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CONSTRUCTIONS OF STABLE EQUIVALENCES OF MORITA TYPE FOR FINITE-DIMENSIONAL ALGEBRAS III

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Dedicated to Professor Zhexian Wan on the occasion of his 80th birthday

Abstract

In this paper, we provide a new method to produce stable equivalences of Morita type. Our main results can be stated as follows. Let A and B be two finite-dimensional k-algebras over a field k. Suppose that two bimodules ${}_AM_B$ and ${}_BN_A$ define a stable equivalence of Morita type between A and B and that R is a generator for A-modules. Then there is a stable equivalence of Morita type defined by X and Y between the endomorphism algebra $\mathrm{End}_A(R)$ of the module R and the endomorphism algebra $\mathrm{End}_B(N \otimes_A R)$ of the module $N \otimes_A R$. If M and N satisfy the property that both $(N \otimes_A -, M \otimes_B -)$ and $(M \otimes_B -, N \otimes_A -)$ are adjoint pairs of functors, then so do the modules X and Y. Moreover, we show that the self-injective dimension and the Gorenstein property are invariant under stable equivalences of Morita type with the above-mentioned adjoint property.

1. *Introduction*

This is a continuation of our study on constructions of stable equivalences of Morita type for general finite-dimensional algebras started in [**15**, **16**]. In the present paper we focus mainly on equivalences of Morita type with two adjoint pairs, including the stable equivalences of Morita type between self-injective algebras, where Broue's conjecture applies (see, for instance, [**8**]).

Concerning the importance of stable equivalences of Morita type and the connection of these equivalences with Broué's abelian defect group conjecture, we refer to [**21**] and the first paper of this series as well as the references therein. Here we remind the reader that for the class of general finite-dimensional algebras the notion of derived equivalence and the notion of a stable equivalence of Morita type are independent, though for the class of self-injective algebras, the former implies the latter by a result of Rickard.

The present paper has two purposes: first, we want to create a more general method to produce stable equivalences of Morita type with two natural adjoint pairs; secondly, we investigate properties of such stable equivalences. The main results of this paper are the following theorems.

Theorem 1.1. *Let A and B be two finite-dimensional k-algebras over a field k. Suppose that two bimodules ${}_AM_B$ and ${}_BN_A$ define a stable equivalence of Morita type between A and B. If R is an A-module such that $\mathrm{add}({}_AA) \subseteq \mathrm{add}(R)$, then there is a stable equivalence of Morita type between the endomorphism algebras $\mathrm{End}_A(R)$ and $\mathrm{End}_B(N \otimes_A R)$.*

As a corollary of Theorem 1.1, we have the following result.

Corollary 1.2. *If A is a self-injective algebra, then, for any A-module X, the algebras $\mathrm{End}_A(A \oplus X)$ and $\mathrm{End}_A(A \oplus \Omega_A^n(X))$ are stably equivalent of Morita type for all $n \in \mathbb{Z}$, and the algebras $\mathrm{End}_A(A \oplus X)$ and $\mathrm{End}_A(A \oplus \tau_A^n(X))$ are stably equivalent of Morita type for all*

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$n \in \mathbb{Z}$, where Ω^n stands for the nth syzygy operator, and τ stands for the Auslander–Reiten translation. In particular, they have the same representation dimension.

Rickard proved that if A and B are symmetric algebras, then a stable equivalence of Morita type between A and B, defined by M and N, usually has the property that both $(N \otimes_A -, M \otimes_B -)$ and $(M \otimes_B -, N \otimes_A -)$ are adjoint pairs of functors. This was then extended to self-injective algebras in [**16**]. However, except for these cases, no example of stable equivalences of Morita type with the above property between non-self-injective algebras was known. The following theorem remedies this situation. In fact, it provides a machinery to produce such stable equivalences.

THEOREM 1.3. *Let A and B be two finite-dimensional k-algebras over a field k. Suppose that two bimodules ${}_AM_B$ and ${}_BN_A$ define a stable equivalences of Morita type between A and B. Let R be an A-module such that the subcategory* add(R) *of A*-mod *contains the regular A-module A. If both $(N \otimes_A -, M \otimes_B -)$ and $(M \otimes_B -, N \otimes_A -)$ are adjoint pairs of functors, then*

(1) $\text{inj.dim}({}_AA) = \text{inj.dim}({}_BB)$*;*
(2) *there is a stable equivalence of Morita type between $\Lambda = \text{End}_A(R)$ and $\Gamma = \text{End}_B(N \otimes_A R)$ defined by ${}_\Lambda\overline{M}_\Gamma$ and ${}_\Gamma\overline{N}_\Lambda$ such that $(\overline{M} \otimes_\Lambda -, \overline{N} \otimes_\Gamma -)$ and $(\overline{N} \otimes_\Lambda -, \overline{M} \otimes_\Gamma -)$ are again adjoint pairs of functors.*

Note that Theorem 1.1 generalizes the construction of stable equivalences of Morita type for Auslander algebras in [**16**]. Theorem 1.3 states that the repeated construction by Theorem 1.1 preserves the property that both $(N \otimes_A -, M \otimes_B -)$ and $(M \otimes_B -, N \otimes_A -)$ are adjoint pairs of functors. This is the advantage of our construction in this paper. Moreover, the new stable equivalence of Morita type constructed in Theorem 1.1 can be considered as an extension of the given one. Similarly, one can construct a stable equivalence of Morita type between quotient algebras such that the property of preserving adjoint pairs is retained, as Proposition 3.7 shows.

All proofs of the results above will be given in Section 3 and Section 4 after we recall some basic facts in Section 2. In Section 5, we shall illustrate the main results of this paper, and pose some questions.

2. *Preliminaries*

In this section, we shall fix notation and recall very briefly the definitions and basic facts required needed in the proofs of our main results. For all other notation and facts which are not explained in this paper, we refer the reader to [**13**, **14**].

Throughout this paper, k will stand for a fixed field. All categories will be k-categories and all functors are k-functors; all categories are closed under isomorphisms and direct summands. Furthermore, we assume that all the algebras considered are finite-dimensional k-algebras with identity. Unless stated otherwise, by module we shall mean a finitely generated left module. The composition of two morphisms $f : X \to Y$ and $g : Y \to Z$ between modules will be denoted by fg.

Given an algebra A, we denote by A-mod the category of finitely generated A-modules. For an A-module X in A-mod, we denote by add(X) the full subcategory of A-mod, the objects of which are summands of finite sums of copies of X. The k-duality $\text{Hom}_k(-, k)$ will be denoted by D.

Now let us recall the definition of a stable equivalence of Morita type, which is a combination of the notion of a Morita equivalence and a stable equivalence.

Definition 2.1. Let A and B be two (arbitrary) k-algebras. We say that A and B are stably equivalent of Morita type if there exist an A–B-bimodule ${}_AM_B$ and a B–A-bimodule ${}_BN_A$ such that:

(1) M and N are projective as one-sided modules, and
(2) $M \otimes_B N \simeq A \oplus P$ as A–A-bimodules for some projective A–A-bimodule P, and $N \otimes_A M \simeq B \oplus Q$ as B–B-bimodules for some projective B–B-bimodule Q.

In this case, we say that M and N define a stable equivalence of Morita type between two algebras A and B.

The notion of a stable equivalence of Morita type was first introduced by Broué [8]. It has been used to study blocks in the representation theory of finite groups, or more generally, finite-dimensional self-injective algebras [14, 20, 21].

Note that if A and B are stably equivalent of Morita type then their opposite algebras A^{op} and B^{op} are similarly equivalent of Morita type.

Suppose that two algebras A and B are stably equivalent of Morita type. We can define functors $T_N : A\text{-mod} \to B\text{-mod}$ by $X \mapsto N \otimes_A X$ and $T_M : B\text{-mod} \to A\text{-mod}$ by $Y \mapsto M \otimes_B Y$. Similarly, we have functors T_P and T_Q.

Lemma 2.2 (see [23]). *(1) T_M, T_N, T_P and T_Q are exact functors.*
(2) $T_M \circ T_N \to \mathrm{id}_{A-\mathrm{mod}} \oplus T_P$ and $T_N \circ T_M \to \mathrm{id}_{B-\mathrm{mod}} \oplus T_Q$ are natural isomorphisms.
(3) The images of T_P and T_Q consist of projective modules.

To describe the module category over an endomorphism algebra of a module, we adapt the technique of the morphism category.

Given an Artin algebra A and a full subcategory $\mathcal{C}$ of A-mod, the morphism category, denoted by $\mathrm{Morph}(\mathcal{C})$, is the category of $\mathcal{C}$ in which the objects are all morphisms $f : C_2 \to C_1$ in $\mathcal{C}$ and the morphisms from an object $f : C_2 \to C_1$ to another object $f' : C_2' \to C_1'$ are pairs (g_1, g_2), where $g_i : C_i \to C_i'$ is a homomorphism in $\mathcal{C}$ for $i = 1, 2$ such that $fg_1 = g_2 f'$. The composition of two morphisms is defined in a natural way.

Of special interest is the case of the morphism category of the full subcategory $\mathcal{P}(A)$ of A-mod consisting of all projective A-modules. We may define a relation on $\mathrm{Morph}(\mathcal{P}(A))$ as follows: for objects $f : P_2 \longrightarrow P_1$ and $f' : P_2' \longrightarrow P_1'$ in $\mathrm{Morph}(\mathcal{P}(A))$, we define $\mathcal{R}_A(f, f') = \{(g_1, g_2) : f \to f' \mid$ such that there is an $h : P_1 \to P_2'$ such that $hf' = g_1\}$. Using this relation, we define a factor category $\mathrm{Morph}(\mathcal{P}(A))/\mathcal{R}_A$. The objects of $\mathrm{Morph}(\mathcal{P}(A))/\mathcal{R}_A$ are the same as those of $\mathrm{Morph}(\mathcal{P}(A))$. The morphisms from f to f' in $\mathrm{Morph}(\mathcal{P}(A))/\mathcal{R}_A$ are the elements in $\mathrm{Hom}(f, f')/\mathcal{R}_A(f, f')$. There is a natural functor $\Sigma_A : \mathrm{Morph}(\mathcal{P}(A)) \longrightarrow A\text{-mod}$, which sends each f to the cokernel of f. Note that the functor Σ_A is full and dense. The following result in [6, Proposition 1.2, p. 102] characterizes the connection of the module category with the morphism category.

