Milnor squares of algebras, I: derived equivalences

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Abstract

Derived equivalences for Artin algebras (and almost ν-stable derived equivalences for finite-dimensional algebras) are constructed from Milnor squares of algebras. Particularly, three operations of gluing vertices, unifying arrows and identifying socle elements on derived equivalent algebras are presented to produce new derived equivalences of the resulting algebras from the given ones. As a byproduct, we construct a series of derived equivalences, showing that derived equivalences may change Frobenius type of algebras in general, though both tilting procedure and almost ν-stable derived equivalences do preserve Frobenius type of algebras.

Contents

1 Introduction

Pullback algebras (specially, Milnor squares of algebras) appear in many aspects in mathematics. For example, in algebraic K-theory, Milnor established a Mayer-Vietoris sequence of K-groups for pullback rings (see [14, Theorem 6.4]); in representation theory, Burban and Drozd classified indecomposable objects of the derived category of Harish-Chardara modules over $SL(\mathbb{R})$ via a special pullback algebra (see [2]), and Herbara and Prihoda studied infinitely generated projective modules via pullback rings (see [5]); and in homological algebra, Kirkman and Kuzmanovich investigated homological dimensions for pullback algebras (see [10]). One of the important ingredients in these investigations is a characterization of projective modules over a

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pullback algebra in terms of the ones over its constituent algebras (see [14, Chapter 2]). In the famous Morita theory of derived categories for rings and algebras developed by Rickard (see [15, 16]), the key notion of tilting complexes involves just a kind of complexes of finitely generated projective modules. This motivates us to consider whether it is possible to get tilting complexes over a pullback algebra through the ones over its constituent algebras. In other words, can we construct derived equivalences by forming pullback algebras?

In this paper, we shall show that under certain conditions, derived equivalences of Artin algebras are preserved by forming pullbacks (see Theorem 3.1). Moreover, if all given derived equivalences are almost ν-stable then so is the induced derived equivalence between pullback algebras (see Corollary 3.4). To apply our result to algebras presented by quivers with relations, we introduce three local operations (gluing vertices, unifying arrows and identifying longest elements) on quivers, so that taking each of them on derived equivalent algebras will produce another derived equivalence of the resulting algebras (see Theorems 4.1, 4.5 and 4.8). All of these operations fit well into our framework of constructing derived equivalences for pullback algebras, and can be combined with each other and employed repeatedly. As an application of these techniques, we investigate behaviors of Frobenius parts of derived equivalent algebras and show that derived equivalences may change Frobenius type of algebras in general.

In Section 2, we fix some notation and recall basic facts needed in later proofs. Particularly, we recall the results on change of rings and on description of projective modules over pullback algebras from [14]. Also, we prove some results on images of simple modules under derived equivalences and on tilting complexes and their endomorphism rings.

In Section 3, we first state our main result, Theorem 3.1, which asserts, roughly speaking, that if A is a pullback of homomorphisms $A_1 \to A_0 \leftarrow A_2$ of Artin algebras with one homomorphism surjective and if B_i is an Artin algebra derived equivalent to A_i for $i = 0, 1, 2$, then there are homomorphisms $B_1 \to B_0 \leftarrow B_2$ of algebras such that their pullback algebra B is derived equivalent to A. After some preparations, we then prove the main result and deduce its corollaries. Also, we investigate almost ν-stable derived equivalences which induce stable equivalences of Morita type (see [8]), and show that, under certain additional conditions, almost ν-stable equivalences between finite-dimensional algebras over an algebraically closed field can be constructed by taking pullback algebras (see Corollary 3.4 for precise statement).

In Section 4, we introduce three operations, called gluing vertices, unifying arrows and identifying longest elements, on algebras presented by quivers with relations, and prove that they can produce new derived equivalences from given ones (see Theorems 4.1, 4.5 and 4.8). These operations are actually some of effective applications of our main result.

In Section 5, we study, as another application of the main result, the question of whether derived equivalences preserve Frobenius type of algebras. Recall that Frobenius type of algebras means the representation type of their Frobenius parts which have been employed in [9] to lift stable equivalences of Morita type to derived equivalences and in [13] to reduce the Auslander-Reiten conjecture (or Alperin-Auslander conjecture referred in [17]) on stable equivalences. In this section, we first point out that Frobenius type is preserved under tilting procedure and almost ν-stable derived equivalences, and then apply our constructions in Section 4 to show that derived equivalences may change Frobenius type of algebras in general.

In the second part of this work, we will deal with constructions of stable equivalences of Morita type from pullback algebras.

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2 Preliminaries

In this section, we fix some notation, recall some basic results on derived equivalences and on projective modules over pullback algebras, and then prove a few results concerning derived equivalences and tilting complexes. All results in this section will serve as preparations for the proof of the main result, Theorem 3.1.

2.1 Derived equivalences

Let $\mathcal{C}$ be an additive category.

Given two morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}$, the composite of f with g is written as fg, which is a morphism from X to Z. But for two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ of categories, their composition is denoted by GF.

For an object X in $\mathcal{C}$, we denote by $\mathrm{add}(X)$ the full subcategory of $\mathcal{C}$ consisting of all direct summands of finite direct sums of copies of X. The object X is said to be *basic* if $X = \bigoplus_{i\in I} X_i$ with I an index set and X_i an indecomposable object for all $i \in I$ such that $X_i \not\simeq X_j$ for $i \neq j$.

By a complex $X^\bullet$ over $\mathcal{C}$ we mean a sequence of morphisms d_X^i between objects X^i in $\mathcal{C}: \cdots \to X^{i-1} \xrightarrow{d_X^{i-1}} X^i \xrightarrow{d_X^i} X^{i+1} \xrightarrow{d_X^{i+1}} \cdots$, with $d_X^i d_X^{i+1} = 0$ for all $i \in \mathbb{Z}$, and write $X^\bullet = (X^i, d_X^i)$. The morphism d_X^i is then called the i-th differential of $X^\bullet$. The complex $X^\bullet$ is said to be *radical* if each of its differentials is a radical morphism. By $X^\bullet[n]$ we denote the n-th shift of $X^\bullet$, that is a complex with the i-th term X^{i+n} and differential $(-1)^n d_X^{i+n}$.

We write $\mathscr{C}(\mathcal{C})$ for the category of all complexes over $\mathcal{C}$, and $\mathscr{K}(\mathcal{C})$ for the homotopy category of $\mathscr{C}(\mathcal{C})$. When $\mathcal{C}$ is an abelian category, we write $\mathscr{D}(\mathcal{C})$ for the derived category of $\mathcal{C}$. As usual, let $\mathscr{C}^{\mathrm{b}}(\mathcal{C}), \mathscr{K}^{\mathrm{b}}(\mathcal{C})$ and $\mathscr{D}^{\mathrm{b}}(\mathcal{C})$ denote the relevant full subcategories consisting of bounded complexes, respectively; and let $\mathscr{C}^{-}(\mathcal{C}), \mathscr{K}^{-}(\mathcal{C})$ and $\mathscr{D}^{-}(\mathcal{C})$ denote the corresponding full subcategories consisting of complexes bounded above. Analogously, $\mathscr{C}^{+}(\mathcal{C}), \mathscr{K}^{+}(\mathcal{C})$ and $\mathscr{D}^{+}(\mathcal{C})$ stand for the corresponding full subcategories consisting of complexes bounded below, respectively.

Let Λ be an Artin algebra over a commutative Artin ring. We denote by $\Lambda\text{-mod}$ the category of finitely generated left Λ-modules, and by $\Lambda\text{-proj}$ the full subcategory of $\Lambda\text{-mod}$ consisting of finitely generated projective Λ-modules. For simplicity, we write $\mathscr{C}(\Lambda), \mathscr{K}(\Lambda)$ and $\mathscr{D}(\Lambda)$ for $\mathscr{C}(\Lambda\text{-mod}), \mathscr{K}(\Lambda\text{-mod})$ and $\mathscr{D}(\Lambda\text{-mod})$, respectively. Similarly, we have abbreviations $\mathscr{C}^{\mathrm{b}}(\Lambda)$, $\mathscr{K}^{\mathrm{b}}(\Lambda)$ and $\mathscr{D}^{\mathrm{b}}(\Lambda)$. In this paper, $\mathscr{D}^{\mathrm{b}}(\Lambda)$ is called the *derived category* of Λ.

It is well known that the homotopy and derived categories of an Artin algebra (or more generally, a ring) are triangulated categories. For basic results on triangulated categories, we refer the reader to the book [4].

Two Artin algebras Λ and Γ are said to be *derived equivalent* if their derived categories are equivalent as triangulated categories. It follows from Rickard's Morita theory for derived categories [16] that two Artin algebras Λ and Γ are derived equivalent if and only if there is a complex $T^\bullet$ in $\mathscr{K}^{\mathrm{b}}(\Lambda\text{-proj})$ satisfying

(1) $T^\bullet$ is self-orthogonal, that is, $\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(\Lambda\text{-proj})}(T^\bullet, T^\bullet[n]) = 0$ for all integers $n \neq 0$;

(2) $\mathrm{add}(T^\bullet)$ generates $\mathscr{K}^{\mathrm{b}}(\Lambda\text{-proj})$ as a triangulated category; and

(3) $\Gamma \simeq \mathrm{End}_{\mathscr{K}^{\mathrm{b}}(\Lambda\text{-proj})}(T^\bullet)$.

For more details on derived equivalences, we refer the reader to the papers [15, 16], and for some constructions of derived equivalences, we refer the reader to recent papers [7, 3].

A complex $T^\bullet$ in $\mathscr{K}^{\mathrm{b}}(\Lambda\text{-proj})$ satisfying the above conditions (1) and (2) is called a *tilting complex* over Λ. It is readily to see that for each tilting complex $T^\bullet$ there is a basic, radical tilting complex $T_0^\bullet \in \mathrm{add}(T^\bullet)$ such that $\mathrm{End}_{\mathscr{K}^{\mathrm{b}}(\Lambda\text{-proj})}(T_0^\bullet)$ is Morita equivalent to $\mathrm{End}_{\mathscr{K}^{\mathrm{b}}(\Lambda\text{-proj})}(T^\bullet)$.

For any derived equivalence $F : \mathscr{D}^{\mathrm{b}}(\Lambda) \to \mathscr{D}^{\mathrm{b}}(\Gamma)$, there is a unique (up to isomorphism) tilting complex $T^\bullet$ over Λ such that $F(T^\bullet)$ is isomorphic in $\mathscr{D}^{\mathrm{b}}(\Gamma)$ to Γ. This complex $T^\bullet$ is called a tilting complex *associated to* F.

Finally, we recall two operations on complexes, which will be used frequently in the paper.

Let $X^\bullet = (X^i, d_X^i)_{i\in\mathbb{Z}}$ be a complex in $\mathscr{C}(\Lambda^{\mathrm{op}})$ and $Y^\bullet = (Y^i, d_Y^i)_{i\in\mathbb{Z}}$ a complex in $\mathscr{C}(\Lambda)$. By $X^\bullet \otimes_\Lambda^\bullet Y^\bullet$ we mean the total complex of the double complex with (i,j)-term $X^i \otimes_\Lambda Y^j$. That is, the n-th term of the complex $X^\bullet \otimes_\Lambda^\bullet Y^\bullet$ is $\bigoplus_{p+q=n} X^p \otimes_\Lambda Y^q = \bigoplus_{q\in\mathbb{Z}} X^{n-q} \otimes_\Lambda Y^q$, and the n-th differential is given by $x \otimes y \mapsto x \otimes (y)d_Y^q + (-1)^q (x) d_X^{n-q} \otimes y$ for $x \in X^{n-q}$ and $y \in Y^q$.

Let $X^\bullet$ and $Y^\bullet$ be two complexes in $\mathscr{C}(\Lambda)$. By $\mathrm{Hom}_\Lambda^\bullet(X^\bullet, Y^\bullet)$ we denote the total complex of the double complex with (i,j)-term $\mathrm{Hom}_\Lambda(X^{-i}, Y^j)$. Thus the n-th term of the complex $\mathrm{Hom}_\Lambda^\bullet(X^\bullet, Y^\bullet)$ is $\prod_{p\in\mathbb{Z}} \mathrm{Hom}_\Lambda(X^p, Y^{n+p})$, and the n-th differential is given by $(\alpha^p)_{p\in\mathbb{Z}} \mapsto (\alpha^p d_Y^{n+p} - (-1)^n d_X^p \alpha^{p+1})_{p\in\mathbb{Z}}$ for $\alpha^p \in \mathrm{Hom}_\Lambda(X^p, Y^{n+p})$.

2.2 Complexes under change of rings

Let $f : \Lambda \to \Gamma$ be a homomorphism of R-algebras, where R is a commutative ring with identity. Then every Γ-module U can be viewed as a Λ-module by defining $a \cdot u := (a)fu$ for all $a \in \Lambda$ and $u \in U$. Thus we get the so-called restriction functor ${}_\Lambda(-) : \Gamma\text{-mod} \to \Lambda\text{-mod}$. Moreover, there is an adjoint pair $(\Gamma \otimes_\Lambda -, {}_\Lambda(-))$ of functors whose unit is the canonical homomorphism of Λ-modules:

$$f^* : X \longrightarrow {}_\Lambda\Gamma \otimes_\Lambda X, \quad x \mapsto 1 \otimes x \text{ for } x \in X.$$

The following lemma tells us about change of projective modules.

Lemma 2.1. *Let $f : \Lambda \to \Gamma$ be a surjective homomorphism of Artin algebras. Then $\Gamma \otimes_\Lambda -$ gives a one-one correspondence between the set of isomorphism classes of indecomposable projective Λ-modules X with $\Gamma \otimes_\Lambda X \neq 0$ and the set of isomorphism classes of indecomposable projective Γ-modules.*

The following lemma is standard for change of rings .

Lemma 2.2. *Let $f : \Lambda \to \Gamma$ be a homomorphism of Artin algebras, and let X be a Λ-module and U a Γ-module.*

(1) *If f is surjective, then so is $f^* : X \to \Gamma \otimes_\Lambda X$.*

(2) *There is a natural isomorphism $\mathrm{Hom}_\Gamma(\Gamma \otimes_\Lambda X, U) \to \mathrm{Hom}_\Lambda(X, U)$ sending g to f^*g.*

Using Lemma 2.2, we can extend results on modules to complexes. Let $f : \Lambda \to \Gamma$ be a homomorphism of Artin algebras. Then we have a functor $\Gamma \otimes^\bullet_\Lambda - : \mathscr{C}(\Lambda) \to \mathscr{C}(\Gamma)$, which has the restriction functor as its right adjoint functor. So, the unit of this adjoint pair of functors provides a natural chain map $f^* : X^\bullet \to \Gamma \otimes^\bullet_\Lambda X^\bullet$ for $X^\bullet \in \mathscr{C}(\Lambda)$. More precisely, f^* is defined by $f^i : X^i \to \Gamma \otimes_\Lambda X^i$ for all integers i. As in the case of modules, we have the following lemma for complexes. Its proof is just a consequence of the universal properties of units of adjoint functors.

