The relative Auslander-Reiten theory of modules

Changchang Xi

School of Mathematical Sciences, Beijing Normal University,
100875 Beijing, P. R. China;
E-mail: xicc@bnu.edu.cn

Abstract

Let A be an Artin algebra. As we know, the construction of the well-known Auslander-Reiten sequence is based on the natural (A, A)-bimodule A and the induced transpose, where the functor $\mathrm{Hom}_A(-, {}_AA_A)$ plays an important role. In this paper we develop a more general transpose, called the relative transpose, by exploiting the canonical $(A, \mathrm{End}({}_AT))$-bimodule ${}_AT_{\mathrm{End}({}_AT)}$ for an arbitrary A-module T and prove some fundamental results, especially, many useful homological exact sequences are generalized. We hope with this consideration the tilting theory and the Auslander-Reiten theory could be unified. The feature of our discussion differs from those on generalizing tilting theory in the literature is that we do not impose any requirement on ${}_AT$. Using our discussion we obtain among other things certain homologically finite subcategories and results on both representation dimensions and finitistic dimensions. In particular, we show that (1) if e is an idempotent in an Artin algebra A such that $Ae \otimes_{eAe} Y \simeq \mathrm{Hom}_{eAe}(eA, Y)$ for Y an eAe-module, then the representation dimension of eAe is bounded above by the representation dimension of A; (2) if a subcategory $\mathcal{C}$ of $\mathrm{App}({}_AT)$ is covariantly finite in A-mod, then the subcategory of the relative transposes of modules in $\mathcal{C}$ is covariantly finite in mod-$\mathrm{End}({}_AT)$.

1 Introduction

In the representation theory of Artin algebras the existence of almost split sequences is very important for the understanding of the structure of the whole category of finitely generated modules. As a key ingredient in this theory, the transpose plays a central role, here for a given Artin algebra A the natural bimodule ${}_AA_A$ and the induced functor $\mathrm{Hom}_A(-, A)$ were mainly involved. In this paper we try to replace ${}_AA_A$ by a more general bimodule, namely, we take an A-module T and consider the induced (A, B)-bimodule ${}_AT_B$ with $B = \mathrm{End}({}_AT)$, and introduce the relative transpose with respect to this bimodule ${}_AT_B$. In this way we have a bridge between the category of finitely generated left A-modules and the category of finitely generated right B-modules via the functor $\mathrm{Hom}_A(-, T)$ applying to T-presentations. As in the usual case, we hope that the relative transpose could help us further to understand the Auslander-Reiten theory. There are another two reasons for considering the relative transpose. The first one is the well-known tilting theory which deals with a bimodule ${}_AT_B$, but emphases strong homological requirements on ${}_AT$, so we try to weaken the conditions. The second one is motivated by the study of the representation dimension of finite dimensional algebras, where the estimations of global dimensions of endomorphism algebras of certain modules ${}_AT$ are considered. The idea in this paper is to try to have a unified method to handle the above mentioned three aspects. Our interest concentrates mainly on the homological topics related to the relative transpose.

It turns out that many important results on the usual transpose are still valid in our general case. Moreover, our consideration yields also homologically finite subcategories in A-mod and in B-mod. As an application, we show that, for a projective A-module P with certain restrictions, the representation dimension of the endomorphism algebra of P is less than or equal to the representation dimension of A.

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The contents of the paper are arranged as follows. In Section 2 we recall some definitions and known homological results, and deduce some basic facts for the later use, thus many useful homological identities are displayed. This seems to be of interest for other investigations. In Section 3 we define the relative transpose and develop its homological properties. It is shown that many important results on the usual transpose are generalized to the relative transpose, especially, such as some long exact sequences and the Auslander-Reiten formula in which the transpose are involved. Section 4 is designed to illustrate the scope of our consideration by several examples. In Section 5 we compare the involved subcategories in A-mod with that in B-mod, and establish their equivalences via the canonical functor $\mathrm{Hom}_A(T,-)$. In Section 6 we discuss how the homological finiteness of a subcategory in A-mod is related to that of the corresponding subcategories in B-mod. This is based on a generalization of a result in [8]. The special case of T being equal to Ae provides a more pleasant example. Also, in this section we consider the finitistic dimensions of A and $\mathrm{End}(_AT)$. In the last section we mainly consider the relationship of representation dimensions between the algebra A and eAe with e an idempotent element in A, but our setup is more general, namely, we first give a statement in terms of functor language, and then apply it to algebras A and eAe.

2 Preliminaries

In this section we give some definitions in our terminology and collect facts which are often used in the paper.

2.1 Definitions and notation

Let A be an Artin R-algebra, that is, R is a commutative Artin ring and A is an R-algebra which is finitely generated as an R-module. We denote by A-mod the category of all finitely generated left A-modules. The set of homomorphisms between two modules X and Y will be denoted by $\mathrm{Hom}_A(X,Y)$, but for simplicity we shall write this set very often as (X,Y). Given two homomorphisms of modules, say $f: L \longrightarrow M$ and $g: M \longrightarrow N$, the composition of f and g is a homomorphism from L to N and will be denoted by fg in this paper. Let E be an injective envelope of the R-module $R/\mathrm{rad}(R)$, where $\mathrm{rad}(R)$ is the Jacobson radical of R. We denote by D the usual duality $\mathrm{Hom}_R(-,E)$ between the category A-mod and the category mod-A of finitely generated right A-modules. The R-length of an R-module M will be denoted by $l_R(M)$.

Throughout this paper, we assume that all modules are finitely generated, and all rings are Artin algebras.

Let $\mathcal{X}$ be a full subcategory of A-mod and M an A-module. A homomorphism $f: X \longrightarrow M$ with $X \in \mathcal{X}$ is called an $\mathcal{X}$-precover of M if the induced sequence $\mathrm{Hom}_A(Y,X) \longrightarrow \mathrm{Hom}_A(Y,M) \longrightarrow 0$ is exact for all $Y \in \mathrm{add}(\mathcal{X})$, here $\mathrm{add}(\mathcal{X})$ stands for the additive category generated by $\mathcal{X}$. If, in addition, the homomorphism f has the property that every endomorphism $g: X \longrightarrow X$ with $f = gf$ is an automorphism, then f is called an $\mathcal{X}$-cover of M. Similarly, we can define the notion of $\mathcal{X}$-preenvelopes and $\mathcal{X}$-envelopes. Note that if $f: X \longrightarrow M$ is an $\mathcal{X}$-precover (or an $\mathcal{X}$-cover) of M, then $f: X \longrightarrow \mathrm{Im}(f)$ is an $\mathcal{X}$- precover (or an $\mathcal{X}$-cover) of the image $\mathrm{Im}(f)$ of f. An $\mathcal{X}$-resolution of M is a complex

$$... \longrightarrow T_m \xrightarrow{f_m} ... \xrightarrow{f_2} T_1 \xrightarrow{f_1} T_0 \xrightarrow{f_0} M \longrightarrow 0$$

with all $T_i \in \mathcal{X}$, such that

$$... \longrightarrow (X,T_m) \longrightarrow ... \longrightarrow (X,T_1) \longrightarrow (X,T_0) \longrightarrow (X,M) \longrightarrow 0$$

is exact for all $X \in \mathcal{X}$. Such a sequence is called a minimal $\mathcal{X}$-resolution of M if $T_i \longrightarrow \mathrm{Ker}(f_{i-1})$ is an $\mathcal{X}$-cover for all $i \geq 1$. Here we do not require the exactness of the complex.

We say that M has $\mathcal{X}$-dimension $\leq m$, denoted by $\mathcal{X}-\dim(M) \leq m$, if there exists an $\mathcal{X}$-resolution of M of the form $0 \longrightarrow T_m \longrightarrow ... \longrightarrow T_1 \longrightarrow T_0 \longrightarrow M \longrightarrow 0$. If there is no such sequence, we say that M has no $\mathcal{X}$-dimension. If such a sequence exists for M and m is the least, then we say that the $\mathcal{X}$-dimension of M is m, denoted by $\mathcal{X}\text{-dim}(M) = m$; and if there is no such m which is the least, we define $\mathcal{X}\text{-dim}(M) = \infty$.

It is clear that for any module ${}_AT \in A\text{-mod}$ and any module ${}_AM \in A\text{-mod}$ there is an $\text{add}({}_AT)$-precover and $\text{add}({}_AT)$-cover. If $f : T_0 \longrightarrow M$ is an $\text{add}({}_AT)$-cover of M, then we decree that the T-syzygy of M is the kernel of f, denoted by $\Omega_T(M)$. This is well defined. Note that f is not necessarily surjective.

We remark that in [9] and in [7] the $\mathcal{X}$-precover and $\mathcal{X}$-cover of M were called right $\mathcal{X}$-approximation and minimal right $\mathcal{X}$-approximation of M, respectively. Since we prefer to a brief expression, we adopt these traditional terminology, as was already used, for example, in the book [15].

Let ${}_AT$ be a module in A-mod. We denote by B the endomorphism algebra of T, thus T is an A-B bimodule in the natural manner. Throughout the paper, we shall fix such a triple $(A, {}_AT, B)$. Now let us introduce the following full subcategories of A-mod:

$$\begin{array}{rl} \text{Gen}({}_AT) = & \{M \in A\text{-mod} \mid \text{there is a surjective morphism from } T^m \text{ to } M, m \in \mathbb{N}\ \}, \\ \text{Pre}\ ({}_AT) = & \{M \in A\text{-mod} \mid \text{there is an exact sequence } T_1 \longrightarrow T_0 \longrightarrow M \longrightarrow 0 \\ & \text{with } T_i \in \text{add}({}_AT) \text{ for } i = 0, 1\ \}, \\ \text{App}({}_AT) = & \{M \in A\text{-mod} \mid \text{there is an exact sequence } T_1 \xrightarrow{f_1} T_0 \xrightarrow{f_0} M \longrightarrow 0 \\ & \text{such that } \text{Ker}(f_0) \in \text{Gen}({}_AT) \text{ and } f_0 \text{ is an add}({}_AT)\text{-precover of } M\ \}. \end{array}$$

Note that the modules in $\text{App}({}_AT)$ are precisely the modules admitting a minimal $\text{add}_AT)$-presentation.

Dually, we can define the subcategory $\text{Cogen}({}_AT)$ whose objects are the A-modules M which are cogenerated by ${}_AT$, that is, M is a submodule of a finite direct sum of ${}_AT$, and the subcategory $\text{Copre}({}_AT)$ whose objects are those modules M which posses an exact sequence of the form $0 \to M \to T_0 \to T_1$ with $T_i \in \text{add}({}_AT)$. Similarly, we use the notion of preenvelope and envelope to define $\text{Coapp}({}_AT)$, that is, a module M belongs to $\text{Coapp}({}_AT)$ if there is an exact sequence $0 \to M \xrightarrow{f} T_0 \to T_1$ with $T_i \in \text{add}({}_AT)$ such that f is an $\text{add}({}_AT)$-preenvelope of M.

If $\mathcal{X}$ and $\mathcal{Y}$ are two full subcategories of A-mod such that $\mathcal{Y}$ is a full subcategory of $\mathcal{X}$, we denote by $\mathcal{X}/\mathcal{Y}$ the factor category of $\mathcal{X}$ modulo $\mathcal{Y}$, that is, the category with the same objects as $\mathcal{X}$, but the set of homomorphisms from an object X to another object Y is the quotient of $\text{Hom}_A(X, Y)$ modulo the morphisms which factor through a module in $\text{add}(\mathcal{Y})$. For simplicity, we denote by $(M, \mathcal{Y}, N)$ the set of all morphisms from M to N which factor through a module in $\text{add}(\mathcal{Y})$. Clearly, there is a functor from $\mathcal{X}$ to $\mathcal{X}/\mathcal{Y}$ which is identity on objects and sends a homomorphism f in $\text{Hom}_A(M, N)$ to its canonical image $\underline{f}$ in $\text{Hom}_A(M, N)/(M, \mathcal{Y}, N)$.

If $\mathcal{Y}$ is equal to the subcategory $\mathcal{P}({}_AA)$ of all projective A-modules and $\mathcal{X} = A\text{-mod}$, we write $A\text{-}\underline{\text{mod}}$ for $A\text{-mod}/\mathcal{P}({}_AA)$.

2.2 Homological facts

In this subsection we shall prepare some homological results and deduce their consequences, which are needed in the course of our discussion.

Lemma 2.1 *Let M be arbitrary A-module. Then:*

(1) *Let X_B be a right B-module. The natural homomorphism*

$$\delta : \quad X \otimes_B Hom_A(T, M) \longrightarrow Hom_B(Hom_A(M, T), X)$$

given by $x \otimes f \mapsto \delta_{x \otimes f}$ with $\delta_{x\otimes f}(g) = (x)(fg)$, is an isomorphism if $M \in add({}_AT)$.

(2) *If $X \in add({}_AT)$, or $M \in add({}_AT)$, then the composition map $m : (X, T) \otimes_B (T, M) \longrightarrow (X, M)$ given by $f \otimes_B g \mapsto fg$ is bijective.*

(3) *If M is in $Gen({}_AT)$, then the evaluation map $e_M : T \otimes_B (T, M) \longrightarrow M$ is surjective. If M is in $App({}_AT)$, then e_M is bijective. Conversely, if e_M is bijective, then $M \in App({}_AT)$.*

Proof. (1) we note that the statement is true for $M =_A T$. This implies that it is true also for direct summands of T and hence for $M \in \mathrm{add}(_AT)$.

(2) is true for $X = {_AT}$ or $M = {_AT}$, this implies that (2) is true for X or M being a direct summand of $_AT$. Hence (2) holds for $X \in \mathrm{add}(_AT)$ or $X \in \mathrm{add}(_AT)$.

(3) the first statement of (3) is obvious. We prove the second one of (3). Note that e_Y is bijective if $Y \in \mathrm{add}(_AT)$. Now, let M be in $\mathrm{App}(_AT)$. Then we have a minimal $\mathrm{add}(_AT)$-presentation of M:

$$T_1 \xrightarrow{f_1} T_0 \xrightarrow{f_0} M \longrightarrow 0.$$

This gives us an exact sequence

$$(T, T_1) \xrightarrow{(T,f_1)} (T, T_0) \xrightarrow{(T,f_0)} (T, M) \longrightarrow 0$$

which induces the following exact commutative diagram:

$$\begin{tikzcd}
T \otimes_B (T, T_1) \arrow[r] \arrow[d, "e_{T_1}"] & T \otimes_B (T, T_0) \arrow[r] \arrow[d, "e_{T_0}"] & T \otimes_B (T, M) \arrow[r] \arrow[d, "e_M"] & 0 \\
T_1 \arrow[r, "f_1"] & T_0 \arrow[r, "f_0"] & M \arrow[r] & 0.
\end{tikzcd}$$

Since $T_i \in \mathrm{add}(_AT)$, the first two vertical maps are isomorphisms. Thus the third one is also an isomorphism.