Lemma 2.3. *The functor Σ_A: $\mathrm{Morph}(\mathcal{P}(A)) \longrightarrow A\text{-mod}$ induces an equivalence of categories: $\mathrm{Morph}(\mathcal{P}(A))/\mathcal{R}_A \longrightarrow A\text{-mod}$.*

Now let R be an A-module and let Λ be the endomorphism algebra $\mathrm{End}_A(R)$ of R. By [6, Proposition 2.1, p. 33], $\mathrm{Hom}_A(R, -) : A\text{-mod} \longrightarrow \Lambda\text{-mod}$ induces an equivalence: $\mathrm{add}(R) \to \mathcal{P}(\Lambda)$. It follows that $\mathrm{Hom}_A(R, -)$ induces an equivalence of categories: $\mathrm{Morph}(\mathrm{add}(R)) \longrightarrow \mathrm{Morph}(\mathcal{P}(\Lambda))$, which is explicitly described as follows. Given $f : U_2 \to U_1$, one defines $\mathrm{Hom}_A(R, f) : \mathrm{Hom}_A(R, U_2) \to \mathrm{Hom}_A(R, U_1)$ by $\alpha \mapsto \alpha f$ for all $\alpha \in \mathrm{Hom}_A(R, U_2)$. For simplicity, we shall denote $\mathrm{Hom}_A(R, X)$ just by (R, X), if there is no danger of confusion.

The following lemma is proved in [16].

LEMMA 2.4. *Let R be an A-module and let Λ be the endomorphism algebra* $\mathrm{End}_A(R)$ *of R. Then the composition functor* $\Sigma_\Lambda \circ (R, -) : \mathrm{Morph}(\mathrm{add}(R)) \longrightarrow \Lambda\text{-mod}$ *induces an equivalence* $H_A : \mathrm{Morph}(\mathrm{add}(R))/\mathcal{R}'_A \longrightarrow \Lambda\text{-mod}$, *where $\mathcal{R}'_A$ is the relation on* Morph(add(*R*)) *defined by* $\mathcal{R}'_A(f, g) = \{(\alpha_1, \alpha_2) : f \to g \mid$ *there is a homomorphism* $\gamma : U_1 \to V_2$ *such that* $\gamma g = \alpha_1\}$ *for objects* $f : U_2 \to U_1$ *and* $g : V_2 \to V_1$.

For the convenience of the reader, we recall the following two homological results which were stated in [**16**].

LEMMA 2.5. *Let C, D and E be three k-algebras.*

(1) *Suppose ${}_CX_D$ and ${}_DY_E$ are bimodules with X_D projective. Then the natural morphism* ϕ: ${}_CX \otimes_D Y_E \longrightarrow \mathrm{Hom}_D({}_DX^*{}_C, {}_DY_E)$, *where* $X^* = \mathrm{Hom}_D(X, D)$ *and* $\phi(x \otimes y)(f) = f(x)y$ *for $x \in X, y \in Y$, and $f \in X^*$ is an isomorphism of C–E-bimodules.*

(2) *For every triple module* $({}_CX_D, {}_CY, Z_E)$, *there is a D–E-bimodule isomorphism* $\psi : \mathrm{Hom}_C({}_CX_D, {}_CY) \otimes_k Z_E \to \mathrm{Hom}_C({}_CX_D, {}_CY \otimes_k Z_E)$ *defined by* $\psi(f \otimes z)(x) = f(x) \otimes z$ *for $x \in X, z \in Z$ and $f \in \mathrm{Hom}_C(X, Y)$.*

Finally, let us recall the definitions of the finitistic dimension and the representation dimension.

DEFINITION 2.6. Let A be an Artin algebra.

(1) The finitistic dimension of A, denoted by fin.dim(A), is defined as

$$\mathrm{fin.dim}(A) = \sup\{\mathrm{proj.dim}({}_AM) \mid M \in A\text{-mod and } \mathrm{proj.dim}({}_AM) < \infty\}.$$

(2) The representation dimension of A, denoted by rep.dim(A), is defined as

$$\mathrm{rep.dim}(A) = \inf\{\mathrm{gl.dim}(\mathrm{End(M)}) \mid M \in A\text{-mod with } \mathrm{add}({}_AA \oplus D(A)) \subseteq \mathrm{add}(M)\}.$$

The notion of representation dimension was introduced by Auslander in [**3**] to measure homologically how far away an Artin algebra is from being representation-finite. Concerning the notion of finitistic dimension, there is a celebrated conjecture, namely the finitistic dimension conjecture, which says that $\mathrm{fin.dim}(A) < \infty$ for any Artin algebra A. This conjecture was proposed 45 years ago and is still open. For more information and new developments on finitistic dimension and representation dimension, we refer the reader to [**23**, **25**, **26**] and the references therein. Concerning the relationship of the finitistic dimension conjecture with other homological conjectures, we refer the reader to [**27**].

3. *Proofs of the main results*

This section is devoted to the proofs of our main results. The first part of the proof to Theorem 1.1 is similar to that in [**16**], but the second part of the proof is completely new. For the first part, we just indicate where the argument might differ from the proof of [**16**, Theorem 1.1]. From now on, we assume that A, B, M, N, P and Q are fixed, as in Definition 2.1. Furthermore, we assume that R is an A-module such that the subcategory add(R) of A-mod contains the regular A-module A. Finally, we denote by Λ and Γ the endomorphism algebras of R and $T_N(R)$, respectively.

LEMMA 3.1. *$T_N(R)$ is a generator for B-mod, that is,* $\mathrm{add}({}_BB) \subseteq \mathrm{add}(T_N(R))$.

Proof. We only need to show that $\mathrm{add}(T_N(R))$ contains each projective B-module. Let X be a projective B-module. Then $T_M(X)$ is a projective A-module and $T_M(X) \in \mathrm{add}(R)$. It follows that $T_N \circ T_M(X) \cong X \oplus T_Q(X) \in \mathrm{add}(T_N(R))$. Thus $X \in \mathrm{add}(T_N(R))$. □

Note that $T_M(\mathrm{add}(T_N(R))) \subseteq \mathrm{add}(T_M \circ T_N(R)) = \mathrm{add}(R \oplus T_P(R)) = \mathrm{add}(R)$ and $T_N(\mathrm{add}(R)) \subseteq \mathrm{add}(T_N(R))$.

Recall that we have the following equivalences of categories:

$$H_A : \mathrm{Morph}(\mathrm{add}(R))/\mathcal{R}'_A \longrightarrow \Lambda\text{−mod}$$

and

$$H_B : \mathrm{Morph}(\mathrm{add}(T_N(R)))/\mathcal{R}'_B \longrightarrow \Gamma\text{−mod}.$$

In order to link the two categories Λ-mod and Γ-mod together, we define two functors

$$\widetilde{T_N} : \mathrm{Morph}(\mathrm{add}(R))/\mathcal{R}'_A \longrightarrow \mathrm{Morph}(\mathrm{add}(T_N(R)))/\mathcal{R}'_B$$

and

$$\widetilde{T_M} : \mathrm{Morph}(\mathrm{add}(T_N(R)))/\mathcal{R}'_B \longrightarrow \mathrm{Morph}(\mathrm{add}(R))/\mathcal{R}'_A.$$

For $f : U_2 \to U_1$ in $\mathrm{Morph}(\mathrm{add}(R))/\mathcal{R}'_A$, we define $\widetilde{T_N}(f) = T_N(f) : T_N(U_2) \to T_N(U_1)$. For a morphism $(\alpha_1, \alpha_2) + \mathcal{R}'_A(f, g) : f \to g$, we set

$$\widetilde{T_N}((\alpha_1, \alpha_2) + \mathcal{R}'_A(f, g)) = (T_N(\alpha_1), T_N(\alpha_2)) + \mathcal{R}'_B(T_N(f), T_N(g)) : \widetilde{T_N}(f) \to \widetilde{T_N}(g).$$

Since $\widetilde{T_N}(\mathcal{R}'_A(f, g)) \subseteq \mathcal{R}'_B(T_N(f), T_N(g))$, it is easy to see that $\widetilde{T_N}$ is well defined. The functor $\widetilde{T_M}$ can be defined similarly.