Lemma 2.3. *Let $f : \Lambda \to \Gamma$ be a homomorphism between Artin algebras Λ and Γ. Then, for any $X^\bullet \in \mathscr{C}(\Lambda)$ and $U^\bullet \in \mathscr{C}(\Gamma)$, we have the following:*

(1) *The morphism $\mathrm{Hom}_{\mathscr{C}(\Gamma)}(\Gamma \otimes^\bullet_\Lambda X^\bullet, U^\bullet) \to \mathrm{Hom}_{\mathscr{C}(\Lambda)}(X^\bullet, U^\bullet)$ sending $h^\bullet$ to $f^*h^\bullet$ is a natural isomorphism.*

(2) *The morphism $\mathrm{Hom}_{\mathscr{K}(\Gamma)}(\Gamma \otimes^\bullet_\Lambda X^\bullet, U^\bullet) \to \mathrm{Hom}_{\mathscr{K}(\Lambda)}(X^\bullet, U^\bullet)$ sending $h^\bullet$ to $f^*h^\bullet$ is a natural isomorphism.*

(3) *If $U^\bullet \simeq \Gamma \otimes^\bullet_\Lambda X^\bullet$ in $\mathscr{C}(\Gamma)$, then, for each epimorphism $g^\bullet : X^\bullet \to {}_\Lambda U^\bullet$ in $\mathscr{C}(\Lambda)$, there is an isomorphism $h^\bullet : \Gamma \otimes^\bullet_\Lambda X^\bullet \to U^\bullet$ in $\mathscr{C}(\Gamma)$ such that $g^\bullet = f^*h^\bullet$.*

Let $X^\bullet \in \mathscr{C}(\Lambda)$, $U^\bullet \in \mathscr{C}(\Gamma)$ and $g^\bullet : X^\bullet \to {}_\Lambda U^\bullet$ be a chain map in $\mathscr{C}(\Lambda)$. If, for each morphism $\alpha^\bullet : X^\bullet \to X^\bullet$ in $\mathscr{K}(\Lambda)$, there is a unique morphism $\beta^\bullet : U^\bullet \to U^\bullet$ in $\mathscr{K}(\Gamma)$ such that $g^\bullet\beta^\bullet = \alpha^\bullet g^\bullet$ in $\mathscr{K}(\Lambda)$, then the map

$$\mathrm{End}_{\mathscr{K}(\Lambda)}(X^\bullet) \longrightarrow \mathrm{End}_{\mathscr{K}(\Gamma)}(U^\bullet)$$

sending $\alpha^\bullet$ to $\beta^\bullet$ is a homomorphism of algebras, which is called the *algebra homomorphism determined by* $g^\bullet$. According to Lemma 2.3(2), the morphism $f^* : X^\bullet \to \Gamma \otimes^\bullet_\Lambda X^\bullet$ determines an algebra homomorphism:

$$\mathrm{End}_{\mathscr{K}(\Lambda)}(X^\bullet) \longrightarrow \mathrm{End}_{\mathscr{K}(\Gamma)}(\Gamma \otimes^\bullet_\Lambda X^\bullet).$$

By universal property of units of adjoint functors, we know that the above homomorphism of algebras is actually given by $\alpha \mapsto \Gamma \otimes^\bullet_\Lambda \alpha$ for $\alpha \in \mathrm{End}_{\mathscr{K}(\Lambda)}(X^\bullet)$.

2.3 Simple modules under derived equivalences

Let Λ be an Artin algebra and Y be an indecomposable Λ-module. For each Λ-module X, we decompose X into a direct sum of indecomposable modules, say $X = \bigoplus_{i=1}^n X_i$, and let $[X : Y]$ be the multiplicity of Y as a direct summand of X, that is, the number of those X_j with $X_j \simeq Y$. Note that $[X : Y]$ is independent of the choice of the decomposition of X. For a bounded complex $X^\bullet$ over Λ-mod, we define

$$[X^\bullet : Y] := \sum_{i \in \mathbb{Z}} [X^i : Y].$$

Note that $[X^\bullet : Y]$ is well defined in $\mathscr{C}^{\mathrm{b}}(\Lambda)$ by the Krull-Remak-Schmidt Theorem. In the following, we denote by S_P the top of a projective module P.

Lemma 2.4. *Let $T^\bullet$ be a basic, radical tilting complex over Λ, and let $\Gamma := \mathrm{End}_{\mathscr{K}^{\mathrm{b}}(\Lambda)}(T^\bullet)$. Suppose that $F : \mathscr{D}^{\mathrm{b}}(\Lambda) \to \mathscr{D}^{\mathrm{b}}(\Gamma)$ is a derived equivalence induced by $T^\bullet$ and that P is an indecomposable projective Λ-module. Then $F(S_P)$ is isomorphic in $\mathscr{D}^{\mathrm{b}}(\Gamma)$ to $S[n]$ for some simple Γ-module S and some integer n if and only if $[T^\bullet : P] = 1$.*

Proof. Suppose $[T^\bullet : P] = 1$. Then there is some integer n such that $\mathrm{Hom}_{\mathscr{K}(\Lambda)}(T^\bullet, S_P[i]) = 0$ for all $i \neq -n$. Hence $F(S_P[-n])$ is isomorphic in $\mathscr{D}^{\mathrm{b}}(\Gamma)$ to a Γ-module X. Now we prove that X is simple. Since $[T^\bullet : P] = 1$, there is only one indecomposable direct summand $T_P^\bullet$ of $T^\bullet$ such that P occurs in $T_P^\bullet$. Let $\bar{P}$ be the indecomposable projective Γ-module $F(T_P^\bullet)$. Then $\mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(\Lambda)}(T^\bullet, S_P[-n]) \simeq \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(\Lambda)}(T_P^\bullet, S_P[-n])$, or equivalently $\mathrm{Hom}_\Gamma(\Gamma, X) \simeq \mathrm{Hom}_\Gamma(\bar{P}, X)$. This means that X only contains composition factors isomorphic to $S_{\bar{P}}$. Moreover, $\mathrm{End}_\Gamma(X) \simeq \mathrm{End}_\Lambda(S_P)$ is a division algebra. Hence X must be simple. Note that in the foregoing proof we only need $T^\bullet$ to be a tilting complex.

Conversely, suppose that $F(S_P[k])$ is isomorphic to a simple Γ-module S for some integer k. Then, by assumption, Γ is a basic algebra and S is a 1-dimensional module over $D := \mathrm{End}_\Gamma(S)$. Thus $\mathrm{Hom}_{\mathscr{K}(\Lambda)}(T^\bullet, S_P[i])$ is zero for all $i \neq k$, and 1-dimensional over D for $i = k$. Since $T^\bullet$ is a radical complex,

$$\mathrm{Hom}_{\mathscr{K}(\Lambda)}(T^\bullet, S_P[i]) \simeq \mathrm{Hom}_\Lambda(T^{-i}, S_P)$$

for all integers i. This implies that the indecomposable projective module P occurs in $T^\bullet$ only in degree $-k$ with the multiplicity 1. Hence $[T^\bullet : P] = 1$. $\square$

As an immediate consequence of the proof of Lemma 2.4, we get the following corollary for tilting modules.

Corollary 2.5. *If $T = P \oplus P'$ is a basic titling Λ-module, where P is projective and P' has a minimal projective resolution $Q^\bullet = (Q^i, d^i)_{i \leq 0}$ such that each indecomposable direct summand of P does not appear in $\bigoplus_{i \geq 0} Q^{-i}$, then there exists a derived equivalence $F : \mathscr{D}^{\mathrm{b}}(\Lambda) \to \mathscr{D}^{\mathrm{b}}(\mathrm{End}_\Lambda(T))$ such that $F(S)$ is isomorphic to a simple $\mathrm{End}_\Lambda(T)$-module for all simple modules $S \in \mathrm{add}(S_P)$.*

The following lemma is very useful in our later proofs.

Lemma 2.6. *Let $\{U_1, \cdots, U_s, V_1, \cdots, V_r\}$ be a complete set of pairwise non-isomorphic indecomposable projective Λ-modules and let $U := \bigoplus_{i=1}^s U_i$. Suppose that $T^\bullet$ is a basic, radical tilting complex over Λ with $[T^\bullet : V_i] = 1$ for all $1 \leq i \leq r$. Then $T^\bullet$ can be written as a direct sum (in $\mathscr{K}^{\mathrm{b}}(\Lambda)$)*

$$T^\bullet \simeq U^\bullet \oplus V_1^\bullet \oplus \cdots \oplus V_r^\bullet$$

of complexes $U^\bullet$ and $V_i^\bullet$ with $1 \leq i \leq r$, satisfying the following properties:

(a) $[V_i^\bullet : V_j] = 1$ *for $i = j$, and zero otherwise. Moreover, all $V_i^\bullet$ are indecomposable complexes.*

(b) $U^\bullet \in \mathscr{K}^{\mathrm{b}}(\mathrm{add}(U))$, *and* $\mathrm{add}(U^\bullet)$ *generates* $\mathscr{K}^{\mathrm{b}}(\mathrm{add}(U))$ *as a triangulated category.*

Proof. Let $\Gamma := \mathrm{End}_{\mathscr{K}(\Lambda)}(T^\bullet)$ and $F : \mathscr{D}^{\mathrm{b}}(\Lambda) \to \mathscr{D}^{\mathrm{b}}(\Gamma)$ be a derived equivalence induced by the tilting complex $T^\bullet$. By Lemma 2.4, there are pairwise non-isomorphic indecomposable projective Γ-modules $\bar{V}_1, \cdots, \bar{V}_r$ such that $F(\mathrm{top}(V_i)) \simeq \mathrm{top}(\bar{V}_i)[n_i]$ for some n_i with $1 \leq i \leq r$. Let $\bar{U}_1, \cdots, \bar{U}_s$ be indecomposable projective Γ-modules such that $\{\bar{U}_1, \cdots, \bar{U}_s, \bar{V}_1, \cdots, \bar{V}_r\}$ is a complete set of pairwise non-isomorphic indecomposable projective Γ-modules and set $\bar{U} := \bigoplus_{i=1}^s \bar{U}_i$. Since $T^\bullet$ is a basic tilting complex, Γ is a basic algebra, and therefore

$${}_\Gamma\Gamma \simeq \bar{U} \oplus \bar{V}_1 \oplus \cdots \oplus \bar{V}_r.$$

By definition, $F(T^\bullet) \simeq {}_\Gamma\Gamma$. Now, let $U^\bullet$ be a direct summand of $T^\bullet$ such that $F(U^\bullet) \simeq \bar{U}$ and let $V_i^\bullet$ be a direct summand of $T^\bullet$ such that $F(V_i^\bullet) \simeq \bar{V}_i$ for $1 \leq i \leq r$. Then $F(U^\bullet \oplus V_1^\bullet \oplus \cdots \oplus V_r^\bullet) \simeq {}_\Gamma\Gamma \simeq F(T^\bullet)$, and consequently

$$T^\bullet \simeq U^\bullet \oplus V_1^\bullet \oplus \cdots \oplus V_r^\bullet \text{ in } \mathscr{D}^{\mathrm{b}}(\Lambda).$$

This implies $T^\bullet \simeq U^\bullet \oplus V_1^\bullet \oplus \cdots \oplus V_r^\bullet$ in $\mathscr{K}^{\mathrm{b}}(\Lambda)$. Now we have

$$\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(\Lambda)}(V_i^\bullet, \mathrm{top}(V_j)[k]) \simeq \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(\Lambda)}(V_i^\bullet, \mathrm{top}(V_j)[k]) \simeq \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(\Gamma)}(\bar{V}_i, \mathrm{top}(\bar{V}_j)[k + n_j]) = 0$$

whenever $i \neq j$ or $k \neq -n_j$. By assumption, $[T^\bullet : V_i] = 1$ for $1 \leq i \leq r$. This forces that the projective module V_i only occurs in the $(-n_i)$-th degree of $V_i^\bullet$.

Now, it is easy to see that all complexes $V_i^\bullet$ can be chosen to be indecomposable. This proves (a).

By (a) and our assumption $[T^\bullet : V_i] = 1$ for all i, the complex $U^\bullet$ is clearly in $\mathscr{K}^{\mathrm{b}}(\mathrm{add}(U))$. Now we show that F induces a triangle equivalence between $\mathscr{K}^{\mathrm{b}}(\mathrm{add}(U))$ and $\mathscr{K}^{\mathrm{b}}(\mathrm{add}(\bar{U}))$. In fact, a complex $P^\bullet$ from $\mathscr{K}^{\mathrm{b}}(\Lambda\text{-proj})$ lies in $\mathscr{K}^{\mathrm{b}}(\mathrm{add}(U))$ if and only if $\mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(\Lambda)}(P^\bullet, \mathrm{top}(V_i)[k]) = 0$ for all $1 \leq i \leq r$ and all $k \in \mathbb{Z}$. However, this is equivalent to $\mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(\Gamma)}(F(P^\bullet), \mathrm{top}(\bar{V}_i)[k + n_i]) = 0$ for all $1 \leq i \leq r$ and all $k \in \mathbb{Z}$, that is, $F(P^\bullet)$ belongs to $\mathscr{K}^{\mathrm{b}}(\mathrm{add}(\bar{U}))$. Hence F induces a triangle equivalence between $\mathscr{K}^{\mathrm{b}}(\mathrm{add}(U))$ and $\mathscr{K}^{\mathrm{b}}(\mathrm{add}(\bar{U}))$. Since $\mathrm{add}(\bar{U})$ generates $\mathscr{K}^{\mathrm{b}}(\mathrm{add}(\bar{U}))$ as a triangulated category, $\mathrm{add}(U^\bullet)$ generates $\mathscr{K}^{\mathrm{b}}(\mathrm{add}(U))$ as a triangulated category. This proves (b). $\square$

2.4 Projective modules over Milnor squares of algebras

Let A_0, A_1 and A_2 be rings with identity. Given two homomorphisms $\pi_i : A_i \to A_0$ of rings, the *pullback ring* A of π_1 and π_2 is defined by $A := \{(x,y) \in A_1 \times A_2 \mid (x)\pi_1 = (y)\pi_2\}$. Transparently, there is a commutative diagram of ring homomorphisms

$$\begin{tikzcd} A \arrow[r, "\lambda_1"] \arrow[d, "\lambda_2"'] & A_1 \arrow[d, "\pi_1"] \\ A_2 \arrow[r, "\pi_2"] & A_0 \end{tikzcd}$$

where λ_i is the canonical projections from A to A_i for $i = 1,2$. The above pullback diagram has a universal property: For any ring homomorphisms $i_1 : B \to A_1$ and $i_2 : B \to A_2$ with $i_1\pi_1 = i_2\pi_2$, there is a unique ring homomorphism $\theta : B \to A$ such that $\theta\lambda_j = i_j$ for $j = 1,2$. Note that if π_1 is surjective then so is λ_2.

If, additionally, one of π_1 and π_2 is surjective, then the above square is called a *Milnor square* of rings (see [14]). For a Milnor square of rings, there is a nice description of projective A-modules via projective A_i-modules in [14]. Let us recall it right now.

Given a projective A_1-module X_1, a projective A_2-module X_2 and an isomorphism $h : A_0 \otimes_{A_1} X_1 \to A_0 \otimes_{A_2} X_2$ of A_0-modules, the *Milnor patching* of the triple (X_1, X_2, h) is defined by

$$M(X_1,X_2,h) := \big\{(x_1,x_2) \in X_1 \oplus X_2 \mid (x_1)\pi_1^* h = (x_2)\pi_2^*\big\} = \{(x_1,x_2) \in X_1 \oplus X_2 \mid (1\otimes x_1)h = 1 \otimes x_2\}.$$

Let $p_i : M(X_1,X_2,h) \to X_i$ be the canonical projection. Note that $M(X_1,X_2,h)$ has an A-module structure: For $a \in A$,

$$a \cdot (x_1,x_2) = \big((a)\lambda_1 \cdot x_1, (a)\lambda_2 \cdot x_2\big) \text{ for } x_1 \in X_1, x_2 \in X_2.$$

Now, we state the following description of projective A-modules given in [14, Chapter 2].

Lemma 2.7. *Suppose that π_1 is surjective, X_i is a projective A_i-module for $i = 1,2$, and $h : A_0 \otimes_{A_1} X_1 \to A_0 \otimes_{A_2} X_2$ is an isomorphism of A_0-modules. Then we have the following:*

(1) *The module $M(X_1,X_2,h)$ is a projective A-module. Furthermore, if, in addition, X_1 and X_2 are finitely generated over A_1 and A_2, respectively, then $M(X_1,X_2,h)$ is finitely generated over A.*

(2) *Every projective A-module is isomorphic to $M(X_1,X_2,h)$ for some suitably chosen X_1, X_2 and h.*

(3) *For $i \in \{1,2\}$, there is a natural isomorphism*

$$\mu_i : A_i \otimes_A M(X_1,X_2,h) \longrightarrow X_i$$

sending $a_i \otimes (x_1,x_2)$ to $a_i x_i$, and the canonical projection $p_i : M(X_1,X_2,h) \to X_i$ is equal to $\lambda_i^*\mu_i$.

