ {\scriptstyle F^sG\delta(p-1,r,L)}\downarrow & & \downarrow{\scriptstyle GF^s\delta(p-1,r,L)} \\ F^sGG^{p-1}F^rL & \xrightarrow{\delta(1,s,G^{p-1}F^rL)} & GF^sG^{p-1}F^rL \end{array}$$

Hence,

$$\begin{aligned} &\delta(p, r+s, L) \\ &\quad = (F^s\delta(1,r,G^{p-1}L))(\delta(1,s,F^rG^{p-1}L)(GF^s\delta(p-1,r,L)))(G\delta(p-1,s,F^rL)) \\ &\quad = (F^s\delta(1,r,G^{p-1}L))((F^sG\delta(p-1,r,L))\delta(1,s,G^{p-1}F^rL))(G\delta(p-1,s,F^rL)) \\ &\quad = F^s(\delta(1,r,G^{p-1}L)(G\delta(p-1,r,L)))(\delta(1,s,G^{p-1}F^rL)(G\delta(p-1,s,F^rL))) \\ &\quad = (F^s\delta(p,r,L))\delta(p,s,F^rL). \end{aligned}$$

This proves (ii). □

REMARK A.2. If, in addition, F and G are auto-isomorphisms, then lemma A.1 remains valid for i, j, p, q, r and s any integers.

Let $\mathcal{D}$ be a full subcategory of $\mathcal{T}$ and let X be an object of $\mathcal{T}$. A morphism $f\colon X \to D$ with $D \in \mathcal{D}$ is called a *left* $(\mathcal{D}, F, G, \Phi)$*-approximation* of X if

$$\mathrm{Hom}_{\mathcal{T}}(f, G^iF^jD')\colon \mathrm{Hom}_{\mathcal{T}}(D, G^iF^jD') \to \mathrm{Hom}_{\mathcal{T}}(X, G^iF^jD')$$

is surjective for every object $D' \in \mathcal{D}$ and $(i,j) \in \Phi$. Dually, we define the right $(\mathcal{D}, F, G, \Phi)$-approximation of X.

Given a triangle

$$0 \to X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \xrightarrow{w} X[1] \quad \text{in } \mathcal{T}$$

with $M_1 \in \mathrm{add}(M)$ for a fixed $M \in \mathcal{T}$, we define $\tilde{w}[-1] = (-w[-1], 0)\colon Y[-1] \to X \oplus M$, $\bar{w} = (0, w)^{\mathrm{T}}$, where $(0, w)^{\mathrm{T}}$ stands for the transpose of the matrix $(0, w)$, and

$$\begin{aligned} I := \{x = (x_{i,j}) \in \mathbb{E}_{\mathcal{T}}^{F,G,\Phi}(X \oplus M) \mid x_{i,j} = 0 \text{ for } (0,0) \neq (i,j) \in \Phi, \\ \text{and } x_{0,0} \text{ factors through } \mathrm{add}(M) \text{ and } \tilde{w}[-1]\}, \\ J := \{y = (y_{i,j}) \in \mathbb{E}_{\mathcal{T}}^{F,G,\Phi}(M \oplus Y) \mid y_{i,j} = 0 \text{ for } (0,0) \neq (i,j) \in \Phi, \\ \text{and } y_{0,0} \text{ factors through } \mathrm{add}(M) \text{ and } \bar{w}\}. \end{aligned}$$

Now, with a proof similar to theorem 3.1, one can get the following result with two functors.

THEOREM A.3. *Let Φ be an admissible subset of $\mathbb{Z}\times\mathbb{Z}$, $\mathcal{T}$ be a triangulated R-category and M be an object in $\mathcal{T}$. Assume that there are two triangle auto-isomorphisms F and G from $\mathcal{T}$ to itself such that FG is naturally isomorphic to GF by $\delta\colon FG \to GF$, Suppose that $X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \xrightarrow{w} X[1]$ is a triangle in $\mathcal{T}$ such that α is* *a left* $(\mathrm{add}(M), F, G, \Phi)$*-approximation of* X *and* β *is a right* $(\mathrm{add}(M), F, G, -(\Phi))$*-approximation of* Y*. If*

$$\mathrm{Hom}_{\mathcal{T}}(M, G^i F^j X) = 0 = \mathrm{Hom}_{\mathcal{T}}(Y, G^i F^j(M)) \quad \textit{for } (0,0) \neq (i,j) \in \Phi,$$

then $\mathbb{E}_{\mathcal{T}}^{F,G,\Phi}(X \oplus M)/I$ *and* $\mathbb{E}_{\mathcal{T}}^{F,G,\Phi}(M \oplus Y)/J$ *are derived equivalent.*

Taking $G = [1]$ and $F = \mathrm{id}$ in a derived module category yields a result on Φ-Auslander–Yoneda algebras. Taking $G = \mathrm{id}$, we recover theorem 3.1 for the case of F being an arbitrary auto-isomorphism.

Outline of the proof of theorem A.3. Clearly, as in the proof of lemma 3.3, we can use lemma 3.2 to show that I and J are ideals in

$$\mathbb{E}_{\mathcal{T}}^{F,G,\Phi}(X \oplus M) \quad \text{and} \quad \mathbb{E}_{\mathcal{T}}^{F,G,\Phi}(M \oplus Y),$$

respectively. The next step is to check that the complex $T^{\bullet}$, which can be defined analogously to lemma 3.5, is a tilting complex. Finally, one needs to prove the isomorphism described in lemma 3.7. However, this verification follows the proof of lemma 3.7 verbatim. □

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