As in [**16**], we define two functors F and G between Λ-mod and Γ-mod. Let $F : \Lambda\text{-mod} \longrightarrow \Gamma\text{-mod}$ be the compositions:

$$\Lambda\text{-mod} \xrightarrow{H_A^{-1}} \mathrm{Morph}(\mathrm{add}(R))/\mathcal{R}'_A \xrightarrow{\widetilde{T_N}} \mathrm{Morph}(\mathrm{add}(T_N(R)))/\mathcal{R}'_B \xrightarrow{H_B} \Gamma\text{-mod},$$

where H_A^{-1} is the inverse of H_A. Similarly, we define $G : \Gamma\text{-mod} \longrightarrow \Lambda\text{-mod}$ as the compositions:

$$\Gamma\text{-mod} \xrightarrow{H_B^{-1}} \mathrm{Morph}(\mathrm{add}(T_N(R)))/\mathcal{R}'_B \xrightarrow{\widetilde{T_M}} \mathrm{Morph}(\mathrm{add}(R))/\mathcal{R}'_A \xrightarrow{H_A} \Lambda\text{-mod},$$

where H_B^{-1} is the inverse of H_B. Therefore we have the following situation

$$\begin{tikzcd}
\Lambda\text{-mod} \arrow[r, "F"] \arrow[d, "H_A^{-1}"] & \Gamma\text{-mod} \\
\mathrm{Morph}(\mathrm{add}(R))/\mathcal{R}'_A \arrow[r, "\widetilde{T_N}"] & \mathrm{Morph}(\mathrm{add}(T_N(R)))/\mathcal{R}'_B \arrow[u, "H_B"]
\end{tikzcd}$$

$$\begin{tikzcd}
\Gamma\text{-mod} \arrow[r, "G"] \arrow[d, "H_B^{-1}"] & \Lambda\text{-mod} \\
\mathrm{Morph}(\mathrm{add}(T_N(R)))/\mathcal{R}'_B \arrow[r, "\widetilde{T_M}"] & \mathrm{Morph}(\mathrm{add}(R))/\mathcal{R}'_A \arrow[u, "H_A"]
\end{tikzcd}$$

We claim that F and G take projective modules to projective modules. Indeed, let $X \simeq \mathrm{Hom}_A(R, U_0)$ be a projective Λ-module with $U_0 \in \mathrm{add}(R)$. Then $H_A^{-1}(X) \simeq (0 \to U_0)$ in $\mathrm{Morph}(\mathrm{add}(R))/\mathcal{R}'_A$ and $\widetilde{T_N} H_A^{-1}(X) \simeq (0 \to T_N(U_0))$ in $\mathrm{Morph}(\mathrm{add}(T_N(R)))/\mathcal{R}'_B$ with $T_N(U_0) \in \mathrm{add}(T_N(R))$. Therefore $F(X) \simeq \mathrm{Hom}_B(T_N(R), T_N(U_0))$ is a projective Γ-module. This implies that the functor F takes projective modules to projective modules. Similarly, the functor G takes projective modules to projective modules. As in [16], we shall prove that F and G are right exact functors.

LEMMA 3.2. *The functors F and G are right exact and faithful.*

Proof. (1) First we show that F and G are right exact. Here, we only prove that F is a right exact functor since a similar argument works for the functor G.

Let $\delta : 0 \to X \to Y \to Z \to 0$ be a short exact sequence in Λ-mod. By the Horseshoe lemma (see [22, lemma 6.20, p. 187]), we have an exact commutative diagram in Λ-mod:

$$\begin{array}{ccccccccc}
\varepsilon : 0 & \longrightarrow & P_1 & \longrightarrow & P_1 \oplus Q_1 & \longrightarrow & Q_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\eta : 0 & \longrightarrow & P_0 & \longrightarrow & P_0 \oplus Q_0 & \longrightarrow & Q_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\delta : 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where P_i and Q_i are projective Λ-modules for $i = 0, 1$, and the short exact sequences ε and η are canonical split exact sequences. Since $\mathrm{Hom}_A(R, -) : A\text{-mod} \to \Lambda\text{-mod}$ induces an equivalence between $\mathrm{add}(R)$ and $\mathcal{P}(\Lambda)$, the exact commutative diagram in Λ-mod

$$\begin{array}{ccccccccc}
\varepsilon : 0 & \longrightarrow & P_1 & \longrightarrow & P_1 \oplus Q_1 & \longrightarrow & Q_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\eta : 0 & \longrightarrow & P_0 & \longrightarrow & P_0 \oplus Q_0 & \longrightarrow & Q_0 & \longrightarrow & 0
\end{array}$$

corresponds to a commutative diagram in A-mod:

$$\begin{array}{ccccccccc}
\varepsilon' : 0 & \longrightarrow & U_1 & \longrightarrow & U_1 \oplus V_1 & \longrightarrow & V_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\eta' : 0 & \longrightarrow & U_0 & \longrightarrow & U_0 \oplus V_0 & \longrightarrow & V_0 & \longrightarrow & 0
\end{array}$$

where U_i and V_i are in $\mathrm{add}(R)$ for $i = 0, 1$, and the short exact sequences ε' and η' are canonical split exact sequences. Using the functor T_N, we get the following commutative diagram in B-mod:

$$\begin{array}{ccccccccc}
\varepsilon'' : 0 & \longrightarrow & T_N(U_1) & \longrightarrow & T_N(U_1) \oplus T_N(V_1) & \longrightarrow & T_N(V_1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\eta'' : 0 & \longrightarrow & T_N(U_0) & \longrightarrow & T_N(U_0) \oplus T_N(V_0) & \longrightarrow & T_N(V_0) & \longrightarrow & 0
\end{array}$$

where ε'' and η'' are canonical split exact sequences. Note that in the above diagram all the B-modules $T_N(U_i)$ and $T_N(V_i)$ are in $\mathrm{add}(T_N(R))$. Applying the functor $\mathrm{Hom}_B(T_N(R), -) :$ $B\text{-mod} \to \Gamma\text{-mod}$ to the above diagram and taking the cokernels of the columns, we get an exact commutative diagram in Γ-mod:

$$
\begin{array}{cccccccc}
0 \to (T_N(R), T_N(U_1)) & \to & (T_N(R), T_N(U_1)) \oplus (T_N(R), T_N(V_1)) & \to & (T_N(R), T_N(V_1)) & \to 0 \\
\downarrow f & & \downarrow g & & \downarrow h & \\
0 \to (T_N(R), T_N(U_0)) & \to & (T_N(R), T_N(U_0)) \oplus (T_N(R), T_N(V_0)) & \to & (T_N(R), T_N(V_0)) & \to 0 \\
\downarrow & & \downarrow & & \downarrow & \\
\widetilde{\delta}: \quad \operatorname{cok}(f) & \longrightarrow & \operatorname{cok}(g) & \longrightarrow & \operatorname{cok}(h) & \to 0 \\
\downarrow & & \downarrow & & \downarrow & \\
0 & & 0 & & 0 &
\end{array}
$$

where $(T_N(R), *)$ denotes $\operatorname{Hom}_B(T_N(R), *)$. On the other hand, we know by definition that the row $\widetilde{\delta}$ is just the image of δ under the functor F. It follows that F is a right exact functor.

(2) Now we show that F and G are faithful. It is sufficient to prove that the additive functors

$$\widetilde{T_N} : \operatorname{Morph}(\operatorname{add}(R))/\mathcal{R}'_A \longrightarrow \operatorname{Morph}(\operatorname{add}(T_N(R)))/\mathcal{R}'_B$$

and

$$\widetilde{T_M} : \operatorname{Morph}(\operatorname{add}(T_N(R)))/\mathcal{R}'_B \longrightarrow \operatorname{Morph}(\operatorname{add}(R))/\mathcal{R}'_A$$

are faithful.

Given two objects $f : U_2 \to U_1$ and $g : V_2 \to V_1$ in the morphism category $\operatorname{Morph}(\operatorname{add}(R))$ of $\operatorname{add}(R)$. Suppose that there is a morphism $(\alpha_1, \alpha_2) : f \to g$ between f and g such that $(T_N(\alpha_1), T_N(\alpha_2)) : T_N(f) \to T_N(g)$ is in $\mathcal{R}'_B(T_N(f), T_N(g))$; we want to show that (α_1, α_2) is in $\mathcal{R}'_A(f, g)$. Since the morphism $(T_N(\alpha_1), T_N(\alpha_2)) : T_N(f) \to T_N(g)$ is in $\mathcal{R}'_B(T_N(f), T_N(g))$, there is a morphism $h : T_N(U_1) \to T_N(V_2)$ such that $hT_N(g) = T_N(\alpha_1)$, which gives rise to the diagram:

$$
\begin{tikzcd}
T_N(U_2) \arrow[r, "T_N(f)"] \arrow[d, "T_N(\alpha_2)"'] & T_N(U_1) \arrow[d, "T_N(\alpha_1)"] \arrow[dl, "h"'] \\
T_N(V_2) \arrow[r, "T_N(g)"'] & T_N(V_1)
\end{tikzcd}
$$

with $T_N(\alpha_2)T_N(g) = T_N(f)T_N(\alpha_1)$ and $hT_N(g) = T_N(\alpha_1)$. Applying the functor T_M we get the following diagram:

$$
\begin{tikzcd}
T_M T_N(U_2) \arrow[r, "T_M T_N(f)"] \arrow[d, "T_M T_N(\alpha_2)"'] & T_M T_N(U_1) \arrow[d, "T_M T_N(\alpha_1)"] \arrow[dl, "T_M(h)"'] \\
T_M T_N(V_2) \arrow[r, "T_M T_N(g)"'] & T_M T_N(V_1)
\end{tikzcd}
$$

with

$$(T_M T_N(\alpha_2))(T_M T_N(g)) = (T_M T_N(f))(T_M T_N(\alpha_1))$$

and

$$T_M(h)(T_M T_N(g)) = T_M T_N(\alpha_1).$$

By Lemma 2.2, we have a natural isomorphism $T_M \circ T_N \to \mathrm{id}_{A-\mathrm{mod}} \oplus T_P$ of functors. It follows that the above diagram corresponds to a diagram

$$\begin{tikzcd}[column sep=huge, row sep=huge]
U_2 \oplus T_P(U_2) \arrow[r, "\begin{pmatrix} f & 0 \\ 0 & T_P(f) \end{pmatrix}"] \arrow[d, "\begin{pmatrix} \alpha_2 & 0 \\ 0 & T_P(\alpha_2) \end{pmatrix}"'] & U_1 \oplus T_P(U_1) \arrow[d, "\begin{pmatrix} \alpha_1 & 0 \\ 0 & T_P(\alpha_1) \end{pmatrix}"] \arrow[dl, "h'"'] \\
V_2 \oplus T_P(V_2) \arrow[r, "\begin{pmatrix} g & 0 \\ 0 & T_P(g) \end{pmatrix}"'] & V_1 \oplus T_P(V_1)
\end{tikzcd}$$

with

$$\begin{pmatrix} \alpha_2 & 0 \\ 0 & T_P(\alpha_2) \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & T_P(g) \end{pmatrix} = \begin{pmatrix} f & 0 \\ 0 & T_P(f) \end{pmatrix} \begin{pmatrix} \alpha_1 & 0 \\ 0 & T_P(\alpha_1) \end{pmatrix}$$

and

$$h' \begin{pmatrix} g & 0 \\ 0 & T_P(g) \end{pmatrix} = \begin{pmatrix} \alpha_1 & 0 \\ 0 & T_P(\alpha_1) \end{pmatrix},$$

where $h' = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is the morphism which corresponds to $T_M(h)$. Clearly, the above diagram gives rise to the diagram

$$\begin{tikzcd}
U_2 \arrow[r, "f"] \arrow[d, "\alpha_2"'] & U_1 \arrow[d, "\alpha_1"] \arrow[dl, "a"'] \\
V_2 \arrow[r, "g"'] & V_1
\end{tikzcd}$$

with $\alpha_2 g = f\alpha_1$ and $ag = \alpha_1$, which indicates that the morphism (α_1, α_2) is in $\mathcal{R}'_A(f,g)$. This shows that the functor $\widetilde{T_N}$ gives a monomorphism between $\mathrm{Hom}(f,g)/\mathcal{R}_A(f,g)$ and $\mathrm{Hom}(\widetilde{T_N}(f), \widetilde{T_N}(g))/\mathcal{R}_B(\widetilde{T_N}(f), \widetilde{T_N}(g))$ for any two objects f and g, and therefore $\widetilde{T_N}$ is a faithful functor. Similarly, one can show that $\widetilde{T_M}$ is a faithful functor. $\square$

