Forum Math. **20** (2008), 81–97
DOI 10.1515/FORUM.2008.004

Forum Mathematicum

Stable equivalences of adjoint type

Changchang Xi

Dedicated to Professor Gus Lehrer on the occasion of his 60th birthday

(Communicated by Frederick R. Cohen)

Abstract. In this paper we define a class of stable equivalences, namely, the stable equivalences of adjoint type, and study the Hochschild cohomology groups of algebras that are linked by a stable equivalence of adjoint type. This notion of adjoint type is a special case of Morita type, covers the stable equivalence of Morita type for self-injective algebras, and thus includes the case where Broué's conjecture was made (see for instance [5]). The main results in this paper are: Let A and B be two artin k-algebras such that A and B are projective over k, and let $H^n(A)$ and $H^n(B)$ be the n-th Hochschild cohomology groups of A and B, respectively. (1) If A and B are stably equivalent of adjoint type, then $H^n(A) \simeq H^n(B)$ for all $n \geq 1$. (2) If A and B are stably equivalent of Morita type, then the absolute values of Cartan determinants of A and B are equal. In particular, two cellular algebras over a field have the same Cartan determinant if they are stably equivalent of Morita type.

2000 Mathematics Subject Classification: 16G10, 16E30; 16G70, 18G05, 20J05.

1 Introduction

Stable equivalences of Morita type are of particular interest in representation theory of finite groups and associative algebras (see [1, 5, 13, 14, 17, 18, 19, 20, 21]); they enter into derived equivalences by a result of Rickard [18], which says that any derived equivalence between self-injective algebras (especially, the block algebras of group algebras) induces a stable equivalence of Morita type. In this way, the notion of stable equivalence of Morita type is then related to Broué's abelian defect group conjecture which says that the module categories of a block algebra A of a finite group algebra and its Brauer correspondent B should have equivalent derived categories if their defect group is abelian (see [18]). When starting to understand stable equivalences of Morita type in general situation, we realize that the stable equivalences of Morita type between self-injective algebras have a special property, namely, the two bimodules that define the stable equivalences of Morita type supply us always with two natural adjoint pairs of functors between module categories (see [14]). Such a stable equivalence will be called a stable equivalence of adjoint type in this paper. One might think this kind of stable equivalences would be rare. But, in fact, it is not the case. Surprisingly, even beyond the scope of self-injective algebras there are many stable equivalences of adjoint type. At moment we do not know any example of two algebras which are stably equivalent of Morita type, but not of adjoint type. In [15] one may find a machinery to construct stable equivalences of adjoint type. Moreover, it was shown in [15] that stable equivalences of adjoint type preserve self-injective dimension and Gorenstein property. It seems that stable equivalences of adjoint type behavior very nicely in transferring information from one algebra to the other. To understand the adjoint type, a natural question is: which possible properties could distinguish the adjoint type from Morita type?

In the present note we shall prove that stable equivalences of adjoint type preserve Hochschild cohomology groups, and that the absolute value of Cartan determinant is invariant under stable equivalences of Morita type. The former generalizes a result in [17] (see also [14]), and the latter extends a result in [16] and in [2] in different direction.

Theorem 1.1. *Let A and B be two artin k-algebras such that A and B are projective k-modules, and let $H^n(A)$ and $H^n(B)$ be the n-th Hochschild cohomology groups of A and B, respectively.*

(1) *If A and B are stably equivalent of adjoint type, then $H^n(A) \simeq H^n(B)$ for all $n \geq 1$.*

(2) *If A and B are stably equivalent of Morita type, then the absolute values of Cartan determinants of A and B are equal. In particular, two cellular algebras have the same Cartan determinant if they are stably equivalent of Morita type.*

The proof of this result is given in Section 4 and Section 5. The main ingredient to the proof of the first statement is the use of a spectral sequence which provides us a homological identity.

2 Preliminaries

Throughout this paper, k will stand for a commutative artin ring with identity. All categories will be k-categories and all functors are k-functors; and all categories are closed under isomorphisms and direct summands. Furthermore, we assume that all algebras A considered are artin k-algebras with identity, that is, A is a finitely generated k-module. Unless stated otherwise, by a module we shall mean a finitely generated left module. The composition of two morphisms $f : X \to Y$ and $g : Y \to Z$ between modules will be denoted by fg.

Given an algebra A, we denote by A-mod the category of finitely generated A-modules. Dually, we denote by mod-A the category of finitely generated right A-modules. A left A- and right B-bimodule X will be denoted by ${}_AX_B$, or by ${}_{A-B^{\mathrm{op}}}X$ and $X_{A^{\mathrm{op}}-B}$. The usual dual of artin k-algebras will be denoted by D.

Let us first collect some homological facts which we need in the later proofs.

Lemma 2.1. (1) *Let A, B and E be three artin k-algebras and ${}_AX_B$ and ${}_BY_E$ bimodules, where X_B is projective. Then the natural morphism $\phi : {}_AX \otimes_B Y_E \to$*

$\mathrm{Hom}_B({}_BX^*{}_A, {}_BY_E)$, *where* $X^* = \mathrm{Hom}_B(X, B)$ *and* $\phi(x \otimes y)(f) = f(x)y$ *for* $x \in X$, $y \in Y$ *and* $f \in X^*$ *is an isomorphism of A-E-bimodules.*

(2) *In the situation* $(P_A, X_B, {}_AU_B)$, *if* P_A *is projective, or if* X_B *is projective, then*

$$P \otimes_A \mathrm{Hom}_B(X_B, {}_AU_B) \simeq \mathrm{Hom}_B(X_B, P \otimes_A U_B).$$

Dually, in the situation $({}_AP, {}_BX, {}_BU_A)$, *if* ${}_AP$ *is projective, or if* ${}_BX$ *is projective, then*

$$\mathrm{Hom}_B({}_BX, {}_BU_A) \otimes_A P \simeq \mathrm{Hom}_B({}_BX, {}_BU \otimes_A P).$$

(3) *In the situation* $({}_AX_B, {}_AY)$, *if* X_B *is projective and* ${}_AY$ *is injective, then* $\mathrm{Hom}_A({}_AX_B, {}_AY)$ *is an injective B-module.* □

As an immediate consequence of Lemma 2.1(1), we have

Corollary 2.2. *Let A and B be two artin k-algebras, and let* ${}_AX$ *and* ${}_BY$ *be two modules. Then* $Y \otimes_k D(X) \simeq \mathrm{Hom}_k(X, Y)$ *as right A- left B-bimodules.* □

The following lemma provides a way to get projective bimodules. For a proof we refer to [3].

Lemma 2.3. *Let A, B and C be three artin algebras. Suppose P is a projective A-B-bimodule.*

(1) *If M is a C-A-bimodule such that* ${}_CM$ *and* M_A *are projective modules, then* $M \otimes_A P$ *is a projective C-B-bimodule. Similarly, if M is a B-C-bimodule such that* ${}_BM$ *and* M_C *are projective modules, then* $P \otimes_B M$ *is a projective A-C-bimodule.*

(2) $\mathrm{Hom}_A(P, X)$ *is an injective B-module for any module* ${}_AX$ *in* A-mod*. Similarly,* $\mathrm{Hom}_B(P_B, Y_B)$ *is an injective right A-module for any right B-module Y.* □

The following Grothendieck spectral theorem can be found in text books of homological algebra (see [10, theorem 9.3, p. 299], for example).