(4) *There is an exact sequence of A-modules:*

$$0 \longrightarrow M(X_1,X_2,h) \xrightarrow{[p_1,p_2]} X_1 \oplus X_2 \xrightarrow{\begin{bmatrix} \pi_1^* h \\ -\pi_2^* \end{bmatrix}} A_0 \otimes_{A_2} X_2 \longrightarrow 0 \ .$$

Proof. The statements (1), (2) and (3) are just [14, Theorems 2.1, 2.2, and 2.3, p. 20]. The statement (4) follows easily from the definition of $M(X_1,X_2,h)$ and the fact that π_1^* is surjective. $\square$

For the rest of this section, we shall assume that A_0, A_1 and A_2 are Artin algebras and that π_1 is surjective. Thus we have an exact sequence of A-bimodules:

$$(*) \quad 0 \longrightarrow A \xrightarrow{[\lambda_1,\lambda_2]} A_1 \oplus A_2 \xrightarrow{\begin{bmatrix} \pi_1 \\ -\pi_2 \end{bmatrix}} A_0 \longrightarrow 0.$$

In the following, we shall give a partition of indecomposable projective A-modules.

Let P_1 be a direct sum of all non-isomorphic indecomposable projective A_1-modules X such that $A_0 \otimes_{A_1} X = 0$, and let Q_1 be a direct sum of all non-isomorphic indecomposable projective A_1-modules Y such that $A_0 \otimes_{A_1} Y \neq 0$. Thus $A_1\text{-proj} = \mathrm{add}(P_1 \oplus Q_1)$. Similarly, we define projective A_2-modules P_2 and Q_2, and get $A_2\text{-proj} = \mathrm{add}(P_2 \oplus Q_2)$.

Since π_1 is surjective, λ_2 is also surjective. Therefore, if X is an indecomposable projective A-module with $A_2 \otimes_A X \neq 0$, then $A_2 \otimes_A X$ is an indecomposable projective A_2-module by Lemma 2.1. Hence, for an indecomposable projective A-module X, only the following three cases occur:

- Case 1: $A_2 \otimes_A X = 0$.
- Case 2: $0 \neq A_2 \otimes_A X \in \mathrm{add}(P_2)$.
- Case 3: $0 \neq A_2 \otimes_A X \in \mathrm{add}(Q_2)$.

According to the three cases, we have a partition of indecomposable projective A-modules: For $1 \leq i \leq 3$, let F_i be the direct sum of all non-isomorphic indecomposable projective A-modules X corresponding to Case i. Then $A\text{-proj} = \mathrm{add}(F_1 \oplus F_2 \oplus F_3)$.

Lemma 2.8. *With the above notation, we have the following:*

(1) *The functor $A_i \otimes_A -$ and the restriction functor ${}_A(-)$ induce mutually inverse equivalences between* $\mathrm{add}(F_i)$ *and* $\mathrm{add}(P_i)$ *for $i = 1,2$.*

(2) *Let $i \in \{1,2\}$ and $X \in \mathrm{add}(P_i)$. Then the natural map* $\mathrm{Hom}_A({}_AX, A) \to \mathrm{Hom}_{A_i}(X, A_i)$, *sending α to $\alpha\lambda_i$, is an isomorphism of right A-modules.*

(3) *Let $i \in \{1,2\}$ and $X \in \mathrm{add}(P_i)$. If* $\mathrm{add}({}_{A_i}X) = \mathrm{add}(\nu_{A_i}X)$, *then* $\mathrm{add}({}_AX) = \mathrm{add}(\nu_AX)$, *where ν_A is the Nakayama functor $D\mathrm{Hom}_A(-, {}_AA)$ of A.*

Proof. (1) We prove the case $i = 1$. For each X in $\mathrm{add}(F_1)$, we have $A_2 \otimes_A X = 0$, and therefore $A_0 \otimes_{A_1} A_1 \otimes_A X \simeq A_0 \otimes_{A_2} A_2 \otimes_A X = 0$ and $A_1 \otimes_A X \in \mathrm{add}(P_1)$. Thus $X \simeq M(A_1 \otimes_A X, 0, 0)$ and the map $\lambda_1^* : X \to A_1 \otimes_A X$ is a bijection by the definition of $M(A_1 \otimes_A X, 0, 0)$. It follows from the statements (1) and (3) in Lemma 2.7 that, for X and Y in $\mathrm{add}(F_1)$, the functor $A_1 \otimes_A -$ induces an isomorphism from $\mathrm{Hom}_A(X,Y)$ to $\mathrm{Hom}_{A_1}(A_1 \otimes_A X, A_1 \otimes_A Y)$. Moreover, for each U in $\mathrm{add}(P_1)$, the module $M(U,0,0)$ is a projective module in $\mathrm{add}(F_1)$ such that $A_1 \otimes_A M(U,0,0) \simeq U$. This shows that the functor $A_1 \otimes_A - : \mathrm{add}(F_1) \to \mathrm{add}(P_1)$ is an equivalence. Clearly, the restriction functor ${}_A(-)$ is right adjoint to $A_1 \otimes_A -$ by Lemma 2.2(2), and therefore a quasi-inverse of $A_1 \otimes_A -$. This proves (1) for $i = 1$. The case $i = 2$ can be shown similarly.

(2) Assume both $i = 1$ and $X \in \mathrm{add}(P_1)$. By Lemma 2.2, $\mathrm{Hom}_A({}_AX, A_i) \simeq \mathrm{Hom}_{A_i}(A_i \otimes_A X, A_i)$ for $0 \leq i \leq 2$. Since $X \in \mathrm{add}(P_1)$, we have ${}_AX \in \mathrm{add}(F_1)$ by (1), and consequently $A_2 \otimes_A X = 0$ and $A_0 \otimes_A X \simeq A_0 \otimes_{A_2} A_2 \otimes_A X = 0$. Therefore $\mathrm{Hom}_A({}_AX, A_0) = 0 = \mathrm{Hom}_A({}_AX, A_2)$. Applying $\mathrm{Hom}_A({}_AX, -)$ to the exact sequence $(*)$ of A-bimodules, we get an isomorphism of right A-modules:

$$\mathrm{Hom}_A({}_AX, A) \longrightarrow \mathrm{Hom}_A({}_AX, {}_AA_1),$$

which sends α to $\alpha\lambda_1$. For the case $i = 2$, a proof can be demonstrated similarly.

(3) Without loss of generality, we can assume that the module X is basic. Then it follows from $\mathrm{add}({}_{A_i}X) = \mathrm{add}(\nu_{A_i}X)$ that $\nu_{A_i}X \simeq X$. This together with (2) implies the following isomorphisms:

$$\nu_AX = D\mathrm{Hom}_A({}_AX, A) \simeq D(\mathrm{Hom}_{A_i}(X, A_i)_A) = {}_A(\nu_{A_i}X) \simeq {}_AX.$$

Thus (3) follows. $\square$

The next lemma describes indecomposable projective A-modules in $\mathrm{add}(F_3)$.

Lemma 2.9. (1) *For each indecomposable A_2-module V in $\mathrm{add}(Q_2)$, there is an A_1-module W (unique up to isomorphism) in $\mathrm{add}(Q_1)$ with an isomorphism $h : A_0 \otimes_{A_1} W \to A_0 \otimes_{A_2} V$ such that $M(W,V,h)$ is an indecomposable projective A-module in $\mathrm{add}(F_3)$.*

(2) *Let $\{V_1, \cdots, V_s\}$ be a complete set of pairwise non-isomorphic indecomposable projective A_2-modules in $\mathrm{add}(Q_2)$, and let $W_i \in \mathrm{add}(Q_1)$ be the projective A_1-module determined by V_i in (1) for $1 \leq i \leq s$. Then $\{M(W_i, V_i, h_i) \mid 1 \leq i \leq s\}$ is a complete set of pairwise non-isomorphic indecomposable projective A-modules in $\mathrm{add}(F_3)$.*

Proof. (1) Since π_1 is surjective, it follows from Lemma 2.1 that there is an A_1-module W (unique up to isomorphism) and an isomorphism $h : A_0 \otimes_{A_1} W \to A_0 \otimes_{A_2} V$. We need to show that $M(W,V,h)$ is in $\mathrm{add}(F_3)$. Let X be an indecomposable direct summand of $M(W,V,h)$. Then there are two possibilities: $A_2 \otimes_A X \neq 0$ or $A_2 \otimes_A X = 0$. If $A_2 \otimes_A X \neq 0$, then $A_2 \otimes_A X$ is a direct summand of V. Since V is indecomposable, we have $A_2 \otimes_A X \simeq V$. By definition, $X \in \mathrm{add}(F_3)$. Now, we exclude the case $A_2 \otimes_A X = 0$. If this happens, then $A_1 \otimes_A X \neq 0$. Otherwise $X \simeq M(A_1 \otimes_A X, A_2 \otimes_A X, g) = 0$. So $A_1 \otimes_A X$ is a nonzero direct summand of W. However, by definition, $X \in \mathrm{add}(F_1)$. It follows from Lemma 2.8(1) that the module $A_1 \otimes_A X$ lies in

$\operatorname{add}(P_1)$. This is a contradiction. Thus $M(W,V,h) \in \operatorname{add}(F_3)$. Since $A_2 \otimes_A M(W,V,h) \simeq V$ is indecomposable, the module $M(W,V,h)$ is indecomposable by Lemma 2.1.

(2) It follows from (1) that $M(W_i,V_i,h_i) \in \operatorname{add}(F_3)$ is indecomposable for all $1 \le i \le s$. Now, let X be an indecomposable A-module in $\operatorname{add}(F_3)$. Then the A_2-module $A_2 \otimes_A X$ is indecomposable since λ_2 is surjective. Thus, there is some V_i such that $A_2 \otimes_A X \simeq V_i \simeq A_2 \otimes_A M(W_i,V_i,h_i)$. By Lemma 2.1, we have $X \simeq M(W_i,V_i,h_i)$. This finishes the proof. □

Finally, we extend previous facts on modules to the ones on complexes of modules.

Given a complex $X_1^\bullet$ in $\mathscr{C}^{\mathrm{b}}(A_1\text{-proj})$ and a complex $X_2^\bullet$ in $\mathscr{C}^{\mathrm{b}}(A_2\text{-proj})$ together with an isomorphism $h^\bullet : A_0 \otimes_{A_1}^\bullet X_1^\bullet \to A_0 \otimes_{A_2}^\bullet X_2^\bullet$ in $\mathscr{C}(A_0)$, we define a complex $M(X_1^\bullet, X_2^\bullet, h^\bullet) := (M(X_1^i, X_2^i, h^i), d^i)_{i\in\mathbb{Z}}$, where the differential is induced by the exact sequence given in Lemma 2.7(4). For this complex, we have the following results similar to Lemma 2.7.

Lemma 2.10. *Suppose $X_1^\bullet \in \mathscr{C}^{\mathrm{b}}(A_1\text{-proj})$, $X_2^\bullet \in \mathscr{C}^{\mathrm{b}}(A_2\text{-proj})$ and $h^\bullet : A_0 \otimes_{A_1}^\bullet X_1^\bullet \to A_0 \otimes_{A_2}^\bullet X_2^\bullet$ is an isomorphism in $\mathscr{C}(A_0)$. Then the following hold:*

(1) *The complex $M(X_1^\bullet, X_2^\bullet, h^\bullet)$ is a bounded complex over A-proj.*

(2) *For $i \in \{1,2\}$, there is a natural isomorphism of complexes*

$$\mu_i^\bullet : A_i \otimes_A^\bullet M(X_1^\bullet, X_2^\bullet, h^\bullet) \longrightarrow X_i^\bullet$$

sending $a_i \otimes (x_1^j, x_2^j)$ to $a_i x_i^j$, and the canonical projection $p_i^\bullet : M(X_1^\bullet, X_2^\bullet, h^\bullet) \to X_i^\bullet$ is equal to $\lambda_i^* \mu_i^\bullet$.

(3) *There is an exact sequence of complexes of A-modules:*

$$0 \longrightarrow M(X_1^\bullet, X_2^\bullet, h^\bullet) \xrightarrow{[p_1^\bullet, p_2^\bullet]} X_1^\bullet \oplus X_2^\bullet \xrightarrow{\begin{bmatrix} \pi_1^* h^\bullet \\ -\pi_2^* \end{bmatrix}} A_0 \otimes_{A_2}^\bullet X_2^\bullet \longrightarrow 0,$$

whers $p_i^\bullet$ is induced by the canonical projection p_i for $i = 1,2$.

(4) *Set $X^\bullet := M(X_1^\bullet, X_2^\bullet, h^\bullet)$ and $X_0^\bullet := A_0 \otimes_{A_2}^\bullet X_2^\bullet$. If $\operatorname{Hom}_{\mathscr{K}(A_0)}(X_0^\bullet, X_0^\bullet[-1]) = 0$, then there exists a pullback diagram of algebras:*

$$\begin{tikzcd}
\operatorname{End}_{\mathscr{K}(A)}(X^\bullet) \arrow[r, "\varepsilon_1"] \arrow[d, "\varepsilon_2"'] & \operatorname{End}_{\mathscr{K}(A_1)}(X_1^\bullet) \arrow[d, "\eta_1"] \\
\operatorname{End}_{\mathscr{K}(A_2)}(X_2^\bullet) \arrow[r, "\eta_2"] & \operatorname{End}_{\mathscr{K}(A_0)}(X_0^\bullet),
\end{tikzcd}$$

where $\varepsilon_1, \varepsilon_2, \eta_1$ and η_2 are homomorphisms of algebras, determined by $p_1^\bullet$, $p_2^\bullet$, $\pi_1^* h^\bullet$ and π_2^*, respectively.

Proof. The statements (1)-(3) follow immediately from the definition of $M(X_1^\bullet, X_2^\bullet, h^\bullet)$ and Lemma 2.7(1)-(3). Now, we prove (4). Since $X^\bullet \in \mathscr{K}^{\mathrm{b}}(A\text{-proj})$, it follows from the triangle

$$X^\bullet \xrightarrow{[p_1^\bullet, p_2^\bullet]} X_1^\bullet \oplus X_2^\bullet \xrightarrow{\begin{bmatrix} \pi_1^* h^\bullet \\ -\pi_2^* \end{bmatrix}} X_0^\bullet \longrightarrow X^\bullet[1]$$

in $\mathscr{D}^{\mathrm{b}}(A)$ that the following long sequence is exact:

$$\cdots \to \operatorname{Hom}_{\mathscr{K}(A)}(X^\bullet, X_0^\bullet[-1]) \to \operatorname{Hom}_{\mathscr{K}(A)}(X^\bullet, X^\bullet) \to \operatorname{Hom}_{\mathscr{K}(A)}(X^\bullet, X_1^\bullet \oplus X_2^\bullet) \to \operatorname{Hom}_{\mathscr{K}(A)}(X^\bullet, X_0^\bullet) \to \cdots.$$

Note that $\operatorname{Hom}_{\mathscr{K}(A)}(X^\bullet, X_i^\bullet[j]) \simeq \operatorname{Hom}_{\mathscr{K}(A_i)}(A_i \otimes_A^\bullet X^\bullet, X_i^\bullet[j]) \simeq \operatorname{Hom}_{\mathscr{K}(A_i)}(X_i^\bullet, X_i^\bullet[j])$ for $j \in \mathbb{Z}$, it follows from the assumption in (4) that $\operatorname{Hom}_{\mathscr{K}(A)}(X^\bullet, X_0^\bullet[-1]) = 0$ and the above sequence is then isomorphic to

$$0 \longrightarrow \operatorname{End}_{\mathscr{K}(A)}(X^\bullet) \xrightarrow{[\varepsilon_1, \varepsilon_2]} \operatorname{End}_{\mathscr{K}(A_1)}(X_1^\bullet) \oplus \operatorname{End}_{\mathscr{K}(A_2)}(X_2^\bullet) \xrightarrow{\begin{bmatrix} \eta_1 \\ -\eta_2 \end{bmatrix}} \operatorname{End}_{\mathscr{K}(A_0)}(X_0^\bullet).$$

This proves (4). □

3 Derived equivalences for Milnor squares of algebras

In this section, we first state and prove our main result, Theorem 3.1, on general derived equivalences, and then turn to almost ν-stable derived equivalences (see Corollary 3.4). These derived equivalences induce stable equivalences of Morita type (see [8]), while the latter is of interest in an approach to Broué's abelian defect group conjecture (see [9, 17]).