Proof of Theorem 1.1. We shall show that the functors F and G define a stable equivalence of Morita type between Λ and Γ. By Lemma 3.2, the functors $F : \Lambda\text{-mod} \to \Gamma\text{-mod}$ and $G : \Gamma\text{-mod} \to \Lambda\text{-mod}$ are right exact. Clearly F and G are additive functors; it follows from Watts theorem (see [**22**, Theorem 3.33, p. 77]) that $F \simeq {}_\Gamma F(\Lambda) \otimes_\Lambda -$ and $G \simeq {}_\Lambda G(\Gamma) \otimes_\Gamma -$, where the right Λ-module structure on $F(\Lambda)$ is induced by the right multiplication of Λ and the right Γ-module structure on $G(\Gamma)$ is induced by the right multiplication of Γ, respectively. Note that ${}_\Gamma F(\Lambda)_\Lambda = \mathrm{Hom}_B((N \otimes_A R)_\Gamma, N \otimes_A R_\Lambda)$ and ${}_\Lambda G(\Gamma)_\Gamma = \mathrm{Hom}_A(R_\Lambda, M \otimes_B (N \otimes_A R)_\Gamma)$. Since F and G take projective modules to projective modules, $F(\Lambda) \simeq F(\Lambda) \otimes_\Lambda \Lambda$ and $G(\Gamma) \simeq G(\Gamma) \otimes_\Gamma \Gamma$ are projective as left modules.

Since the composition of right exact functors is right exact, the functor $G \circ F : \Lambda\text{-mod} \to \Lambda\text{-mod}$ is a right exact functor. Thus ${}_\Lambda G(F(\Lambda))_\Gamma = \mathrm{Hom}_A(R_\Lambda, M \otimes_B N \otimes_A R_\Lambda)$ is a Λ–Λ-bimodule. It can easily be verified that the Λ–Λ-bimodule structure on $G(F(\Lambda))$ arises naturally from the Hom structure. Since we have an A–A-bimodule isomorphism $\rho = (\rho_1, \rho_2) : M \otimes_B N \simeq A \oplus P$, it follows that the natural isomorphism

$$\overline{\rho} : \mathrm{Hom}_A(R_\Lambda, M \otimes_B N \otimes_A R_\Lambda) \simeq \mathrm{Hom}_A(R_\Lambda, R_\Lambda) \oplus \mathrm{Hom}_A(R_\Lambda, P \otimes_A R_\Lambda)$$

is an isomorphism of Λ–Λ-bimodules, where $\overline{\rho}(f) = (f(\rho_1 \otimes 1_R)\mu, f(\rho_2 \otimes 1_R))$ and $\mu : A \otimes_A R \to R$ is the multiplication map. Note that $\mathrm{Hom}_A({}_AR_\Lambda, {}_AR_\Lambda) = \Lambda$ is the regular Λ–Λ-bimodule. Now we claim that $\mathrm{Hom}_A(R_\Lambda, P \otimes_A R_\Lambda)$ is a projective Λ–Λ-bimodule.

Indeed, by our assumption, P is a projective A–A-bimodule. Thus P is isomorphic to a direct summand of a free A–A-bimodule $(A \otimes_k A^{\mathrm{op}})^m$ for some positive number m. By Lemma 2.5, we have the following Λ–Λ-bimodule isomorphisms:

$$\begin{aligned}\mathrm{Hom}_A(R_\Lambda, A \otimes_k A^{\mathrm{op}} \otimes_A R_\Lambda) &\simeq \mathrm{Hom}_A(R_\Lambda, A) \otimes_k A^{\mathrm{op}} \otimes_A R_\Lambda \\ &\simeq \mathrm{Hom}_A(R_\Lambda, A) \otimes_k \mathrm{Hom}_A(\mathrm{Hom}_A(A^{\mathrm{op}}, A_A), R_\Lambda).\end{aligned}$$

Clearly, $\mathrm{Hom}_A(R_\Lambda, A) \otimes_k \mathrm{Hom}_A(\mathrm{Hom}_A(A^{\mathrm{op}}, A), R_\Lambda)$ is a projective Λ–Λ-bimodule since $\mathrm{add}({}_AA) \subseteq \mathrm{add}(R)$ implies that $\mathrm{Hom}_A(R_\Lambda, A)$ is a projective left Λ-module and that $\mathrm{Hom}_A(\mathrm{Hom}_A(A^{\mathrm{op}}, A), R_\Lambda)$ is a projective right Λ-module. This shows that for any free A–A-bimodule W the Λ–Λ-bimodule $\mathrm{Hom}_A(R_\Lambda, W \otimes_A R_\Lambda)$ is projective, and therefore $\mathrm{Hom}_A(R_\Lambda, P \otimes_A R_\Lambda)$ is projective for any projective A–A-bimodule P.

Since $G(F(\Lambda)) \simeq G(\Gamma) \otimes_\Gamma F(\Lambda)$ as Λ–Λ-bimodules, we have proved that $G(\Gamma) \otimes_\Gamma F(\Lambda) \simeq \Lambda \oplus \mathrm{Hom}_A(R, P \otimes_A R)$ as Λ–Λ-bimodules, where $\mathrm{Hom}_A(R, P \otimes_A R)$ is a projective Λ–Λ-bimodule. Similarly, we have the Γ–Γ-bimodule isomorphism $F(\Lambda) \otimes_\Lambda G(\Gamma) \simeq \Gamma \oplus \mathrm{Hom}_B(T_N(R), Q \otimes_B T_N(R))$, where $\mathrm{Hom}_B(T_N(R), Q \otimes_B T_N(R))$ is a projective Γ–Γ-bimodule since Q is a projective B–B-bimodule.

To complete the proof of Theorem 1.1, we have to show that $F(\Lambda)$ and $G(\Gamma)$ are projective as right modules. This is equivalent to showing that $F \simeq F(\Lambda) \otimes_\Lambda -$ and $G \simeq G(\Gamma) \otimes_\Gamma -$ are exact functors. Suppose $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ is an exact sequence in Λ-mod, and we want to show that $0 \longrightarrow FX \longrightarrow FY \longrightarrow FZ \longrightarrow 0$ is an exact sequence in Γ-mod. Note that the composition $G \circ F = (G(\Gamma) \otimes_\Gamma -) \circ (F(\Lambda) \otimes_\Lambda -) \simeq (\Lambda \otimes_\Lambda -) \oplus (\mathrm{Hom}_A(R, P \otimes_A R) \otimes_\Lambda -)$ is an exact functor since both Λ and $\mathrm{Hom}_A(R, P \otimes_A R)$ are right projective Λ-modules. Hence we have an exact sequence $0 \longrightarrow GFX \longrightarrow GFY \longrightarrow GFZ \longrightarrow 0$. By Lemma 3.2, the functor G is a faithful functor between two abelian categories. Thus G reflects exact sequences (see [**11**, Proposition 3, p. 94]). This implies that the sequence $0 \longrightarrow FX \longrightarrow FY \longrightarrow FZ \longrightarrow 0$ is exact, as desired. Thus the proof is completed. □

In the following, we shall point out that the stable equivalence of Morita type between Λ and Γ is an extension of the one between A and B.

Proposition 3.3. *Under the assumption of Theorem* 1.1, *the stable equivalence of Morita type between the endomorphism algebras* Λ *and* Γ *extends the original one between* A *and* B. *In particular, if* $N \otimes_A - : A$-mod $\to B$-mod *is not a Morita equivalence, then* $F : \Lambda$-mod $\to \Gamma$-mod *is not a Morita equivalence.*