Conversely, suppose that e_M is an isomorphism. This implies that M is generated by $_AT$. It is easy to see that there is an $\mathrm{add}(_AT)$-cover of M by [9, proposition 4.2], say $T_0 \xrightarrow{f} M \longrightarrow 0$. We need to show that the kernel K of f is generated by $_AT$, that is, we have to show that the map e_K is surjective. However, this follows from the following exact commutative diagram immediately by applying the snake lemma:

$$\begin{tikzcd}
 & T \otimes_B (T, K) \arrow[r] \arrow[d, "e_K"] & T \otimes_B (T, T_0) \arrow[r] \arrow[d, "\wr"] & T \otimes_B (T, M) \arrow[r] \arrow[d, "e_M"] & 0 \\
0 \arrow[r] & K \arrow[r] & T_0 \arrow[r, "f"] & M \arrow[r] & 0. \square
\end{tikzcd}$$

The first statement of the following lemma is a generalization of [6, proposition 2.1].

Lemma 2.2 (1) *Let $_AT$ be an arbitrary A-module. If $X \in App(_AT)$, then for any A-module Y there is an isomorphism $Hom_A(X, Y) \simeq Hom_B((_AT, {_AX}), (_AT, {_AY}))$ as R-modules, and this isomorphism is functorial in X and Y. Dually, if $Y \in Coapp(_AT)$, then $Hom_A(X, Y) \simeq Hom_B((_AY, {_AT}), (_AX, {_AT}))$ as R-modules, which is functorial in Y and X .*

(2) *The natural homomorphism*

$$\alpha_M : M \longrightarrow Hom_B(Hom_A(M, T), T)$$

is an isomorphism if and only if there exists an exact sequence

$$0 \longrightarrow M \xrightarrow{f} T^m \longrightarrow T^n,$$

where f is an add$(_AT)$-preenvelope of M.

Proof. (2) is a result in [11, proposition 2.1]. We prove (1). Since $_AX \in \mathrm{App}(_AT)$, we get the isomorphism $_AX \simeq {_AT} \otimes_B (T, X)$ as A-modules by Lemma 2.1(3). Thus it follows from the adjoint isomorphism that

$$\mathrm{Hom}_B((T, X), (T, Y)) \simeq \mathrm{Hom}_A(T \otimes_B (T, X), Y) \simeq \mathrm{Hom}_A(X, Y),$$

as desired. For the second statement, we use Lemma 2.2(2) and the following well-known identity (see [1, proposition 20.7]):

$$\mathrm{Hom}_R(_RM, \mathrm{Hom}_S(N_S, {_RU_S})) \simeq \mathrm{Hom}_S(N_S, \mathrm{Hom}_R(_RM, {_RU_S})),$$

where R and S are rings, M, N and U are modules as indicated. Thus we have

$$((Y,T),(X,T)) \simeq (X,((Y,T),T)) \simeq (X,Y),$$

and the lemma is proved. $\Box$

Concerning the natural homomorphism α_M, there is the notion of a T-torsionless module. Recall that an A-module ${}_AM$ is said to be T-torsionless if α_M is injective, and T-reflexive if α_M is bijective. One has the following basic facts on T-torsionless modules.

Lemma 2.3 (1) *$Ker(\alpha_M) = \bigcap_{f:M\to T} Ker(f)$.*

(2) *$Ker(\alpha_M) = 0$ if and only if for any $0 \neq m \in M$ there is a map $f \in Hom_A(M,T)$ such that $(m)f \neq 0$.*

(3) *${}_AM$ is T-torsionless if and only if there is an exact sequence of the form $0 \longrightarrow M \xrightarrow{f} T^m$.*

(4) *Every submodule of a T-torsionless module is T-torsionless.* $\Box$

The following homological identity is well-known (see, for example, [1, proposition 20.11]).

Lemma 2.4 *Let R and S be rings, P a finitely generated right projective R-module, U an (S,R)-bimodule, and N a left S-module. Then the morphism*

$$\nu: \ P \otimes_R Hom_S({}_SU_R, N) \longrightarrow Hom_S(Hom_R(P,U), N)$$

defined by $\nu(p\otimes\gamma) : \delta \mapsto (\delta(p))\gamma$ is an isomorphism. $\Box$

The following result is taken from [15, p.79].

Lemma 2.5 (1) *Let R and S be rings, M a left R-module, U an (R,S)-module and C a left S-flat module. Then the natural map $\tau : Hom_R(M,U)\otimes_S C \longrightarrow Hom_R(M, U\otimes_S C)$ defined by $\tau(f\otimes c)(a) = f(a)\otimes c$ is an isomorphism. If R is additionally left noetherian, then*

$$Ext^i_R(M,U)\otimes_S C \simeq Ext^i_R(M, U\otimes_S C).$$

for all $i \geq 0$.

(2) *Let R and S be rings, M a left R-module, U an (R,S)-module and C an injective right S-module. Then*

$$Tor_i^R(Hom_S(U,C), M) \simeq Hom_S(Ext^i_R(M,U), C)$$

for all $i \geq 0$.

(3) *Let R and S be rings, M a right R-module, U an (R,S)-module and C an injective right S-module. Then*

$$Ext^i_R(M_R, Hom_S({}_RU_S, C_S)) \simeq Hom_S(Tor_i^R(M_R, {}_RU_S), C_S)$$

for all $i \geq 0$. $\Box$

From Lemma 2.5 we can deduce the following result which is useful when dealing with partial tilting modules, or modules with small projective dimensions.

Proposition 2.6 *Let R and S be Artin algebras, and M an R-module, T an (R,S)-bimodule with $Ext^i_R(M,T) = 0$ for $i = 1,2$. Then for any S-module C with the projective dimension at most one, there is an exact sequence of S-modules*

$$0 \to Ext^1_R({}_RM, Tor_1^S(T_S, C)) \to (M,T)\otimes_S C \to ({}_RM,\ {}_RT\otimes_S C) \to Ext^2_R(M, Tor_1^S(T,C)) \to 0.$$

Proof. Since the projective dimension of C is at most one, there is an exact sequence $0 \to P_1 \to P_0 \to C \to 0$, where P_i are projective S-modules. This yields a new exact sequence

$$0 \to \mathrm{Tor}_1^S(T, C) \to T \otimes_S P_1 \to T \otimes_S P_0 \to T \otimes_S C \to 0$$

Since $\mathrm{Ext}^1_R(M, T) = 0 = \mathrm{Ext}^2_R(M, T)$, we have by Lemma 2.5 that $\mathrm{Ext}^j_R(M, T \otimes_S P_i) = 0$ for $j = 1, 2$. Thus we obtain the following commutative diagram with exact rows:

$$\begin{tikzcd}
 & (M,T) \otimes_S P_1 \arrow[r] \arrow[d, "\tau_1", "\wr"'] & (M,T) \otimes_S P_0 \arrow[r] \arrow[dd, "\tau_2"] & (M,T) \otimes_S C \arrow[r] \arrow[dd, "\tau_3"] & 0 & \\
 & (M, T \otimes_S P_1) \arrow[d] & & & & \\
0 \arrow[r] & (M,Q) \arrow[r] & (M, T \otimes_S P_0) \arrow[r] & (M, T \otimes_S C) \arrow[r] & \mathrm{Ext}^1_R(M,Q) \arrow[r] & 0,
\end{tikzcd}$$

where Q is the image of the map $T \otimes_S P_1 \longrightarrow T \otimes_S P_0$. Note that τ_1 and τ_2 are isomorphisms by Lemma 2.5. The cokernel of the first vertical map is isomorphic to $\mathrm{Ext}^1_R(M, \mathrm{Tor}_1^S(T, C))$ by Lemma 2.5. We also have that $\mathrm{Ext}^1_R(M, Q) \simeq \mathrm{Ext}^2_R(M, \mathrm{Tor}_1^S(T, C))$ again by Lemma 2.5. Now the result follows from the snake lemma. $\square$

A special case of 2.6 is the following result, here we relax the condition on the module C.

Proposition 2.7 *Let R and S be Artin algebras and T an (R, S)-bimodule. Suppose that M is an R-module such that $Ext^1_R(M, T) = 0$ and proj.dim ${}_RM \leq 1$. Then for any S-module C of finite projective dimension, there is an exact sequence of S-modules*

$$0 \to Ext^1_R({}_RM, Tor_1^S(T_S, C)) \to (M, T) \otimes_S C \to ({}_RM,\ {}_RT \otimes_S C) \to 0.$$

Proof. We prove this by induction on the projective dimension n of C. Clearly, the proposition is true for $n = 0, 1$. Assume that $n > 1$. As in the above case, we take an exact sequence $P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} C \to 0$ with P_i projective. Then we have an exact commutative diagram

$$\begin{tikzcd}
 & (M,T) \otimes_S P_1 \arrow[r] \arrow[d, "h"'] & (M,T) \otimes_S P_0 \arrow[r] \arrow[d, "\tau_2"] & (M,T) \otimes_S C \arrow[r] \arrow[d, "\tau_3"] & 0 & \\
0 \arrow[r] & (M,Q) \arrow[r] & (M, T \otimes_S P_0) \arrow[r] & (M, T \otimes_S C) \arrow[r] & \mathrm{Ext}^1_R(M,Q) \arrow[r] & 0,
\end{tikzcd}$$

where Q is the image of the map $T \otimes_S P_1 \longrightarrow T \otimes_S P_0$. Let K be the kernel of f_0. It follows from the first commutative square that h is a composition of $(M, T) \otimes_S P_1 \to (M, T) \otimes_S K \xrightarrow{\tau} (M, T \otimes_S K) \to (M, Q)$. Thus we have another exact commutative diagram:

$$\begin{tikzcd}
(M,T) \otimes_S P_1 \arrow[r, "h"] \arrow[d] & (M,Q) \arrow[r] \arrow[d, equal] & \mathrm{Cok}(h) \arrow[r] \arrow[d, "g"] & 0 \\
(M, T \otimes_S K) \arrow[r] & (M,Q) \arrow[r] & \mathrm{Ext}^1_R(M, \mathrm{Tor}_1^S(T, C)) \arrow[r] & \mathrm{Ext}^1_R(M, T \otimes_S K).
\end{tikzcd}$$

Note that $T \otimes_S K$ is generated by T. Thus $\mathrm{Ext}^1_R(M, T \otimes_S K) = 0$ since proj.dim ${}_RM \leq 1$. This implies also that $\mathrm{Ext}^1_R(M, Q) = 0$. Since proj.dim ${}_SK \leq n - 1$, we know by induction that τ is surjective, and therefore the first vertical map in the above diagram is surjective. Hence $\mathrm{Cok}(h) \simeq \mathrm{Ext}^1_R(M, \mathrm{Tor}_1^S(T, C))$. Now our result follows from the first commutative diagram. $\square$

Proposition 2.8 *Let R and S be Artin algebras, and M an R-module, U an (R, S)-bimodule Then*

(1) $Ext^i_R(U\otimes_S P, {}_RM) \simeq Hom_S({}_SP, Ext^i_R({}_RU, {}_RM))$ *for all* $i \geq 0$ *and any projective* S*-module* ${}_SP$.

(2) *If* ${}_RU$ *is a generator for* R*-mod, that is,* $add({}_RR) \subseteq add({}_RU)$*, then for any* S*-module* Y*, there is an exact sequence*

$$0 \longrightarrow Ext^1_S({}_SY, Hom_R({}_RU, {}_RM)) \longrightarrow Ext^1_R(U\otimes_S Y, M) \longrightarrow Hom_S({}_SY, Ext^1_R({}_RU, M)).$$

Proof. (1) can be proved by induction on i. To prove (2), we take an injective hull of ${}_RM : 0 \to M \to I \to C \to 0$, where I is injective. This induces two exact sequences:

$$0 \longrightarrow (U, M) \longrightarrow (U, I) \xrightarrow{d} (U, C) \longrightarrow \mathrm{Ext}^1_R(U, M) \longrightarrow 0,$$

$$0 \longrightarrow (U\otimes_S Y, M) \longrightarrow (U\otimes_S Y, I) \longrightarrow (U\otimes_S Y, C) \longrightarrow \mathrm{Ext}^1_R(U\otimes_S Y, M) \longrightarrow 0.$$

Let Q be the image of d. Then we have the following exact commutative diagram:

$$\begin{array}{ccccccccc}
 & & (U\otimes_S Y, I) & \longrightarrow & (U\otimes_S Y, C) & \longrightarrow & \mathrm{Ext}^1_R(U\otimes_S Y, M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (Y, Q) & \longrightarrow & (Y, (U, C)) & \longrightarrow & (Y, \mathrm{Ext}^1_R(U, M)). & &
\end{array}$$

Note that the cokernel of the first vertical map is $\mathrm{Ext}^1_S(Y, (U, M))$ since ${}_RU$ is a generator and (U, I) is injective, and that the second vertical map is an adjoint isomorphism. Now (2) follows from the snake lemma. $\square$

Remark. The above result (2) describes the kernel of the map $\mathrm{Ext}^1_R(U \otimes_S Y, M) \to \mathrm{Hom}_S({}_SY, \mathrm{Ext}^1({}_RU, M))$. This can be extended to a general case: $0 \longrightarrow \mathrm{Ext}^1_S({}_SY, \mathrm{Hom}_R({}_RU, \Omega^{-i+1}(M)) \longrightarrow \mathrm{Ext}^i_R(U\otimes_S Y, M) \longrightarrow \mathrm{Hom}_S({}_SY, \mathrm{Ext}^i_R({}_RU, M))$, where $\Omega^{-i}(M)$ is the i-cosyzygy of M with $i \geq 1$. It would be nice to have a description of the cokernel of this map.

3 The relative transpose and AR-sequences

In this section we shall introduce the relative T-transpose which is defined in a way similar to the usual one defined by Auslander in 1962. The surprise thing is that almost all of the main results about the usual transpose hold true for T-transpose. As a by-product we also produce many indecomposable modules over the endomorphism algebra B of ${}_AT$. We hope that our discussion will be useful for the study of representations of the algebra B.

Let M be an arbitrary A-module in $\mathrm{Pre}({}_AT)$. Then we have an exact sequence

$$(*) \quad T_1 \xrightarrow{f_1} T_0 \xrightarrow{f_0} M \longrightarrow 0.$$

Now applying $\mathrm{Hom}_A(-, {}_AT)$ to $(*)$, we have an exact sequence in mod-B:

$$0 \longrightarrow (M, T) \longrightarrow (T_0, T) \longrightarrow (T_1, T) \longrightarrow \Sigma_T(M) \longrightarrow 0,$$

where $\Sigma_T(M)$ stands for the cokernel of $\mathrm{Hom}(f_1, T)$. From this we get the following new exact sequence in B-mod:

$$0 \longrightarrow D\Sigma_T(M) \longrightarrow D(T_1, T) \longrightarrow D(T_0, T) \longrightarrow D(M, T)) \longrightarrow 0.$$

We call the module $\Sigma_T(M)$ the **relative transpose** of M with respect to T, or T-transpose of M. Note that the T-transpose of a left A-module is a right B-module, and depends on the exact sequence $(*)$. Observe that $\Omega_B(T, Z) \simeq (T, \Omega_T(Z))$ for all A-module Z, and that $\Sigma_T(M) = 0$ if and only if $M \in \mathrm{add}({}_AT)$.

Note that if we take ${}_AT = {}_AA$ then we get the usual transpose Tr which was defined in [5]. The relative transpose generalizes also the case discussed in [20], where ${}_AT$ is demanded to have the Ext-group vanishing property and the double centralizer property.

Suppose we are given a map $f_1 : T_1 \longrightarrow T_0$ and the exact sequence $(*)$, the following observation describes the kernel and the cokernel of f_1.