3.1 General result

The main result of this paper is the following theorem.

Theorem 3.1. *Suppose that $A_1 \xrightarrow{\pi_1} A_0 \xleftarrow{\pi_2} A_2$ are homomorphisms of Artin algebras with π_1 surjective. Let $T_i^\bullet$ be a basic, radical tilting complex over A_i with $B_i := \mathrm{End}_{\mathscr{K}^{\mathrm{b}}(A_i)}(T_i^\bullet)$ for $0 \le i \le 2$. If $T_0^\bullet$ is a direct sum of shifts of projective A_0-modules and there is an isomorphism $A_0 \otimes^\bullet_{A_i} T_i^\bullet \simeq T_0^\bullet$ of complexes for $i = 1, 2$, then there exist homomorphisms $B_1 \xrightarrow{\eta_1} B_0 \xleftarrow{\eta_2} B_2$ of Artin algebras with η_1 surjective such that the pullback algebra B of η_1 and η_2 is derived equivalent to the pullback algebra A of π_1 and π_2.*

Thus, it follows immediately from derived invariants that the algebras A and B in Theorem 3.1 share many common properties. For instance, they have the same Hochschild (co)homology rings, Coxeter polynomials, and algebraic K-theory. For a list of derived invariants, see, for example, [19] and the references therein.

Remark that if A_0 is a product of local algebras, or a self-injective algebra with radical-square zero, then every tilting complex over A_0 is a direct sum of shifts of projective A_0-modules.

To prove Theorem 3.1, we first show the following lemma.

Lemma 3.2. *Let $f : \Lambda \to \Gamma$ be a surjective homomorphism between Artin algebras Λ and Γ. If $T^\bullet$ is a basic, radical tilting complex over Λ such that $\Gamma \otimes^\bullet_\Lambda T^\bullet$ is a basic tilting complex over Γ of the form $\bigoplus_{i=1}^r X_i[n_i]$, where $\{X_1, \cdots, X_r\}$ is a complete set of non-isomorphic indecomposable projective Γ-modules, then the induced morphism*

$$\mathrm{Hom}_{\mathscr{K}(\Lambda)}(T^\bullet, f^*) : \mathrm{Hom}_{\mathscr{K}(\Lambda)}(T^\bullet, T^\bullet) \longrightarrow \mathrm{Hom}_{\mathscr{K}(\Lambda)}(T^\bullet, {}_\Lambda\Gamma \otimes^\bullet_\Lambda T^\bullet)$$

is surjective.

Proof. By Lemma 2.1, we can assume that there is a complete set $\{V_1, \cdots, V_r, U_1, \cdots, U_s\}$ of pairwise non-isomorphic indecomposable projective Λ-modules such that $\Gamma \otimes_\Lambda V_i \simeq X_i$ for all $i = 1, \cdots, r$, and that $\Gamma \otimes_\Lambda U_i = 0$ for all $i = 1, \cdots, s$. Set $U := \bigoplus_{i=1}^s U_i$. By our assumption, $[\Gamma \otimes_\Lambda T^\bullet : X_i] = 1$ for all $1 \le i \le r$. This implies that $[T^\bullet : V_i] = 1$ for $1 \le i \le r$. So, by Lemma 2.6, we can write $T^\bullet$ as

$$T^\bullet := U^\bullet \oplus V_1^\bullet \oplus \cdots \oplus V_r^\bullet$$

such that $U^\bullet \in \mathscr{K}^{\mathrm{b}}(\mathrm{add}(U))$, and $[V_i^\bullet : V_j] = 1$ for $i = j$ and zero otherwise. Thus $\Gamma \otimes_\Lambda U^\bullet = 0$ and $\Gamma \otimes^\bullet_\Lambda V_i^\bullet \simeq (\Gamma \otimes_\Lambda V_i)[n_i] \simeq X_i[n_i]$ for some integer n_i. It is sufficient to prove that

$$\mathrm{Hom}_{\mathscr{K}(\Lambda)}(T^\bullet, f^*) : \mathrm{Hom}_{\mathscr{K}(\Lambda)}(T^\bullet, V_i^\bullet) \longrightarrow \mathrm{Hom}_{\mathscr{K}(\Lambda)}(T^\bullet, {}_\Lambda\Gamma \otimes^\bullet_\Lambda V_i^\bullet)$$

is surjective for all $i = 1, \cdots, r$.

In the following, we set $\Sigma := \mathrm{End}_{\mathscr{K}(\Lambda)}(T^\bullet)$. Since

$$\begin{aligned} \mathrm{Hom}_\Sigma\big(\mathrm{Hom}_{\mathscr{K}(\Lambda)}(T^\bullet, U^\bullet), \mathrm{Hom}_{\mathscr{K}(\Lambda)}(T^\bullet, \Gamma \otimes^\bullet_\Lambda V_i^\bullet)\big) &\simeq \mathrm{Hom}_{\mathscr{K}(\Lambda)}\big(U^\bullet, \Gamma \otimes^\bullet_\Lambda V_i^\bullet)\big) \\ &\simeq \mathrm{Hom}_{\mathscr{K}(\Gamma)}(\Gamma \otimes^\bullet_\Lambda U^\bullet, \Gamma \otimes^\bullet_\Lambda V_i^\bullet) = 0, \end{aligned}$$

the Σ-module $\mathrm{Hom}_{\mathscr{K}(\Lambda)}(T^\bullet, \Gamma \otimes^\bullet_\Lambda V_i^\bullet)$ has no composition factors in $\mathrm{add}\big(\mathrm{top}(\mathrm{Hom}_{\mathscr{K}(\Lambda)}(T^\bullet, U^\bullet))\big)$.

Now, for $1 \le k \le r$, let S_k denote the top of V_k, and $\bar{S}_k$ the top of the Σ-module $\mathrm{Hom}_{\mathscr{K}(\Lambda)}(T^\bullet, V_k^\bullet)$. Let $G : \mathscr{D}^{\mathrm{b}}(\Lambda) \to \mathscr{D}^{\mathrm{b}}(\Sigma)$ be the derived equivalence induced by $T^\bullet$. Then, by the proof of Lemma 2.4, we have $G(S_k) \simeq \bar{S}_k[-n_k]$ for $1 \le k \le r$. Since $\Gamma \otimes^\bullet_\Lambda T^\bullet$ is a tilting complex over Γ,

$$\mathrm{Hom}_{\mathscr{K}(\Lambda)}(T^\bullet, (\Gamma \otimes^\bullet_\Lambda V_k^\bullet)[n]) \simeq \mathrm{Hom}_{\mathscr{K}(\Gamma)}(\Gamma \otimes^\bullet_\Lambda T^\bullet, (\Gamma \otimes^\bullet_\Lambda V_k^\bullet)[n]) = 0$$

for all $1 \leq k \leq r$ and all $n \neq 0$, and $\mathrm{Hom}_{\mathscr{K}(\Lambda)}(T^\bullet, \Gamma \otimes^\bullet_\Lambda V_k^\bullet) \simeq G(\Gamma \otimes^\bullet_\Lambda V_k^\bullet)$ for all $1 \leq k \leq r$. Hence

$$\begin{aligned} \mathrm{Hom}_\Sigma(\mathrm{Hom}_{\mathscr{K}(\Lambda)}(T^\bullet, \Gamma \otimes^\bullet_\Lambda V_i^\bullet), \bar{S}_k) &\simeq \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(\Sigma)}(G(\Gamma \otimes^\bullet_\Lambda V_i^\bullet), G(S_k[n_k])) \\ &\simeq \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(\Lambda)}(\Gamma \otimes^\bullet_\Lambda V_i^\bullet, S_k[n_k]) \\ &\simeq \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(\Lambda)}(X_i[n_i], S_k[n_k]) \end{aligned}$$

is zero for all $k \neq i$, and is one-dimensional over $\mathrm{End}_\Lambda(S_k)$ for $k = i$. Hence the top of the Σ-module $\mathrm{Hom}_{\mathscr{K}(\Lambda)}(T^\bullet, \Gamma \otimes^\bullet_\Lambda V_i^\bullet)$ is $\bar{S}_i$, and there is projective cover

$$\varepsilon : \mathrm{Hom}_{\mathscr{K}(\Lambda)}(T^\bullet, V_i^\bullet) \longrightarrow \mathrm{Hom}_{\mathscr{K}(\Lambda)}(T^\bullet, \Gamma \otimes^\bullet_\Lambda V_i^\bullet).$$

Clearly, such an epimorphism is given by $\mathrm{Hom}_{\mathscr{K}(\Lambda)}(T^\bullet, g^\bullet)$ for some morphism $g^\bullet : V_i^\bullet \to \Gamma \otimes^\bullet_\Lambda V_i^\bullet$. By Lemma 2.3(2), there is a morphism $u^\bullet$ from $\Gamma \otimes^\bullet_\Lambda V_i^\bullet$ to $\Gamma \otimes^\bullet_\Lambda V_i^\bullet$ such that $g^\bullet = f^* u^\bullet$. It follows that

$$\varepsilon = \mathrm{Hom}_{\mathscr{K}(\Lambda)}(T^\bullet, f^*) \cdot \mathrm{Hom}_{\mathscr{K}(\Lambda)}(T^\bullet, u^\bullet).$$

Hence the endomorphism $\mathrm{Hom}_{\mathscr{K}(\Lambda)}(T^\bullet, u^\bullet)$ of the Σ-module $\mathrm{Hom}_{\mathscr{K}(\Lambda)}(T^\bullet, \Gamma \otimes^\bullet_\Lambda V_i^\bullet)$ is surjective, and therefore is an isomorphism. Consequently, $\mathrm{Hom}_{\mathscr{K}(\Lambda)}(T^\bullet, f^*)$ is surjective. This finishes the proof. $\square$

Proof of Theorem 3.1. We have the following pullback diagram of homomorphisms of algebras:

$$\begin{tikzcd} A \arrow[r, "\lambda_1"] \arrow[d, "\lambda_2"'] & A_1 \arrow[d, "\pi_1"] \\ A_2 \arrow[r, "\pi_2"] & A_0 \end{tikzcd}$$

By the assumptions of Theorem 3.1, the tilting complex $T_0^\bullet$ is of the form $T_0^\bullet = \bigoplus_{i=1}^m U_i[n_i]$ with U_i projective A_0-modules such that $n_i \neq n_j$ whenever $i \neq j$. Thus $\mathrm{Hom}(U_i[n_i], U_j[n_j]) = 0$ for all $i \neq j$, and $\bigoplus_{i=1}^m U_i$ is a basic, projective generator for A_0-mod.

Recall from Subsection 2.4 that $A_j\text{-proj} = \mathrm{add}(P_j \oplus Q_j)$ for $j = 1, 2$, where $A_0 \otimes_{A_j} P_j = 0$ and $A_0 \otimes_{A_j} Y \neq 0$ for each indecomposable direct summand Y of Q_j. Let $\{V_1, \cdots, V_r\}$ and $\{W_1, \cdots, W_s\}$ be complete sets of pairwise non-isomorphic indecomposable projective modules in $\mathrm{add}(Q_1)$ and $\mathrm{add}(Q_2)$, respectively. Since $h_i^\bullet : A_0 \otimes^\bullet_{A_i} T_i^\bullet \simeq T_0^\bullet$ in $\mathscr{C}(A_0)$ for $i = 1, 2$, and since each indecomposable projective A_0-module occurs in $T_0^\bullet$ only once, we deduce $[T_1^\bullet : V_i] = 1 = [T_2^\bullet : W_j]$ for all i, j. By Lemma 2.6, we can write

$$T_1^\bullet = P_1^\bullet \oplus V_1^\bullet \oplus \cdots \oplus V_r^\bullet \quad \text{ and } \quad T_2^\bullet = P_2^\bullet \oplus W_1{}^\bullet \oplus \cdots \oplus W_s{}^\bullet,$$

such that

(1) $P_i^\bullet \in \mathscr{K}^{\mathrm{b}}(\mathrm{add}(P_i))$, and $\mathrm{add}(P_i^\bullet)$ generates $\mathscr{K}^{\mathrm{b}}(\mathrm{add}(P_i))$ as a triangulated category for $i = 1, 2$, and

(2) $[V_i^\bullet : V_j] = \delta_{ij}$ and $[W_k^\bullet : W_l] = \delta_{kl}$, where δ_{ij} is the Kronecker symbol.

Note that $A_0 \otimes_{A_1} P_1 = 0$ and $A_0 \otimes^\bullet_{A_1} V_i^\bullet = (A_0 \otimes_{A_1} V_i)[n_{V_i}]$ for some integer n_{V_i} with $1 \leq i \leq r$. By assumption, we have an isomorphism of complexes:

$$h_1^\bullet : \quad \bigoplus_{i=1}^r (A_0 \otimes_{A_1} V_i)[n_{V_i}] \simeq \bigoplus_{i=1}^m U_i[n_i].$$

This gives rise to a partition $\sigma = \{\sigma_1, \cdots, \sigma_m\}$ of $\{1, \cdots, r\}$ with $\sigma_i := \{j \mid n_{V_j} = n_i\}$. Now we define

$$V_{\sigma_i} := \bigoplus_{j \in \sigma_i} V_j, \quad \text{ and } \quad V_{\sigma_i}^\bullet := \bigoplus_{j \in \sigma_i} V_j^\bullet.$$

for all $1 \leq i \leq m$. This partition means that we collect terms of the left-hand side of $h_1^\bullet$ according to the position n_i of terms in $T_0^\bullet$. Thus,

$$A_0 \otimes^\bullet_{A_1} T_1^\bullet = \big(A_0 \otimes^\bullet_{A_1} P_1^\bullet\big) \oplus \big(\bigoplus_{i=1}^r A_0 \otimes^\bullet_{A_1} V_i^\bullet\big) = \bigoplus_{i=1}^m A_0 \otimes^\bullet_{A_1} V_{\sigma_i}^\bullet.$$

Since $\mathrm{Hom}(U_i[n_i], U_j[n_j]) = 0$ for all $i \neq j$, the isomorphism $h_1^\bullet : A_0 \otimes^\bullet_{A_1} T_1^\bullet \to T_0^\bullet$ can be rewritten as

$$\mathrm{diag}[g_1, \cdots, g_m] : \bigoplus_{i=1}^m (A_0 \otimes^\bullet_{A_1} V_{\sigma_i}^\bullet) \longrightarrow \bigoplus_{i=1}^m U_i[n_i] = T_0^\bullet,$$

where $g_i : A_0\otimes^{\bullet}_{A_1} V^{\bullet}_{\sigma_i} \to U_i[n_i]$ is an isomorphism in $\mathscr{C}(A_0)$ for all i. By repeating the above procedure, we get a partition $\tau := \{\tau_1,\cdots,\tau_m\}$ of $\{1,\cdots,s\}$ with

$$\tau_i := \{k\in\{1,\cdots,s\} \mid A_0\otimes^{\bullet}_{A_2} W^{\bullet}_k \simeq (A_0\otimes_{A_2} W_k)[n_{W_k}] \text{ and } n_{W_k} = n_i\}.$$

Define

$$W_{\tau_i} := \bigoplus_{k\in\tau_i} W_k, \quad \text{and} \quad W^{\bullet}_{\tau_i} := \bigoplus_{k\in\tau_i} W^{\bullet}_k.$$

The isomorphism $h^{\bullet}_2$ can be rewritten as

$$\mathrm{diag}[f_1,\cdots,f_m] : \bigoplus_{i=1}^{m} A_0\otimes^{\bullet}_{A_2} W^{\bullet}_{\tau_i} \longrightarrow \bigoplus_{i=1}^{m} U_i[n_i] = T^{\bullet}_0.$$

Now, we define $T^{\bullet} := M(T^{\bullet}_1, T^{\bullet}_2, h^{\bullet}_1 h^{\bullet\,-1}_2)$, that is, $T^{\bullet} = M(P^{\bullet}_1,0,0)\oplus M(0,P^{\bullet}_2,0)\oplus\bigoplus_{i=1}^{m} M(V^{\bullet}_{\sigma_i}, W^{\bullet}_{\tau_i}, g_i f_i^{-1})$. In the sequel, we shall show that $T^{\bullet}$ is a tilting complex over A.