Proof. We have a projective Λ-module $P' = \mathrm{Hom}_A(R, A)$ and a projective Γ-module $Q' = \mathrm{Hom}_B(N \otimes_A R, B)$. Let $\mathrm{Pre}(P')$ be the full subcategory of Λ-mod, the objects of which are those X in Λ-mod which have a projective presentation $P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0$ with the P_i in $\mathrm{add}(P')$ for $i = 0, 1$. Similarly, we have the full subcategory $\mathrm{Pre}(Q')$ of Γ-mod. It is well known that the functor $P' \otimes_A - : A$-mod $\longrightarrow \Lambda$-mod induces an equivalence from A-mod to $\mathrm{Pre}(P')$ with the inverse functor $\mathrm{Hom}_\Lambda(P', -) : \mathrm{Pre}(P') \longrightarrow A$-mod. Now let X be an A-module. On the one hand, under the stable equivalence of Morita type between A and B, X corresponds to the B-module $N \otimes_A X$. On the other hand, X corresponds to the Λ-module $P' \otimes_A X$ under the equivalence between A-mod and $\mathrm{Pre}(P')$. In the following, we show that the Λ-module $P' \otimes_A X$ corresponds to the Γ-module $Q' \otimes_B (N \otimes_A X)$ under a stable equivalence of Morita type between Λ and Γ. We take a projective presentation

$$A_1 \xrightarrow{f} A_0 \longrightarrow X \longrightarrow 0$$

of the A-module X with A_i a projective A-module for $i = 0, 1$. Then the sequence

$$P' \otimes_A A_1 \xrightarrow{1\otimes_A f} P' \otimes_A A_0 \longrightarrow P' \otimes_A X \longrightarrow 0$$

is a projective presentation of the Λ-module $P' \otimes_A X$. We have the following commutative diagram in Λ-module:

$$\begin{tikzcd}
\mathrm{Hom}_A(R, A) \otimes_A A_1 \arrow[r, "1\otimes_A f"] \arrow[d, "\simeq"'] & \mathrm{Hom}_A(R, A) \otimes_A A_0 \arrow[d, "\simeq"'] \\
\mathrm{Hom}_A(R, A_1) \arrow[r, "{_A(1,f)}"] & \mathrm{Hom}_A(R, A_0)
\end{tikzcd}$$

where the columns are Λ-isomorphisms by [24, Lemma 2.1]. Thus, by our construction of the stable equivalence of Morita type between Λ and Γ, we know that the Λ-module $P' \otimes_A X$ corresponds to the cokernel of the Γ-morphism

$$\mathrm{Hom}_B(N \otimes_A R, N \otimes_A A_1) \xrightarrow{_B(1,1\otimes_A f)} \mathrm{Hom}_B(N \otimes_A R, N \otimes_A A_0).$$

Now, it follows from the following commutative diagram in Γ-module:

$$\begin{tikzcd}
\mathrm{Hom}_B(N \otimes_A R, B) \otimes_B (N \otimes_A A_1) \arrow[r, "1\otimes_B 1\otimes_A f"] \arrow[d, "\simeq"'] & \mathrm{Hom}_B(N \otimes_A R, B) \otimes_B (N \otimes_A A_0) \arrow[d, "\simeq"'] \\
\mathrm{Hom}_B(N \otimes_A R, N \otimes_A A_1) \arrow[r, "{_B(1,1\otimes_A f)}"] & \mathrm{Hom}_B(N \otimes_A R, N \otimes_A A_0)
\end{tikzcd}$$

that the cokernel of the above Γ-morphism is isomorphic to the Γ-module $Q' \otimes_B (N \otimes_A X)$. Summarizing the above discussion, we get the following commutative diagram up to natural isomorphisms:

$$\begin{tikzcd}
A\text{-mod} \arrow[r, "P'\otimes_A -"] \arrow[d, "N\otimes_A -"'] & \mathrm{Pre}(P') \arrow[r] \arrow[d, "F"] & \Lambda\text{-mod} \arrow[d, "F"] \\
B\text{-mod} \arrow[r, "Q'\otimes_B -"] & \mathrm{Pre}(Q') \arrow[r] & \Gamma\text{-mod}
\end{tikzcd}$$

which indicates that the stable equivalence of Morita type F between Λ and Γ extends the original one between A and B. In particular, if $N \otimes_A - : A\text{-mod} \to B\text{-mod}$ is not a Morita equivalence, then $F : \Lambda\text{-mod} \to \Gamma\text{-mod}$ is not a Morita equivalence. $\square$

The following corollary is a consequence of Theorem 1.1.

Corollary 3.4. *If A is a self-injective algebra, then, for any A-module X, the algebras $\mathrm{End}_A(A \oplus X)$ and $\mathrm{End}_A(A \oplus \Omega^n_A(X))$ are stably equivalent of Morita type for all $n \in \mathbb{Z}$, and the algebras $\mathrm{End}_A(A \oplus X)$ and $\mathrm{End}_A(A \oplus \tau^n_A(X))$ are stably equivalent of Morita type for all $n \in \mathbb{Z}$, where Ω^n stands for the nth syzygy operator and τ stands for the Auslander–Reiten translation. In particular, they have the same representation dimension.*

Proof. Let A be a self-injective algebra. Then it is well known (see [2], for example) that the A–A-bimodule N, given by

$$0 \longrightarrow N \longrightarrow A \otimes_k A^{\mathrm{op}} \xrightarrow{\mu} A \longrightarrow 0$$

defines a stable equivalence of Morita type between A and itself, where μ is the multiplication map. It follows from this exact sequence that for any A-module X, there is an exact sequence

$$0 \longrightarrow N \otimes_A X \longrightarrow A \otimes_k A^{\mathrm{op}} \otimes_A X \longrightarrow X \longrightarrow 0$$

and a projective A-module U such that $N \otimes_A X \simeq \Omega_A(X) \oplus U$. Since the left A-module ${}_AN$ is a generator for A-mod, the category of all finitely generated left A-modules, we see that $\mathrm{add}(N \otimes_A (A \oplus X)) = \mathrm{add}(A \oplus \Omega_A(X))$. For the functor $\tau : A\text{-mod} \longrightarrow A\text{-mod}$, there is a similar statement. Thus the corollary follows directly from Theorem 1.1. □

As another consequence of Theorem 1.1, we re-obtain a result of [**16**].

COROLLARY 3.5. *Suppose that A and B are representation-finite algebras. If A and B are stably equivalent of Morita type, then the Auslander algebras of A and B are also similarly stably equivalent of Morita type.*

Proof. Let R be the direct sum of all non-isomorphic indecomposable A-modules. Then the condition in Theorem 1.1 is clearly satisfied for R, so the corollary follows. □

Next, we consider the question of whether the condition of adjoint pairs on modules M and N can be transferred to the bimodules, which define the stable equivalence of Morita type between Λ and Γ.

THEOREM 3.6. *Let A and B be two finite-dimensional k-algebras over a field k. Suppose that two bimodules ${}_AM_B$ and ${}_BN_A$ define a stable equivalence of Morita type between A and B. Let R be an A-module such that the subcategory* $\mathrm{add}(R)$ *of* A-mod *contains the regular A-module A. We define $\Lambda = \mathrm{End}({}_AR)$ and $\Gamma = \mathrm{End}({}_BN \otimes_A R)$. If both $(N \otimes_A -, M \otimes_B -)$ and $(M \otimes_B -, N \otimes_A -)$ are adjoint pairs of functors, then there is a stable equivalence of Morita type between the endomorphism algebras $\mathrm{End}_A(R)$ and $\mathrm{End}_B(N \otimes_A R)$ defined by ${}_\Lambda\overline{M}_\Gamma$ and ${}_\Gamma\overline{N}_\Lambda$ with the property that $(\overline{M} \otimes_\Lambda -, \overline{N} \otimes_\Gamma -)$ and $(\overline{N} \otimes_\Lambda -, \overline{M} \otimes_\Gamma -)$ are again adjoint pairs of functors.*

Proof. By the proof of Theorem 1.1, we may define ${}_\Gamma\overline{N}_\Lambda = \mathrm{Hom}_B({}_B(N \otimes_A R)_\Gamma, N \otimes_A R_\Lambda)$ and ${}_\Lambda\overline{M}_\Gamma = \mathrm{Hom}_A(R_\Lambda, M \otimes_B (N \otimes_A R)_\Gamma)$. Since N_A is a generator for mod-A, we see that there is an embedding from $\mathrm{End}({}_AR)$ to $\mathrm{End}({}_BN \otimes_A R)$ by sending $f \mapsto \mathrm{id}_N \otimes f$ for $f \in \mathrm{End}({}_AR)$. Then ${}_\Gamma\overline{N}_\Lambda \simeq {}_\Gamma\Gamma_\Lambda$ and ${}_\Lambda\overline{M}_\Gamma \simeq {}_\Lambda\Gamma_\Gamma$ since $(N \otimes_A -, M \otimes_B -)$ is an adjoint pair and since

$$ {}_\Lambda\overline{M}_\Gamma = \mathrm{Hom}_A(R_\Lambda, M \otimes_B (N \otimes_A R)_\Gamma) \simeq \mathrm{Hom}_B(N \otimes_A R, (N \otimes_A R)_\Gamma). $$

Thus it is sufficient to show that $({}_\Lambda\Gamma \otimes_\Gamma -, {}_\Gamma\Gamma \otimes_\Lambda -)$ and $({}_\Gamma\Gamma \otimes_\Lambda -, {}_\Lambda\Gamma \otimes_\Gamma -)$ are adjoint pairs. This is equivalent to showing that ${}_\Gamma\Gamma \otimes_\Lambda - \simeq \mathrm{Hom}_\Lambda({}_\Lambda\Gamma_\Gamma, -)$ and ${}_\Lambda\Gamma \otimes_\Gamma - \simeq \mathrm{Hom}_\Gamma({}_\Gamma\Gamma_\Lambda, -)$. The latter is clear. To prove the former, we show that:

(1) Γ is a projective left Λ-module, and
(2) ${}_\Gamma\Gamma_\Lambda$ and $\mathrm{Hom}_\Lambda(\Gamma, \Lambda)$ is isomorphic as Γ–Λ-bimodules.

Once this is done, our result follows then by [**6**, Proposition III.4.12, pp. 92–93]. Part (1) is proved in the proof of Theorem 1.1 since $\overline{M}$ is projective as a left Λ-module. For (2), we have the following isomorphisms:

$$ \begin{aligned} {}_\Gamma\Gamma_\Lambda &= \mathrm{Hom}_B((N \otimes_A R)_\Gamma, N \otimes_A R_\Lambda) \\ &\simeq \mathrm{Hom}_A({}_AM \otimes_B (N \otimes_A R)_\Gamma, {}_AR_\Lambda) \\ &\simeq \mathrm{Hom}_\Lambda\big({}_\Lambda\mathrm{Hom}_A({}_AR, {}_AM \otimes_B (N \otimes_A R)_\Gamma), {}_\Lambda\mathrm{Hom}_A({}_AR_\Lambda, R_\Lambda)\big)_\Lambda \\ &= \mathrm{Hom}_\Lambda({}_\Lambda\Gamma_\Gamma, \Lambda). \end{aligned} $$

Note that the second isomorphism follows from [**24**, Lemma 2.2] and the fact that $M \otimes_B N \otimes R \in \mathrm{add}({}_AR)$. Thus we complete the proof. □

Theorem 3.6 allows us to repeatedly construct stable equivalences of Morita type with the adjoint pair property by applying the method of Theorem 1.1. For instance, one can start with a stable equivalence of Morita type between two self-injective algebras, and then apply Theorem 3.6 as many times as possible to get many stable equivalences of Morita type between algebras that are no longer self-injective in general. For information on stable equivalences of Morita type between self-injective algebras, we refer the reader to [**8**, **14**, **19**, **21**] and the references therein.