Lemma 3.1 *If we are given the homomorphism f_1 and the exact sequence $(*)$ with M the cokernel of f_1, then $Ker(f_1) \simeq Hom_B(\Sigma_T M, {}_AT_B)$ and $M \simeq T \otimes_B X$, where X is the cokernel of $(T, T_1) \to (T, T_0)$.* $\square$

Also, there is another construction in which minimal projective presentation is used. Take a minimal projective presentation

$$P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

of M, we define $\Gamma_T(M)$ as follows:

$$0 \to (M, T) \to (P_0, T) \to (P_1, T) \to \Gamma_T(M) \to 0.$$

If $T = A$, then $\Gamma_T(M) = \mathrm{Tr}_A(M)$. Furthermore, we have the following connection between Γ_T and Tr_A.

Lemma 3.2 *For any A-module M we have $\Gamma_T(M) \simeq Tr_A(M) \otimes_A T$.*

Proof. From the construction of $\Gamma_T(M)$ we have the following exact commutative diagram:

$$\begin{array}{ccccccc}
(P_0, A) \otimes_A T & \longrightarrow & (P_1, A) \otimes_A T & \longrightarrow & \mathrm{Tr}_A(M) \otimes_A T & \longrightarrow & 0 \\
\downarrow & & \downarrow & & & & \\
(P_0, T) & \longrightarrow & (P_1, T) & \longrightarrow & \Gamma_T(M) & \longrightarrow & 0.
\end{array}$$

By Lemma 2.1 (2), the first two vertical maps are isomorphisms. Hence we have that $\Gamma_T(M) \simeq \mathrm{Tr}_A(M) \otimes_A T$. $\square$

Let M be in $\mathrm{App}({}_AT)$. Then we have a minimal $\mathrm{add}({}_AT)$-presentation of M: $T_1 \xrightarrow{f_1} T_0 \xrightarrow{f_0} M \to 0$. Note that such a sequence for M is unique up to isomorphisms. Since for any T' in $\mathrm{add}({}_AT)$ the map $\alpha_{T'} : T' \to ((T', T), T)$ is an isomorphism by Lemma 2.2, we see that if M has no non-zero direct summand in $\mathrm{add}({}_AT)$, then the right B-module $\Sigma_T(M)$ has no non-zero projective direct summand, and therefore $(T_0, T) \to (T_1, T) \to \Sigma_T(M) \to 0$ is a minimal projective presentation of the right B-module $\Sigma_T(M)$.

Let M and N be in $\mathrm{App}({}_AT)$ such that both M and N have no direct summand isomorphic to a module in $\mathrm{add}({}_AT)$, and $h : M \longrightarrow N$ a homomorphism. Since M and N are in $\mathrm{App}({}_AT)$, we can take a minimal $\mathrm{add}({}_AT)$-presentation for M and N respectively. Then there exists the following exact commutative diagram with $Q_i \in \mathrm{add}({}_AT)$:

$$\begin{array}{ccccccc}
T_1 & \xrightarrow{f_1} & T_0 & \xrightarrow{f_0} & M & \longrightarrow & 0 \\
\downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_0} & & \downarrow{\scriptstyle h} & & \\
Q_1 & \xrightarrow{g_1} & Q_0 & \xrightarrow{g_0} & N & \longrightarrow & 0
\end{array}$$

and consequently a unique morphism $\Sigma_{(T,h_1,h_0)}(g) : \Sigma_T(N) \longrightarrow \Sigma_T(M)$ such that the diagram

$$\begin{array}{ccccccc}
(Q_0, T) & \xrightarrow{(g_1,T)} & (Q_1, T) & \longrightarrow & \Sigma_T(N) & \longrightarrow & 0 \\
\downarrow{\scriptstyle (h_0,T)} & & \downarrow{\scriptstyle (h_1,T)} & & \downarrow{\scriptstyle \Sigma_{(T,h_1,h_0)}(h)} & & \\
(T_0, T) & \xrightarrow{(f_1,T)} & (T_1, T) & \longrightarrow & \Sigma_T(M) & \longrightarrow & 0,
\end{array}$$

is commutative. Clearly, this morphism depends upon the choice of the map (h_1, h_0). If (h_1', h_0') is another choice, then in the category $\underline{\text{mod}}$-B we have $\underline{\Sigma_{(T,h_1',h_0')}(h)} = \underline{\Sigma_{(T,h_1,h_0)}(h)}$. Hence

we have a contravariant functor $\Sigma_T : \text{App}({}_AT) \longrightarrow \underline{\text{mod}}$-$B$ which sends $h \in \text{Hom}_B(M, N)$ to $\Sigma_T(h) = \underline{\Sigma_{(T,h_1,h_0)}(h)} \in \underline{\text{Hom}}_B(\Sigma_T(N), \Sigma_T(M))$. We can also see that $\Sigma_T(h) = 0$ if and only if h factors through a module in $\text{add}({}_AT)$. Thus we have defined in fact a contravariant functor $\Sigma_T : \text{App}({}_AT)/\text{add}({}_AT) \longrightarrow \underline{\text{mod}}$-$B$. This contravariant functor is faithful. In particular, we have the following result on the endomorphism ring of $\Sigma_T(M)$.

Proposition 3.3 *Let M be an indecomposable module in App$({}_AT)$ which is not in add$({}_AT)$. Then $\Sigma_T(M)$ is indecomposable, and moreover, End$({}_AM)/(M, add({}_AT), M)$ is isomorphic to $\underline{End}$ $(\Sigma_T(M)_B)$.*

Proof. The second statement follows from the faithfulness of the functor Σ_T and Lemma 2.2(1). Now we show that $\Sigma_T(M)$ is indecomposable. In order to show this, we show that a map $h \in \text{End}({}_AM)$ is an isomorphism if and only if so is $\Sigma_{(T,h_1,h_0)}(h)$. If h is an isomorphism, then the maps h_0 and h_1 are automorphisms by the definition of minimal add$({}_AT)$-presentation. Thus $\Sigma_{(T,h_1,h_0)}(h)$ is an isomorphism. Conversely, if $\Sigma_{(T,h_1,h_0)}(h)$ is an isomorphism, then (h_0, T) and (h_1, T) are automorphism since the sequence $(T_0, T) \longrightarrow (T_1, T) \longrightarrow \Sigma_T(M) \longrightarrow 0$ is a minimal B-projective presentation of $\Sigma_T(M)$. By Lemma 2.2(1), h_0 and h_1 are automorphisms. This finishes the proof. $\square$

The relationship between Γ_T and Σ_T as well as Tr is shown as follows:

Lemma 3.4 (1) *For an indecomposable M in App$({}_AT)$ such that $M \notin$ add$({}_AT)$, we have $\Gamma_T(\Sigma_T(M)) \simeq M$. In particular, if M_1 and M_2 are indecomposable in App$({}_AT)$ such that $M_i \notin$ add$({}_AT)$, then $M_1 \simeq M_2$ if and only if $\Sigma_T(M_1) \simeq \Sigma_T(M_2)$.*

(2) *$\Gamma_T(X \oplus Y) \simeq \Gamma_T(X) \oplus \Gamma_T(Y)$ for any A-modules X and Y.*

(3) *$\Sigma_T(X \oplus Y) \simeq \Sigma_T(X) \oplus \Sigma_T(Y)$ for all A-modules X, Y in Pre$({}_AT)$.*

(4) *For any A-module X, there is an exact sequence*

$$0 \to Ext^1_A({}_AX, {}_AT) \to \Gamma_T(X) \to (\Omega^2_A(X), {}_AT) \to Ext^2_A({}_AX, {}_AT) \to 0,$$

where Ω^2_A is the second syzygy operator.

(5) *For any A-module X, there is an exact sequence*

$$0 \to Ext^1_A(Tr_A({}_AT), Tr_A(X)) \to \Gamma_T(X) \to (({}_AT, A), Tr_A(X)) \to Ext^2_A(Tr_A(T), Tr_A(X)) \to 0.$$

Proof. (1) Let $T_1 \to T_0 \to M \to 0$ be a minimal add$({}_AT)$-presentation of M. Then $(T_0, T) \to (T_1, T) \to \Sigma_T(M) \to 0$ is a minimal projective presentation of the right B-module $\Sigma_T(M)$. By the definition of Γ_T and Lemma 2.2, we have the following exact commutative diagram which implies (1):

$$\begin{array}{ccccccc}
((T_1,T),T) & \longrightarrow & ((T_0,T),T) & \longrightarrow & \Gamma_T(\Sigma_T(M)) & \longrightarrow & 0 \\
\downarrow \wr & & \downarrow \wr & & & & \\
T_1 & \longrightarrow & T_0 & \longrightarrow & M & \longrightarrow & 0.
\end{array}$$

(2) and (3) are obvious.

(4) follows from [24]. For completeness we include here a proof. We take a minimal projective presentation $P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0$ of the A-module with P_i projective. This gives the following exact sequence

$$0 \longrightarrow \Omega^2_A(X) \longrightarrow P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0.$$

Furthermore, we have the following exact commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (X,T) & \longrightarrow & (P_0,T) & \longrightarrow & (\Omega_A(X),T) & \longrightarrow & \text{Ext}^1_A(X,T) & \longrightarrow & 0 \\
 & & \| & & \| & & \downarrow f & & \downarrow g & & \\
0 & \longrightarrow & (X,T) & \longrightarrow & (P_0,T) & \longrightarrow & (P_1,T) & \longrightarrow & \Gamma_T(X) & \longrightarrow & 0.
\end{array}$$

Note that the last vertical map is injective because f is injective, and that the cokernel of f is isomorphic to the cokernel of g. On the other hand, we have an exact sequence

$$0 \longrightarrow (\Omega_A(X), T) \xrightarrow{f} (P_1, T) \longrightarrow (\Omega_A^2(X), T) \longrightarrow \mathrm{Ext}_A^1(\Omega_A(X), T) \longrightarrow 0.$$

Thus, combining the snake lemma with the above exact sequence, we have a long exact sequence

$$0 \longrightarrow \mathrm{Ext}_A^1(X, T) \longrightarrow \Gamma_T(X) \longrightarrow (\Omega_A^2(X), T) \longrightarrow \mathrm{Ext}_A^2(X, T) \longrightarrow 0.$$

(5) By [6, proposition 3.2, p.123], we have an exact sequence

$$0 \to \mathrm{Ext}_A^1(\mathrm{Tr}_A({}_AT), \mathrm{Tr}_A(X)) \to \mathrm{Tr}_A(X) \otimes_A T \to (({}_AT, A), Tr_A(X)) \to \mathrm{Ext}_A^2(\mathrm{Tr}_A(T), \mathrm{Tr}_A(X)) \to 0,$$

which yields the sequence in (5) by Lemma 3.2. $\square$

The following result shows that there is a nice relationship between the T-transpose of M and the B-module (T, M). For $T = A$ we get the usual almost split sequence in A-mod.

Theorem 3.5 *Let M be an indecomposable module in $App({}_AT)$. If $M \in add(T)$, then $\Sigma_T(M) = 0$. If $M \notin add(T)$, then there is an Auslander-Reiten sequence in B-mod of the form*

$$0 \longrightarrow D\Sigma_T(M) \longrightarrow X \longrightarrow (T, M) \longrightarrow 0.$$

Proof. The proof of this result can be done by following the idea of Auslander-Reiten for the module category. Here we provide a proof that uses the idea from [17]: Let M be an indecomposable A-module. Then there are two functors from A-mod to the category of abelian groups given by $(-, M)$ and $S_M := (-, M)/\mathrm{rad}(-, M)$. Suppose M is non-projective. An exact sequence

$$0 \longrightarrow N \longrightarrow X \longrightarrow M \longrightarrow 0$$

is an Auslander-Reiten sequence if the induced sequence

$$0 \longrightarrow (-, N) \longrightarrow (-, X) \longrightarrow (-, M) \longrightarrow S_M \longrightarrow 0$$

is a minimal projective resolution of S_M in the category of functors from A-mod to the category of abelian groups.

Now assume that M is indecomposable and $M \notin \mathrm{add}({}_AT)$. We start with a minimal $\mathrm{add}({}_AT)$-presentation of M:

$$T_1 \xrightarrow{f_1} T_0 \xrightarrow{f_0} M \longrightarrow 0.$$

This induces the following three exact sequences

$$(T, T_1) \longrightarrow (T, T_0) \longrightarrow (T, M) \longrightarrow 0,$$

$$0 \longrightarrow D\Sigma_T(M) \longrightarrow D(T_1, T) \xrightarrow{g} D(T_0, T),$$

$$D((T, T_1), -) \longrightarrow D((T, T_0), -) \longrightarrow D((T, M), -) \longrightarrow 0.$$

By Lemma 2.2 and Lemma 2.4, we have the following series of isomorphisms: $D((T, T_i), -) \simeq D((T_i, T), (T, T)), -) \simeq D((T_i, T) \otimes_B (B, -)) \simeq D((T_i, T) \otimes_B -) \simeq (-, D(T_i, T))$. Since $(-, (T, M))$ is a projective functor, the homomorphism $(-, (T, M)) \longrightarrow S_{(T,M)} \longrightarrow D((T, M), -)$ factors through $(-, D(T_0, T))$, where $D((T, M), -)$ is the injective functor with the simple socle $S_{(T,M)}$. Clearly, the morphism $(-, (T, M)) \longrightarrow (-, D(T_0, T))$ is induced from a homomorphism $g' : (T, M) \longrightarrow D(T_0, T)$. Hence we have the following exact commutative diagram:

$$\begin{array}{ccccccc}
 & & D((T,T_1),-) & \longrightarrow & D((T,T_0),-) & \longrightarrow & D((T,M),-) & \longrightarrow 0 \\
 & & \wr| & & \wr| & & \uparrow & \\
0 \longrightarrow (-, D\Sigma_T(M)) \longrightarrow & & (-, D(T_1,T)) & \longrightarrow & (-, D(T_0,T)) & & | & \\
\| & & \uparrow & & \uparrow & & | & \\
0 \longrightarrow (-, D\Sigma_T(M)) \longrightarrow & & (-, X) & \longrightarrow & (-, (T,M)) & \longrightarrow & S_{(T,M)} & \longrightarrow 0,
\end{array}$$

where X is a pullback of g and g'. Since $D\Sigma_T(M)$ is indecomposable, we see that the lower sequence is a minimal projective resolution of $S_{(T,M)}$ in the functor category. Thus we have a desired exact sequence. $\square$

Let us remark that the role of the minimal add$({}_AT)$-presentation of M is to guarantee the existence of the exact sequence $(T, T_1) \longrightarrow (T, T_0) \longrightarrow (T, M) \longrightarrow 0$. In general, for a module M in Pre$({}_AT)$ such a exact sequence may not exist.

The above theorem reveals a connection between Σ_T and the usual transpose Tr as indicated in the next corollary.

Corollary 3.6 *If M is in App$({}_AT)$, Then $\Sigma_T(M) \simeq Tr_B Hom_A(T, M)$.*

This corollary describes the transpose of the Hom-functor $\mathrm{Hom}_A({}_AT, -)$. The next observation gives a characterization of the relative transpose of the tensor functor $T \otimes_B -$.