First, we show that $\mathrm{add}(T^{\bullet})$ generates $\mathscr{K}^{\mathrm{b}}(A\text{-proj})$ as a triangulated category.

For simplicity, we write $Z^{\bullet}_i$ for $M(V^{\bullet}_{\sigma_i}, W^{\bullet}_{\tau_i}, g_i f_i^{-1})$ for $1\le i\le m$. By definition, for each integer k, $Z^k_i := M(V^k_{\sigma_i}, W^k_{\tau_i}, g_i f_i^{-1})$. For $k\neq -n_i$, the term $V^k_{\sigma_i}$ is in $\mathrm{add}(P_1)$, and the term $W^k_{\tau_i}$ is in $\mathrm{add}(P_2)$. Hence $A_0\otimes_{A_1} V^k_{\sigma_i} = 0 = A_0\otimes_{A_2} W^k_{\tau_i}$, and $Z^k_i \simeq {}_A V^k_{\sigma_i}\oplus {}_A W^k_{\tau_i}\in \mathrm{add}(F_1\oplus F_2)$ for all $k\neq -n_i$. Since V_{σ_i} is a direct summand of $V^{-n_i}_{\sigma_i}$ and since W_{τ_i} is a direct summand of $W^{-n_i}_{\tau_i}$, the A-module $M(V_{\sigma_i}, W_{\tau_i}, g_i f_i^{-1})$ is a direct summand of $Z^{-n_i}_i$. By Lemma 2.8(1), the functor ${}_A(-) : \mathrm{add}(P_1)\to\mathrm{add}(F_1)$ is an equivalence, and consequently induces a triangle equivalence between $\mathscr{K}^{\mathrm{b}}(\mathrm{add}(P_1))$ and $\mathscr{K}^{\mathrm{b}}(\mathrm{add}(F_1))$. Since $\mathrm{add}(P^{\bullet}_1)$ generates $\mathscr{K}^{\mathrm{b}}(\mathrm{add}(P_1))$ as a triangulated category, $\mathrm{add}(M(P^{\bullet}_1,0,0)) = \mathrm{add}({}_A P^{\bullet}_1)$ generates $\mathscr{K}^{\mathrm{b}}(\mathrm{add}(F_1))$ as a triangulated category. Similarly, $\mathrm{add}(M(0,P^{\bullet}_2,0))$ generates $\mathscr{K}^{\mathrm{b}}(\mathrm{add}(F_2))$ as a triangulated category. As all terms Z^k_i with $k\neq -n_i$ are in $\mathrm{add}(F_1\oplus F_2)$, the term $Z^{-n_i}_i$ is in the triangulated full subcategory of $\mathscr{K}^{\mathrm{b}}(A\text{-proj})$ generated by $\mathrm{add}(T^{\bullet})$. Thus, the module $F_1\oplus F_2\oplus(\bigoplus_{i=1}^{m} Z^{-n_i}_i)$ is in the triangulated full subcategory of $\mathscr{K}^{\mathrm{b}}(A\text{-proj})$ generated by $\mathrm{add}(T^{\bullet})$. By Lemma 2.9(2), the direct sum

$$\bigoplus_{i=1}^{m} M(V_{\sigma_i}, W_{\tau_i}, g_i f_i^{-1})$$

is a basic, additive generator of F_3. Recall that $M(V_{\sigma_i}, W_{\tau_i}, g_i f_i^{-1})$ is a direct summand of $Z^{-n_i}_i$ for all $1\le i\le m$. It follows that $F_1\oplus F_2\oplus F_3$ is generated by $\mathrm{add}(T^{\bullet})$ in the triangulated full subcategory of $\mathscr{K}^{\mathrm{b}}(A\text{-proj})$ generated by $\mathrm{add}(T^{\bullet})$. As $F_1\oplus F_2\oplus F_3$ is an additive generator of A-proj, $\mathrm{add}(T^{\bullet})$ generates $\mathscr{K}^{\mathrm{b}}(A\text{-proj})$ as a triangulated category.

Next, we prove that $T^{\bullet}$ is self-orthogonal, that is, $\mathrm{Hom}_{\mathscr{K}(A)}(T^{\bullet}, T^{\bullet}[n]) = 0$ for all $n\neq 0$.

By the construction of $T^{\bullet}$, there is an exact sequence of complexes of A-modules (see Lemma 2.10(3)):

$$0 \longrightarrow T^{\bullet} \xrightarrow{[p^{\bullet}_1, p^{\bullet}_2]} T^{\bullet}_1\oplus T^{\bullet}_2 \xrightarrow{\begin{bmatrix}\pi_1^* h^{\bullet}_1\\ -\pi_2^* h^{\bullet}_2\end{bmatrix}} T_0 \longrightarrow 0,$$

which yields a triangle in $\mathscr{D}^{\mathrm{b}}(A)$. Applying $\mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A)}(T^{\bullet},-)$ to this triangle, we get the following commutative diagram with exact rows for each integer i:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A)}(T^{\bullet}, T^{\bullet}_0[i-1]) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A)}(T^{\bullet}, T^{\bullet}[i]) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A)}(T^{\bullet}, \bigoplus_{k=1}^{2} T^{\bullet}_k[i]) \arrow[d, "\simeq"] \\
\mathrm{Hom}_{\mathscr{K}(A)}(T^{\bullet}, T^{\bullet}_0[i-1]) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Hom}_{\mathscr{K}(A)}(T^{\bullet}, T^{\bullet}[i]) \arrow[r] \arrow[d, equal] & \bigoplus_{k=1}^{2}\mathrm{Hom}_{\mathscr{K}(A)}(T^{\bullet}, T^{\bullet}_k[i]) \arrow[d, "\simeq"] \\
\mathrm{Hom}_{\mathscr{K}(A_0)}(T^{\bullet}_0, T^{\bullet}_0[i-1]) \arrow[r] & \mathrm{Hom}_{\mathscr{K}(A)}(T^{\bullet}, T^{\bullet}[i]) \arrow[r] & \bigoplus_{k=1}^{2}\mathrm{Hom}_{\mathscr{K}(A_k)}(T^{\bullet}_k, T^{\bullet}_k[i])
\end{tikzcd} \qquad (**)$$

Here we use the following natural isomorphisms:

$$\mathrm{Hom}_{\mathscr{K}(A)}(T^{\bullet}, T^{\bullet}_k[i]) \simeq \mathrm{Hom}_{\mathscr{K}(A_k)}(A_k\otimes_A T^{\bullet}, T^{\bullet}_k[i]) \simeq \mathrm{Hom}_{\mathscr{K}(A_k)}(T^{\bullet}_k, T^{\bullet}_k[i])$$

for $0 \le k \le 2$, where the last isomorphism is due to Lemma 2.10(2). Since $\mathrm{Hom}_{\mathscr{K}(A_0)}(T_0^\bullet, T_0^\bullet[i-1]) = 0$ for all $i \neq 1$ and since $\mathrm{Hom}_{\mathscr{K}(A_k)}(T_k^\bullet, T_k^\bullet[i]) = 0$ for all $i \neq 0$ and all $0 \le k \le 2$, we have $\mathrm{Hom}_{\mathscr{K}(A)}(T^\bullet, T^\bullet[i]) = 0$ for all $i \neq 0, 1$. It follows from Lemma 3.2 that the morphism $\eta_1 : \mathrm{Hom}_{\mathscr{K}(A_1)}(T_1^\bullet, T_1^\bullet) \to \mathrm{Hom}_{\mathscr{K}(A_0)}(T_0^\bullet, T_0^\bullet)$ determined by $\pi_1^* h_1^\bullet$ is surjective. Consequently, from the long exact sequence $(**)$, we get $\mathrm{Hom}_{\mathscr{K}(A)}(T^\bullet, T^\bullet[1])$ = 0. Thus $T^\bullet$ is self-orthogonal. Altogether, we have shown that $T^\bullet$ is a tilting complex over A.

To finish the proof of Theorem 3.1, we consider the endomorphism algebra of $T^\bullet$. By Lemma 2.10(4), there exists a pullback diagram of homomorphisms of algebras:

$$\begin{tikzcd}
\mathrm{End}_{\mathscr{K}(A)}(T^\bullet) \arrow[r, "\varepsilon_1"] \arrow[d, "\varepsilon_2"] & \mathrm{End}_{\mathscr{K}(A_1)}(T_1^\bullet) \arrow[d, "\eta_1"] \\
\mathrm{End}_{\mathscr{K}(A_2)}(T_2^\bullet) \arrow[r, "\eta_2"] & \mathrm{End}_{\mathscr{K}(A_0)}(T_0^\bullet),
\end{tikzcd}$$

where η_1 and η_2 are determined by $\pi_1^* h_1^\bullet$ and $\pi_2^* h_2^\bullet$, respectively, and where ε_1 and ε_2 are determined by the projections from $T^\bullet$ to $T_1^\bullet$ and $T_2^\bullet$, respectively. This completes the proof of Theorem 3.1. $\square$

An immediate consequence of Theorem 3.1 is the following result.

Corollary 3.3. *Let A be an Artin algebra and T a basic, radical tilting complex over A. Suppose that I is an ideal in A such that* $\mathrm{rad}(A) \subseteq I$, $\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A)}\big(T^\bullet, IT^\bullet[i]\big) = 0$ *for all $i \neq 0$ and* $\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A)}(T^\bullet/IT^\bullet, (T^\bullet/IT^\bullet)[-1])$ = 0. *Let $B := \mathrm{End}_{\mathscr{K}^{\mathrm{b}}(A)}(T^\bullet)$ and J be the ideal of B consisting of all those endomorphisms of $T^\bullet$ that factorize through the injection $IT^\bullet \to T^\bullet$. If $T^\bullet/IT^\bullet$ is a basic, radical complex, then the algebras*

$$\Lambda := \{(a,a') \in A \times A \mid a - a' \in I\} \ \ \textit{and} \ \ \Gamma := \{(b,b') \in B \times B \mid b - b' \in J\}$$

are derived equivalent.

Proof. By the assumptions on I and [7, Theorem 4.2], we see that the complex $T^\bullet/IT^\bullet$ is a tilting complex over A/I and induces a derived equivalence between A/I and B/J. Since the algebra A/I is semisimple, the complex $T^\bullet/IT^\bullet$ satisfies the conditions of Theorem 3.1 for $T_0^\bullet$. Thus the pullback algebras of $A \to A/I \leftarrow A$ and $B \to B/J \leftarrow B$ are derived equivalent, that is, Λ and Γ are derived equivalent. $\square$

3.2 Special case: iterated almost ν-stable derived equivalences

A special class of derived equivalences is the one of almost ν-stable derived equivalences which induce stable equivalences of Morita type, while such stable equivalences play a significant role in an approach to Broué's abelian defect group conjecture (see [17, 9]). Thus it is quite natural to ask if almost ν-stable derived equivalences can be constructed from Milnor squares. In this section, we show that it is the case for finite-dimensional algebras over an algebraically closed field (see Corollary 3.4).

Throughout this section all algebras are finite-dimensional over a fixed field.

Let $F : \mathscr{D}^{\mathrm{b}}(A) \to \mathscr{D}^{\mathrm{b}}(B)$ be a derived equivalence of algebras A and B. Suppose that $Q^\bullet$ and $\bar{Q}^\bullet$ are radical tilting complexes associated to F and the quasi-inverse F^{-1} of F, respectively. By applying the shift function if necessary, we may assume that $Q^\bullet$ is of the form

$$0 \longrightarrow Q^{-n} \longrightarrow \cdots \longrightarrow Q^{-1} \longrightarrow Q^0 \longrightarrow 0$$

and $\bar{Q}^\bullet$ is of the form

$$0 \longrightarrow \bar{Q}^0 \longrightarrow \bar{Q}^1 \longrightarrow \cdots \longrightarrow \bar{Q}^n \longrightarrow 0.$$

Let $Q := \bigoplus_{i=1}^n Q^{-i}$ and $\bar{Q} := \bigoplus_{i=1}^n \bar{Q}^n$. The derived equivalence F is called *almost ν-stable* provided that $\mathrm{add}({}_AQ) = \mathrm{add}(\nu_A Q)$ and $\mathrm{add}({}_B\bar{Q}) = \mathrm{add}(\nu_B \bar{Q})$. The composite of finitely many almost ν-stable derived equivalences or their quasi-inverses is called an *iterated almost ν-stable derived equivalence*. Such a derived equivalence of finite-dimensional algebras over a field always induces a stable equivalence of Morita type (see [8] and [6]).

A module $P \in A\text{-mod}$ is said to be *ν-stably projective* if $\nu_A^i P$ is projective for all $i \ge 0$, where ν_A is the Nakayama functor $D\mathrm{Hom}_A(-,A) \simeq D(A) \otimes_A - : A\text{-mod} \to A\text{-mod}$. We denote by A-stp the full subcategory of A-proj consisting of all ν-stably projective A-modules.

For finite-dimensional algebras, Theorem 3.1 can be strengthened as the following corollary which is the main result in this subsection.

Corollary 3.4. *Keep the assumptions in Theorem 3.1, and further assume the following conditions:*

(1) *A_0, A_1 and A_2 are finite-dimensional algebras over an algebraically closed field k.*

(2) *$T_1^\bullet$ and $T_2^\bullet$ induce iterated almost ν-stable derived equivalences.*

(3) *$T_0^\bullet$ is a stalk complex concentrated in degree zero.*

Then the derived equivalence between the pullback algebras in Theorem 3.1 is iterated almost ν-stable.

Thus the pullback algebras in Corollary 3.4 have many common nice properties: the same global, finitistic and dominant dimensions, and the same numbers of non-isomorphic, non-projective simple modules, that is the Auslander-Reiten conjecture holds true for the two stably equivalent algebras (see [8, 9]).

For the proof of Corollary 3.4, we have to prepare a few lemmas. Recall that S_X denotes the top of an indecomposable projective module X.

Lemma 3.5. *If P is an indecomposable module in A-stp, then there is an exact sequence of A-modules*

$$(\star) \quad 0 \longrightarrow R_P \longrightarrow \nu_A S_P \longrightarrow S_{\nu P} \longrightarrow 0$$

such that the composition factors of R_P are of the form S_X for some indecomposable projective $X \notin A$-stp.