Lemma 2.4. *Given two additive covariant functors* $F : \mathscr{A} \to \mathscr{B}$ *and* $G : \mathscr{B} \to \mathscr{C}$ *between abelian categories with enough injective objects. Suppose for any injective object* $I \in \mathscr{A}$, *the object FI is right G-acyclic, that is, for the right derived functor* $R^p(G)$ *of G we have* $(R^pG)(FI) = 0$ *for any* $p \geq 1$, *and* $(R^pG)(FI) = G(FI)$ *for* $p = 0$. *Then there is a spectral sequence* $\{E_n(A)\}$ *corresponding to each object A of* $\mathscr{A}$, *such that*

$$E_1^{p,q} = (R^pG)(R^qF)(A) \Rightarrow R^{p+q}(GF)(A),$$

which converges finitely to the graded object associated with $\{R^{p+q}(GF)(A)\}$, *suitably filtered.* □

We shall use this to prove the following homological identity.

Theorem 2.5. *Let R, S and T be artin k-algebras. In the situation* $({}_{R-S}X, Y_{T-S}, {}_RZ_T)$ *we assume that* ${}_RX$, Y_T *and* Y_S *are projective. Then*

$$\mathrm{Ext}^n_{R\otimes_k S}({}_{R-S}X, \mathrm{Hom}_T(Y_{T-S}, {}_RZ_T)) \simeq \mathrm{Ext}^n_{T\otimes_k S}(Y_{T-S}, \mathrm{Hom}_R({}_{R-S}X, Z))$$

for all $n \geq 0$.

Before starting the proof of this result, we state some facts from [6]. The first statement of the following lemma is [6, exercise 1, p. 360], and the second one is [6, proposition 2.3a, p. 166].

Lemma 2.6. (1) *In the situation* $({}_{R-S}X, Y_{T-S}, {}_RZ_T)$, *where R, S and T are k-algebras with k a commutative ring with identity, there is an isomorphism*

$$\mathrm{Hom}_{R\otimes_k S}(X, \mathrm{Hom}_T(Y, Z)) \simeq \mathrm{Hom}_{T\otimes_k S}(Y, \mathrm{Hom}_R(X, Z)).$$

(2) *In the situation* $({}_RX_S, Y_{T-S})$ *assume that* ${}_RX$ *is a projective R-module and Y is an injective* $(T \otimes_k S)$*-module. Then* $\mathrm{Hom}_S({}_RX_S, Y_{T-S})$ *is injective as a right* $(R \otimes_k T)$*-module.* $\square$

Proof of Theorem 2.5. Let $\mathscr{A} = (R \otimes_k T^{\mathrm{op}})$-mod, $\mathscr{B} = (R \otimes_k S)$-mod and $\mathscr{C} = k$-mod. We define two additive covariant functors

$$F = \mathrm{Hom}_T(Y_{T-S}, -) : \mathscr{A} \to \mathscr{B}; \quad G = \mathrm{Hom}_{R\otimes_k S}({}_{R-S}X, -) : \mathscr{B} \to \mathscr{C}.$$

If ${}_RI_T$ is an injective R-T-bimodule, we show that $FI = \mathrm{Hom}_T(Y_{T-S}, {}_RI_T)$ is right G-acyclic, that is, $\mathrm{Ext}^p_{R\otimes_k S}(X, FI) = 0$ for all $p \geq 1$. For this, it is sufficient to show that FI is an injective $(R \otimes_k S)$-module if Y_S is projective. However, this follows from Lemma 2.6(2) directly. To apply Lemma 2.4, we need to know the right derived functors of GF. But, by Lemma 2.6(1), it is easy to see that $R^p(GF) \simeq \mathrm{Ext}^p_{T\otimes_k S}(Y_{T-S}, \mathrm{Hom}_R({}_{R-S}X, -))$ since we assume that ${}_RX$ is projective. Now it follows from Lemma 2.4 that there is a spectral sequence $\{E_n(Z)\}$ for each $Z \in \mathscr{A}$ such that

$$\mathrm{Ext}^p_{R\otimes_k S}(X, \mathrm{Ext}^q_T(Y, {}_RZ_T)) \Rightarrow \mathrm{Ext}^n_{T\otimes_k S}(Y_{T-S}, \mathrm{Hom}_R({}_{R-S}X, {}_RZ_T)),$$

with $n = p + q$. Since Y_T is projective, this spectral sequence collapses. Thus we have

$$\mathrm{Ext}^n_{R\otimes_k S}(X, \mathrm{Hom}_T(Y, Z)) \simeq \mathrm{Ext}^n_{T\otimes_k S}(Y_{T-S}, \mathrm{Hom}_R({}_{R-S}X, {}_RZ_T))$$

for all $n \geq 0$. This finishes the proof of Theorem 2.5. $\square$

We need also the following result whose proof can be found in [6, theorem 2.8, 2.8a, p. 167]. Note that the statement (3) below is a dual version of (2) for left modules.

Lemma 2.7 (see [6, p. 167]). (1) *Let* Λ, Γ *and* Σ *be three* k*-algebras that are projective over* k. *In the situation* $(X_{\Lambda-\Gamma}, {}_{\Lambda}Y_{\Sigma}, {}_{\Gamma-\Sigma}Z)$ *assume that* $\mathrm{Tor}_n^{\Lambda}(X, Y) = 0 = \mathrm{Tor}_n^{\Sigma}(Y, Z)$ *for* $n \geq 1$. *Then there is an isomorphism*

$$\mathrm{Tor}^{\Gamma\otimes_k\Sigma}(X \otimes_{\Lambda} Y, Z) \simeq \mathrm{Tor}^{\Lambda\otimes_k\Gamma}(X, Y \otimes_{\Sigma} Z).$$

(2) *Let* Λ, Γ *and* Σ *be three* k*-algebras that are projective over* k. *In the situation* $(X_{\Lambda-\Gamma}, {}_{\Lambda}Y_{\Sigma}, Z_{\Gamma-\Sigma})$ *assume that* $\mathrm{Tor}_n^{\Lambda}(X, Y) = 0 = \mathrm{Ext}_{\Sigma}^n(Y, Z)$ *for* $n \geq 1$. *Then there is an isomorphism*

$$\mathrm{Ext}_{\Gamma\otimes_k\Sigma}(X \otimes_{\Lambda} Y, Z) \simeq \mathrm{Ext}_{\Lambda\otimes_k\Gamma}(X, \mathrm{Hom}_{\Sigma}(Y, Z)).$$

(3) *Let* Λ, Γ *and* Σ *be three* k*-algebras that are projective over* k. *In the situation* $({}_{\Lambda-\Gamma}X, {}_{\Sigma}Y_{\Lambda}, {}_{\Gamma-\Sigma}Z)$ *assume that* $\mathrm{Tor}_n^{\Lambda}(Y, X) = 0 = \mathrm{Ext}_{\Sigma}^n(Y, Z)$ *for* $n \geq 1$. *Then there is an isomorphism*

$$\mathrm{Ext}_{\Gamma\otimes_k\Sigma}(Y \otimes_{\Lambda} X, {}_{\Gamma-\Sigma}Z) \simeq \mathrm{Ext}_{\Lambda\otimes_k\Gamma}({}_{\Lambda-\Gamma}X, \mathrm{Hom}_{\Sigma}(Y, Z)). \qquad \square$$

3 Stable equivalences of adjoint type

In this section we define the notion of a stable equivalence of adjoint type and develop its basic properties. Now let us first recall the definition of a stable equivalence of Morita type introduced by Broué [5], which is a combination of the notion of a Morita equivalence with the one of a stable equivalence.