Proposition 3.7 *Let ${}_BX$ be an arbitrary B-module and let $P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0$ be a projective presentation of ${}_BX$. Then we have an add$({}_AT)$-presentation of ${}_AT \otimes_B X$*

$$T \otimes_B P_1 \longrightarrow T \otimes_B P_0 \longrightarrow T \otimes_B X \longrightarrow 0$$

and $Tr_B(X) \simeq \Sigma_T(T \otimes_B X)$, where Σ_T is defined by the above add$({}_AT)$-presentation of $T \otimes_B X$.

Proof. Note that $T \otimes_B P_i$ lies in add$({}_AT)$. We consider the following exact commutative diagram:

$$\begin{array}{ccccccc} (T \otimes_B P_0, T) & \longrightarrow & (T \otimes_B P_1, T) & \longrightarrow & \Sigma_T(T \otimes_B X) & \longrightarrow 0 \\ \downarrow & & \downarrow & & & \\ (P_0, (T,T)) & \longrightarrow & (P_1, (T,T)) & \longrightarrow & \mathrm{Tr}_B(X) & \longrightarrow 0, \end{array}$$

where the first two vertical maps are the adjoint isomorphisms. Now the result follows immediately from this diagram. $\square$

We also have the following generalization of a result on the usual transpose.

Theorem 3.8 *Let M be an A-module in App$({}_AT)$, and let X_B be an arbitrary right B-module. Then there is an exact sequence:*

$$0 \longrightarrow Ext_B^1(\Sigma_T(M), X) \longrightarrow X \otimes_B (T, M) \stackrel{\delta_{X,M}}{\longrightarrow} Hom_B((M,T), X) \longrightarrow Ext_B^2(\Sigma_T(M), X) \longrightarrow 0,$$

where $\delta_{X,M}$ sends $x \otimes f$ to a morphism in $Hom_B((M,T), X)$ which maps each $\alpha \in (M,T)$ to $x(f\alpha)$ for $x \in X$ and $f \in (T, M)$.

Proof. Since M lies in App$({}_AT)$, there is a minimal add$({}_AT)$-presentation of M:

$$T_1 \stackrel{f_1}{\longrightarrow} T_0 \stackrel{f_0}{\longrightarrow} M \longrightarrow 0,$$

and an exact sequence $(T, T_1) \longrightarrow (T, T_0) \longrightarrow (T, M) \longrightarrow 0$, which yields the following exact sequence

$$X \otimes_B (T, T_1) \longrightarrow X \otimes_B (T, T_0) \longrightarrow X \otimes_B (T, M) \to 0.$$

On the other hand, we have an exact sequence

$$0 \to (M, T) \to (T_0, T) \to (T_1, T) \to \Sigma_T(M) \to 0.$$

Let Q be the cokernel of the morphism $(M, T) \to (T_0, T)$. Then we know that $\mathrm{Ext}_B^1(Q_B, X_B) \simeq \mathrm{Ext}_B^2(\Sigma_T(M), X_B)$ because Q is the first syzygy of $\Sigma_T(M)$. Since the sequence $((T_1, T), X) \to$

$(Q,X) \to \mathrm{Ext}^1_B(\Sigma_T(M),X) \to 0$ is exact, the desired exact sequence follows now from the following commutative diagram with the first and third row exact:

$$\begin{tikzcd}
 & X\otimes_B (T,T_1) \arrow[r] \arrow[d, "\simeq"] & X\otimes_B (T,T_0) \arrow[r] \arrow[d, "\simeq"] & X\otimes_B (T,M) \arrow[r] \arrow[d, "\delta_{X,M}"] & 0 & \\
 & ((T_1,T),X) \arrow[r] \arrow[d] & ((T_0,T),X) \arrow[r] \arrow[d, equal] & ((M,T),X) \arrow[d, equal] & & \\
0 \arrow[r] & (Q,X) \arrow[r] & ((T_0,T),X) \arrow[r] & ((M,T),X) \arrow[r] & \mathrm{Ext}^2_B(\Sigma_T(M),X) \arrow[r] & 0,
\end{tikzcd}$$

where the first two isomorphisms follows from Lemma 2.1(1). □

We have the following exact sequence, which is an immediate consequence of Theorem 3.8 if M belongs to $\mathrm{App}({}_AT)$, and also a generalization of [20, 2.3].

Theorem 3.9 *If M lies in Pre$({}_AT)$, then we have an exact sequence*

$$0 \longrightarrow Ext^1_B(\Sigma_T(M),T_B) \longrightarrow M \stackrel{\alpha_M}{\longrightarrow} Hom_B((M,T),T_B) \longrightarrow Ext^2_B(\Sigma_T(M),T_B) \longrightarrow 0,$$

where α_M is the natural homomorphism.

Proof. Since M lies in $\mathrm{Pre}({}_AT)$, we have an exact sequence

$$T_1 \stackrel{f_1}{\longrightarrow} T_0 \stackrel{f_0}{\longrightarrow} M \longrightarrow 0$$

with $T_i \in \mathrm{add}({}_AT)$. This yields a new exact sequence

$$0 \longrightarrow (M,T) \longrightarrow (T_0,T) \stackrel{\bar{f_1}}{\longrightarrow} (T_1,T) \longrightarrow \Sigma_T M \longrightarrow 0,$$

where $\bar{f_1}$ stands for the induced map of f_1. Let Q be the cokernel of the morphism $(M,T) \to (T_0,T)$. Let $p_2 : (T_0,T) \longrightarrow Q$ and $q_2 : Q \longrightarrow (T_1,T)$ be the canonical projection and inclusion respectively. Then there is a homomorphism g which makes following diagram commutative:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}(f_0) \arrow[r, "q_1"] \arrow[d, "g"] & T_0 \arrow[r, "f_0"] \arrow[d, "\alpha_{T_0}"] & M \arrow[r] \arrow[d, "\alpha_M"] & 0 \\
0 \arrow[r] & (Q,T) \arrow[r, "\bar{p}_2"] & ((T_0,T),T) \arrow[r] & ((M,T),T) \arrow[r] & \mathrm{Ext}^1_B(Q,T) \longrightarrow 0,
\end{tikzcd}$$

where $p_1 : T_1 \to \mathrm{Ker}(f_0)$ and $q_1 : \mathrm{Ker}(f_0) \to T_0$ are the components of the canonical decomposition of f_1. Let $\bar{p}_i$ denote the map (p_i,T) induced from p_i. Then we have the following commutative diagram:

$$\begin{tikzcd}
T_1 \arrow[rr, "f_1"] \arrow[dr, "p_1"] \arrow[ddd, "\alpha_{T_1}"'] & & T_0 \arrow[ddd, "\alpha_{T_0}"] \\
 & \mathrm{Ker}(f_0) \arrow[ur, hook, "q_1"] \arrow[d, "g"] & \\
 & (Q,T) \arrow[dr, "\bar{p}_2"] & \\
(T_1,T),T) \arrow[ur, "\bar{q}_2"] \arrow[rr] & & ((T_0,T),T).
\end{tikzcd}$$

Since α_{T_1} and p_1 are surjective, the cokernel of g is isomorphic to the cokernel of $\bar{q}_2$, and the latter is clearly isomorphic to $\mathrm{Ext}^1_B(\Sigma_T(M),T)$. Note that $\mathrm{Ext}^1_B(Q,T) \simeq \mathrm{Ext}^2_B(\Sigma_T(M),T)$. Now, the desired exact sequence follows from the first commutative diagrams and the snake lemma. □

The following result is an easy consequence of 3.9, which generalizes the usual case for $T = A$.

Corollary 3.10 *An A-module M in Pre$({}_AT)$ is ${}_AT$-reflexive if and only if $Ext^i_B(\Sigma_T(M),T_B) = 0$ for $i = 1,2$.*

As a consequence of Theorem 3.9, we have the following description of the kernel and cokernel of α_M when M is the cokernel of a morphism in $\mathrm{App}({}_AT)$. This may be useful for detecting whether the cokernel of a homomorphism is T-reflexive.

Corollary 3.11 *If $f\colon M \to N$ is homomorphism with M in $Pre({}_AT)$ and N in $App({}_AT)$, then there is a module X_B such that the kernel and cokernel of $\alpha : Cok(f) \longrightarrow ((Cok(f),T),T)$ are isomorphic to $Ext^1_B(X_B,T_B)$ and $Ext^2_B(X_B,T_B)$, respectively.*

Proof. If we can prove, under the above assumption, that the module $\mathrm{Cok}(f)$ lies in $\mathrm{Pre}({}_AT)$, then Theorem 3.9 will imply the corollary immediately.

Take a presentation $T_1 \xrightarrow{d_1} T_0 \xrightarrow{d_0} M \to 0$ for M and an minimal $\mathrm{add}({}_AT)$-presentation $T_1' \xrightarrow{d_1'} T_0' \xrightarrow{d_0'} N \to 0$ for N, where T_i and T_i' are in $\mathrm{add}({}_AT)$. Then, by definition, there are two homomorphisms $h_1 : T_1 \to T_1'$ and $h_0 : T_0 \to T_0'$ such that the following diagram is commutative:

$$\begin{array}{ccccccc} 0 & \longrightarrow & T_1 & \xrightarrow{d_1} & T_0 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_0} & & \\ 0 & \longrightarrow & T_1' & \xrightarrow{d_1'} & T_0' & \longrightarrow & 0. \end{array}$$

In this way we have a map h from the complex $C : 0 \to T_1 \to T_0 \to 0$ to the complex C' : $0 \to T_1' \to T_0' \to 0$, and thus an exact sequence of complexes: $0 \to C' \to \mathrm{Con}(h) \to C \to 0$, where $\mathrm{Con}(h)$ is the mapping cone of h defined by $\mathrm{Con}(h)_i = C_{i-1} \oplus C_i'$ and the differential from $\mathrm{Con}(h)_i$ to $\mathrm{Con}(h)_{i-1}$ is given by $(-d_C, d_{C'} + h)$. Hence it follows from homological algebra that we have an exact sequence of homology groups (see [13, chap. IV, exercise 3, p. 73]):

$$H_1(C') \to H_1(\mathrm{Con}(h)) \to H_0(C) \to H_0(C') \to H_0(\mathrm{Con}(h)) \to 0,$$

where $H_0(C) \simeq M$ and $H_0(C') \simeq N$ and the map $H_0(C) \to H_0(C')$ is f. Thus $H_0(\mathrm{Con}(h)) \simeq \mathrm{Cok}(f)$. Now our desired exact sequence follows from the complex Con(h): $T_0 \oplus T_1' \longrightarrow T_0' \longrightarrow H_0(\mathrm{Con}(h)) \longrightarrow 0$. $\square$

As another consequence of Theorem 3.8 and Theorem 3.9, we can reobtained the second statement of Lemma 2.1(1). The details of the proof are left to the reader.

Dually, we have also an exact sequence involving Tor and Σ_T, which is a generalization of both Lemma 2.1 (2) and a result in [2, proposition 7.1] .

Theorem 3.12 *Suppose M is a module in $Pre({}_AT)$, that is, there is an exact sequence $T_1 \to T_0 \to M \to 0$ with $T_i \in add({}_AT)$, and this defines a module $\Sigma_T(M)$. For each A-module ${}_AX$ there is an exact sequence*

$$0 \to Tor_2^B(\Sigma_T(M),(T,X)) \to (M,T) \otimes_B (T,X) \to (M,X) \to Tor_1^B(\Sigma_T(M),(T,X)) \to 0.$$

Proof. By construction, we have the following exact sequence

$$0 \to (M,T) \to (T_0,T) \to (T_1,T) \to \Sigma_T(M) \to 0.$$

Let Q be the image of the map $(T_0,T) \to (T_1,T)$. Then $\mathrm{Tor}_1^B(Q,(T,X)) \simeq \mathrm{Tor}_2^B(\Sigma_T(M),(T,X))$ and the kernel of the map $Q \otimes_B (T,X) \longrightarrow (T_1,T) \otimes_B (T,X)$ is $\mathrm{Tor}_1^B(\Sigma_T(M),(T,X))$. From the above exact sequence one gets the following exact commutative diagram:

$$\begin{array}{ccccccc} 0 \to \mathrm{Tor}_2^B(\Sigma_T(M),(T,X)) \to & (M,T) \otimes_B (T,X) & \to & (T_0,T) \otimes_B (T,X) & \to & Q \otimes_B (T,X) & \to 0 \\ & \downarrow{\scriptstyle m_1} & & \downarrow{\scriptstyle m_2} & & \downarrow{\scriptstyle m_3'} & \\ 0 \longrightarrow & (M,X) & \longrightarrow & (T_0,X) & \longrightarrow & (T_1,X), & \end{array}$$

where m_1 and m_2 are the multiplication maps, and m_3' is the composition of the map $Q \otimes_B (T, X) \longrightarrow (T_1, T) \otimes_B (T, X)$ with the multiplication map $m_3 : (T_1, T) \otimes_B (T, X) \longrightarrow (T_1, X)$. By Lemma 2.1(2), the maps m_2 and m_3 are isomorphisms. Thus the kernel of m_3' is $\mathrm{Tor}_1^B(\Sigma_T(M), (T, X))$ and the kernel of m_1 is $\mathrm{Tor}_2^B(\Sigma_T(M), (T, X))$. Now our desired exact sequence follows immediately from the snake lemma. $\square$

A similar, but different, version of Theorem 3.8 is the following result.

Proposition 3.13 *Suppose M is a module in $App({}_AT)$. For each A-module ${}_AX$ with $Ext_A^1(X, T) = 0$ there is an exact sequence*

$$0 \to Ext_A^1(X, (\Sigma_T(M), T)) \to (X, T) \otimes_B (T, M) \to (X, M) \to Ext_A^2(X, (\Sigma_T(M), T)) \to 0.$$

Proof. There is a minimal $\mathrm{add}({}_AT)$-presentation $T_1 \xrightarrow{f} T_0 \to M \to 0$ for M and an exact sequence $(T, T_1) \to (T, T_0) \to (T, M) \to 0$. This defines uniquely the module $\Sigma_T(M)$. Let K be the image of f. Then we obtain the following exact commutative diagram:

$$\begin{tikzcd}
(X,T)\otimes_B(T,T_1) \arrow[rr] \arrow[dd, "m_1"'] \arrow[dr] & & (X,T)\otimes_B(T,T_0) \arrow[r] \arrow[dd, "m_0"] & (X,T)\otimes_B(T,M) \arrow[r] \arrow[dd] & 0 \\
& (X,T)\otimes_B(T,K) \arrow[ur] \arrow[dd, "m_3"] & & & \\
(X,T_1) \arrow[rr] \arrow[dr] & & (X,T_0) \arrow[r] & (X,M)\ , & \\
& (X,K) \arrow[ur, hook] & & &
\end{tikzcd}$$

where the multiplication maps m_1 and m_0 are isomorphisms by Lemma 2.1(2). Note that $\mathrm{Ker}(f) \simeq (\Sigma_T(M), T)$ by Lemma 3.1, and that the cokernel of m_3 is isomorphic to the cokernel of the map $(X, T_1) \longrightarrow (X, K)$. It follows from $\mathrm{Ext}^1(X, T) = 0$ that the cokernel of m_3 is $\mathrm{Ext}_A^1(X, (\Sigma_T(M), T))$ and the cokernel of the map $(X, T_0) \to (X, M)$ is $\mathrm{Ext}_A^1(X, K)$ which is isomorphic to $\mathrm{Ext}_A^2(X, (\Sigma_T(M), T))$. Now the desired exact sequence follows from the snake lemma. $\square$

The following is a generalization of Theorem 3.8 in a deferent direction. Consider a long **exact** sequence

$$T_n \xrightarrow{f_n} T_{n-1} \xrightarrow{f_{n-1}} \dots \longrightarrow T_1 \xrightarrow{f_1} T_0 \xrightarrow{f_0} M \longrightarrow 0$$

such that $f_i : T_i \to \mathrm{Im}(f_i)$ is an $\mathrm{add}({}_AT)$-cover. We denote the kernel of f_{i-1} by $\Omega_T^i(M)$ for $i = 1, 2, ..., n$, and $\Omega_T^0(M) = M$.