Proof. Since S_P is the top of P, the module $\nu_A S_P$ is a quotient of $\nu_A P \in A$-stp, while $\nu_A P$ is an indecomposable projective module in A-stp and has $S_{\nu_A P}$ as its top. Thus $\nu_A S_P$ is an indecomposable module with a simple top $S_{\nu_A P}$. Hence there is an exact sequence of A-modules:

$$0 \longrightarrow R_P \longrightarrow \nu_A S_P \longrightarrow S_{\nu P} \longrightarrow 0.$$

For each indecomposable module $Y \in A$-stp, the multiplicity of S_Y as a composition factor of $\nu_A S_P$ is the length of $\mathrm{Hom}_A(Y, \nu_A S_P)$ as an $\mathrm{End}_A(S_Y)$-module. However,

$$\mathrm{Hom}_A(Y, \nu_A S_P) \simeq \mathrm{Hom}_A(Y, D(A) \otimes_A S_P) \simeq \mathrm{Hom}_A(Y, D(A)) \otimes_A S_P \simeq D(Y) \otimes_A S_P \simeq \mathrm{Hom}_A(\nu_A^{-1} Y, S_P)$$

is zero if $Y \not\simeq \nu_A P$, and has length 1 if $Y \simeq \nu_A P$. Hence $\nu_A S_P$ has the composition factor $S_{\nu P}$ at top with $[\nu_A S_P : S_{\nu P}] = 1$, and other composition factors of the form S_X with X an indecomposable projective module not in A-stp. $\square$

Lemma 3.6. [6, Theorem 1.1] *Let $F : \mathscr{D}^{\mathrm{b}}(A) \to \mathscr{D}^{\mathrm{b}}(B)$ be a derived equivalence between algebras A and B over an algebraically closed field, and let $T^\bullet$ and $\bar{T}^\bullet$ be tilting complexes associated to F and F^{-1}, respectively. Set $T^\pm := \oplus_{i \neq 0} T^i$ and $\bar{T}^\pm := \oplus_{j \neq 0} \bar{T}^j$. Then the following are equivalent:*

(1) *The functor F is an iterated almost ν-stable derived equivalence;*

(2) $\mathrm{add}(T^\pm) = \mathrm{add}(\nu_A T^\pm)$ *and* $\mathrm{add}(\bar{T}^\pm) = \mathrm{add}(\nu_B \bar{T}^\pm)$*;*

(3) $T^\pm \in A$-stp *and* $\bar{T}^\pm \in B$-stp*;*

(4) *For each indecomposable projective A-module $X \notin A$-stp, $F(S_X)$ is isomorphic in $\mathscr{D}^{\mathrm{b}}(B)$ to a simple B-module;*

(5) *For each indecomposable projective A-module $X \notin A$-stp, there hold $X \notin \mathrm{add}(T^\pm)$ and $[U^0 : X] = 1$, where $U^\bullet = (U^i, d_U)$ is the direct sum of all non-isomorphic indecomposable direct summands of $T^\bullet$.*

Moreover, if one of (1)-(5) is satisfied, then A and B are stably equivalent of Morita type.

Thus a derived equivalence F is iterated almost ν-stable if and only if so is its quasi-inverse F^{-1} by (2). For the definition of stable equivalences of Morita type, the reader is referred to, for instance, [8].

Lemma 3.7. *Let Λ and Γ be algebras over an algebraically closed field and $F : \mathscr{D}^{\mathrm{b}}(\Lambda) \to \mathscr{D}^{\mathrm{b}}(\Gamma)$ be an iterated almost ν-stable derived equivalence. Suppose that P is an indecomposable projective Λ-module in Λ-stp.*

(1) *If $F(S_P)$ is isomorphic to a simple Γ-module $S_{P'}$, then so is $F(S_{\nu_\Lambda P})$. Moreover, P' must be in Γ-stp.*

(2) *If $F(S_P)$ is not isomorphic to a simple Γ-module, then neither is $F(S_{\nu_\Lambda P})$.*

Proof. (1) We may assume that the given derived equivalence F is almost ν-stable with $Q^\bullet$ and $\bar{Q}^\bullet$ being radical tilting complexes associated to F and F^{-1}, respectively. Let $Q := \bigoplus_{i>0} Q^{-i}$ and $\bar{Q} := \bigoplus_{i>0} \bar{Q}^i$. Then, by definition, $\mathrm{add}(\nu_\Lambda Q) = \mathrm{add}(Q)$ and $\mathrm{add}(\nu_\Gamma \bar{Q}) = \mathrm{add}(\bar{Q})$.

By [8, Lemma 5.2], there is a radical, two-sided tilting complex ${}_\Gamma\Delta_\Lambda^\bullet$:

$$0 \longrightarrow \Delta^0 \longrightarrow \Delta^1 \longrightarrow \cdots \longrightarrow \Delta^n \longrightarrow 0$$

such that $F(X^\bullet) \simeq \Delta^\bullet \otimes^\bullet_\Lambda X^\bullet$ with $\Delta^i \in \mathrm{add}(\bar{Q} \otimes_k Q^*)$ for all $i > 0$. Here, $Q^* = \mathrm{Hom}_\Lambda(Q, \Lambda)$ is the Λ-duality of ${}_\Lambda Q$. Let $\Theta^\bullet := \mathrm{Hom}^\bullet_\Gamma(\Delta^\bullet, \Gamma)$, the an inverse of $\Delta^\bullet$. Then the bimodules Δ^0 and Θ^0 define a stable equivalence of Morita type between Λ and Γ (see the proof of Theorem 5.3 in [8]). Here, we stress that $\Delta^0 \otimes_\Lambda -$ is both a left and right adjoint to $\Theta^0 \otimes_\Gamma -$. Indeed, $\Theta^0 := \mathrm{Hom}_\Gamma(\Delta^0, \Gamma)$ implies that $\Delta^0 \otimes_\Lambda -$ is a left adjoint to $\Theta^0 \otimes_\Gamma -$. Note that there is an isomorphism $\Delta^\bullet \simeq \mathrm{Hom}^\bullet_\Lambda(\Theta^\bullet, \Lambda)$ in $\mathscr{D}^{\mathrm{b}}(\Gamma \otimes_k \Lambda^{\mathrm{op}})$, due to the fact that $\Delta^\bullet$ is an inverse of $\Theta^\bullet$. The isomorphism can be regarded as in $\mathscr{K}^{\mathrm{b}}(\Gamma \otimes_k \Lambda^{\mathrm{op}})$ by [8, Lemma 2.1]. Since both complexes $\Delta^\bullet$ and $\mathrm{Hom}^\bullet_\Lambda(\Theta^\bullet, \Lambda)$ are radical, they are even isomorphic in $\mathscr{C}^{\mathrm{b}}(\Gamma \otimes_k \Lambda^{\mathrm{op}})$ by [8, (b), p.113]. It follows that $\Delta^0 \simeq \mathrm{Hom}_\Lambda(\Theta^0, \Lambda)$ and $\Delta^0 \otimes_\Lambda -$ is a right adjoint to $\Theta^0 \otimes_\Gamma -$.

Suppose $F(S_P) \simeq S_{P'}$ in $\mathscr{D}^{\mathrm{b}}(\Gamma)$ for an indecomposable projective Γ-module P'. Then $P' \in \Gamma$-stp. In fact, if $P' \notin \Gamma$-stp, then $\mathrm{Hom}_\Lambda(P, S_P) \simeq \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(\Gamma)}(F(P), S_{P'})$ would vanish since $F(P)$ is isomorphic to a complex in $\mathscr{K}^{\mathrm{b}}(\Gamma\text{-stp})$ by [8, Lemma 3.9]. This is a contradiction.

To prove (1), we show $F(S_{\nu P}) \simeq S_{\nu P'}$.

Indeed, since $F(S_P)$ is simple, $\mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(\Lambda)}(T^\bullet, S_P[i]) \simeq \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(\Gamma)}(\Gamma, F(S_P)[i]) = 0$ for all $i \neq 0$. It follows that $Q^* \otimes_\Lambda S_P \simeq \mathrm{Hom}_\Lambda(Q, S_P) = 0$, and thus $\Delta^i \otimes_\Lambda S_P = 0$ for all $i > 0$. Hence $F(S_P) \simeq \Delta^\bullet \otimes^\bullet_\Lambda S_P = \Delta^0 \otimes_\Lambda S_P \simeq S_{P'}$.

For $P \in \Lambda$-stp, there is the following exact sequence of Λ-modules by Lemma 3.5:

$$(\star) \quad 0 \longrightarrow R_P \longrightarrow \nu_\Lambda S_P \longrightarrow S_{\nu P} \longrightarrow 0$$

Now, applying $\Delta^0 \otimes_\Lambda -$ to $(\star)$, we get an exact sequence of Γ-modules

$$(\star\star) \quad 0 \longrightarrow \Delta^0 \otimes_\Lambda R_P \longrightarrow \Delta^0 \otimes_\Lambda \nu_\Lambda S_P \longrightarrow \Delta^0 \otimes_\Lambda S_{\nu P} \longrightarrow 0.$$

Note that $\Delta^0 \otimes_\Lambda \nu_\Lambda S_P \simeq \nu_\Gamma(\Delta^0 \otimes_\Lambda S_P)$ by a property of stable equivalences of Morita type (see (b) in the proof of [9, Lemma 3.1]. Note that (b) holds without any additional assumptions in [9, Lemma 3.1] because $\Delta^0 \otimes_\Lambda -$ is both a left and right adjoint to $\Theta^0 \otimes_\Gamma -$). Recall that $F(S_P) \simeq \Delta^0 \otimes_\Lambda S_P \simeq S_{P'}$. It follows that $\nu_\Gamma(\Delta^0 \otimes_\Lambda S_P) \simeq \nu_\Gamma(F(S_P)) \simeq \nu_\Gamma(S_{P'})$. Hence $\Delta^0 \otimes_\Lambda \nu_\Lambda S_P \simeq \nu_\Gamma(S_{P'})$. Due to $\mathrm{Hom}_\Lambda(Q, S_P) = 0$, we get $P \notin \mathrm{add}(Q)$ and $\nu_\Lambda P \notin \mathrm{add}(\nu_\Lambda Q) = \mathrm{add}(Q)$. This implies $\mathrm{Hom}_\Lambda(Q, S_{\nu P}) = 0$. Hence $\Delta^i \otimes_\Lambda S_{\nu P} = 0$ for $i \neq 0$ and $F(S_{\nu P}) \simeq \Delta^\bullet \otimes_\Lambda S_{\nu P} \simeq \Delta^0 \otimes_\Lambda S_{\nu P}$. Thus we assume $F(S_{\nu P}) = \Delta^0 \otimes_\Lambda S_{\nu P} \in \Gamma\text{-mod}$ and rewrite $(\star\star)$ as

$$0 \longrightarrow \Delta^0 \otimes_\Lambda R_P \longrightarrow \nu_\Gamma S_{P'} \longrightarrow F(S_{\nu P}) \longrightarrow 0$$

Note that both $\nu_\Gamma S_{P'}$ and $F(S_{\nu P})$ have a simple top isomorphic to $S_{\nu P'}$ and that $\nu_\Gamma S_{P'}$ has other composition factors of the form $S_{X'}$ with $X' \notin \Gamma$-stp indecomposable by Lemma 3.5. So, to prove that $F(S_{\nu P})$ is simple, we only have to show that $F(S_{\nu P})$ does not have any submodule isomorphic to $S_{X'}$ for all indecomposable projective Γ-modules $X' \notin \Gamma$-stp. This is equivalent to showing $\mathrm{Hom}_\Gamma(S_{X'}, F(S_{\nu P})) = 0$ for all indecomposable projective modules $X' \notin \Gamma$-stp. Indeed, by definition, F is iterated almost ν-stable if and only if F^{-1} is iterated almost ν-stable. Hence, by Lemma 3.6(4), for each indecomposable projective Γ-module $X' \notin \Gamma$-stp, there is an indecomposable projective Λ-module $X \notin \Lambda$-stp such that $F(S_X) \simeq S_{X'}$. Thus $\mathrm{Hom}_\Gamma(S_{X'}, F(S_{\nu P})) \simeq \mathrm{Hom}_\Lambda(S_X, S_{\nu P}) = 0$. Consequently, $F(S_{\nu P})$ has a unique composition factor $S_{\nu P'}$, that is, $F(S_{\nu P}) \simeq S_{\nu P'}$.

(2) follows from (1). $\square$

Proof of Corollary 3.4. We keep the notations in the proof of Theorem 3.1. The tilting complex $T^\bullet$ induces a derived equivalence between the pullback algebras. To prove that $T^\bullet$ induces an iterated almost ν-stable derived equivalence, we show the following statements:

(a) $T^i \in A$-stp for all $i \neq 0$

In fact, by assumption, the complex $T^\bullet_0$ is a stalk complex concentrated in degree 0 and $A_0 \otimes_{A_i} T^\bullet_i \simeq T^\bullet_0$ for $i = 1, 2$. It follows that $T^m_i \in \mathrm{add}(P_i)$ for $i = 1, 2$ and $m \neq 0$, where P_i is as defined in Subsection 2.4. Thus, by the construction of $T^\bullet$, the term T^m is equal to $M(T^m_1, 0, 0) \oplus M(0, T^m_2, 0)$ for $m \neq 0$. By Lemma 3.6, for $i \in \{1, 2\}$, the A_i-module $T^\pm_i := \bigoplus_{m \neq 0} T^m_i$ satisfies $\mathrm{add}(\nu_{A_i} T^\pm_i) = \mathrm{add}(T^\pm_i)$. It follows from Lemma 2.8(3) that $T^\pm := \bigoplus_{m \neq 0} T^m$ satisfies $\mathrm{add}(\nu_A T^\pm) = \mathrm{add}(T^\pm)$. Hence $T^m \in A$-stp for all $m \neq 0$.

(b) $[T^0 : X] = 1$ for each indecomposable projective A-module $X \notin A$-stp.

Let X be an indecomposable projective A-module and $X \notin A$-stp. We need to show $[T^0 : X] = 1$. Suppose contrarily $[T^0 : X] = r > 1$. Clearly, from the construction of $T^\bullet$, we have $T^0 \simeq M(T^0_1, T^0_2, h^0)$ with $h^0 : A_0 \otimes_{A_1} T^0_1 \to A_0 \otimes_{A_2} T^0_2$ an isomorphism of A_0-modules. Also, $X \simeq M(X_1, X_2, h_X)$ for $X_1 = A_1 \otimes_A X$, $X_2 = A_2 \otimes_A X$ and an A_0-module isomorphism $h_X : A_0 \otimes_{A_1} X_1 \to A_0 \otimes_{A_2} X_2$. If $h_X \neq 0$, then $A_0 \otimes_{A_1} X_1 = A_0 \otimes_{A_1} A_1 \otimes_A X$

is a direct summand of $A_0\otimes_{A_1}A_1\otimes_A T^0\simeq A_0\otimes_{A_1}T_1^0\simeq T_0^0$ with the multiplicity at least r. This contradicts to the assumption that $T_0^{\bullet}$ is a basic projective generator of A_0-modules. Hence $h_X=0$, $A_0\otimes_{A_1}X_i=0$ for $i=1,2$, and $X\simeq M(X_1,0,0)\oplus M(0,X_2,0)=X_1\oplus X_2$. It follows that $X_i\in \mathrm{add}(P_i)$ for $i=1,2$, and either $X_1=0$ or $X_2=0$. Without loss of generality, we assume $X_1\neq 0$. Then $[T_1^0:X_1]\geq r$ since $A_1\otimes_A T^0\simeq T_1^0$, and consequently $X_1\in A_1$-stp by Lemma 3.6(5), and the image of $\mathrm{top}(X_1)$ of the indecomposable projective A_1-module X_1 under the derived equivalence induced by $T_1^{\bullet}$ is not isomorphic to a simple module by Lemma 2.4.

To finish the proof of (b), we show the following:

$(\divideontimes)$ Let Y_1 be the direct sum of all indecomposable projective A_1-modules Y such that the image of $\mathrm{top}(Y)$ under the derived equivalence induced by $T_1^{\bullet}$ is not isomorphic to a simple module. Then $Y_1\in\mathrm{add}(P_1)$ and $\mathrm{add}(\nu_{A_1}Y_1)=\mathrm{add}(Y_1)$.

Indeed, let Y be an indecomposable A_1-module such that the image of $\mathrm{top}(Y)$ under the derived equivalence induced by $T_1^{\bullet}$ is not isomorphic to a simple module. If Y is a direct summand of T_1^m for some $m\neq 0$, then $Y\in\mathrm{add}(P_1)$ since $T_1^m\in\mathrm{add}(P_1)$. Now, assume that Y only occurs, as a direct summand, in T_1^0. Then $[T_1^0:Y]>1$ by Lemma 2.4. If $Y\notin\mathrm{add}(P_1)$, then $0\neq A_0\otimes_{A_1}Y$ is a direct summand of T_0^0. It follows from $T_0^0\simeq A_0\otimes_{A_1}T_1^0$ with $[T_0^0:A_0\otimes_{A_1}T_1^0]>1$ that $T_0^0=T_0^{\bullet}$ is not a basic A_0-module. Consequently, Y cannot occur in T_1^0, and therefore $Y\in\mathrm{add}(P_1)$ and $Y_1\in\mathrm{add}(P_1)$. By assumption and Lemma 3.6(4), $Y_1\in A_1$-stp. Now, it follows from Lemma 3.7(2) that, for each indecomposable $Y\in\mathrm{add}(Y_1)$, the module $\nu_{A_1}Y$ is again in $\mathrm{add}(Y_1)$. Hence $\mathrm{add}(\nu_{A_1}Y_1)=\mathrm{add}(Y_1)$.