Definition 3.1. Let A and B be two (arbitrary) artin k-algebras. We say that A and B are stably equivalent of *Morita type* if there exists an A-B-bimodule ${}_AM_B$ and a B-A-bimodule ${}_BN_A$ such that

(1) M and N are projective as one-sided modules, and

(2) $M \otimes_B N \simeq A \oplus P$ as A-A-bimodules for some projective A-A-bimodule ${}_AP_A$, and $N \otimes_A M \simeq B \oplus Q$ as B-B-bimodules for some projective B-B-bimodule ${}_BQ_B$.

In the case of Definition 3.1 we say that M and N define a stable equivalence of Morita type between two algebras A and B. In [14] we proved that if M and N define a stable equivalence of Morita type between two self-injective algebras, then one always has two adjoint pairs $(M' \otimes_B -, N' \otimes_A -)$ and $(N' \otimes_A, M' \otimes_B -)$ of functors between module categories over A and B. Motivated by this phenomenon, we introduce the following notion.

Definition 3.2. If a stable equivalence of Morita type between two algebras A and B defined by M and N satisfies that both $(N \otimes_A -, M \otimes_B -)$ and $(M \otimes_B -, N \otimes_A -)$ are adjoint pairs of functors, then it is called a stable equivalence of *adjoint type*.

It is clear that a stable equivalence of Morita type defined by M and N is of adjoint type if and only if $N \simeq \mathrm{Hom}_A(M, A)$ and $M \simeq \mathrm{Hom}_B(N, B)$ as bimodules. Typical examples of stable equivalences of adjoint type are the stable equivalences of Morita type between self-injective algebras (see [14]). We stress that there are plenty examples of stable equivalences of adjoint type outside the scope of self-injective algebras, according to the construction in [15].

The following is an easy property of adjoint type.

Lemma 3.3. *Suppose a stable equivalence of Morita type between A and B is defined by M and N. If $(N\otimes_A -, M\otimes_B -)$ and $(M\otimes_B -, N\otimes_A -)$ are adjoint pairs of functors between A-*mod *and B-*mod*, then $(-\otimes_A M, -\otimes_B N)$ and $(-\otimes_B N, -\otimes_A M)$ are adjoint pairs of functors between* mod-A *and* mod-B. *Thus ${}_AM_B \simeq \mathrm{Hom}_A({}_BN_A, {}_AA_A)$ and ${}_BN_A \simeq \mathrm{Hom}_B({}_AM_B, {}_BB_B)$.*

Proof. Since ${}_BN$ is projective, we have a canonical isomorphism of B-modules: ${}_BN \to \mathrm{Hom}_B(\mathrm{Hom}_B({}_BN, {}_BB_B), {}_BB_B)$. One can check that this is also a right A-homomorphism. Thus ${}_BN_A \simeq \mathrm{Hom}_B({}_AM_B, {}_BB_B)$. We shall use dimension shifting and the projectivity of M_B to show that $\alpha : X\otimes_B \mathrm{Hom}_B({}_AM_B, {}_BB_B) \to \mathrm{Hom}_B({}_AM_B, X_B)$ is an isomorphism as right A-modules, where α sends $x\otimes\phi$ to a morphism from M to X by $m \mapsto x\phi(m)$. In fact, the α is an isomorphism for $X_B = B_B$, and thus for any projective right B-module. Now we take a projective presentation of X_B:

$$P_1 \to P_0 \to X_B \to 0.$$

This gives the following commutative exact diagram:

$$\begin{array}{ccccccc}
P_1\otimes_B \mathrm{Hom}_B(M_B, {}_BB_B) & \longrightarrow & P_0\otimes_B \mathrm{Hom}_B(M_B, {}_BB_B) & \longrightarrow & X\otimes_B \mathrm{Hom}_B(M_B, {}_BB_B) & \longrightarrow & 0 \\
{\scriptstyle\alpha}\big\downarrow & & {\scriptstyle\alpha}\big\downarrow & & {\scriptstyle\alpha}\big\downarrow & & \\
\mathrm{Hom}_B(M_B, P_1) & \longrightarrow & \mathrm{Hom}_B(M_B, P_0) & \longrightarrow & \mathrm{Hom}_B(M_B, X) & \longrightarrow & 0,
\end{array}$$

where the exactness of the lower row follows from the projectivity of M_B. Since the first two vertical maps are isomophisms, this implies that the α in the third column is also an isomorphism. One can also check that this isomorphism is natural in X. Thus $-\otimes_B N_A \simeq \mathrm{Hom}_B({}_AM_B, -)$; and therefore $(-\otimes_A M, -\otimes_B N)$ is an adjoint pair. Similarly, we can show that $(-\otimes_B N, -\otimes_A M)$ is an adjoint pair. □

Proposition 3.4. *Suppose k is a perfect field. Let A and B be two k-algebras without two-sided semisimple direct summands. Suppose ${}_AM_B$ and ${}_BN_A$ define a stable equivalence of Morita type between A and B, where M and N have no direct summands of projective bimodules. Then M and N define a stable equivalence of adjoint type between A and B if and only if $\mathrm{Hom}_A({}_AM_B, {}_AM_B) \simeq {}_BB_B \oplus Q'$ as bimodules with Q' a projective B-B-bimodule, and $\mathrm{Hom}_B({}_BN_A, {}_BN_A) \simeq {}_AA_A \oplus P'$ as bimodules with P' a projective A-A-bimodule.*

Proof. Suppose that $\mathrm{Hom}_A({}_AM_B, {}_AM_B) \simeq {}_BB_B \oplus Q'$ as bimodules with Q' a projective B-B-bimodule, and $\mathrm{Hom}_B({}_BN_A, {}_BN_A) \simeq {}_AA_A \oplus P'$ as bimodules with P' a projective A-A-bimodule. By these assumptions we have the following bimodule isomorphisms:

$$(*)\qquad \begin{aligned} M \oplus P' \otimes_A M_B &\simeq \mathrm{Hom}_B({}_BN_A, {}_BN_A) \otimes_A M_B \\ &\simeq \mathrm{Hom}_B({}_BN_A, {}_BN \otimes_A M_B) \\ &\simeq \mathrm{Hom}_B({}_BN_A, {}_BB_B) \oplus \mathrm{Hom}_B({}_BN_A, {}_BQ_B). \end{aligned}$$