If we assume that the module ${}_AT$ has no self-extension, that is, $\mathrm{Ext}_A^1(T, T) = 0$, then we can describe also the kernel of a natural morphism $\mathrm{Tor}_i^B(-, (T, M)) \to (\ \mathrm{Ext}_A^i(M, T), -)$. This is shown by the following result.

Theorem 3.14 *Suppose that $Ext_A^1({}_AT, {}_AT) = 0$. Let M be an A-module with a long exact sequence*

$$T_n \xrightarrow{f_n} T_{n-1} \xrightarrow{f_{n-1}} \dots \longrightarrow T_1 \xrightarrow{f_1} T_0 \xrightarrow{f_0} M \longrightarrow 0$$

such that $f_i : T_i \to Im(f_i)$ is an $add({}_AT)$-cover for all $0 \le i \le n$. Then there is an exact sequence:

$$\begin{aligned} 0 \longrightarrow \quad & Ext_B^1(\Sigma_T \Omega_T^i(M), X_B) \quad \longrightarrow \quad Tor_i^B(X_B, (T, M)) \\ & \longrightarrow (Ext_A^1(\Omega_T^{i-1}(M), T), X_B) \longrightarrow Ext_B^2(\Sigma_T \Omega_T^i(M), X_B) \end{aligned}$$

for all $1 \le i \le n-1$. In particular, if $Ext_A^j({}_AT, {}_AT) = 0$ for all $j \ge 1$, we can replace $Ext_A^1((\Omega_T^{i-1}(M), T)$ in the above sequence by $Ext_A^i(M, T)$.

Proof. By assumption, we have a projective resolution of the B-module (T, M):

$$(T, T_n) \longrightarrow (T, T_{n-1}) \longrightarrow ... \longrightarrow (T, T_1) \longrightarrow (T, T_0) \longrightarrow (T, M) \longrightarrow 0.$$

By the definition of Tor_i, the following sequence is exact:

$$0 \longrightarrow \mathrm{Tor}_i^B(X, (T, M)) \longrightarrow X \otimes_B (T, \Omega_T^i(M)) \longrightarrow X \otimes_B (T, T_{i-1}).$$

Since $\mathrm{Ext}_A^1(T, T) = 0$, the exact sequence $0 \to \Omega_T^i(M) \to T_{i-1} \to \Omega_T^{i-1}(M) \to 0$ induces an exact sequence

$$(T_{i-1}, T) \longrightarrow (\Omega_T^i(M), T) \longrightarrow Ext_A^1(\Omega_T^{i-1}(M), T) \longrightarrow 0.$$

Thus we may form the following exact commutative diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & Tor_i^B(X,(T,M)) & \xrightarrow{f} & (Ext^1(\Omega_T^{i-1}(M),T),X_B) & & \\
 & & \downarrow & & \downarrow & & \\
0 \to Ext_B^1(\Sigma_T\Omega_T^i(M),X_B) & \longrightarrow & X \otimes_B (T,\Omega_T^i(M)) & \to & ((\Omega_T^i(M),T),X_B) & \longrightarrow & Ext_B^2(\Sigma_T\Omega_T^i(M),X_B) \to 0 \\
 & & \downarrow & & \downarrow & & \\
 & & X \otimes_B (T,T_{i-1}) & \simeq & ((T_{i-1},T),X_B) & & \\
 & & \downarrow & & & & \\
 & & X \otimes_B (T,\Omega_T^{i-1}(M)) & & & & \\
 & & \downarrow & & & & \\
 & & 0, & & & &
\end{array}$$

which yields our desired result by the well-known snake lemma since the kernel of f is isomorphic to $\mathrm{Ext}_B^1(\Sigma_T\Omega_T^i(M), X_B)$.

Now, if $\mathrm{Ext}_A^j(T, T) = 0$ for all $j \geq 1$, then it follows from the exact sequence $0 \to \Omega_T^{i-2}(M) \longrightarrow T_{i-1} \longrightarrow \Omega_T^{i-1}(M) \longrightarrow 0$ that $\mathrm{Ext}_A^j(\Omega_T^{i-1}(M), T) \simeq \mathrm{Ext}_A^{j+1}(\Omega_T^{i-2}(M), T)$ for $j \geq 1$ and $i \geq 2$. This yields that $\mathrm{Ext}_A^1(\Omega_T^{i-1}(M), T) \simeq \mathrm{Ext}_A^i(M, T)$ for all $i \geq 1$. Thus we have

$$\begin{array}{llll}
0 \longrightarrow & Ext_B^1(\Sigma_T\Omega_T^i(M), X_B) & \longrightarrow & Tor_i^B(X_B, (T, M)) \\
\longrightarrow & (Ext_A^i(M, T), X_B) & \longrightarrow & Ext_B^2(\Sigma_T\Omega_T^i(M), X_B),
\end{array}$$

this finishes the proof. □

Remark. If we take T to be the algebra A, then for any injective right A-module X and positive integer i, we have that proj.dim $(X_A) < i$ if and only if $\mathrm{Hom}_A(\mathrm{Ext}_A^i(M, A), X_A) = 0$ for all M in A-mod. This kind of results is useful for studying k-Gorenstein algebras.

There is another exact sequence involving the T-transpose as the following shows.

Proposition 3.15 *If M is in Pre$({}_AT)$, that is, there is an exact sequence $T_1 \to T_0 \to M \to 0$ with T_i in add$({}_AT)$, then, for any A-module Z, there exists the following exact sequence*

$$0 \to (M, Z) \to (T_0, Z) \to (T_1, Z) \to \Sigma_T(M) \otimes_B (T, Z) \to 0.$$

Proof. Given the exact sequence $T_1 \to T_0 \to M \to 0$, we may get a new exact sequence $(T_0, T) \to (T_1, T) \to \Sigma_T(M) \to 0$ of right B-modules, which yields the following exact commutative diagram:

$$\begin{array}{cccccc}
 & (T_0, T) \otimes_B (T, Z) & \longrightarrow & (T_1, T) \otimes_B (T, Z) & \longrightarrow \Sigma_T(M) \otimes_B (T, Z) \longrightarrow 0 \\
 & \downarrow & & \downarrow & \\
0 \longrightarrow (M, Z) \longrightarrow & (T_0, Z) & \longrightarrow & (T_1, Z), &
\end{array}$$

where the vertical maps are isomorphic by Lemma 2.1. □

As an easy consequence we have the following corollary.

Corollary 3.16 *Let A be an Artin algebra over R. Suppose M is in $Pre(_AT)$ with an exact sequence $T_1 \to T_0 \to M \to 0$ such that T_i in $add(_AT)$. Then, for any A-module Z, we have*

$$l_R(M, Z) - l_R(Hom_B((T, Z), D\Sigma_T(M)) = l_R(T_0, Z) - l_R(T_1, Z),$$

where $l_R(X)$ stands for the length of an R-module X.

In the following we shall prove that there is also an Auslander-Reiten formula for computing Ext-groups. Before we do this, we first prove a result which might be considered as an analogue of the defect of exact sequences.

Proposition 3.17 *Let $\delta : 0 \to X \to Y \to Z \to 0$ be an exact sequence in A-mod such that $0 \to (T, X) \to (T, Y) \to (T, Z) \to 0$ is exact. For an A-module M and B-module N, define $\delta^*(M)$ and $\delta_T(N)$ as follows:*

$$0 \to (M, X) \to (M, Y) \to (M, Z) \to \delta^*(M) \to 0$$

$$0 \to ((T, Z), N) \to ((T, Y), N) \to ((T, X), N) \to \delta_T(N) \to 0.$$

Then the R-lengths of $\delta_T(D\Sigma_T(M))$ and $\delta^*(M)$ are equal for all M in $Pre(_AT)$.

Proof. This proposition follows directly from 3.15. □

Now we have the following generalization of Auslander-Reiten formula [6, VI.4, proposition 4.5].

Theorem 3.18 *For any A-module Z in $Gen(_AT)$ and M in $Pre(_AT)$, we have*

$$l_R(Ext^1_B((T, Z), D\Sigma_T(M))) = l_R((M, Z)/(M, add(_AT), Z)) = l_R(Tor^B_1(\Sigma_T(M), (T, Z))),$$

where $(M, add(_AT), Z)$ stands for the set of all homomorphisms from M to Z which factor through a module in $add(_AT)$.

Proof. Take an $\mathrm{add}(_AT)$-precover of Z, say $f : T_0 \to Z$, and denote by K the kernel of f. Then we have an exact sequence $\delta : 0 \to K \to T_0 \to Z \to 0$ which induces another exact sequence $\delta' : 0 \to (T, K) \to (T, T_0) \to (T, Z) \to 0$. By 3.17, the R-length of $\delta^*(M)$ is the same as that of $\delta_T(D\Sigma_T(M))$. It is clear that the R-length of $\delta^*(M)$ is the same as that of $(M, Z)/(M, \mathrm{add}(_AT), Z)$. On the other hand, by tensoring $\Sigma_T(M)$ to the sequence δ' and using the adjunction we have an exact sequence

$$0 \to ((T, Z), D\Sigma_T(M)) \to ((T, T_0), D\Sigma_T(M)) \to ((T, K), D\Sigma_T(M)) \to D\mathrm{Tor}^B_1(\Sigma_T(M), (T, Z)) \to 0,$$

this shows that the length of $\delta_T(D\Sigma_T(M))$ is the same as that of $D\mathrm{Tor}^B_1(\Sigma_T(M), (T, Z))$. But if we apply $(-, D\Sigma_T(M))$ to the sequence δ', we get that this number is also equal to the R-length of $\mathrm{Ext}^1_B((T, Z), D\Sigma_T(M))$. Thus the theorem has been proved. □

Remark. In the above formula, the second equality holds true even for $M \in \mathrm{Pre}(_AT)$ and arbitrary $Z \in A$-mod. This can be seen from 3.12 because the image of the map $(M, T) \otimes_B (T, Z) \longrightarrow (M, Z)$ is just the morphisms which factor through a module in $\mathrm{add}(_AT)$.

Now let us consider Σ_T with $\mathrm{Ext}^1_A(T, T) = 0$.

Proposition 3.19 *Suppose $Ext^1_A(_AT, {_AT}) = 0$. If $_AM$ is a module such that there is an exact sequence of the form $0 \to T_1 \to T_0 \to M \to 0$ with T_i in $add(_AT)$, then $\Sigma_T(M) \simeq Ext^1_A(M, T)$ as right B-modules.*

Proof. By the assumption, we have an exact sequence $0 \to T_1 \to T_0 \to M \to 0$ with $T_i \in \mathrm{add}(_AT)$. This yields the following exact sequence

$$0 \to (M, T) \longrightarrow (T_0, T) \longrightarrow (T_1, T) \longrightarrow \mathrm{Ext}^1_A(M, T) \to 0$$

since $\mathrm{Ext}^1_A(T_0, T) = 0$. Thus the isomorphism follows. □

The next result tells us a relationship of $\Sigma_T(M)$ and $\mathrm{add}(_AT)$-$\dim(M)$.

Proposition 3.20 *Let M be an arbitrary A-module in App$({}_AT)$. Then Hom$_B(\Sigma_T(M), B_B) = 0$ if add$({}_AT)$-dim$(M) \leq 1$.*

Proof. It is easy to see that add$({}_AT)$-dim$(M) \leq 1$ implies that proj.dim$_B(T, M) \leq 1$. The latter is equivalent to $\mathrm{Hom}_B(D(B), D\mathrm{Tr}_B(T, M)) = 0$. Thus, by 3.6, we have

$$0 = \mathrm{Hom}_B(D(B), D\Sigma_T(M)) \simeq \mathrm{Hom}_B(\Sigma_T(M), B).$$

This is what we want to prove. □

4 Examples

Now let us consider some special cases which show that the previous consideration covers a large variety of important examples. We consider a triple $(A, {}_AT, B)$ with $B = \mathrm{End}({}_AT)$.

(1) The case ${}_AT = Ae$

Suppose that we are given the triple (A, Ae, eAe), where e is an idempotent element in the R-algebra A. In this case we have the following description on the Auslander-Reiten translation.

Proposition 4.1 *For a non-projective A-module M in App(Ae), we have $\Sigma_{Ae}(M) = (Tr(M))e$, that is, $DTr_{eAe}(eM) = D(Tr(M)e)$, where Tr stands for the usual transpose of A-mod.*

Proof. Take an exact sequence of the form $T_1 \to T_0 \to M \to 0$ with $T_i \in \mathrm{add}(Ae)$. Clearly, this is also a projective presentation of M. Note that $A = Ae \oplus A(1-e)$. The statement follows immediately from the construction of the usual transpose Tr. □

As a consequence of Theorem 3.8, we have the following isomorphism.

Corollary 4.2 *For a module M in App(Ae), we have $eM \simeq Hom_{eAe}(Hom_A({}_AM, Ae), eAe)$.*

For a detailed investigation of the case of ${}_AT$ being projective we refer to [3].

(2) The case ${}_AT = {}_A(A/I)_{A/I}$

If I is an ideal in A, then we have a natural $(A, A/I)$-bimodule structure on A/I. In fact, in this case we have a triple $(A, A/I, A/I)$. Note that $\mathrm{App}({}_AT) = A/I\text{-mod} = \{M \in A\text{-mod} \mid IM = 0\}$, thus for an A/I-module M, the relative transpose $\Sigma_T(M)$ is just the usual transpose of the A/I-modules, that is, $D\Sigma_T(M)$ coincides with the Auslander-Reiten translation of the algebra A/I.

(3) ${}_AT$ is a tilting module

Recall that an A-module ${}_AT$ is called a classical tilting module if the projective dimension of T is at most one, $\mathrm{Ext}^1_A(T, T)$ vanishes and the number of the non-isomorphic indecomposable direct summands of T is the number of non-isomorphic simple A-modules. In this case we know that $\mathrm{Gen}({}_AT) = \{M \in A\text{-mod} \mid \mathrm{Ext}^1_A(T, M) = 0\}$. Thus $\mathrm{Gen}({}_AT)$=$\mathrm{App}({}_AT)$, and the classical tilting theory (see, for example, [25, chapt. 4]) is included in our setting.

Let us remark that the classical tilting theory was generalized in different directions by many authors, where ${}_AT$ was usually assumed to have strong homological properties (see [23], [18], [14], [12] and others).