Now, one of the constructions in [**15**] provides another way to get stable equivalences of Morita type with two adjoint pairs.

Proposition 3.7. *Let A and B be two finite-dimensional k-algebras over a field k. Suppose that two bimodules ${}_AM_B$ and ${}_BN_A$ define a stable equivalence of Morita type between A and B. Suppose that I is an ideal of A and J is an ideal of B such that $JN = NI$ and $IM = MJ$. If M and N satisfy the property that both $(N \otimes_A -, M \otimes_B -)$ and $(M \otimes_B -, N \otimes_A -)$ are adjoint pairs of functors, then the bimodules $(A/I) \otimes_A M \otimes_B (B/J)$ and $(B/J) \otimes_B N \otimes_A (A/I)$ satisfy the property and define a stable equivalence of Morita type between A/I and B/J.*

Proof. By [**15**, Theorem 1.1], what we need to prove is that the pair $((M/IM) \otimes_{B/J} -, (N/NI) \otimes_{A/I} -)$ and the pair $((N/NI) \otimes_{A/I} -, (M/IM) \otimes_{B/J} -)$ are adjoint pairs if we are given the adjoint pairs $(M \otimes_B -, N \otimes_A -)$ and $(N \otimes_A -, M \otimes_B -)$. Note that for any B/J-module X and for any A/I-module Y, we have

$$ {}_{A/I}(M/IM) \otimes_{B/J} X \simeq_A M \otimes_B (B/J) \otimes_{B/J} X \simeq M \otimes_B X $$

as A/I-modules and $N \otimes_A Y \simeq N \otimes_A (A/I) \otimes_{A/I} Y \simeq (N/NI) \otimes_{A/I} Y$ as B/J-modules. Now it follows that

$$ \begin{aligned} \mathrm{Hom}_{A/I}((M/IM) \otimes_{B/J} X, Y) &\simeq \mathrm{Hom}_A((M/IM) \otimes_{B/J} X, Y) \\ &\simeq \mathrm{Hom}_A(M \otimes_B X, Y) \\ &\simeq \mathrm{Hom}_B(X, N \otimes_A Y) \simeq \mathrm{Hom}_{B/J}(X, (N/NI) \otimes_{A/I} Y). \end{aligned} $$

Similarly, we have that $({}_{B/J}(N/NI) \otimes_{A/I} -, (M/IM) \otimes_{B/J} -)$ is an adjoint pair. This completes the proof. □

The proof of Theorem 3.6 suggests that we need to understand the stable equivalence of Morita type between algebras A and B with B a subalgebra of A. In fact, every stable equivalence of Morita type supplies us a stable equivalence of Morita type between A and C with A a subalgebra of C. This can be seen as follows.

Given a stable equivalence of Morita type between A and B defined by ${}_AM_B$ and ${}_BN_A$, let $C = \mathrm{End}({}_BN)$. Then N is a B–C-bimodule. Since N_A is a projective generator for mod-A, there is an injective algebra homomorphism $\varphi : A \longrightarrow C$ by sending $a \in A$ to $\varphi_a : n \mapsto na$ for all $n \in N$. Thus we may consider the algebra A as a subalgebra of C. Note that ${}_BN$ is a projective generator for B-mod. Therefore C is equivalent of Morita type to B. Since the composition of two stable equivalences of Morita type is again a stable equivalence of Morita type, there is a stable equivalence of Morita type between A and C with A a subalgebra of C. It would be interesting to investigate such stable equivalences of Morita type between A and C, where the equivalences are defined by the natural bimodules ${}_AC_C$ and ${}_CC_A$.

Now, our result can be used to construct a family of algebras that are stably equivalent of Morita type to each other, but not themselves Morita equivalent.

Proposition 3.8. *Suppose that k is a field. There is an infinite series of finite-dimensional k-algebras A_i, $i \in \mathbb{N}$ such that:*
(1) $\dim_k(A_i) < \dim_k(A_{i+1})$ *for all i;*
(2) *all A_i are stably equivalent of Morita type;*
(3) *A_i is not Morita equivalent to A_j for all $i \neq j$; and*
(4) *all A_i have the same finitistic dimension and the same representation dimension.*

Proof. Let A be the algebra $k[x,y]/(x^2,y^2)$. Then the unique simple module S has the property that $\dim_k \Omega^i(S) = 2i+1$ and $\dim_k \operatorname{End}({}_A\Omega^i(S)) = i(i+1)+1$ for $i \geqslant 1$. If we put $A_i = \operatorname{End}({}_AA \oplus \Omega^i(S))$, then all A_i are stably equivalent of Morita type by Corollary 3.4, and not equivalent of Morita type since they are basic and pairwise non-isomorphic. Since a stable equivalence of Morita type preserves the finitistic and representation dimensions [23], the last property follows. □

4. *Homological dimensions of modules and algebras*

As we know, the global, finitistic and representation dimensions are preserved under stable equivalences of Morita type. In this section we shall point out that some other homological dimensions and properties are also preserved under stable equivalences of Morita type with the condition that both $(N \otimes_A -, M \otimes_B -)$ and $(M \otimes_B -, N \otimes_A -)$ are adjoint pairs of functors. The first result in this direction is the following proposition which says that the finiteness of injective dimension of the regular representation is invariant. Our interest in this proposition lies in the fact that if $\operatorname{inj.dim}({}_AA) < \infty$, then $\operatorname{fin.dim}(A) < \infty$, that is, the finitistic dimension conjecture for A is true. For convenience, we may call $\operatorname{inj.dim}({}_AA)$ the *left self-injective dimension* of A. Similarly, we have the notion of right self-injective dimension. The Gorenstein symmetry conjecture says that for any Artin algebra A the finiteness of the left self-injective dimension implies the finiteness of the right self-injective dimension. This conjecture is still open.

Proposition 4.1. *Let A and B be two Artin algebras. If two bimodules ${}_AM_B$ and ${}_BN_A$ define a stable equivalence of Morita type between A and B and if both $(N \otimes_A -, M \otimes_B -)$ and $(M \otimes_B -, N \otimes_A -)$ are adjoint pairs of functors, then* $\operatorname{inj.dim}({}_AA) = \operatorname{inj.dim}({}_BB)$. *In particular, under the assumption, we have* $\operatorname{inj.dim}({}_AA) < \infty$ *if and only if* $\operatorname{inj.dim}({}_BB) < \infty$.

This proposition is a direct consequence of the following more general formulation because the modules M and N are projective generators as one-sided modules. Recall that the dominant dimension of an A-module X is the maximal number s in a minimal injective resolution

$$0 \longrightarrow X \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_{s-1} \longrightarrow \cdots$$

of X with all $I_0, \dots, I_{s-1}$ projective-injective. We denote the dominant dimension of X by $\operatorname{dom.dim}(X)$.

Lemma 4.2. *Let A and B be two Artin algebras. If two bimodules ${}_AM_B$ and ${}_BN_A$ define a stable equivalence of Morita type between A and B with both $(N \otimes_A -, M \otimes_B -)$ and $(M \otimes_B -, N \otimes_A -)$ being adjoint pairs, then we have the following.*
(1) $\operatorname{inj.dim}({}_AX) = \operatorname{inj.dim}_B(N \otimes_A X)$ *for any $X \in A$-mod. In particular,* $\operatorname{inj.dim}({}_AA) = \operatorname{inj.dim}({}_BB)$.
(2) $\operatorname{dom.dim}({}_AX) = \operatorname{dom.dim}_B(N \otimes_A X)$ *for any $X \in A$-mod. In particular,* $\operatorname{dom.dim}({}_AA) = \operatorname{dom.dim}({}_BB)$.

Proof. We first show that for any injective A-module I, the B-module $N \otimes_A I$ is injective. Since $(M \otimes_B -, N \otimes_A -)$ is an adjoint pair, we have $\operatorname{Hom}_A(M, -) \simeq N \otimes_A -$ as functors. It follows from [9, Proposition 1.4, p. 107] that for any injective A-module I, the B-module $N \otimes_A I \simeq \mathrm{Hom}_A(M, I)$ is injective. Similarly, since $(N \otimes_A -, M \otimes_B -)$ is an adjoint pair, we know that for any injective B-module I', the A-module $M \otimes_B I'$ is injective.

Now let X be an A-module. Suppose that the sequence

$$0 \longrightarrow X \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_n \longrightarrow \cdots$$

is a minimal injective resolution of the A-module X with I_i injective. Then

$$0 \longrightarrow N \otimes_A X \longrightarrow N \otimes_A I_0 \longrightarrow N \otimes_A I_1 \longrightarrow \cdots \longrightarrow N \otimes_A I_n \longrightarrow \cdots$$

is an injective resolution of the B-module $N \otimes_A X$ by the foregoing fact. It follows that $\mathrm{inj.dim}({}_B N \otimes_A X) \leqslant \mathrm{inj.dim}({}_A X)$. Similarly, if

$$0 \longrightarrow N \otimes_A X \longrightarrow I'_0 \longrightarrow I'_1 \longrightarrow \cdots \longrightarrow I'_n \longrightarrow \cdots$$

is a minimal injective resolution of the B-module $N \otimes_A X$ with all I'_j injective, then

$$0 \longrightarrow M \otimes_B N \otimes_A X \longrightarrow M \otimes_B I'_0 \longrightarrow M \otimes_B I'_1 \longrightarrow \cdots \longrightarrow M \otimes_B I'_n \longrightarrow \cdots$$

is an injective resolution of the A-module $M \otimes_B N \otimes_A X \simeq X \oplus P \otimes_A X$. This implies that $\mathrm{inj.dim}({}_A X) \leqslant \mathrm{inj.dim}({}_B N \otimes_A X)$. Therefore $\mathrm{inj.dim}({}_A X) = \mathrm{inj.dim}_B(N \otimes_A X)$. This proves (1).