Note that $P' \otimes_A M$ is a projective bimodule by Lemma 2.3. Thus $\mathrm{Hom}_B({}_BN, {}_BB)$ is a projective A-module, and therefore any direct summand $\mathrm{Hom}_B({}_BN_A, {}_BBe)$, with $e = e^2 \in B$, of $\mathrm{Hom}_B({}_BN, {}_BB)$ is also a projective A-module. This implies that the bimodule $\mathrm{Hom}_B({}_BN_A, {}_BQ_B)$ is a projective bimodule since if we write $Q = \bigoplus_i (Be_i \otimes_k f_iB)$ for some $e_i^2 = e_i$, $f_i^2 = f_i \in B$ then $\mathrm{Hom}_B({}_BN_A, {}_BQ_B) \simeq \bigoplus_i \mathrm{Hom}_B({}_BN_A, {}_BBe_i \otimes_k f_iB) \simeq \bigoplus_i \mathrm{Hom}_B({}_BN_A, Be) \otimes_k fB$ by Lemma 2.1(2). Now we show that $\mathrm{Hom}_B({}_BN, {}_BB_B)$ has no summands of projective bimodules. Suppose $\mathrm{Hom}_B({}_BN, {}_BB_B) = {}_AY_B \oplus {}_AY'_B$, where Y has no projective summands and Y' is a projective A-B-bimodule. Applying $\mathrm{Hom}_B(-, B_B)$ to this decomposition, we get

$${}_BN_A \simeq \mathrm{Hom}_B({}_AY_B, {}_BB_B) \oplus \mathrm{Hom}_B({}_AY'_B, {}_BB_B).$$

Thus $\mathrm{Hom}_B({}_AY'_B, B_B)$ is a projective right A-module since N_A is projective. Moreover, we may suppose $Y' = Ae \otimes_k fB$ with e an idempotent in A and f an idempotent in B. Then we have $\mathrm{Hom}_B({}_AY'_B, {}_BB_B) \simeq \mathrm{Hom}_B(Ae \otimes_k fB, {}_BB_B) \simeq \mathrm{Hom}_k(Ae, Bf) \simeq Bf \otimes_k D(Ae)$ by Lemma 2.2. Thus the right A-projectivity of $\mathrm{Hom}_B({}_AY'_B, {}_BB_B)$ shows that $D(Ae)$ is also projective. Thus $\mathrm{Hom}_B({}_AY'_B, {}_BB_B)$ is a projective B-A-bimodule since it is a direct sum of modules of the form $Bf \otimes_k D(Ae)$ with $D(Ae)$ a projective right A-module. But this contradicts to the assumption for N, thus $\mathrm{Hom}_B({}_AY'_B, {}_BB_B) = 0$, that is, $Y' = 0$. By comparison of the non-projective part of the both sides in $(*)$, we have that $M \simeq \mathrm{Hom}_B({}_BN, {}_BB)$ as bimodules.

Similarly, we can show that $N \simeq \mathrm{Hom}_A({}_AM, {}_AA)$ as bimodules. Thus M and N define a stable equivalence of adjoint type.

Conversely, if M and N define a stable equivalence of adjoint type, then ${}_BN_A \simeq \mathrm{Hom}_A({}_AM_B, {}_AA_A)$ and $M \simeq \mathrm{Hom}_B({}_BN_A, {}_BB_B)$. Hence $\mathrm{Hom}_A({}_AM_B, {}_AM_B) \simeq \mathrm{Hom}_A({}_AM_B, {}_AA_A) \otimes_A M_B$ by Lemma 2.1(2), which is then isomorphic to ${}_BN \otimes_A M_B \simeq B \oplus Q$ as bimodules. Similarly, $\mathrm{Hom}_B({}_BN, {}_BN) \simeq A \oplus P$ as bimodules. This finishes the proof. □

Proposition 3.5. *Suppose k is a perfect field. Let A and B be two k-algebras without two-sided semisimple direct summands. Suppose ${}_AM_B$ and ${}_BN_A$ define a stable equivalence of Morita type between A and B, where M and N have no direct summands of projective bimodules. If $M \otimes_B N \simeq M \otimes_B \mathrm{Hom}_A({}_AM, {}_AA)$ and $N \otimes_A M_B \simeq N \otimes_A \mathrm{Hom}_B({}_BN, {}_BB)$ as bimodules, then M and N define a stable equivalence of adjoint type.*

Proof. We have that $\mathrm{Hom}_A({}_AM, {}_AA) \otimes_A M \simeq \mathrm{Hom}_A(M, M)$ as B-B-bimodules by Lemma 2.1(2). It follows from the assumption that

$$N \otimes_B M \otimes_B \mathrm{Hom}_A(M, A) \otimes_A M \simeq N \otimes_A M \otimes_B N \otimes_A M.$$

From this we get the following isomorphisms of bimodules:

$$\begin{aligned} &\mathrm{Hom}_A(M, M) \oplus Q \otimes_B \mathrm{Hom}_A(M, M) \\ &\quad \simeq (B \oplus Q) \otimes_B (B \oplus Q) \simeq B \oplus Q \oplus Q \oplus Q \otimes_B Q. \end{aligned}$$

Thus $\mathrm{Hom}_A(M, M)$ is projective as one-sided modules since the right hand side of the above isomorphism is projective as one-sided modules. This implies that $Q \otimes_B \mathrm{Hom}_A(M, M)$ is a projective bimodule by Lemma 2.3. Hence we have a bimodule isomorphism: $\mathrm{Hom}_A(M, M) \simeq B \oplus Q'$ with Q' a projective B-B-bimodule. Similarly, we can show that $\mathrm{Hom}_B(N, N) \simeq A \oplus P'$ with P' a projective A-A-bimodule. Then, by Proposition 3.4, M and N define a stable equivalence of Morita type. □

Proposition 3.6. *Suppose k is a perfect field. Let A and B be two k-algebras without two-sided semisimple direct summands. Suppose ${}_AM_B$ and ${}_BN_A$ define a stable equivalence of Morita type between A and B, where M and N have no direct summands of projective bimodules, such that $M \otimes_B N \simeq \mathrm{Hom}_B({}_BN, {}_BB) \otimes_B \mathrm{Hom}_A({}_AM, {}_AA)$ and $N \otimes_A M_B \simeq \mathrm{Hom}_A({}_AM_B, {}_AA) \otimes_A \mathrm{Hom}_B({}_BN, {}_BB)$ as bimodules. If the evaluation maps $M \otimes_B \mathrm{Hom}(M, A) \to A$ and $N \otimes_A \mathrm{Hom}(N, B) \to B$ are split as an A-A-bimodule and a B-B-bimodule homomorphism, respectively, then M and N define a stable equivalence of adjoint type.*

Proof. As in the proof of 3.4, we can show that $\mathrm{Hom}_B({}_BN, {}_BB)$ and $\mathrm{Hom}_A({}_AM, A)$ contain no projective bimodules as a direct summand.

Since the evaluation map $ev : M \otimes_B \mathrm{Hom}(M, A) \to A$ is a split A-A-bimodule homomorphism, we know that the induced map $M \otimes_B \mathrm{Hom}(M, A) \otimes_A \mathrm{Hom}_B(N, B) \to A \otimes_A \mathrm{Hom}_B(N, B)$ is a split A-B-bimodule homomorphism. Thus $\mathrm{Hom}_B(N, B)$ is a direct summand of $M \otimes_B \mathrm{Hom}(M, A) \otimes_A \mathrm{Hom}_B(N, B) \simeq M \oplus M \otimes_B Q$. We know that $M \otimes_B Q$ is a projective A-B-bimodule. This implies that $\mathrm{Hom}_B(N, B)$ is a direct summand of M. Similarly, $\mathrm{Hom}_A(M, A)$ is a direct summand of N. We may assume that $M = \mathrm{Hom}_B(N, B) \oplus X$ and $N = \mathrm{Hom}_A(M, A) \oplus Y$. Then $M \otimes_B N = \mathrm{Hom}_B(N, B) \otimes_B \mathrm{Hom}_A(M, A) \oplus \mathrm{Hom}(N, B) \otimes_B Y \oplus \mathrm{Hom}_A(M, A) \otimes_A X \oplus X \otimes_B Y$. By assumption, we must have $\mathrm{Hom}_A(M, X) \simeq \mathrm{Hom}_A(M, A) \otimes_A X = 0 = \mathrm{Hom}_B(N, B) \otimes_B Y \simeq \mathrm{Hom}_B(N, Y)$. Since ${}_AM$ and ${}_BN$ are generators for A-mod and B-mod, respectively, we get that $X = 0 = Y$. Thus $N \simeq \mathrm{Hom}_A({}_AM, A)$ and $M \simeq \mathrm{Hom}_B({}_BN, {}_BB)$ as bimodules. This implies that M and N define a stable equivalence of adjoint type. □

Related to stable equivalences of adjoint type, we have the following unsolved basic question.