Thus, by $(\divideontimes)$ and Lemma 2.8(3), $X=X_1$ lies in A-stp. This is a contradiction and shows $[T^0:X]=1$.

Altogether, we have shown $\mathrm{add}(\nu_A T^{\pm})=\mathrm{add}(T^{\pm})$, $T^{\pm}\in A$-stp and $[T^0:X]=1$ for every indecomposable projective A-module $X\notin A$-stp. Note that if $X\notin A$-stp then $X\notin\mathrm{add}(T^{\pm})$. Now, by Lemma 3.6(5), $T^{\bullet}$ induces an iterated almost ν-stable derived equivalence. $\square$

4 Some realizations by quivers with relations

In this section, we shall realize the main result, Theorem 3.1, by three "local" operations on derived equivalent algebras presented by quivers with relations. They are facilitated by gluing vertices, unifying arrows and identifying socle elements. The details are given in Theorems 4.1, 4.5 and 4.8, respectively. Note that these operations can be combined with each other and applied repeatedly.

Let $Q=(Q_0,Q_1)$ be a quiver with Q_0 the set of vertices and Q_1 the set of arrows between vertices. For $m>1$, let Q_m be the set of all paths in Q of length m. The starting and ending vertices of a path p are denoted by $s(p)$ and $e(p)$, respectively. As usual, the trivial path corresponding to a vertex $i\in Q_0$ is denoted by e_i.

We fix a field k and denote by kQ the path algebra of Q over k. The composition of two paths p and q in kQ is written as pq if $e(p)=s(q)$, and zero otherwise. A *relation* ω on Q is a k-linear combination of paths: $\omega=\lambda_1p_1+\lambda_2p_2+\cdots+\lambda_np_n$ with $0\neq\lambda_i\in k$, $e(p_1)=\cdots=e(p_n)$ and $s(p_1)=\cdots=s(p_n)$. Here, we assume that the length of each p_i, that is the number of arrows in p_i, is at least 2. If $n=1$ in ω, then ω is called a *monomial* relation.

Let ρ be a set of relations in kQ and $\langle\rho\rangle$ be the ideal of kQ generated by ρ. Then an algebra of the form $kQ/\langle\rho\rangle$ is said to be presented by the quiver Q with relations ρ. Clearly, $\langle\rho\rangle\subseteq\langle Q_2\rangle$. Note that for any ideal $I\subseteq\langle Q_2\rangle$ of kQ such that kQ/I is finite-dimensional, there is a set ρ of relations such that $\langle\rho\rangle=I$.

4.1 Derived equivalences from gluing vertices

In this subsection, we shall construct derived equivalences from given ones by gluing vertices of quivers. This also gives a way to get derived equivalences for subalgebras from the ones for given algebras.

Let $A=kQ/\langle\rho\rangle$ be a finite-dimensional algebra over a field k. For a subset $X\subseteq Q_0$, we denote by e_X the idempotent element $\sum_{i\in X}e_i$ in A. Let X be a subset of Q_0 and $\sigma=\{\sigma_1,\cdots,\sigma_m\}$ be a partition of X, that is, $X=\bigcup_i\sigma_i$ and $\sigma_i\cap\sigma_j=\emptyset$ for $i\neq j$. Let Q^{σ} be the quiver obtained from Q by just gluing the vertices in σ_t into one vertex, also denoted by σ_t, for all t, and keeping all arrows. Thus the vertex set of Q^{σ} is the union of $\{\sigma_1,\sigma_2,\cdots,\sigma_m\}$ and $Q_0\setminus X$, and the arrow set of Q^{σ} is Q_1. Then there is a natural homomorphism of algebras:

$$\lambda_{\sigma}:kQ^{\sigma}\longrightarrow kQ/\langle\rho\rangle$$

which sends e_i to e_i for $i \notin X$, e_{σ_t} to $\sum_{i \in \sigma_t} e_i$ for $1 \leq t \leq m$ and preserves all arrows. Clearly, the kernel of λ_σ is contained in $\langle Q_2^\sigma \rangle$ in kQ^σ. Let ρ^σ be a set of relations on Q^σ such that $\langle \rho^\sigma \rangle = \mathrm{Ker}(\lambda_\sigma)$. The relations ρ^σ can be obtained in the following way: For each t, let ρ^{σ_t} be the set of relations on Q^σ consisting of all $\alpha\beta$ with α, β being arrows such that $e(\alpha)$ and $s(\beta)$ are different vertices in σ_t. Then $\rho^\sigma = \rho \cup \rho^{\sigma_1} \cup \cdots \cup \rho^{\sigma_m}$. The algebra $A^\sigma := kQ^\sigma / \langle \rho^\sigma \rangle$ is called the σ-*gluing algebra* of A. The above homomorphism λ_σ induces a homomorphism from A^σ to A, denoted again by λ_σ. Observe that $\lambda_\sigma : A^\sigma \to A$ is injective, and the image of λ_σ is the subalgebra of A generated by all the arrows in Q, the idempotents $e_{\sigma_1}, \cdots, e_{\sigma_m}$ and $\{e_i \mid i \in Q_0 \backslash X\}$. Note that the Jacobson radicals of A^σ and A are equal. This construction has been used in the study of the finitistic dimension conjecture (for example, see [18]).

Now, we illustrate the above procedure by an example. Let A be a k-algebra presented by the quiver Q

$$\begin{tikzcd}[row sep=small, column sep=small]
 & \overset{2}{\bullet} \arrow[dr, "\delta"] & & & \\
\overset{1}{\bullet} \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & \overset{4}{\bullet} & \overset{5}{\bullet} \arrow[r, "\eta"] & \overset{6}{\bullet} \\
 & \underset{3}{\bullet} \arrow[ur, "\gamma"'] & & &
\end{tikzcd}$$

with the relation $\alpha\delta - \beta\gamma$. Let $X := \{1,2,3,4,5\}$ and $\sigma := \big\{\{1,2,3\},\{4,5\}\big\}$ be a partition of X. Then the σ-gluing algebra A^σ of A is presented by the quiver Q^σ

$$\begin{tikzcd}
\bullet \arrow[loop above, "\alpha"] \arrow[loop below, "\beta"] \arrow[r, shift left, "\delta"] \arrow[r, shift right, "\gamma"'] & \bullet \arrow[r, "\eta"] & \bullet
\end{tikzcd}$$

with relations $\{\alpha\delta - \beta\gamma\} \cup \rho^{\sigma_1} \cup \rho^{\sigma_2} = \{\alpha\delta - \beta\gamma, \alpha^2, \alpha\beta, \alpha\gamma, \beta\alpha, \beta^2, \beta\delta, \delta\eta, \gamma\eta\}$.

In the following, we shall interpret the procedure of a σ-gluing as a pullback of algebras. We define

$$k^X := \bigoplus_{i \in X} k$$

to be the path algebra of the quiver with isolated vertices indexed by X. Considering σ as a set, we have the algebra k^σ which is just the σ-gluing algebra of k^X. There is an embedding $\lambda_\sigma : k^\sigma \to k^X$ sending e_{σ_i} to $\sum_{j \in \sigma_i} e_j$ for $1 \leq i \leq m$. Also, note that there is a canonical algebra homomorphism

$$\pi : kQ/\langle \rho \rangle \longrightarrow k^X$$

sending e_i to e_i for $i \in X$, and all other idempotents and all arrows to zero. Similarly, there is a canonical, surjective algebra homomorphism $\pi : kQ^\sigma/\langle \rho^\sigma \rangle \to k^\sigma$. Then we have the following commutative diagram of algebra homomorphisms:

$$\begin{tikzcd}
kQ^\sigma/\langle \rho^\sigma \rangle \arrow[r, "\lambda_\sigma"] \arrow[d, "\pi"'] & kQ/\langle \rho \rangle \arrow[d, "\pi"] \\
k^\sigma \arrow[r, "\lambda_\sigma"] & k^X
\end{tikzcd}$$

Moreover, $\dim_k A + \dim_k k^\sigma = \dim_k A^\sigma + \dim_k k^X$. This implies that the above commutative diagram is a pullback diagram.

Theorem 4.1. *Suppose that F is a derived equivalence between algebras $A := kQ/\langle \rho \rangle$ and $A' := kQ'/\langle \rho' \rangle$. Let X be a subset of Q_0 such that the simple A-modules corresponding to the vertices in X are sent by F to simple A'-modules. Let X' be the set of vertices in Q'_0 corresponding to these simple A'-modules. Let σ be a partition of X and σ' be the corresponding partition of X'. Then the algebras A^σ and $A'^{\sigma'}$ are derived equivalent.*

Proof. By assumption, there is a basic, radical tilting complex $T^\bullet$ over A such that $F(T^\bullet) \simeq A'$ in $\mathscr{D}^{\mathrm{b}}(A')$. By Lemmas 2.4 and 2.6, we can rewrite $T^\bullet$ as $T^\bullet = U^\bullet \oplus \bigoplus_{i \in X} V_i^\bullet$ such that $U^\bullet \in \mathscr{K}^{\mathrm{b}}(\mathrm{add}(\bigoplus_{i \in Q_0 \backslash X} Ae_i))$ and $V_i^\bullet$ is indecomposable with $[V_i^\bullet : Ae_j] = \delta_{ij}$ for all $i, j \in X$. Moreover, for each $i \in X$, the projective A-module Ae_i occurs as a direct summand of V_i^0 with the multiplicity 1 (see the proof of Lemma 2.4). By the definition of $\pi : A \to k^X$, we have $k^X \otimes_A Ae_i = 0$ for $i \notin X$ and $k^X \otimes_A Ae_i \simeq k^X e_i$ for $i \in X$. Thus there is an isomorphism in $\mathscr{C}(k^X)$:

$$h^\bullet : k^X \otimes_A T^\bullet \longrightarrow k^X.$$

Clearly, $k^X \otimes_{k^\sigma} k^\sigma \simeq k^X$. Let $\eta_1 : \mathrm{End}_{\mathscr{K}(A)}(T^\bullet) \to \mathrm{End}_{k^X}(k^X)$ be the algebra homomorphism determined by the composite $\pi^* h_1^\bullet : T^\bullet \to k^X \otimes_A T^\bullet \to k^X$, and let $\eta_2 : \mathrm{End}_{k^\sigma}(k^\sigma) \to \mathrm{End}_{k^X}(k^X)$ be the algebra homomorphism determined by λ_σ. By Theorem 3.1, the pullback algebra of η_1 and η_2 is derived equivalent to the pullback algebra A^σ of $\pi : A \to k^X$ and $\lambda_\sigma : k^\sigma \to k^X$. It remains to show that $A'^{\sigma'}$ is isomorphic to the pullback algebra of η_1 and η_2.

For each x in Q_0 (respectively, Q_0'), we denote by S_x (respectively, S_x') the simple A-module (respectively, A'-module) corresponding to the vertex x. By relabeling the vertices if necessary, we can assume that

$$X = \{1, \cdots, m\} = X'$$

such that $F(S_i) \simeq S_i'$ for $1 \le i \le m$. In this case, σ and σ' are the same partition of $\{1, \cdots, m\}$. For $i, j \in \{1, \cdots, m\}$, the Hom-space

$$\mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A')}(F(V_i^\bullet), S_j')) \simeq \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A')}(F(V_i^\bullet), F(S_j)) \simeq \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A)}(V_i^\bullet, S_j)$$

is 1-dimensional for $i = j$, and zero for $i \ne j$. Thus it follows from the indecomposability of $F(V_i^\bullet)$ that there exists an isomorphism $g_i : F(V_i^\bullet) \to A'e_i$ for $1 \le i \le m$. Let $f := \sum_{j \in Q_0' \setminus X'} e_j \in A'$. Then there is an isomorphism $g : F(U^\bullet) \to A'f$. Thus we obtain an isomorphism

$$\mathrm{diag}[g, g_1, \cdots, g_m] : F(T^\bullet) \longrightarrow A',$$

which induces an isomorphism $\tilde{g} : \mathrm{End}_{\mathscr{D}^{\mathrm{b}}(A')}(F(T^\bullet)) \to \mathrm{End}_{A'}(A')$. Let s be the composite of the following maps

$$\mathrm{End}_{\mathscr{K}(A)}(T^\bullet) \simeq \mathrm{End}_{\mathscr{D}^{\mathrm{b}}(A)}(T^\bullet) \longrightarrow \mathrm{End}_{\mathscr{D}^{\mathrm{b}}(A')}(F(T^\bullet)) \xrightarrow{\tilde{g}} \mathrm{End}_{A'}(A') \longrightarrow A'.$$

Then, for each $i \in \{1, \cdots, m\}$, the map s sends the primitive idempotent corresponding to the direct summand $V_i^\bullet$ to e_i. According to this fact, it is easy to check that the following diagram is commutative.

$$\begin{tikzcd}
\mathrm{End}_{\mathscr{K}(A)}(T^\bullet) \arrow[r, "\eta_1"] \arrow[d, "s"', "\simeq"] & \mathrm{End}_{k^X}(k^X) \arrow[d, "\simeq"] & \mathrm{End}_{k^\sigma}(k^\sigma) \arrow[l, "\eta_2"'] \arrow[d, "\simeq"] \\
A' \arrow[r, "\pi"] & k^{X'} & k^{\sigma'} \arrow[l, "\lambda_{\sigma'}"']
\end{tikzcd}$$

Note that the unlabeled vertical isomorphisms are the canonical ones. This diagram shows that $A'^{\sigma'}$, which is the pullback of π and $\lambda_{\sigma'}$, is isomorphic to the pullback algebra of η_1 and η_2, and finishes the proof. $\square$

Remark. In Theorem 4.1, the indecomposable projective A^σ-module corresponding to a part of the partition σ occurs only once (in degree zero) in the tilting complex that induces a derived equivalence between A^σ and $A'^{\sigma'}$ (see the proof of Theorem 3.1). Therefore, by Lemma 2.4, this derived equivalence sends the simple modules corresponding to parts of σ to the simple modules corresponding to parts of σ'. Thus Theorem 4.1 can be applied repeatedly.

Theorem 4.1 also provides a way to construct a new derived equivalence from two given derived equivalences.

Corollary 4.2. *Let F be a derived equivalence between two algebras $A := kQ/\langle\rho\rangle$ and $A' := kQ'/\langle\rho'\rangle$, and let G be a derived equivalence between $B := k\Gamma/\langle\phi\rangle$ and $B' := k\Gamma'/\langle\phi'\rangle$. Suppose that $\bar{Q}_0$ (respectively, $\bar{\Gamma}_0$) be a subset of Q_0 (respectively, Γ_0) such that the simple modules corresponding to the vertices in $\bar{Q}_0$ (respectively, $\bar{\Gamma}_0$) are sent by F (respectively, G) to simple modules corresponding to the vertices in $\bar{Q}_0'$ (respectively, $\bar{\Gamma}_0'$) and that $|\bar{Q}_0| = |\bar{Q}_0'|$ and $|\bar{\Gamma}_0| = |\bar{\Gamma}_0'|$. Let σ be a partition of the set $\bar{Q}_0 \cup \bar{\Gamma}_0$ and σ' be the corresponding partition of $\bar{Q}_0' \cup \bar{\Gamma}_0'$. Then the algebras $(A \times B)^\sigma$ and $(A' \times B')^{\sigma'}$ are derived equivalent.*

Proof. Taking coproducts of algebras, we can get a derived equivalence between $A \times B$ and $A' \times B'$, which sends the simple modules corresponding to the vertices in $\bar{Q}_0 \cup \bar{\Gamma}_0$ to the simple modules corresponding to the vertices in $\bar{Q}_0' \cup \bar{\Gamma}_0'$. Thus the corollary follows immediately from Theorem 4.1. $\square$

A special case of Corollary 4.2 is $B = A^{\mathrm{op}}$ and $B' = A'^{\mathrm{op}}$. In this case we can get derived equivalence between $(A \times A^{\mathrm{op}})^\sigma$ and $(A' \times A'^{\mathrm{op}})^{\sigma'}$ since algebras A and A' are derived equivalent if and only if so are their opposite algebras.