(4) ${}_AT$ is a generator for A-mod

Recall that an A-module T is called a generator for A-mod if $\mathrm{add}({}_AA) \subseteq \mathrm{add}(T)$, and a cogenerator for A-mod if $\mathrm{add}(D(A_A)) \subseteq \mathrm{add}(T)$. If ${}_AT$ is a generator for A-mod, then $\mathrm{App}({}_AT)$ coincides with A-mod, and $\mathrm{End}(T_B) \simeq A$. Note that $D({}_AT_B) \simeq \mathrm{Hom}_A({}_AT_B, D(A_A))$. Moreover, if ${}_AT$ is a generator, then the functor $({}_AT, -)$ preserves injective modules, that is, $({}_AT, {}_AI)$ is an injective B-module for any injective A-module ${}_AI$; and $D(T_B)$ lies in $\mathrm{Bild}({}_AT)$ (for definition, see the next section). Finally, if ${}_AT$ is a cogenerator of A-mod, then every A-module is T-reflexive by Lemma 2.2 (2).

If ${}_AT$ is both a generator and a cogenerator of A-mod, then T_B is a faithful projective and injective right B-module since $D({}_AT_B) \simeq \mathrm{Hom}_A({}_AT, D({}_AA_A))$. In this case we have the following result concerning the dominant dimension. Recall that given a B-module ${}_BX$, we say that ${}_BX$ has dominant dimension greater than or equal to n, denoted by dom.dim $({}_BX) \geq n$, if there is an exact sequence $0 \to {}_BX \to I_1 \to I_2 \to \ldots \to I_n$ such that I_i is projective-injective for all $1 \leq i \leq n$. Since $D(T_B)$ is a faithful projective-injective module, we know that each I_i is in $\mathrm{add}(D(T_B))$.

Proposition 4.3 *Let ${}_AT$ be a generator-cogenerator for A-mod and ${}_BX$ a B-module. Then the following are equivalent:*
(1) *dom.dim*$({}_BX) \geq 2$;
(2) ${}_BX \simeq Hom_A({}_AT_B, T \otimes_B X)$.

Proof. (1) $\Rightarrow$ (2): There is a natural morphism $\beta_X : X \longrightarrow ({}_AT_B, T \otimes_B X)$ given by $(t)[(x)\beta)] = t \otimes x$ for all $t \in T$ and $x \in X$. For $X = {}_BB$ the natural map β_X is an isomorphism, in particular, for $X = D(T_B)$, a projective B-module, the map β_X is an isomorphism. Hence β_X is an isomorphism for $X \in \mathrm{add}(D(T_B))$. Since dom.dim ${}_BX \geq 2$, we have an exact sequence $0 \to X \to I_0 \to I_1$ with I_i projective-injective. Now (2) follows from the following commutative diagram:

$$\begin{array}{ccccccc}
0 \longrightarrow & X & \longrightarrow & I_0 & \longrightarrow & I_1 \\
& \big\downarrow{\scriptstyle \beta_X} & & \big\downarrow{\scriptstyle \beta_{I_0}} & & \big\downarrow{\scriptstyle \beta_{I_1}} \\
0 \longrightarrow & (T, T \otimes_B X) & \longrightarrow & (T, T \otimes_B I_0) & \longrightarrow & (T, T \otimes_B I_1).
\end{array}$$

(2) $\Rightarrow$ (1): Let $0 \to X \to I_0(X) \to I_1(X) \to \ldots$ be a minimal injective resolution of ${}_BX$. Note that for each injective A-module ${}_AQ$, the left B-module $\mathrm{Hom}_A({}_AT, {}_AQ)$ is a direct summand of $({}_AT_B, D(A_A))^n$ for some n and $({}_AT_B, D(A_A)) \simeq D(T_B)$. Let $0 \to T \otimes_B X \to Q_0 \to Q_1 \to \ldots$ be a minimal injective resolution of the A-module $T \otimes_B X$. Then, since ${}_BX \simeq ({}_AT_B, T \otimes_B X)$, we know that $I_i(X)$ is a direct summand of $({}_AT, Q_i)$ which belongs to $\mathrm{add}(D(T_B))$. This implies that $\mathrm{dom.dim}({}_BX) \geq 2$. $\square$

5 The functor $\mathrm{Hom}_A({}_AT, -)$

In this section we investigate subcategories of $\mathrm{App}({}_AT)$ and the action of the functor $\mathrm{Hom}_A(T, -)$. First let us describe the image of the functor $\mathrm{Hom}_A(T, -)$. For each B-module ${}_BX$ we have denoted by $\beta_X : X \longrightarrow ({}_AT_B, T \otimes_B X)$ the natural map given by $(t)[(x)\beta)] = t \otimes x$ for all $t \in T$ and $x \in X$. We define

$$\mathrm{Bild}({}_AT) := \{X \in B\text{-mod} \mid \beta_X \text{ is an isomorphism}\}.$$

Note that if T is a generator-cogenerator for A-mod, then $\mathrm{Bild}({}_AT)$ is precisely the B-modules of dominant dimension at least 2 by Proposition 4.3.

Proposition 5.1 *For an arbitrary ${}_AT$, the functor $Hom_A({}_AT_B, -) : \mathrm{App}({}_AT) \longrightarrow \mathrm{Bild}(T_B)$ is an equivalence, its inverse is $T \otimes_B -$.*

Proof. First, notice that the image of $\mathrm{Hom}_A({}_AT_B, -)$ on $\mathrm{App}({}_AT)$ lies in $\mathrm{Bild}(T_B)$ since $\beta_{(T,M)}(T, e_M) = id_{(T,M)}$ for $M \in \mathrm{App}({}_AT)$. Second, we show that if β_X is an isomorphism, then X belongs to the image of $\mathrm{Hom}_A(T, -)$. In fact, since $T \otimes_B X$ is always generated by ${}_AT$, we have a surjective $\mathrm{add}({}_AT)$-cover $f : T_0 \longrightarrow T \otimes_B X$. Put $K = \mathrm{Ker}(f)$, we claim that K is generated by ${}_AT$. Clearly, we have an exact sequence $0 \to (T, K) \to (T, T_0) \to (T, T \otimes_B X) \to 0$. Consider the following exact commutative diagram:

$$\begin{array}{ccccccccc}
& T \otimes_B (T, K) & \longrightarrow & T \otimes_B (T, T_0) & \longrightarrow & T \otimes_B (T, T \otimes_B X) & \longrightarrow & 0 \\
& \big\downarrow{\scriptstyle e_K} & & \big\downarrow{\scriptstyle e_{T_0}} & & \big\downarrow{\scriptstyle e_{T \otimes X}} & & \\
0 \longrightarrow & K & \longrightarrow & T_0 & \longrightarrow & T \otimes_B X & \longrightarrow & 0.
\end{array}$$

Since $1 \otimes \beta_X$ is an isomorphism and $(1 \otimes \beta)e_{T\otimes X} = id_{T\otimes X}$, we see that $e_{T\otimes X}$ is injective. Now the snake lemma shows that e_K is surjective. The module $T \otimes_B (T, K)$ is generated by ${}_AT$, and therefore K is generated by ${}_AT$. This implies that $T \otimes_B X$ is in $\mathrm{App}({}_AT)$ and that X, which is isomorphic to $(T, T \otimes_B X)$, lies in the image of $\mathrm{Hom}_A(T, -)$. By Lemma 2.1, the equivalence of the two categories follows. $\square$

Remark. In [16] many ring theoretic aspects are discussed and there is also a version of Proposition 5.1 (see [16, chapt. 6]).

Now let us deduce some consequences of 5.1. As an easy consequence we re-obtain a result of Auslander in [3, proposition 8.3].

Corollary 5.2 *If ${}_AT$ is a generator for A-mod, then* (1) *A-mod can be regarded as a full subcategory of B-mod. In particular, if B is representation-finite, then A is representation-finite.* (2) *$End({}_AV) \simeq End({}_BHom_A(T, V))$ for any A-module V. In particular, the global dimensions of $End({}_AV)$ and $End({}_BHom_A(T, V))$ are equal.*

Proof. Since T is a generator, we know that $\mathrm{App}({}_AT) = A$-mod. Thus, by Lemma 2.1(2), the evaluation map e_M is bijective for all A-module M. Thus $(T \otimes_B -) \circ \mathrm{Hom}_A(T, -)$ is an identity functor on A-mod. This implies the corollary. $\square$

Corollary 5.3 *If ${}_AT$ is a projective A-module, then the functor in 5.1 is an equivalence from $Pre({}_AT) = App({}_AT)$ to B-mod.*

Proof. In fact, it suffices to show that under the assumption each β_X is an isomorphism for all B-module X. Now we take a projective presentation of ${}_BX$:

$$({}_AT, T_1) \longrightarrow ({}_AT, T_0) \longrightarrow {}_BX \longrightarrow 0$$

with $T_i \in \mathrm{add}({}_AT)$. From this we obtain another exact sequence

$$T \otimes_B ({}_AT, T_1) \longrightarrow T \otimes_B ({}_AT, T_0) \longrightarrow T \otimes_B X \longrightarrow 0.$$

Since ${}_AT$ is projective, we have the following exact commutative diagram:

$$\begin{array}{ccccccc}
(T, T_1) & \longrightarrow & (T, T_0) & \longrightarrow & {}_BX & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
(T, T \otimes_B (T, T_1)) & \longrightarrow & (T, T \otimes_B (T, T_0)) & \longrightarrow & (T, T \otimes_B X) & \longrightarrow & 0,
\end{array}$$

where T_i is isomorphic to $T \otimes_B (T, T_i)$ by the evaluation map in Lemma 2.1. Hence the first two vertical maps are isomorphisms. This means that β_X is an isomorphism. $\square$

The following result is a consequence of 4.3 and 5.1.

Theorem 5.4 *If ${}_AT$ is a generator-cogenerator for A-mod, then A-mod is equivalent to the full subcategory of B-mod whose objects are B-modules with dominant dimension at least 2.*

To understand the behavior of the functor $(T, -)$ in 5.1, we need to know information on the map β_X. In [16] there is a description of β_X in terms of submodules of ${}_BX$. However, it would be interesting to have a characterization of the kernel and cokernel of β_X in terms of long exact sequence similar to 3.9. As a consequence of 2.6 we have the following description of the kernel and cokernel of β_X in a special case.

Lemma 5.5 *Suppose $Ext^1_A(T, T) = 0 = Ext^2_A(T, T)$. If ${}_BX$ is a B-module with proj.dim ${}_BX \leq 1$, then we have the following exact sequence:*

$$0 \to Ext^1_A({}_AT, Tor_1^B(T_B, {}_BX)) \to X \longrightarrow (T, T \otimes_B X) \to Ext^2_A({}_AT, Tor_1^B(T_B, {}_BX)) \longrightarrow 0. \square$$

As we have seen, the category $\mathrm{App}({}_AT)$ could be very large, this depends on the module ${}_AT$. Now let us consider certain subcategories of $\mathrm{App}({}_AT)$ which may lead to an understanding of the whole category $\mathrm{App}({}_AT)$.

We say that an A-module M has a strong $\mathrm{add}({}_AT)$-resolution if there is an exact sequence

$$(*) \qquad \cdots \to T_n \xrightarrow{f_n} T_{n-1} \to \cdots \to T_1 \xrightarrow{f_1} T_0 \xrightarrow{f_0} M \to 0$$

with $T_i \in \mathrm{add}({}_AT)$, such that the canonical map $T_i \to \mathrm{Im}(f_i)$ is an $\mathrm{add}({}_AT)$-cover for all i. If $T_j = 0$ for all $j > n$ and $T_n \neq 0$, we call n the length of the strong $\mathrm{add}({}_AT)$-resolution.

We define

$$\mathrm{App}({}_AT)_n := \{M \in A\text{-mod} \mid M \text{ has a strong } \mathrm{add}({}_AT)\text{-resolution of length at most } n\},$$

$$\mathrm{Bild}({}_AT)_n := \{{}_BX \in \mathrm{Bild}({}_AT) \mid \mathrm{proj.dim}({}_BX) \leq n \text{ and } \mathrm{Tor}_i^B(T, X) = 0 \text{ for all } i \geq 1\}.$$

Note that in the definition of $\mathrm{add}({}_AT)$-dimension we do not require that the sequence $(*)$ is exact. So, in general, the class of all modules with $\mathrm{add}({}_AT)$-dimension at most n is not equal to $\mathrm{App}({}_AT)_n$, but contains $\mathrm{App}({}_AT)_n$.

Clearly, $\mathrm{App}({}_AT)_0 = \mathrm{add}({}_AT)$, and $\mathrm{App}({}_AT)_n \subseteq \mathrm{App}({}_AT)_{n+1}$. Note that if $M \in \mathrm{App}({}_AT)_n$, then $\mathrm{proj.dim}_B({}_AT, {}_AM) \leq n$. The following is a partial converse of this statement which gives another description of modules in $\mathrm{App}({}_AT)_n$.

Proposition 5.6 *The following are equivalent for a module ${}_AM$:*
(1) $M \in \mathrm{App}({}_AT)_n$,
(2) $M \simeq T \otimes_B (T, M)$, *$\mathrm{Tor}_i^B(T_B, (T, M)) = 0$ for all $1 \leq i \leq n$ and $\mathrm{proj.dim}_B(T, M) \leq n$.*

Proof. Suppose we have (1). Then $M \simeq T \otimes_B (T, M)$ by 2.1. Since M has a strong $\mathrm{add}({}_AT)$-resolution of length at most n, say $0 \to T_n \xrightarrow{f_n} T_{n-1} \to \cdots \to T_1 \xrightarrow{f_1} T_0 \xrightarrow{f_0} M \to 0$ with T_i in $\mathrm{add}({}_AT)$, this gives us a projective resolution $0 \to (T, T_n) \to ... \to (T, T_1) \to (T, T_0) \to (T, M) \to 0$ of the B-module (T, M). Furthermore, $\Omega_B^i(T, M) = (T, \Omega_T^i(M))$, where $\Omega_T^i(M)$ is the image of f_i. We have the following exact commutative diagram:

$$\begin{array}{ccccccc}
 & & 0 & \longrightarrow & \Omega_T^i(M) & \longrightarrow & T_{i-1} \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Tor}_1^B(T, (T, \Omega_T^i(M))) & \longrightarrow & T \otimes_B (T, \Omega_T^i(M)) & \longrightarrow & T \otimes_B (T, T_{i-1}).
\end{array}$$

Note that the last two vertical maps are isomorphisms by Lemma 2.1. Hence $\mathrm{Tor}_1^B(T, (T, \Omega_T^i(M))) = 0$ and $\mathrm{Tor}_i^B(T_B, (T, M)) = \mathrm{Tor}_1^B(T, \Omega_B^{i-1}(T, M)) = \mathrm{Tor}_1^B(T, (T, \Omega_T^{i-1}(M))) = 0$ for all $1 \leq i \leq n$. Thus (2) holds.

Now assume (2). As in the proof of 2.1(3), there is an exact sequence $0 \to K \to T_0 \to M \to 0$ which provides a projective cover of the B-module (T, M) with $\Omega_B(T, M) = (T, K)$. Since $\mathrm{Tor}_i^B(T, (T, M)) = 0$ for $1 \leq i \leq n$, we can prove that $K \in \mathrm{App}({}_AT)$. If $n > 1$, we can proceed the above argument and show that there is an exact sequence

$$T_n \longrightarrow T_{n-1} \to \cdots \to T_1 \longrightarrow T_0 \longrightarrow M \to 0$$

with $T_i \in \mathrm{add}({}_AT)$, such that T_i is an $\mathrm{add}({}_AT)$-cover of the image K_i of the map $T_i \longrightarrow T_{i-1}$. Note that $K_i \in \mathrm{App}({}_AT)$ for $0 \leq i \leq n$. Now let N be the kernel of $T_n \longrightarrow T_{n-1}$. Then we have an exact sequence of B-modules:

$$0 \to (T, N) \to (T, T_n) \to ... \to (T, T_1) \to (T, T_0) \to (T, M) \to 0.$$

But $\mathrm{proj.dim}_B(T, M) \leq n$ implies that $(T, N) = 0$. Since K_n lies in $\mathrm{App}({}_AT)$, we know that N is generated by T. Thus $N = 0$, and therefore M belongs to $\mathrm{App}({}_AT)_n$. $\square$

As a consequence of 5.6 and 5.1, we have the following corollary.