Question 1. Are there any two algebras A and B such that they are stably equivalent of Morita type, but between them there is not any stable equivalence of adjoint type?

4 Hochschild cohomologies

In this section we shall use Theorem 2.5 to prove that a stable equivalence of adjoint type preserves the higher Hochschild cohomology groups. For self-injective algebras this was first proved in [17], and then in [14] by a different method. However, both proofs depend heavily on the self-injectivity of the given algebras. Our proof here does not use any self-injectivity; and our result generalizes the one for self-injective algebras.

Now, let us first recall the definition of Hochschild cohomology.

Definition 4.1. Let Λ be an artin k-algebra. If X is a Λ-Λ-bimodule, then the Hochschild homology of Λ with coefficients in X is defined as

$$H_n(\Lambda, X) = \mathrm{Tor}_n^{\Lambda^e}(X, \Lambda)$$

for all $n \geq 0$, where $\Lambda^e = \Lambda \otimes_k \Lambda^{\mathrm{op}}$ is the enveloping algebra of Λ. If $X = \Lambda$ we obtain the Hochschild homology of $\Lambda : H_*(\Lambda) = \mathrm{Tor}_*^{\Lambda^e}(\Lambda, \Lambda)$.

Dually, the Hochschild cohomology of Λ with coefficients in X is defined as

$$H^n(\Lambda, X) = \mathrm{Ext}_{\Lambda^e}^n(\Lambda, X)$$

for all $n \geq 0$. If $X = \Lambda$ we obtain the Hochschild cohomology of $\Lambda : H^*(\Lambda) = \mathrm{Ext}_{\Lambda^e}^*(\Lambda, \Lambda)$.

Note that the low-dimensional Hochschild homology and cohomology have a simple interpretation, namely, for a k-algebra A, $H^0(A)$ is the center of A, that is, $H^0(A) = \{a \in A \mid ax = xa \text{ for all } x \in A\}$; and $H^1(A)$ is isomorphic to $\mathrm{Der}(A)/\mathrm{Inn}(A)$, where $\mathrm{Der}(A)$ stands for the set of all k-linear derivations on A, and $\mathrm{Inn}(A)$ stands for the set of all inner derivations of A. $H_0(A)$ is the quotient of A modulo the k-space $[A, A]$ spanned by all elements of the form $xa - ax$, with $a, x \in A$.

It was proved that the Hochschild homology groups H_n (for $n \geq 1$) are invariant under stable equivalences of Morita type; while it is open in general whether the Hochschild cohomology groups H^n are invariant under stable equivalences of Morita type. However, we show the following

Theorem 4.2. *Let A and B be two artin k-algebras such that A and B are k-projective. If A and B are stably equivalent of adjoint type, then $H^n(A) \simeq H^n(B)$ for all $n \geq 1$.*

Proof. Suppose that the stable equivalence of adjoint type between A and B is defined by the bimodule ${}_AM_B$ and ${}_BN_A$ such that both $(N \otimes_A -, M \otimes_B -)$ and $(M \otimes_B -, N \otimes_A -)$ are adjoint pairs of functors. Then we know from Lemma 3.3 that

${}_BN_A \simeq \mathrm{Hom}_A({}_AM_B, {}_AA_A)$, ${}_BN_A \simeq \mathrm{Hom}_B({}_AM_B, {}_BB_B)$, ${}_AM_B \simeq \mathrm{Hom}_B({}_BN_A, {}_BB_B)$, and ${}_AM_B \simeq \mathrm{Hom}_A({}_BN_A, {}_AA_A)$.

To prove Theorem 4.2, we shall show the following three claims:

(1) $\mathrm{Ext}^i_{B\otimes_k A^{\mathrm{op}}}(N,N) \simeq \mathrm{Ext}^i_{A\otimes_k B^{\mathrm{op}}}(M,M)$ for all $i \geq 0$.

In fact, we may apply Theorem 2.5 to deduce this: In Theorem 2.5 we define $R = A$, $S = B^{\mathrm{op}}$, $T = A$, ${}_{R-S}X = {}_{A-B^{\mathrm{op}}}M$, $Y_{S-T} = N_{B^{\mathrm{op}}-A}$ and ${}_RZ_T = {}_AA_A$. Then ${}_RX = {}_AM$ is projective, $Y_{S-T} \simeq {}_BN_A$ is projective as a left module and as a right module. Thus we get from Theorem 2.5 that for $n \geq 0$,

$$\mathrm{Ext}^n_{A\otimes_k B^{\mathrm{op}}}({}_AM_B, {}_AM_B) = \mathrm{Ext}^n_{A\otimes_k B^{\mathrm{op}}}(M, \mathrm{Hom}_A({}_BN_A, {}_AA_A))$$

$$= \mathrm{Ext}^n_{R\otimes_k S}({}_{R-S}X, \mathrm{Hom}_T(Y_{S-T}, {}_RZ_T)) \simeq \mathrm{Ext}^n_{T\otimes_k S}(Y_{T-S}, \mathrm{Hom}_R({}_{R-S}X, {}_RZ_T))$$

$$= \mathrm{Ext}^n_{B\otimes_k A^{\mathrm{op}}}({}_BN_A, \mathrm{Hom}_A({}_AM_B, {}_AA_A)) \simeq \mathrm{Ext}^n_{B\otimes_k A^{\mathrm{op}}}({}_BN_A, {}_BN_A).$$

(2) As in [14, theorem 4.7], we show that $\mathrm{Ext}^i_{B\otimes_k B^{\mathrm{op}}}(N\otimes_A M, B) \simeq \mathrm{Ext}^i_{A\otimes_k B^{\mathrm{op}}}(M,M)$, and $\mathrm{Ext}^i_{A\otimes_k A^{\mathrm{op}}}(M\otimes_B N, A) \simeq \mathrm{Ext}^i_{B\otimes_k A^{\mathrm{op}}}(N,N)$ for all $i \geq 0$.