Now we prove (2). Let $\mathrm{dom.dim}({}_A X) = n$ (or ∞). If the sequence

$$0 \longrightarrow X \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_n \longrightarrow \cdots$$

is a minimal injective resolution of an A-module X, then I_i is projective for all $i < n$ and I_n is not projective. It follows that

$$0 \longrightarrow N \otimes_A X \longrightarrow N \otimes_A I_0 \longrightarrow N \otimes_A I_1 \longrightarrow \cdots \longrightarrow N \otimes_A I_n \longrightarrow \cdots$$

is an injective resolution of the B-module $N \otimes_A X$ and $N \otimes_A I_i$ is projective for all $i < n$. Thus $\mathrm{dom.dim}_B(N \otimes_A X) \geqslant \mathrm{dom.dim}({}_A X)$. Similarly, let $\mathrm{dom.dim}({}_B N \otimes_A X) = m$ (or ∞). If the sequence

$$0 \longrightarrow N \otimes_A X \longrightarrow I'_0 \longrightarrow I'_1 \longrightarrow \cdots \longrightarrow I'_m \longrightarrow \cdots$$

is a minimal injective resolution of the B-module $N \otimes_A X$, then I'_i is projective for all $i < m$ and I'_m is not projective. It follows that

$$0 \longrightarrow M \otimes_B N \otimes_A X \longrightarrow M \otimes_B I'_0 \longrightarrow M \otimes_B I'_1 \longrightarrow \cdots \longrightarrow M \otimes_B I'_m \longrightarrow \cdots$$

is an injective resolution of the A-module $M \otimes_B N \otimes_A X \simeq X \oplus P \otimes_A X$, and $M \otimes_B I'_i$ is projective for all $i < m$. This implies that $\mathrm{dom.dim}({}_A X) \geqslant \mathrm{dom.dim}({}_B N \otimes_A X)$. Therefore $\mathrm{dom.dim}({}_A X) = \mathrm{dom.dim}({}_B N \otimes_A X)$. □

If the considered algebras A and B have neither nodes nor semi-simple summands, then the global dimension and the dominant dimension are invariant even under stable equivalences (see [18]).

Note that Lemma 4.2 may not be true for derived equivalences. Let us just take A to be a hereditary algebra and B to be a tilted algebra of global dimension 2 (that is, B is the endomorphism algebra of a tilting A-module). Then A and B are derived equivalent, but $\mathrm{inj.dim}({}_A A) = 1$ and $\mathrm{inj.dim}({}_B B) = 2$. Also, we can easily find an example that A and B are stably equivalent and have different finite self-injective dimensions; for example, the k-algebra $k[x]/(x^2)$ and the 2×2 upper triangular matrix algebra over k. The following example shows that under a stable equivalence, one algebra may have finite self-injective dimension and the other may have infinite self-injective dimension. Let us consider the algebra of Igusa, Smalø and Todorov (see [13]). By a result of Martinez-Villa [17], which says that by gluing a simple projective vertex with a simple injective vertex in a quiver and putting zero relations one gets a stable equivalence, we know that the algebra of Igusa, Smalø and Todorov is stably equivalent to the algebra B given by the following quiver and relations:

$$\circ \xleftarrow{\ \gamma\ } \circ \underset{\alpha}{\overset{\beta}{\leftleftarrows}} \circ\,, \qquad \alpha\gamma = 0.$$

Note that these two algebras were also used in [25]. Clearly, the algebra B has the left self-injective dimension 2. In fact, the global dimension of B is also equal to 2. An easy verification shows that the left self-injective dimension of the algebra of Igusa, Smalø and Todorov is infinite. Hence the finiteness of left self-injective dimensions is not preserved under stable equivalences in general.

Next, we consider the Gorenstein dimensions of modules. Let A be an Artin algebra. Recall that an A-module X is reflexive if the canonical map $X \longrightarrow X^{**}$ is an isomorphism, where $X^* = \mathrm{Hom}_A(X, A)$. This is equivalent to the condition that $\mathrm{Ext}^i_A(\mathrm{Tr}(X), A) = 0$ for $i = 1, 2$, where Tr stands for the transpose over A. A module X is said to have Gorenstein dimension zero, written $\text{G-dim}(X) = 0$, if it is reflexive, and $\mathrm{Ext}^i_A(X, A) = 0 = \mathrm{Ext}^i_A(X^*, A)$ for all $i \geqslant 1$. Let $n > 0$ be a natural number. An A-module X is said to have Gorenstein dimension at most n, denoted $\text{G-dim}(X) \leqslant n$, if there is an exact sequence $0 \longrightarrow M_n \longrightarrow \cdots \longrightarrow M_0 \longrightarrow X \longrightarrow 0$ such that $\text{G-dim}(M_j) = 0$ for all $0 \leqslant j \leqslant n$. The minimal such n is denoted by $\text{G-dim}(X)$ if it exists. Otherwise, we say that $\text{G-dim}(X) = \infty$.

Let A and B be two Artin algebras. Suppose that two bimodules ${}_AM_B$ and ${}_BN_A$ define a stable equivalence of Morita type between A and B with both $(N \otimes_A -, M \otimes_B -)$ and $(M \otimes_B -, N \otimes_A -)$ being adjoint pairs. We have the following lemma.

LEMMA 4.3. *For any A-module X, there is a projective right B-module Q' such that $\mathrm{Tr}_B(N \otimes_A X) \oplus Q' \simeq \mathrm{Tr}_A(X) \otimes_A M_B$ as B-modules.*

Proof. Let $P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0$ be a minimal projective presentation of the A-module X with P_i projective. Applying $\mathrm{Hom}_A(-, A)$, we get the exact sequence

$$0 \longrightarrow X^* \longrightarrow P_0^* \longrightarrow P_1^* \longrightarrow \mathrm{Tr}_A(X) \longrightarrow 0$$

and the exact sequence

$$P_0^* \otimes_A M \longrightarrow P_1^* \otimes_A M \longrightarrow \mathrm{Tr}_A(X) \otimes_A M \longrightarrow 0.$$

Since $P_i^* \otimes_A M \simeq (P_i, M)$, we may rewrite the exact sequence above as

$$(P_0, M) \longrightarrow (P_1, M) \longrightarrow \mathrm{Tr}_A(X) \otimes_A M \longrightarrow 0.$$

On the other hand, we have the following exact sequence:

$$(N \otimes_A P_0)^* \longrightarrow (N \otimes_A P_1)^* \longrightarrow \mathrm{Tr}_B(N \otimes_A X) \oplus Q' \longrightarrow 0,$$

where Q' is a projective right B-module. By the canonical adjunction isomorphism, we may rewrite the above exact sequence as

$$(P_0, (N, B)) \longrightarrow (P_1, (N, B)) \longrightarrow \mathrm{Tr}_B(N \otimes_A X) \oplus Q' \longrightarrow 0.$$

Since $(N \otimes_B -, M \otimes_B -)$ is an adjoint pair, we know $\mathrm{Hom}_B({}_BN, -) \simeq M \otimes_B -$ as functors. This implies that $\mathrm{Hom}_B(N, B) \simeq M$. Thus $\mathrm{Tr}_B(N \otimes_A X)$ is the cokernel of the map $(P_0, M) \longrightarrow (P_1, M)$, and therefore $\mathrm{Tr}_B(N \otimes_A X) \oplus Q' \simeq \mathrm{Tr}_A(X) \otimes_A M$. □

In fact, Lemma 4.3 establishes a relation between the Auslander–Reiten translations of algebras A and B.

Corollary 4.4. *Let A and B be two Artin algebras. Suppose that two bimodules ${}_AM_B$ and ${}_BN_A$ define a stable equivalence of Morita type between A and B with both $(N \otimes_A -, M \otimes_B -)$ and $(M \otimes_B -, N \otimes_A -)$ being adjoint pairs. Then, for any A-module X, we have $D\mathrm{Tr}_B(N \otimes_A X) \oplus I \simeq N \otimes_A D\mathrm{Tr}_A(X)$ with I an injective B-module.*

Proof. Let X be an A-module. It follows from Lemma 4.3 that

$$D(\mathrm{Tr}_B(N \otimes_A X) \oplus Q') \simeq D(\mathrm{Tr}_A(X) \otimes_A M) \simeq \mathrm{Hom}_A(M, D\mathrm{Tr}_A(X)) \simeq N \otimes_A D\mathrm{Tr}_A(X).$$

Here the second isomorphism is the adjunction, and the last comes from the adjoint pair $(M \otimes_B -, N \otimes_A -)$. □

With Lemma 4.3 in hand, we have the following result.

Proposition 4.5. *Suppose that X is an A-module. Then* $\text{G-dim}(X) \leqslant n$ *if and only if* $\text{G-dim}(N \otimes_A X) \leqslant n$.

Proof. To show the proposition, we need only to show that $\text{G-dim}(X) = 0$ if and only if $\text{G-dim}(N \otimes_A X) = 0$.