Corollary 5.7 *Under the functor $(T, -)$ the category App$({}_AT)_n$ and Bild$({}_AT)_n$ are equivalent for all $n \geq 0$.*

Proof. It suffices to demonstrate that any ${}_BX$ in $\mathrm{Bild}({}_AT)_n$ has a pre-image in $\mathrm{App}({}_AT)_n$. Since $\mathrm{proj.dim}_B X \leq n$, we have a complex of the form $T_n \longrightarrow T_{n-1} \to \cdots \to T_1 \longrightarrow T_0$ with $T_i \in \mathrm{add}({}_AT)$ such that $0 \to (T, T_n) \to \ldots \to (T, T_1) \to (T, T_0) \to {}_BX \to 0$ is a minimal projective resolution of X. If we tensor this sequence by T, then we get a new exact sequence $0 \to T \otimes_B (T, T_n) \to \ldots \to T \otimes_B (T, T_1) \to T \otimes_B (T, T_0) \to T \otimes_B X \to 0$, here the exactness follows from the condition $\mathrm{Tor}_i^B(T, X) = 0$ for all $1 \leq i \leq n$. By Lemma 2.1(3), this exact sequence is isomorphic to

$$0 \to T_n \to T_{n-1} \to \cdots \to T_1 \to T_0 \to T \otimes_B X \to 0.$$

Hence the complex $0 \to T_n \longrightarrow T_{n-1} \to \cdots \to T_1 \longrightarrow T_0$ is exact. Since $X \simeq (T, T \otimes_B X)$, the module $T \otimes_B X$ is in $\mathrm{App}({}_AT)_n$. $\square$

From the proofs of the above results, we can easily get the following slightly general result.

Theorem 5.8 *The subcategory of A-mod consisting of modules with strong add$({}_AT)$-resolutions is equivalent to the subcategory of B-mod consisting of those B-modules X in Bild$({}_AT)$ with Tor$_i^B(T, X) = 0$ for all $i \geq 1$.*

6 Homological finiteness

In this section we shall compare the homologically finite subcategories in A-mod with that in B-mod. The natural bridge between the two categories is the adjoint pair of functors $T \otimes_B -$ and $({}_AT, -)$. First we generalizes a result of Auslander-Reiten on adjoint functors, and then we turn to a discussion of our subcategories in A-mod and in B-mod.

Recall that a full subcategory $\mathcal{C}'$ in a category $\mathcal{C}$ is called contravariantly finite in $\mathcal{C}$ if each object $C \in \mathcal{C}$ has a $\mathcal{C}'$-precover. Dually, a full subcategory $\mathcal{C}'$ in a category $\mathcal{C}$ is called covariantly finite in $\mathcal{C}$ if each object $C \in \mathcal{C}$ has a $\mathcal{C}'$-preenvelope. A full subcategory $\mathcal{C}'$ of $\mathcal{C}$ is called functorially finite in $\mathcal{C}$ if it is both contravariantly finite and covariantly finite in $\mathcal{C}$.

Let us start with the following lemma which generalizes a result in [8, proposition 1.2]. For completeness we include here a short proof which is essentially the same as that in [8].

Lemma 6.1 *Suppose $\mathcal{C}$ and $\mathcal{D}$ are categories and $F : \mathcal{C} \longrightarrow \mathcal{D}$ and $G : \mathcal{D} \longrightarrow \mathcal{C}$ an adjoint pair of functors with F a left adjoint and G a right adjoint. Then:*

(1) *If a full subcategory $\mathcal{C}'$ of $\mathcal{C}$ is contravariantly finite in $\mathcal{C}$, then the image $F(\mathcal{C}')$ of $\mathcal{C}'$ under F is contravariantly finite in $\mathcal{D}$.*

(2) *If a full subcategory $\mathcal{D}'$ of $\mathcal{D}$ is covariantly finite in $\mathcal{D}$, then the image $G(\mathcal{D}')$ of $\mathcal{D}'$ under G is covariantly finite in $\mathcal{C}$.*

Proof. (1) Since (F, G) is an adjoint pair of functors, we have that $\mathrm{Hom}_{\mathcal{D}}(F(C), D) \overset{\eta}{\simeq} \mathrm{Hom}_{\mathcal{C}}(C, G(D))$ for all $C \in \mathcal{C}$ and $D \in \mathcal{D}$. It is well-known (for example, see [22, proposition 1.1, p.119] that given a morphism $g : F(C) \longrightarrow D$, if $h : C \longrightarrow G(D)$ is the morphism corresponding to g under the map η, and if η_D is the map corresponding to $id_{G(D)}$, then $g = (Fh)\eta_D$. Let D be in $\mathcal{D}$. Then $G(D) \in \mathcal{C}$. Since $\mathcal{C}'$ is contravariantly finite in $\mathcal{C}$, we have a morphism $h : C' \longrightarrow G(D)$ with $C' \in \mathcal{C}'$ such that the induced map $(-, h)$ is surjective on $\mathcal{C}'$. Let $g : F(C') \longrightarrow D$ be the morphism which corresponds to h under η. We claim that g has the property that for any $f : F(X) \longrightarrow D$ with $X \in \mathcal{C}'$ there is a morphism $f' : F(X) \longrightarrow F(C')$ such that $f = f'g$. In fact, given such an f, we have a corresponding morphism $\bar{f} : X \longrightarrow G(D)$ with $f = (F\bar{f})\eta_D$. Since X is in $\mathcal{C}'$, there exists a morphism $j : X \longrightarrow C'$ such that $\bar{f} = jh$. This implies that $F\bar{f} = (Fj)(Fh)$ and $f = (F\bar{f})(Fh)\eta_D = (Fj)g$.

(2) is proved dually. $\square$

Note that if we take $\mathcal{C}'$ to be the category $\mathcal{C}$ and $\mathcal{D}'$ to be the category $\mathcal{D}$, then we get [8, proposition 1.2]. As a direct consequence of 6.1 we have the following proposition.

Proposition 6.2 (1) *If T is a generator for A-mod, then Bild$({}_AT)$ is covariantly finite in B-mod.*

(2) *Let e be an idempotent in A. If $\mathcal{C}$ is a contravariantly (respectively, covariantly) finite subcategory in A-mod, then $e\mathcal{C} := \{eX \mid X \in \mathcal{C}\}$ is contravariantly (respectively, covariantly) finite in eAe-mod. In particular, if a full subcategory $\mathcal{C}$ of A-mod is functorially finite in A-mod, then so is $e\mathcal{C}$ in eAe-mod.*

Proof. Since $F := T \otimes_B - : B\text{-mod} \longrightarrow A\text{-mod}$ and $G := (T, -) : A\text{-mod} \longrightarrow B\text{-mod}$ form an adjoint pair (F, G) of functors, (1) follows immediately from 6.1. To get (2), we note that $eA \otimes_A -$ and $\mathrm{Hom}_{eAe}(eA, -)$ are an adjoint pair. This implies that $e\mathcal{C}$ is contravariantly finite in eAe-mod if so is $\mathcal{C}$ in A-mod. Note also that we have another adjoint pair $(Ae \otimes_{eAe} -, \mathrm{Hom}_A(Ae, -))$, this means that $e\mathcal{C}$ is covariantly finite in eAe-mod if $\mathcal{C}$ is covariantly finite in A-mod. $\square$

The above result can be applied to standardly stratified algebras. For the unexplained definitions on standardly stratified algebras we refer the reader to [30] or the references therein. If A is a standardly stratified algebra over a field k with standard modules $\Delta(i)$, $1 \leq i \leq n$, then it is well-known that the subcategory $\mathcal{F}(\Delta)$ of all Δ-good modules is functorially finite in A-mod, thus for *arbitrary* idempotent $e \in A$ the category $e\mathcal{F}(\Delta)$ is functorially finite in eAe-mod, and therefore has almost split sequences. Note that in general we cannot have $e\mathcal{F}(\Delta) = \mathcal{F}(e\Delta)$.

The following is another application of 6.1.

Proposition 6.3 (1) *If App$({}_AT)_n$ is covariantly finite in A-mod, then Bild$({}_AT)_n$ is covariantly finite in A-mod.*

(2) *If Bild$({}_AT)$ is contravariantly finite in B-mod, then App$({}_AT)$ is contravariantly finite in A-mod.*

Combining 5.8 with 6.1, we have the following corollary.

Corollary 6.4 *If the subcategory of A-mod consisting of A-modules which have strong add$({}_AT)$-resolutions is covariantly finite in A-mod, then the subcategory $\{{}_BX \in Bild({}_AT) \mid Tor_i^B(T, X) = 0 \; for \; i \geq 1\}$ is covariantly finite in B-mod.*

Note that Bild$({}_AT)$ contains projective B-modules, and is closed under direct sums and direct summands. In the following we point out that sometimes it is closed under extensions. Note that if ${}_AT$ is a generator for A-mod, then T_B is a projective right B-module.

Lemma 6.5 *Let $0 \to X \to Y \to Z \to 0$ be an exact sequence in B-mod. If X and Z are in Bild$({}_AT)$ and if $Tor_i^B(T_B, {}_BZ) = 0$ for all $i \geq 1$, then $Y \in$ Bild$({}_AT)$. In particular, if ${}_AT$ is a generator in A-mod, then Bild$({}_AT)$ contains all projective B-modules, and is closed under extensions and kernels of surjective homomorphisms between modules in Bild$({}_AT)$.*

By exploiting Γ_T and Σ_T, we can also get homologically finite subcategories.

Proposition 6.6 *Let $\mathcal{C}$ be an additive full subcategory in A-mod. If $\mathcal{C}$ is covariantly finite in A-mod, then add$(\Gamma_T(\mathcal{C}) \cup T_B)$ is contravriantly finite in mod-B, where add$(\Gamma_T(\mathcal{C}))$:= add$\{\Gamma_T(C) \mid C \in \mathcal{C}\}$.*

Proof. If $\mathcal{C}$ is covariantly finite in A-mod, then $\mathrm{Tr}_A(\mathcal{C}) \cup A_A$ is contravariantly finite in mod-A by [10, proposition 7.2, p.453]. Since $\Gamma_T(C) = \mathrm{Tr}_A(C) \otimes_A T$, we have that add$((\Gamma_T(\mathcal{C}) \cup T_B)$ is contravariantly finite in mod-B by 6.1. $\square$

Similarly, we get the following result.

Proposition 6.7 *Let $\mathcal{C}$ be an additive full subcategory in App$({}_AT)$. If $\mathcal{C}$ is covariantly finite in A-mod, then add$(\Sigma_T(\mathcal{C}))$ is contravriantly finite in mod-B, where add$(\Sigma_T(\mathcal{C}))$:= add$\{\Sigma_T(C) \mid C \in \mathcal{C}\}$.*

Proof. If $\mathcal{C}$ is covariantly finite in A-mod, then $(T, \mathcal{C})$ is covariantly finite in B-mod by 6.1. It follows from [10] that $\mathrm{Tr}_B(T, \mathcal{C})$ is contravariantly finite in mod-B. This implies that $\mathrm{add}(\Sigma_T(\mathcal{C}))$ is contravariantly finite in mod-B by 3.6. $\square$

In the following we shall discuss the projective dimensions of modules in A-mod and their related modules in B-mod. This will be related to finitistic dimensions of A and B.

Recall that the finitistic dimension of an Artin algebra A, denoted by fin.dim(A), is by definition the supremum of the projective dimensions of those modules in A-mod which have finite projective dimension. Let us denote by $\mathcal{P}^{<\infty}(A)$ the full subcategory of A-mod whose objects are modules of finite projective dimension. Then $\mathrm{fin.dim}(A) = \sup\{\mathrm{proj.dim}({}_AM) \mid M \in \mathcal{P}^{<\infty}(A)\}$. The next result may be viewed as a similar formulation of estimation of the global dimensions in tilting theory.

Proposition 6.8 *If ${}_AT$ is a generator for A-mod such that proj.dim$({}_AT) = m < \infty$ and gl.dim$(B) = n < \infty$, then gl.dim$(A) \leq n + m < \infty$.*

Proof. Suppose that ${}_AT$ is a generator for A-mod such that $\mathrm{gl.dim}(B) = n$ and $\mathrm{proj.dim}({}_AT) = m$. Put $P = (T, A)$. Then P is a projective B-module and $\Lambda := \mathrm{End}({}_BP) \simeq A$. Suppose that M is a Λ-module. Let $({}_BP, {}_BP_1) \to ({}_BP, {}_BP_0) \to M \to 0$ be a minimal projective presentation of the Λ-module M, which is induced from a homomorphism $f : {}_BP_1 \to {}_BP_0$ with $P_i \in \mathrm{add}({}_BP)$. Since $n = \mathrm{gl.dim}(B)$ is finite, there is a projective resolution of the B-module $\mathrm{Cok}(f)$:

$$0 \to Q_n \to Q_{n-1} \to \cdots \to Q_2 \to P_1 \to P_0 \to \mathrm{Cok}(f) \to 0,$$

where all Q_j are projective B-modules, but not necessarily in $\mathrm{add}({}_BP)$. This induces the following exact sequence:

$$0 \to ({}_BP, Q_n) \to ({}_BP, Q_{n-1}) \to \cdots \to ({}_BP, Q_2) \to ({}_BP, P_1) \to ({}_BP, P_0) \to {}_\Lambda M \to 0$$

because ${}_BP$ is projective. We claim that $\mathrm{proj.dim}_\Lambda({}_BP, {}_BB) \leq m$. In fact, we have

$$\mathrm{Hom}_B(P, {}_BB) \simeq \mathrm{Hom}_B((T, A), (T, T)) \overset{2.2(1)}{\simeq} \mathrm{Hom}_A(A, T) \simeq {}_AT.$$

This yields the promised claim. Since each indecomposable direct summand of (P, Q_j) is isomorphic to a direct summand of (P, B), we know that $\mathrm{proj.dim}_A(P, Q_j) \leq m$. Hence $\mathrm{proj.dim}({}_AM) \leq 2 + \mathrm{proj.dim}(\Omega_A^2(M)) \leq 2 + (n - 2 + m) = n + m$. Thus $\mathrm{gl.dim}(A) \leq n + m$. $\square$

Proposition 6.9 *Let ${}^\perp T$ be the subcategory of A-mod whose objects are the X such that $\mathrm{Ext}_A^i(X, T) = 0$ for all $i \geq 1$. If $m := \sup\{\mathrm{proj.dim}({}_AX) \mid X \in \mathcal{P}^{<\infty}(A) \cap {}^\perp T\}$ is finite, then fin.dim$(A) \leq m +$ inj.dim${}_AT$.*

Proof. Suppose $\mathrm{inj.dim}_AT = n$. Let ${}_AX$ be an A-module of finite projective dimension n, and let $\Omega^i(X)$ be the i-th syzygy of X. Then $\mathrm{Ext}_A^i(\Omega^n(X), T) = \mathrm{Ext}_A^{n+i}(X, T) = 0$ for all $i \geq 1$. This implies that $\Omega^n(X)$ lies in $\mathcal{P}^{<\infty}(A) \cap {}^\perp T$. Thus $\mathrm{proj.dim}(\Omega^n(X)) \leq m$ and $\mathrm{proj.dim}(X) \leq n + m$. Thus $\mathrm{fin.dim}(A) \leq n + m$. $\square$

Corollary 6.10 *If $\mathcal{P}^{<\infty}(A) \cap {}^\perp T$ is contravariantly finite in A-mod, then fin.dim$(A) \leq m +$ inj.dim${}_AT$, where $m := \sup\{\mathrm{proj.dim}({}_AX) \mid X \in \mathcal{P}^{<\infty}(A) \cap {}^\perp T\}$.*

Proof. Note that the category $\mathcal{P}^{<\infty}(A) \cap {}^\perp T$ is a resolving subcategory of A-mod (for the definition, see [7], for example). Thus under our assumption the above m is a finite number (see [7, corollary 3.9]), and therefore our corollary follows from Proposition 6.9. $\square$

7 Representation dimensions

In this section we shall present an application of our consideration in the previous sections to the representation dimension. As a main result of this section we show that given a projective module P with certain restrictions, the representation dimension of $\mathrm{End}(_AP)$ can not exceed that of A. Thus we solve partially a problem proposed by Auslander thirty years ago in [4]: given an Artin algebra A and a projective A-module P, is rep.dim $\mathrm{End}(_AP) \leq$ rep.dim(A)?