In fact, we let $\Lambda = A$, $\Sigma = B$ and $\Gamma = B^{\mathrm{op}}$, and define $X = {}_AM_B = {}_{\Lambda-\Gamma}M$, $Y = {}_BN_A = {}_\Sigma N_\Lambda$ and $Z = {}_BB_B = {}_{\Gamma-\Sigma}B$ in Lemma 2.7(3). By the definition of a stable equivalence of Morita type, the modules ${}_BN$ and N_A are projective, that is, ${}_\Sigma Y$ and Y_Λ are projective. Thus $\mathrm{Tor}^\Lambda_i(Y,X) = 0 = \mathrm{Ext}^i_\Sigma(Y,Z)$ for $i \geq 1$. Hence there is an isomorphism $\mathrm{Ext}^i_{B\otimes_k B^{\mathrm{op}}}(N\otimes_A M, B) \simeq \mathrm{Ext}^i_{A\otimes_k B^{\mathrm{op}}}(M, \mathrm{Hom}_B(N,B))$. Since $\mathrm{Hom}_B(N,B) \simeq M$, we have $\mathrm{Ext}^i_{B\otimes_k B^{\mathrm{op}}}(N\otimes_A M, B) \simeq \mathrm{Ext}^i_{A\otimes_k B^{\mathrm{op}}}(M,M)$. Similarly, we have $\mathrm{Ext}^i_{A\otimes_k A^{\mathrm{op}}}(M\otimes_B N, A) \simeq \mathrm{Ext}^i_{B\otimes_k A^{\mathrm{op}}}(N,N)$ for all $i \geq 0$.

(3) We have the following isomorphism identities: for all $n \geq 1$,

$$\begin{aligned} H^n(A) = \mathrm{Ext}^n_{A^e}(A,A) &\simeq \mathrm{Ext}^n_{A^e}(A\oplus P, A) \simeq \mathrm{Ext}^n_{A^e}(M\otimes_B N, A) \\ &\simeq \mathrm{Ext}^n_{B\otimes_k A^{\mathrm{op}}}(N,N) \qquad (\text{by } (2)) \\ &\simeq \mathrm{Ext}^n_{A\otimes_k B^{\mathrm{op}}}(M,M) \qquad (\text{by } (1)) \\ &\simeq \mathrm{Ext}^n_{B^e}(N\otimes_A M, B) \qquad (\text{by } (2)) \\ &\simeq \mathrm{Ext}^n_{B^e}(B\oplus Q, B) \simeq \mathrm{Ext}^n_{B^e}(B,B) = H^n(B). \end{aligned}$$

This finishes the proof of Theorem 4.2. □

As a consequence, we re-obtained the following result in [17] (see also [14]).

Corollary 4.3. *Let A and B be finite-dimensional self-injective k-algebras with k a field. If there is a stable equivalence of Morita type between A and B, then for any* $n \geq 1$, $H^n(A) \simeq H^n(B)$. □

In general, we have the following

Proposition 4.4. *Let A and B be two artin k-algebras such that A and B are projective k-modules. If A and B are stably equivalent of adjoint type, which is defined by ${}_AM_B$ and ${}_BN_A$, then, for any A-A-bimodule X and any integer $n \geq 1$, $H^n(B, N \otimes_A X \otimes_A M) \simeq H^n(A, X) \oplus H^n(A, P \otimes_A X)$. In particular, if $X = A$, we have $H^n(A, P) \simeq H^n(B, Q)$.*

Proof. Since M and N define a stable equivalence of adjoint type between A and B, we know that $X \otimes_A M_B \simeq \mathrm{Hom}_A({}_BN_A, X_A)$. Note that if N_A is projective, then ${}_{A^{\mathrm{op}}}N$ is also projective, and $\mathrm{End}({}_{A^{\mathrm{op}}}A) \simeq A^{\mathrm{op}}$. We claim that if X is a C-A-bimodule then $X \otimes_A M_B \simeq \mathrm{Hom}_A({}_BN_A, X_A)$ as left C-modules. This follows from the following isomorphisms of modules:

$$\begin{aligned}
{}_CX \otimes_A M &= {}_CX \otimes_A \mathrm{Hom}_A(N_A, {}_AA_A) \\
&\simeq \mathrm{Hom}_{A^{\mathrm{op}}}({}_{A^{\mathrm{op}}}N, {}_{A^{\mathrm{op}}}A_{A^{\mathrm{op}}}) \otimes_{A^{\mathrm{op}}} X_{C^{\mathrm{op}}} \\
&\simeq \mathrm{Hom}_{A^{\mathrm{op}}}({}_{A^{\mathrm{op}}}N, {}_{A^{\mathrm{op}}}X_{C^{\mathrm{op}}}) \quad \text{(by [22, lemma 2.1(2)])} \\
&\simeq \mathrm{Hom}_A(N_A, {}_CX_A).
\end{aligned}$$

Now we compute the Hochschild cohomology for $n \geq 1$:

$$\begin{aligned}
&H^n(B, N \otimes_A X \otimes_A M) \\
&= \mathrm{Ext}^n_{B^e}(B, N \otimes_A X \otimes_A M) = \mathrm{Ext}^n_{B^e}(N \otimes_A M, N \otimes_A X \otimes_A M) \\
&\simeq \mathrm{Ext}^n_{A \otimes_K B^{\mathrm{op}}}(M, M \otimes_B N \otimes_A X \otimes_A M) \\
&\simeq \mathrm{Ext}^n_{A \otimes_k B^{\mathrm{op}}}(M, \mathrm{Hom}_A(N, M \otimes_B N \otimes_A X)) \\
&\simeq \mathrm{Ext}^n_{A^e}(M \otimes_B N, M \otimes_B N \otimes_A X) \simeq H^n(A, X) \oplus H^n(A, P \otimes_A X).
\end{aligned}$$

The last statement follows from Theorem 4.2. □

Remark. If k is a perfect field, then there is a short proof of Theorem 4.2, which is a direct consequence of Proposition 4.4 and the following lemma.

Lemma 4.5. *Let A and B be two finite-dimensional k-algebras with k a perfect field. Suppose that ${}_AM_B$ and ${}_BN_A$ define a stable equivalence of adjoint type between A and B. Let P and Q be given in Definition 3.1. Then ${}_AP_A$ and ${}_BQ_B$ are projective-injective bimodules.*

Proof. It follows from

$$
\begin{aligned}
{}_AA_A \oplus P &\simeq {}_AM \otimes_B N_A \simeq \mathrm{Hom}_B({}_BN_A, {}_BB_B) \otimes_B N \simeq \mathrm{Hom}_B({}_BN_A, {}_BN_A) \\
&\simeq \mathrm{Hom}_B({}_BN_A, \mathrm{Hom}_A({}_AM_B, {}_AA_A)) \simeq \mathrm{Hom}_A({}_AM \otimes_B N_A, {}_AA) \\
&\simeq {}_AA_A \oplus \mathrm{Hom}_A({}_AP, {}_AA)
\end{aligned}
$$

that ${}_AP_A \simeq \mathrm{Hom}_A({}_AP, {}_AA)$ as A-A-bimodules. Similarly,

$$
\begin{aligned}
{}_AA_A \oplus P &\simeq {}_AM \otimes_B N_A \simeq M \otimes_B \mathrm{Hom}_B(M_B, B_B) \simeq \mathrm{Hom}_B(M_B, M_B) \\
&\simeq \mathrm{Hom}_B(M_B, \mathrm{Hom}_A({}_BN_A, A_A)) \simeq \mathrm{Hom}_A(M \otimes_B N, A_A) \\
&\simeq A \oplus \mathrm{Hom}_A(P_A, A_A)
\end{aligned}
$$

and ${}_AP_A \simeq \mathrm{Hom}_A(P_A, A_A)$ as bimodules. By Lemma 2.3(2), ${}_AP$ and P_A are injective. Since k is a perfect field, we may write $P = \bigoplus_{i=1}^n Ae_i \otimes_k f_iA$ with e_i and f_i idempotents in A. Thus all Ae_i are injective, and all f_iA are injective right A-modules, and therefore all Ae_i and all f_iA are projective-injective. Hence P is a projective-injective A-A-bimodule. Similarly, we know that Q is a projective-injective B-B-bimodule. $\square$

Concerning the invariance of self-injective dimension and Gorenstein property under stable equivalences of adjoint type (or Morita type) we refer to [15] and [4].