Suppose that $\text{G-dim}(X) = 0$. Then, by [4, Proposition 3.8, p. 95], this is equivalent to the condition that $\mathrm{Ext}^i_A(\mathrm{Tr}_A(X), A_A) = 0 = \mathrm{Ext}^i_A(X, {}_AA)$ for all $i \geqslant 1$. Now we calculate $\mathrm{Ext}^i_B(\mathrm{Tr}_B(N \otimes_A X), B_B)$. By Lemma 4.3, we have

$$\mathrm{Ext}^i_B(\mathrm{Tr}_B(N \otimes_A X), B_B) \simeq \mathrm{Ext}^i_B(\mathrm{Tr}_A(X) \otimes_A M, B_B).$$

Since ${}_AM_B$ is projective on both sides, we have

$$\mathrm{Ext}^i_B(\mathrm{Tr}_A(X) \otimes_A M, B_B) \simeq \mathrm{Ext}^i_A(\mathrm{Tr}_A(X), (M, B_B))$$

by [22, Theorem 11.56]. If we could prove that $\mathrm{Hom}_B(M_B, B_B) \simeq {}_BN_A$ as bimodules, then we would get that

$$\mathrm{Ext}^i_A(\mathrm{Tr}_A(X), (M, B)) \simeq \mathrm{Ext}^i_A(\mathrm{Tr}_A(X), N) = 0 \quad \text{for } i \geqslant 1$$

since N_A is a projective generator for mod-A. Thus we would have that $\mathrm{Ext}^i_B(\mathrm{Tr}_B(N \otimes_A X), B_B) = 0$ for all $i \geqslant 1$. In fact, since $(N \otimes_A -, M \otimes_B -)$ is an adjoint pair, we have $M \otimes_B - \simeq \mathrm{Hom}_B({}_BN, -)$ and ${}_AM_B \simeq \mathrm{Hom}_B({}_BN_A, {}_BB_B)$. Since ${}_BN$ is a projective B-module, the canonical map ${}_BN \longrightarrow \mathrm{Hom}_B(\mathrm{Hom}_B({}_BN, {}_BB), {}_BB_B)$ is an isomorphism of B-modules by [24, Lemma 2.2(2)]. It is easy to check that this is also a right A-module homomorphism. Thus we have

$${}_BN_A \simeq \mathrm{Hom}_B(\mathrm{Hom}_B({}_BN_A, {}_BB), {}_BB_B) \simeq \mathrm{Hom}_B({}_AM_B, {}_BB_B).$$

It follows from [22, Theorem 11.56] that

$$\mathrm{Ext}^i_B({}_BN \otimes_A X, {}_BB) \simeq \mathrm{Ext}^i_A(X, ({}_BN, {}_BB)) \simeq \mathrm{Ext}^i_A(X, M) = 0 \quad \text{for all } i \geqslant 1$$

since ${}_AM$ is a projective generator for A-mod. Thus $\text{G-dim}(N \otimes_A X) = 0$ by [4, Proposition 3.8, p. 95]. □

Recall that an Artin algebra A is called *Gorenstein* if both the left self-injective dimension and the right self-injective dimension are finite. As a consequence of our discussion, we obtain the following corollary again which is a special case of a conclusion in [7]. (The authors thank Apostolos Beligannis for pointing out the reference [7]).

Corollary 4.6. *Let A and B be two Artin algebras. Suppose that two bimodules ${}_AM_B$ and ${}_BN_A$ define a stable equivalence of Morita type between A and B with both*

$(N \otimes_A -, M \otimes_B -)$ and $(M \otimes_B -, N \otimes_A -)$ being adjoint pairs. Then A is Gorenstein if and only if B is Gorenstein.

Proof. By Proposition 4.1, it is true that $\mathrm{inj.dim}({}_AA) < \infty$ if and only if $\mathrm{inj.dim}({}_BB) < \infty$. It remains to check the right self-injective dimensions. An argument similar to the proof of Proposition 4.5 shows that $\mathrm{Hom}_A({}_BN_A, {}_AA_A) \simeq {}_AM_B$. It follows from [**9**, Proposition 1.4, p. 107] that $\mathrm{Hom}_A({}_BN_A, I_A)$ is an injective right B-module for any right injective A-module I_A. This implies that $\mathrm{inj.dim}({}_B\mathrm{Hom}_A({}_BN_A, A_A))$ is finite if $\mathrm{inj.dim}(A_A)$ is finite. Thus $\mathrm{inj.dim}(M_B)$ and $\mathrm{inj.dim}(B_B)$ are finite if $\mathrm{inj.dim}(A_A)$ is finite. Similarly, we can show that $\mathrm{inj.dim}(A_A)$ is finite if $\mathrm{inj.dim}(B_B)$ is finite. Thus we have finished the proof of the theorem. $\square$

REMARK. An alternative proof to Corollary 4.6 is the use of Proposition 4.5 together with the characterization of Gorenstein algebras given by Hoshino in [**12**]: An Artin algebra A has the property that $\mathrm{inj.dim}({}_AA) = \mathrm{inj.dim}(A_A) < \infty$ if and only if each finitely generated left A-module has finite Gorenstein dimension. It is known that if $\mathrm{inj.dim}({}_AA) < \infty$ and $\mathrm{inj.dim}(A_A) < \infty$, then $\mathrm{inj.dim}({}_AA) = \mathrm{inj.dim}(A_A)$.

At the end of this section, we consider k-Gorenstein algebras which were introduced by Auslander.

Recall that an Artin algebra A is called a *k-Gorenstein algebra* if in a minimal injective resolution $0 \longrightarrow {}_AA \longrightarrow I_0 \longrightarrow \cdots \longrightarrow I_j \longrightarrow \cdots$ the projective dimension of I_i is at most i for all $i = 0, \cdots, k-1$. The following result follows from the proof of Lemma 4.2.

PROPOSITION 4.7. *Let A and B be two Artin algebras. Suppose that two bimodules ${}_AM_B$ and ${}_BN_A$ define a stable equivalence of Morita type between A and B with both $(N \otimes_A -, M \otimes_B -)$ and $(M \otimes_B -, N \otimes_A -)$ being adjoint pairs. Then A is k-Gorenstein if and only if B is k-Gorenstein.*

Note that Gorenstein algebras may not be k-Gorenstein. It is open whether A being k-Gorenstein for all k implies that A is Gorenstein (see [**5**] for discussion).

5. *An example and some questions*

In the following, we illustrate the main results in the paper with an example.

EXAMPLE. Let A be the algebra $k[X]/(X^3)$ over a field k, and let $W = k[X]/(X^2)$. Then W is an A-module of length 2. The first syzygy of W is the unique simple A-module k. By calculation we know that $\Lambda := \mathrm{End}_A(A \oplus W)$ and $\Gamma := \mathrm{End}_A(A \oplus k)$ are given by the following quivers with relations, respectively:

$$\Lambda : \begin{tikzcd} 1\circ \arrow[r, shift left, "\alpha"] & \circ\, 2 \arrow[l, shift left, "\beta"] \end{tikzcd} \qquad\qquad \Gamma : \begin{tikzcd} \circ \arrow[r, shift left, "x"] & \circ \arrow[l, shift left, "y"] \arrow[loop right, "z"] \end{tikzcd}$$

$$\alpha\beta\alpha\beta = 0 \qquad\qquad xy = xz = zy = z^2 - yx = 0.$$

By Corollary 3.4, these two algebras are stably equivalent of Morita type. Moreover, on the one hand, based on these two algebras, we can use Theorem 3.6 to get another stable equivalence of Morita type; on the other hand, the algebra Λ is a Nakayama algebra. If we take R to be the direct sum of indecomposable modules with the socle isomorphic to $S(2)$, the simple module corresponding to the vertex of 2, then $\mathrm{add}(R)$ satisfies the condition of Theorem 1.1 since $\mathrm{add}(R)$ is closed under the kernels of morphisms in $\mathrm{add}(R)$. In fact, the indecomposable direct summand of the kernel of any morphism $f : U \longrightarrow V$ in $\mathrm{add}(R)$ has a simple socle isomorphic to $S(2)$. Thus we have a stable equivalence of Morita type between the endomorphism algebra

$\mathrm{End}({}_\Lambda R)$ of R and the endomorphism algebra $\mathrm{End}({}_\Gamma N \otimes_\Lambda R)$ of $N \otimes_A R$, where N is the module which defines the stable equivalence of Morita type between Λ and Γ. Therefore, starting with a self-injective algebra, we may produce many stable equivalences of Morita type between non-self-injective algebras.

As to self-injective dimensions, an easy calculation shows that $\mathrm{inj.dim}({}_\Lambda\Lambda) = \mathrm{inj.dim}(\Lambda_\Lambda) = 2 = \mathrm{inj.dim}({}_\Gamma\Gamma) = \mathrm{inj.dim}(\Gamma_\Gamma)$.

Finally, we pose the following questions suggested by the results in this paper.

Question 1. Suppose that two Artin algebras A and B are stably equivalent of Morita type. Is it true that $\mathrm{inj.dim}({}_A A) < \infty$ if and only if $\mathrm{inj.dim}({}_B B) < \infty$?

Note that the positive answer to Question 1 would imply the statement in Proposition 4.1 and the statement in Corollary 4.6. The question below has the opposite feature of Donovan's conjecture on blocks of group algebras (see [1] for information on Donovan's conjecture).

Question 2. Let k be a field. Is there any infinite series of finite-dimensional k-algebras A_i, $i \in \mathbb{N}$, such that

(1) $\dim_k(A_i) = \dim_k(A_{i+1})$ for all i;
(2) all A_i are stably equivalent of Morita type, and
(3) A_i is not Morita equivalent to A_j for all $i \neq j$?

Note that we do not know if there is an infinite series of algebras with the same dimension such that they are all derived equivalent, but not Morita equivalent.

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Note added in proof. Recently, Dugas and Martinez-Villa showed the following result [10]: let A and B be finite-dimensional algebras over a field k such that A and B have no semi-simple blocks and that $A/\mathrm{rad}(A)$ and $B/\mathrm{rad}(B)$ are separable over k. If A and B are stably equivalent of Morita type defined by indecomposable bimodules ${}_A M_B$ and ${}_B N_A$, then $(M \otimes_B -, N \otimes_A -)$ and $(N \otimes_A -, M \otimes_B -)$ are adjoint pairs. Thus, under the assumptions of the paper, the second statement of Theorem 3.6 follows also from this result of Dugas and Martinez-Villa.

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