Given an Artin algebra A, Auslander defined the representation dimension of A, denoted by rep.dim(A), as follows:

$$\begin{aligned}\text{rep.dim}(A) = \quad & \inf\{\text{gl.dim}(\Lambda) \mid \Lambda \text{ is an Artin algebra with dom.dim}(\Lambda) \geq 2 \text{ and End}(_\Lambda T) \text{ is} \\ & \qquad \text{Morita equivalent to } A, \text{where } T \text{ is the maximal injective summand of } \Lambda\}.\end{aligned}$$

As was pointed in [4], this is equivalent to

$$\text{rep.dim}(A) = \inf\{\ \text{gl.dim End}(_AM) \mid M \text{ is a generator-cogenerator for } A\text{-mod}\}.$$

As is known, the representation dimension is closely related to coherent functors. The simple coherent functor is described by the almost split sequences. In [19] Hartshorne used coherent functors to solve problems in algebraic space curves. Recently, there are some advances on representation dimension (see [27], [28], [29] and [21]). Now let us consider the following type of question:

Suppose two algebras are in good relationship, how are their representation dimensions related to each other ?

In this direction let us first prove the following result.

Proposition 7.1 *Let A and B be two Artin algebras. Suppose $F : A$-mod $\longrightarrow B$-mod is a fully exact functor and preserves direct sums. Moreover, suppose that each indecomposable B-module is a direct summand of a module FX for some $X \in A$-mod. If there is a generator-cogenerator M for A-mod such that FM is a generator-cogenerator for B-mod and rep.dim$(A) =$ gl.dim$(End(M))$, then rep.dim$(B) \leq$ rep.dim(A).*

Proof. Let M be a generator-cogenerator for A-mod such that $n =$ rep.dim$(A) =$ gl.dim(End$(_AM)) < \infty$. Then

(1) If $f : M_0 \longrightarrow X$ is an add(M)-cover of an A-module X with $M_0 \in$ add(M), then $Ff : FM \longrightarrow FX$ is an add(FM)-cover of the B-module FX with $FM_0 \in$ add(FM).

In fact, given an indecomposable B-module Y in add(FM) and a homomorphism $g : Y \longrightarrow FX$, we have to find a homomorphism $h : Y \longrightarrow FM_0$ such that $g = h(Ff)$. We may assume that $FM = Y \oplus Y'$. Let p and q be the canonical projection from FM to Y and the canonical inclusion of Y into FM, respectively. Since the functor F is full, there is a homomorphism $g' : M \longrightarrow X$ such that $Fg' = pg$. Since f is a cover, we have a homomorphism $h' : M \longrightarrow M_0$ such that $g' = h'f$. This implies that $g = qpg = q(Fg') = q(Fh')(Ff)$. Thus we have (1).

(2) If Y is an arbitrary B-module, then proj.dim$_{\mathrm{End}(_BFM)}(FM, Y) \leq n-2$.

To see this, let X be an A-module such that Y is a direct summand of FX. Since gl.dim(End$(_AM)) = n$, there is an exact sequence

$$0 \to M_{n-2} \stackrel{d_{n-2}}{\longrightarrow} \cdots \longrightarrow M_0 \stackrel{d_0}{\longrightarrow} X \longrightarrow 0$$

with $M_i \in$ add(M) such that the induced sequence $0 \longrightarrow (M, M_{n-2}) \longrightarrow \cdots \longrightarrow (M, M_0) \longrightarrow (M, X) \longrightarrow 0$ is exact, this means that each map $M_j \longrightarrow$Ker(d_{i-1}) is an add(M)-cover of Ker(d_{i-1}). Since the functor F is exact, we have another exact sequence

$$0 \to FM_{n-2} \longrightarrow \cdots \longrightarrow FM_0 \longrightarrow FX \longrightarrow 0$$

with $FM_i \in$ add(FM). By (1), each map $FM_i \longrightarrow F$Ker(d_{i-1})=Ker(Fd_{i-1}) is an add(FM)-cover of FKer(d_{i-1}). Thus we have an exact sequence

$$0 \longrightarrow (FM, FM_{n-2}) \longrightarrow \cdots \longrightarrow (FM, FM_0) \longrightarrow (FM, FX) \longrightarrow 0.$$

This implies that the projective dimension of the $\mathrm{End}({}_BFM)$-module (FM, FX) is at most $n-2$, and hence the direct summand (FM, Y) of (FM, FX) has projective dimension at most $n-2$. So we have proved (2).

Now let S be a simple $\mathrm{End}({}_BFM)$-module. Then there is a B-homomorphism $g : Y_1 \longrightarrow Y_0$ with $Y_i \in \mathrm{add}(FM)$ such that $(FM, Y_1) \longrightarrow (FM, Y_0) \longrightarrow S \longrightarrow 0$ is a minimal projective presentation of S. Since $\mathrm{proj.dim}_{\mathrm{End}({}_BFM)}(FM, \mathrm{Ker}(g)) \leq n-2$ by (2), it follows from the exact sequence

$$0 \longrightarrow (FM, \mathrm{Ker}(g)) \longrightarrow (FM, Y_1) \longrightarrow (FM, Y_0) \longrightarrow S \longrightarrow 0$$

that $\mathrm{gl.dim}(\mathrm{End}({}_BFM)) \leq n$. This finishes the proof. $\square$

As an application of 7.1, we have the following result.

Theorem 7.2 *Let A be an Artin algebra and e an idempotent in A such that $Ae \otimes_{eAe} Y \simeq Hom_{eAe}(eA, Y)$ as A-modules for all $Y \in eAe$-mod. Then rep.dim$(eAe) \leq$ rep.dim(A).*

Proof. For $Y \in eAe$-mod, we denote by a_Y the natural map $Ae \otimes_{eAe} Y \longrightarrow \mathrm{Hom}_{eAe}(eA, Y)$, which sends $b \otimes y$ with $b \in Ae$ and $y \in Y$ to the eAe-homomorphism from eA to Y defined by mapping c in eA to $(cb)y$. Consider the following statements:

(1) a_Y is bijective for any $Y \in eAe$-mod.

(2) eA is a projective eAe-module, and a_{eAe} is bijective.

(3) $eA \otimes_A$: A-mod $\longrightarrow eAe$-mod is a full functor.

We prove that (1)$\Leftrightarrow$(2), and (2)$\Rightarrow$(3).

(1)$\Rightarrow$(2): Since $(eA \otimes_A -, \mathrm{Hom}_{eAe}(eA, -))$ is an adjoint pair, there is an unit from $Id_{A\text{-mod}} \longrightarrow \mathrm{Hom}_{eAe}(eA, eA \otimes_A -)$. This gives a homomorphism $\epsilon : Ae \longrightarrow \mathrm{Hom}_{eAe}(eA, eAe)$. We can see that a_Y is the composition of $b_Y = \epsilon \otimes id_Y : Ae \otimes_{eAe} Y \longrightarrow \mathrm{Hom}_{eAe}(eA, eAe) \otimes_{eAe} Y$ with m_Y : $\mathrm{Hom}_{eAe}(eA, eAe) \otimes_{eAe} Y \longrightarrow \mathrm{Hom}_{eAe}(eA, Y)$, where m_Y is the multiplication map. Since m_Y is surjective for any Y and a_Y is bijective, we see that m_Y is bijective and $\mathrm{Hom}_{eAe}(eA, -) \simeq \mathrm{Hom}_{eAe}(eA, eAe) \otimes_{eAe} -$ as functors. This means that $\mathrm{Hom}_{eAe}(eA, -)$ preserves surjective homomorphisms and eA is a projective eAe-module.

(2)$\Rightarrow$(1): since eA is a projective eAe-module, the natural A-homomorphism $(eA, eAe) \otimes_{eAe} Y \longrightarrow (eA, Y)$, which is the multiplication map, is bijective by Lemma 2.1(2). Hence

$$(eA, Y) \simeq (eA, eAe) \otimes_{eAe} Y \simeq Ae \otimes_{eAe} Y.$$

(1) and (2)$\Rightarrow$(3): For any $V \in A$-mod, consider the A-homomorphisms $s_V : Ae \otimes_{eAe} eA \otimes_A V \longrightarrow V$, which sends $x \otimes y \otimes z$ to xyz, and $t_V : V \longrightarrow \mathrm{Hom}_{eAe}(eA, eA \otimes_A V)$, $(t_V(z))(y) = y \otimes z$, where $x \in Ae, y \in eA$ and $z \in V$. Then $s_V t_V = a_{(eA \otimes_A V)}$ holds. Thus t_V is a split epimorphism. Hence $\mathrm{Hom}_A(U, V) \stackrel{(-,t_V)}{\longrightarrow} \mathrm{Hom}_A(U, \mathrm{Hom}_{eAe}(eA, eA \otimes_A V)) = \mathrm{Hom}_{eAe}(eA \otimes_A U, eA \otimes_A V)$ is surjective for any $U, V \in A$-mod.

Note that the functor $eA \otimes_A -$ is dense. Thus the theorem follows from Theorem 7.1. $\square$

The above result suggests the following

Conjecture: For any Artin algebra A and any idempotent element $e \in A$, rep.dim$(eAe) \leq$ rep.dim(A).

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References

[1] F.W.Anderson and K.R.Fuller, *Rings and categories of modules.* GTM 13, Springer-Verlag, New York-Heidelberg-Berlin, 1973.

[2] M.Auslander, Coherent functors. Proceedings of the Conference on Categorial Algebra, La Jolla, Springer-Verlag, Berlin-Heidelberg-New York, 189-231 (1966).

[3] M.Auslander, Representation theory of Artin algebras I. Commun. Algebra 1(3), 177-268 (1974).

[4] M.Auslander, *Representation dimension of Artin algebras* . Queen Mary College Mathematics Notes, Queen Mary College, London, 1971.

[5] M.Auslander and M.Bridger, Stable module category. Memoirs Amer. Math. Soc. No. 94, (1969).

[6] M.Auslander, I.Reiten and S.Smalø, *Representation theory of Artin algebras.* Cambridge Studies in Advanced Mathematics 36, Cambridge University Press, 1995.

[7] M.Auslander and I.Reiten, Applications of contravariantly finite subcategories. Adv. Math. 85, 111-152 (1990).

[8] M.Auslander and I.Reiten, Homologically finite subcategories. In: London Math. Soc. Lecture Note Series 168, (Eds.: H.Tachikawa and S.Brenner), 1-42 (1992).

[9] M.Auslander and S.O.Smalø, Preprojective modules over Artin algebras. J. Algebra 66, 61-122 (1980).

[10] M.Auslander and S.O.Smalø, Almost split sequences in subcategories. J. Algebra 69, 426-454 (1981).

[11] M.Auslander and O.Solberg, Gorenstein algebras and algebras with dominant dimension at least 2. Commun. Algebra 21, No.11, 3897-3934 (1993).

[12] M.Auslander and O.Solberg, Relative homology and representation theory III. Commun. Algebra 21, No.3, 3081-3097 (1993).

[13] H.Cartan and S.Eilenberg, *Homological algebra.* Princeton Landmarks in Mathematics, 1973. Originally published in 1956.

[14] R.Colpi and K.R.Fuller, Cotilting modules and bimodules. Pacific J. Math. 192, 275-292 (2000).

[15] E.E.Enochs and O.M.G.Jenda, *Relative homological algebra.* De Gruyter Expositions in Math. 30, Walter de Gruyter-Berlin-New York, 2000.

[16] T.G.Faticoni, Categories of modules over endomorphism rings. Memoirs Amer. Math. Soc. 103, No. 492 (1993).

[17] P.Gabriel, Auslander-Reiten sequences and representation-finite algebras. Lecture Notes in Math. 831, Springer-Verlag, Berlin-Heidelberg-New York, 1-71 (1980).

[18] D.Happel, *Triangulated categories in the representation theory of finite dimensional algebras.* London Math. Soc. Lecture Note Series 119. Cambridge University Press, Cambridge, 1988.

[19] R.Hartshorne, Coherent functors. Adv. Math. 140, 44-94 (1998).

[20] Z.Huang, On a generalization of Auslander-Bridge transpose. Commun. Algbera 27, 5791-5812 (1999).

[21] O.Iyama, Finiteness of representation dimension. Proc. Amer. Math. Soc. 131, No.4, 1011-1014 (2003).

[22] B.Mitchel, *Theory of categories.* New York and London, Academic Press, 1965.

[23] Y. Miyashita, Tilting modules of finite projective dimension. Math. Z. 193, 113-146 (1986).

[24] I.Reiten, Homological theory of noetherian rings. Canad. Math. Soc. Conf. Proceedings 19, 247-259 (1996).

[25] C.M.Ringel, *Tame algebras and integral quadratic forms.* Lecture Notes in Math. 1099, Springer-Verlag, Berlin-Heidelberg-New York, 1984.

[26] A. N. Wiseman, Projective modules over pullback rings. Proc. Camb. Phil. Soc. 97, 399-406 (1985).

[27] C.C.Xi, On the representation dimension of finite dimensional algebras. J. Algebra 226, 332-346 (2000).

[28] C.C.Xi, Representation dimension and quasi-hereditary algebras. Adv. Math.168, No. 2, 193-212 (2002).

[29] C.C.Xi, On the finitistic dimension conjecture II: related to finite global dimension. Adv. Math. (to appear). Preprint is available at http://math.bnu.edu.cn/~ccxi/Papers/Articles/finchain.pdf/.

[30] C.C.Xi, Standardly stratified algebras and cellular algebras. Math. Proc. Camb. Phil. Soc. 133, No.1, 37-53 (2002).