Finally, let us mention the following question.

Question 2. Suppose two artin k-algebras A and B are stably equivalent of Morita type. Is $H^n(A)$ isomorphic to $H^n(B)$ for all $n \geq 1$?

5 Cartan determinants

In this section we consider the behavior of Cartan determinants of algebras which are stably equivalent of Morita type. Here by Cartan determinant we mean the determinant of Cartan matrix.

Let A be an artin k-algebra. We denote by $K_0(A)$ the Grothendieck group of A, that is, it is a quotient group of the free abelian group generated by isomorphism classes $[X]$ of all A-modules X in A-mod modulo the subgroup generated by all elements of the form $[Y] - [X] - [Z]$, where $0 \to X \to Y \to Z \to 0$ is an exact sequence in A-mod. Thus $K_0(A)$ is the free abelian group generated by the isomorphism classes $[E_i]$ of simple A-modules E_i with $i = 1, 2, \ldots, n$. We denote $[E_i]$ by e_i. The Cartan matrix C_A of the algebra A is given by the map $\sigma_A : K_0(A) \to K_0(A)$, $e_i \mapsto p_i = [P_i]$ with P_i the projective cover of E_i. By elementary divisor theory (see [7, Chapter III, p. 91–95]), we may choose two bases for $K_0(A)$ such that the map σ_A with respect to these bases corresponds to a diagonal matrix $\mathrm{diag}\{\delta_1, \delta_2, \ldots, \delta_r, 0, \ldots, 0\}$, with

δ_i positive integers such that $\delta_i | \delta_{i+1}$, and r the rank of C_A. That is, there are two modular matrices X and Y over $\mathbb{Z}$ such that $C_A = X \operatorname{diag}\{\delta_1, \delta_2, \ldots, \delta_r, 0, \ldots, 0\} Y$. Recall that a matrix X over $\mathbb{Z}$ is called modular if $\det(X)$ is a unit in $\mathbb{Z}$. Thus the cokernel $\operatorname{cok}(\sigma_A)$ of σ_A is isomorphic to $\mathbb{Z}/(\delta_1) \oplus \cdots \oplus \mathbb{Z}/(\delta_r) \oplus \mathbb{Z}^{n-r}$. So, $\det(C_A) \neq 0$ if and only if $\operatorname{cok}(\sigma_A)$ is a finite abelian group. In this case, $\det(C_A) = \pm \delta_1 \cdots \delta_n$.

If there is a stable equivalence of Morita type between A and B defined by ${}_A M_B$ and ${}_B N_A$, then we may define a function $f_N : K_0(A) \to K_0(B)$, $e_i \mapsto [N \otimes_A E_i]$. Since the image of p_i under f_N lies in the image of the map σ_B, the function induces a $\mathbb{Z}$-linear map $f_N' : \operatorname{cok}(\sigma_A) \to \operatorname{cok}(\sigma_B)$. Similarly, we have a function $f_M : K_0(B) \to K_0(A)$ which induces a $\mathbb{Z}$-linear map $f_M' : \operatorname{cok}(\sigma_B) \to \operatorname{cok}(\sigma_A)$. By Definition 3.1(2), we see that the composition of f_N' with f_M' is the identity map on $\operatorname{cok}(\sigma_A)$, and the composition of f_M' with f_N' is the identity map on $\operatorname{cok}(\sigma_B)$. Thus f_N' is an isomorphism. This shows that $\det(C_A) \neq 0$ if and only if $\det(C_B) \neq 0$. Thus we have the following proposition which drops the condition "no node and no semisimple summands" in [16].

Proposition 5.1. *If there is a stable equivalence of Morita type between two artin k-algebras A and B, then the Cartan determinants of A and B have the same absolute values.*

Proof. Under the assumption, we may assume that $\det(C_A) \neq 0 \neq \det(C_B)$. We have seen that $\operatorname{cok}(\sigma_A) \simeq \operatorname{cok}(\sigma_B)$ as abelian groups. Suppose $\operatorname{cok}(\sigma_A) = \mathbb{Z}/(\delta_1) \oplus \cdots \oplus \mathbb{Z}/(\delta_n)$ and $\operatorname{cok}(\sigma_B) = \mathbb{Z}/(\tau_1) \oplus \cdots \oplus \mathbb{Z}/(\tau_m)$ where δ_i and τ_j are positive integers. Let $\mathscr{S}_A$ be the collection of elementary divisors d of $\operatorname{diag}\{\delta_1, \ldots, \delta_n\}$ with $d \neq 1$. The isomorphism of the abelian groups shows that $\mathscr{S}_A = \mathscr{S}_B$. Since the product of all numbers in $\mathscr{S}_A$ is $\delta_1 \cdots \delta_n$, we get that $\delta_1 \cdots \delta_n = \tau_1 \cdots \tau_m$. This implies that $\det(C_A)$ and $\det(C_B)$ have the same absolute values. □

Corollary 5.2. *Let k be a field, and let G and H be two finite groups. If there is a stable equivalence of Morita type between a block A of the group algebra kG and a block B of the group algebra kH, then A and B have the same Cartan determinant.*

Proof. By Proposition 5.1, the Cartan determinants of A and B have the same absolute value. Since we know that any block of a group algebra has always the positive Cartan determinant, the corollary follows. □

Similarly, we have the following corollary for cellular algebras. For convenience of the reader, we recall the definition of cellular algebras. For a basis-free definition and some basic facts of cellular algebras we refer to [9] and [11]. For the definition of standardly stratified algebras we may refer to [8], for example.

Definition 5.3 (Graham and Lehrer, [9]). An associative algebra A over a field k is called a *cellular algebra* with cell datum (I, M, C, i) if the following conditions are satisfied:

(C1) The finite set I is partially ordered. Associated with each $\lambda \in I$ there is a finite set $M(\lambda)$. The algebra A has a k-basis $C^{\lambda}_{S,T}$ where (S,T) runs through all elements of $M(\lambda) \times M(\lambda)$ for all $\lambda \in I$.

(C2) The map i is a k-linear anti-automorphism of A with $i^2 = id$ which sends $C^{\lambda}_{S,T}$ to $C^{\lambda}_{T,S}$.

(C3) For each $\lambda \in I$ and $S, T \in M(\lambda)$ and each $a \in A$ the product $aC^{\lambda}_{S,T}$ can be written as $(\sum_{U \in M(\lambda)} r_a(U,S) C^{\lambda}_{U,T}) + r'$ where r' is a linear combination of basis elements with upper index μ strictly smaller than λ, and where the coefficients $r_a(U,S) \in k$ do not depend on T.

Typical examples of cellular algebras include Brauer algebras, Temperley-Lieb algebras, partition algebras, q-Schur algebras and many others.

Corollary 5.4. *Let A and B be two k-algebras with k a field. Suppose there is a stable equivalence of Morita type between A and B.*

(1) *If A and B are cellular, then A and B have the same Cartan determinant.*

(2) *If A and B are standardly stratified, then A and B have the same Cartan determinant.*

Proof. By [12], the Cartan matrix of an arbitrary cellular algebra is positive definite. By [8], the Cartan determinant of a standardly stratified algebra is the product of the dimensions of the endomorphism algebras of standard modules. Thus the corollary follows now from Proposition 5.1 immediately. □

One should note that Proposition 5.1 could be wrong for stable equivalences in general. An easy example is that $k[x]/(x^2)$ is stably equivalent to the path algebra over k of the quiver $\circ \to \circ$. Clearly, the former algebra has Cartan determinant equal to 2, and the latter algebra has Cartan determinant equal to 1.

Finally, we point out that for the so-called "self-injectively free" algebras without nodes and semi-simple summands it was shown in [16] that the Cartan matrices are invariant under stable equivalences. The following example shows that even when two indecomposable algebras are stably equivalent of adjoint type, they may have different Cartan matrices.

Example. Let us consider the algebras in [13, Example 1]. Let A be the algebra given by the quiver

$$\nabla:\quad \begin{tikzcd} 1 \arrow[rr, "\alpha"] & & 2 \arrow[dl, "\beta"] \\ & 3 \arrow[ul, "\gamma"] & \end{tikzcd}$$

with relations

$$\alpha\beta\gamma = \beta\gamma\alpha\beta = 0.$$

Then the Cartan matrix of A is

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Let B be the algebra given by the quiver

$$\Delta:\quad 1 \underset{\rho'}{\overset{\rho}{\rightleftarrows}} 2 \underset{\delta'}{\overset{\delta}{\rightleftarrows}} 3$$

with relations

$$\rho\delta = \rho\rho' = \delta'\rho' = \rho'\rho - \delta\delta' = 0.$$

Then the Cartan matrix of B is

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

It was proved that there is a stable equivalence of Morita type between A and B. Since this stable equivalence is obtained from a stable equivalence of adjoint type by quotients, we know from [15, proposition 3.8] that there is a stable equivalence of adjoint type between A and B. Note that A and B do not contain nodes and semi-simple summands, and do not have the same Cartan matrix, but the same Cartan determinant.

We remark that even for those stable equivalences of adjoint type, which are obtained from derived equivalences between blocks of group algebras, we cannot get the same Cartan matrices, though derived equivalences preserve Cartan determinants for arbitrary algebras. For an example, see [14]. On the other hand, suggested by the above results and many examples, the following question seems to have a positive answer.

Question 3. If there is a stable equivalence of Morita type between A and B, are the determinants of the Cartan matrices of A and B equal?

Acknowledgements. The author thanks Dr. Y. M. Liu and W. Hu at BNU for discussions on mathematics, and acknowledges gratefully the support from CFKSTIP(707004) and the Doctor Program Foundation (20040027002), Ministry of Education of China. As a winner of Bessel Prize of the Alexander von Humboldt Foundation, the author thanks the Humboldt Foundation as well as his teachers and many of his mathematical colleagues for their long standing help and support.

Added in Proof (August 20, 2006). More recently, I learn from Martinez-Villa that in a paper "A note on stable equivalences of Morita type" by Dugas and Martinez-Villa the following result is proved: Let A and B are finite-dimensional algebras over a field such that A and B are indecomposable and that $A/\mathrm{rad}(A)$ and $B/\mathrm{rad}(B)$ are separable. If A and B are stably equivalent of Morita type, then they are stably equivalent of adjoint type. Thus our result on Hochschild cohomology includes this situation.

References

[1] Asashiba H.: On a lift of an individual stable equivalence to a standard derived equivalence for representation-finite selfinjective algebras. Algebras and Representation Theory **6** (2003), 427–447

[2] Auslander M. and Reiten I.: Grothendick groups of algebras and orders. J. Pure Appl. Algebra **39** (1986), 1–51

[3] Auslander M. and Reiten I.: On a theorem of E.Green on the dual of the transpose. Canad. Math. Soc. Conf. Proc. 11 (1991), 53–65

[4] Beligannis A.: Cohen-Macaulay modules, (co)torsion pairs and virtually Gorenstein algebras. J. Algebra **288** (2005), 137–211

[5] Broué M.: Equivalences of blocks of group algberas. In: Finite dimensional algebras and related topics (V. Dlab and L. L. Scott eds.). Kluwer, 1994, 1–26

[6] Cartan H. and Eilenberg S.: Homological Algebra. Princeton Landmarks in Mathematics, 1973. Originally published in 1956

[7] Curtis Ch. W. and Reiner I.: Representation theory of finite groups and associative algebras. John Wiley and Sons, Inc., 1962

[8] Erdmann K. and Saenz C.: On standardly stratified algebras. Comm. Algebra **31** (2003), 3429–3446

[9] Graham J. J. and Lehrer G. I.: Cellular algebras. Invent. Math. **123** (1996), 1–34

[10] Hilton P. J. and Stammbach U.: A Course in Homological Algebra. Graduate Text in Mathematics 4. Springer-Verlag, New York-Heidelberg-Berlin 1971

[11] König S. and Xi C. C.: On the structure of cellular algebras. In: I. Reiten, S. Smalø and Ø. Solberg (Eds.): Algebras and Modules II. Canad. Math. Soc. Conf. Proc. 24 (1998), 365–386

[12] König S. and Xi C. C.: When is a cellular algebra quasi-hereditary? Math. Ann. **315** (1999), 281–293

[13] Liu Y. M. and Xi C. C.: Constructions of stable equivalences of Morita type for finite-dimensional algebras I. Trans. Amer. Math. Soc. **358** (2006), 2537–2560. Preprint is available at http://math.bnu.edu.cn/~ccxi/

[14] Liu Y. M. and Xi C. C.: Constructions of stable equivalences of Morita type for finite-dimensional algebras II. Math. Z. **251** (2005), 21–39. Preprint is available at http://math.bnu.edu.cn/~ccxi/

[15] Liu Y. M. and Xi C. C.: Constructions of stable equivalences of Morita type for finite-dimensional algebras III. Preprint is available at: http://math.bnu.edu.cn/~ccxi/

[16] Martinez-Villa R.: Properties that are left invariant under stable equivalence. Comm. Algebra **18** (1990), 4141–4169

[17] Pogorzały Z.: Invariance of Hochschild cohomology algebras under stable equivalences of Morita type. J. Math. Soc. Japan **53** (2001), 913–918

[18] Rickard J.: Some recent advances in modular representation theory. Canad. Math. Soc. Conf. Proc. 23 (1998), 157–178

[19] Rickard J.: Equivalences of derived categories for symmetric algebras. J. Algebra **257** (2002), 460–481
[20] Xi C. C.: On the finitistic dimension conjecture II: related to finite global dimension. Adv. in Math. **201** (2006), 116–142. Preprint is available at http://math.bnu.edu.cn/~ccxi/
[21] Xi C. C.: Representation dimension and quasi-hereditary algebras. Adv. in Math. **168** (2002), 193–212
[22] Xi C. C.: The relative Auslander-Reiten theory of modules. Preprint is avialable at http://math.bnu.edu.cn/~ccxi/Papers/Articles/rartnew.pdf/

Received March 4, 2006

School of Mathematical Sciences, Beijing Normal University, 100875 Beijing, People's Republic of China
xicc@bnu.edu.cn