Another special case of Corollary 4.2 is to attach an algebra simultaneously to derived equivalent algebras and make the resulting algebras again derived equivalent.

Corollary 4.3. *Let F be a derived equivalence between the algebras $A := kQ/\langle\rho\rangle$ and $A' := kQ'/\langle\rho'\rangle$ such that F sends the simple A-modules corresponding to the vertices in $\bar{Q}_0$ to the simple A'-modules corresponding to the vertices in $\bar{Q}'_0$ and that $|\bar{Q}_0| = |\bar{Q}'_0|$. Suppose that $C := k\Gamma/\langle\rho''\rangle$ is an arbitrary algebra. Let σ be a partition of $\bar{Q}_0 \cup \Gamma_0$ and σ' be the corresponding partition of $\bar{Q}'_0 \cup \Gamma_0$. Then the algebras $(A\times C)^{\sigma}$ and $(A'\times C)^{\sigma'}$ are derived equivalent.*

4.2 Derived equivalences from unifying arrows

In this subsection, we shall construct new derived equivalences from given ones by unifying certain arrows in quivers.

We first fix some notation. Throughout this subsection, Δ is the quiver with the vertex set $\{x, 1, 2, \cdots, n\}$ and n arrows $\alpha_j : x \to j$, $1 \le j \le n$. Here, we understand that the arrows have pairwise distinct ending vertices. We define $E := \{1, \cdots, n\}$. It may happen that the vertex x falls into E. In this case Δ has the vertex set E. Let σ be the partition of E with only one part, and let $\alpha := \{\alpha_1, \cdots, \alpha_n\}$ for simplicity.

Let $A = kQ/\langle\rho\rangle$ be a finite-dimensional k-algebra such that Δ is a subquiver of Q. By the previous discussion, there is an algebra embedding

$$\lambda_\sigma : kQ^{\sigma}/\langle\rho^{\sigma}\rangle \longrightarrow kQ/\langle\rho\rangle.$$

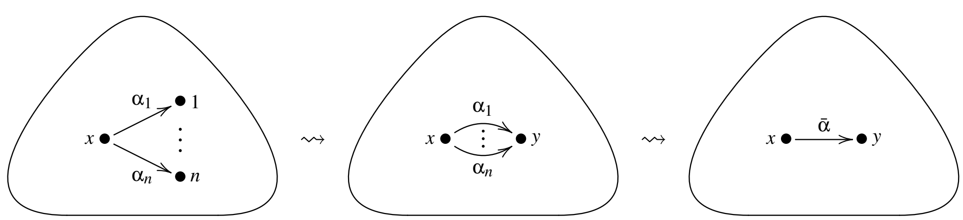
Let Q^{α} be the quiver obtained from Q^{σ} by unifying the arrows $\alpha_1, \cdots, \alpha_n$ into one arrow $\bar{\alpha}$ in Q^{σ}. Thus Q^{α} has the vertex set Q^{σ}, while the set of arrows is $\{\bar{\alpha}\} \cup Q^{\sigma}_1 \backslash \{\alpha_1, \cdots, \alpha_n\}$. Then there is a canonical algebra homomorphism

$$\varphi : kQ^{\alpha} \longrightarrow kQ^{\sigma}/\langle\rho^{\sigma}\rangle$$

sending $\bar{\alpha}$ to $\sum_{i=1}^{n} \alpha_i$, and preserving all other arrows and all vertices. It is easy to see that $\mathrm{Ker}(\varphi)$ is contained in $\langle Q^{\alpha}_2\rangle$. Let ρ^{α} be relations on Q^{α} such that $\langle\rho^{\alpha}\rangle = \mathrm{Ker}(\varphi)$. Then we get a natural embedding

$$\lambda_\alpha : kQ^{\alpha}/\langle\rho^{\alpha}\rangle \longrightarrow kQ^{\sigma}/\langle\rho^{\sigma}\rangle.$$

We define $A^{\alpha} := kQ^{\alpha}/\langle\rho^{\alpha}\rangle$. This is called the *unifying algebra* of A by α. The image of the composite $\lambda_\alpha\lambda_\sigma$ is the subalgebra of A generated by all the arrows $\beta \notin \alpha$, $\sum_{i=1}^{n} \alpha_i$ and idempotents e_E, $e_i, i \in Q_0 \backslash E$. The above procedure can be illustrated visually by the following (local) pictures:



Q $\qquad$ Q^{σ} $\qquad$ Q^{α}

Next, we shall interpret the algebra A^{α} as a pullback algebra. Actually, A^{α} fits into the following pullback diagram of algebra homomorphisms:

$$\begin{tikzcd} A^{\alpha} \arrow[r, "\lambda_\alpha"] \arrow[d, "\pi"'] & A^{\sigma} \arrow[d, "\pi"] \\ k^{\Delta_0^{\sigma}} \arrow[r, "\lambda"] & (k\Delta)^{\sigma}/\langle\sum_{i=1}^{n}\alpha_i\rangle \end{tikzcd}$$

The vertical homomorphisms in the above diagram are obviously defined.

Lemma 4.4. *The algebra $(k\Delta)^{\sigma}/\langle\sum_{i=1}^{n}\alpha_i\rangle$ is radical-square zero.*

Proof. If $x \notin \{1, \cdots, n\}$, then $x \ne y$ and $(k\Delta)^{\sigma}/\langle\sum_{i=1}^{n}\alpha_i\rangle$ is radical-square zero. Without loss of generality, we now assume that α_1 is a loop in the quiver Δ. Then none of $\alpha_2, \cdots, \alpha_n$ is a loop by the assumption that the vertices $1, \cdots, n$ are pairwise distinct. Thus, $\alpha_i\alpha_j = 0$ for all $i \ne 1$ and all $j \in \{1, \cdots, n\}$. Further, for each $j \in \{1, \cdots, n\}$, the path $\alpha_1\alpha_j = (\sum_{i=1}^n \alpha_i)\alpha_j$ is in $\langle \sum_{i=1}^n \alpha_i \rangle$. Altogether, we have shown that all paths in $(k\Delta)^\sigma$ of length 2 belong to $\langle \sum_{i=1}^n \alpha_i \rangle$, and the lemma is then proved. $\square$

Let $kQ/\langle\rho\rangle$ be a finite-dimensional algebra defined by a quiver Q with relations ρ. Let i and j be vertices in Q_0, and let Q_{ij} be the k-vector space with all arrows from i to j as a basis. Then every vector space automorphism $\chi : Q_{ij} \to Q_{ij}$ extends to an algebra automorphism $\phi_\chi : kQ \to kQ$ which sends $\alpha \in Q_{ij}$ to $(\alpha)\chi$ and preserves all other arrows and all vertices. If $(\langle\rho\rangle)\phi_\chi = \langle\rho\rangle$ for all such automorphisms χ on Q_{ij}, then ρ is said to be (i,j)-*invariant*. Let $\Gamma = (\Gamma_0, \Gamma_1)$ be a sub-quiver of Q. We say that ρ is Γ-*invariant* if ρ is (i,j)-invariant for all $i, j \in \Gamma_0$. For example, ρ is Γ-invariant if ρ consists only of monomial relations and there is at most 1 arrow from i to j in Q for any two vertices i, j in Γ_0. Note that ρ is Γ-invariant if and only if ρ^{op} in $kQ^{\mathrm{op}}/\langle\rho^{\mathrm{op}}\rangle$ is Γ^{op}-invariant.

Theorem 4.5. *Let $A := kQ/\langle\rho\rangle$ and $A' := kQ'/\langle\rho'\rangle$ be algebras, and suppose that the given quiver Δ is a sub-quiver of both Q and Q'. Assume that ρ or ρ' is Δ-invariant. If $F : \mathscr{D}^{\mathrm{b}}(A) \to \mathscr{D}^{\mathrm{b}}(A')$ is a derived equivalence such that $F(S_i) \simeq S'_i$ for all $i \in \Delta_0$, then A^α and A'^α are derived equivalent.*

Proof. Without loss of generality, we assume that ρ' is Δ-invariant. Further, we assume that the common starting vertex x of $\alpha_1, \cdots, \alpha_n$ is not in E. The case that $x \in E$ can be proved similarly. Let $\tilde{\Delta}$ be the full sub-quiver of Q defined by Δ_0. Then Δ is a sub-quiver of $\tilde{\Delta}$ with the same vertices and (possibly) less arrows. Let $B := k\tilde{\Delta}/\langle\tilde{\Delta}_2\rangle$, and let $\Lambda := (k\Delta)^\sigma/\langle\sum_{i=1}^n \alpha_i\rangle$. Then, by Lemma 4.4, there is a canonical surjective homomorphism $\pi : B^\sigma \to \Lambda$ of algebras.

Let $T^\bullet$ be a basic, radical tilting complex associated to the derived equivalence F. Set $U := \bigoplus_{i \in Q_0 \setminus \Delta_0} Ae_i$. Since $F(S_i) \simeq S'_i$ for all $i \in \Delta_0$, we can assume

$$T^\bullet = U^\bullet \oplus V_x^\bullet \oplus V_1^\bullet \oplus \cdots \oplus V_n^\bullet$$

by Lemmas 2.4 and 2.6, where $V_i^\bullet$ is a complex in $\mathscr{K}^{\mathrm{b}}(A\text{-proj})$ such that, for each $i \in \Delta_0$, $V_i^0 = Ae_i \oplus U_i$ for some $U_i \in \mathrm{add}(U)$ and $V_i^j \in \mathrm{add}(U)$ for all $j \neq 0$. Note that there is a commutative diagram

$$\begin{tikzcd}
A^\sigma \arrow[r, "\pi"] \arrow[d, "\lambda_\sigma"] & B^\sigma \arrow[r, "\pi"] \arrow[d, "\lambda_\sigma"] & k^\sigma \arrow[d, "\lambda_\sigma"] \\
A \arrow[r, "\pi"] & B \arrow[r, "\pi"] & k^E
\end{tikzcd}$$

where the horizontal maps are the canonical maps. The right-hand square and the entire square are pullback diagrams of algebras. This implies that the left-hand square is also a pullback diagram. It is easy to see $B \otimes_A U = 0$ and that there is an isomorphism of stalk complexes in $\mathscr{C}(B)$:

$$h^\bullet : B \otimes_A T^\bullet = B \otimes_A (Ae_x \oplus \bigoplus_{i=1}^n Ae_i) \longrightarrow B \simeq B \otimes_{B^\sigma} B^\sigma.$$

By the proof of Theorem 3.1, the complex $T_\sigma^\bullet := M(T^\bullet, B^\sigma, h^\bullet)$ is a tilting complex over A^σ with $\mathrm{End}_{\mathscr{K}(A^\sigma)}(T_\sigma^\bullet) \simeq A'^\sigma$. Moreover, there is a pullback diagram

$$\begin{tikzcd}
\mathrm{End}_{\mathscr{K}(A^\sigma)}(T_\sigma^\bullet) \arrow[r, "\varepsilon_1"] \arrow[d, "\varepsilon_2\mu"] & \mathrm{End}_{\mathscr{K}(A)}(T^\bullet) \arrow[d, "\eta\mu"] \\
B^\sigma \arrow[r, "\lambda_\sigma"] & B,
\end{tikzcd}$$

where η is determined by $T^\bullet \to B$, ε_1 and ε_2 are determined by the projections from $T_\sigma^\bullet$ to $T^\bullet$ and B^σ, respectively, and μ is the canonical isomorphism from $\mathrm{End}({}_\Lambda\Lambda)$ to Λ for an algebra Λ.

By assumption, $F(S_i) \simeq S'_i$ for all $i \in \Delta_0$. It follows that $\mathrm{Ext}_A^1(S_i, S_j) \simeq \mathrm{Ext}_{A'}^1(S'_i, S'_j)$ for all $i, j \in \Delta_0$. This indicates that the number of arrows from i to j are equal in both Q and Q'. Hence we can assume that $\tilde{\Delta}$ is also a full sub-quiver of Q' with vertices Δ_0. As a consequence, there is a canonical, surjective homomorphism $\pi : A' \to B$ of algebras.

Let $\theta : A' \to \mathrm{End}_{\mathscr{K}(A)}(T^\bullet)$ be an isomorphism of algebras. Note that $\mathrm{End}_B(B) \simeq B$ is radical-square zero by definition. Thus it is easy to know that $\theta\eta\mu : A' \to B$ sends the kernel of $\pi : A' \to B$ to zero, and that there is an algebra homomorphism $\chi: B\to B$, which fixes all idempotents $e_i, i\in\Delta_0$, such that $\theta\eta\mu=\pi\chi$. Since χ fixes the idempotents e_i with $i\in\Delta_0$, it induces an automorphism of the vector space e_iBe_j which is isomorphic to the vector space Q'_{ij} for all $i,j\in\Delta_0$. Since ρ' is Δ-invariant, there is an automorphism $\phi_\chi: A'\to A'$ extending χ, that is, $\phi_\chi\pi=\pi\chi$. Thus $\theta^{-1}\phi_\chi\pi=\eta\mu$, that is, there is a commutative diagram

$$\begin{tikzcd}
\mathrm{End}_{\mathscr{K}(A)}(T^\bullet) \arrow[r, "\eta\mu"] \arrow[d, "\simeq"', "\theta^{-1}\phi_\chi"] & B \arrow[d, equal] & B^\sigma \arrow[l, "\lambda_\sigma"'] \arrow[d, equal] \\
A' \arrow[r, "\pi"] & B & B^\sigma \arrow[l, "\lambda_\sigma"']
\end{tikzcd}$$

It then follows that there is an isomorphism ψ from the pullback algebra $\mathrm{End}_{\mathscr{K}(A^\sigma)}(T^\bullet_\sigma)$ of $\eta\mu$ and λ_σ to the pullback algebra A'^σ of π and λ_σ such that the following diagram

$$\begin{tikzcd}
\mathrm{End}_{\mathscr{K}(A^\sigma)}(T^\bullet_\sigma) \arrow[r, "\varepsilon_2"] \arrow[d, "\psi"] & \mathrm{End}_{B^\sigma}(B^\sigma) \arrow[d, "\mu"] \\
A'^\sigma \arrow[r, "\pi"] & B^\sigma
\end{tikzcd}$$

is commutative. This diagram can be extended to the following commutative diagram

$$\begin{tikzcd}
\mathrm{End}_{\mathscr{K}(A^\sigma)}(T^\bullet_\sigma) \arrow[r, "\varepsilon_2"] \arrow[d, "\simeq"', "\psi"] & \mathrm{End}_{B^\sigma}(B^\sigma) \arrow[r, "p"] \arrow[d, "\simeq"', "\mu"] & \mathrm{End}_\Lambda(\Lambda) \arrow[d, "\simeq"', "\mu"] & \mathrm{End}_{k^{\Delta_0^\sigma}}(k^{\Delta_0^\sigma}) \arrow[l, "i"'] \arrow[d, "\simeq"', "\mu"] \\
A'^\sigma \arrow[r, "\pi"] & B^\sigma \arrow[r, "\pi"] & \Lambda & k^{\Delta_0^\sigma} \arrow[l, "\lambda"']
\end{tikzcd}$$

where p and i are determined by π and λ, respectively. It then follows that the pullback algebra A'^α of $\pi: A'^\sigma\to\Lambda$ and λ is isomorphic to the pullback algebra of $\varepsilon_2 p$ and i. Note that

$$\Lambda\otimes_{k^{\Delta_0^\sigma}} k^{\Delta_0^\sigma}\simeq\Lambda\simeq\Lambda\otimes_{B^\sigma}B^\sigma\simeq\Lambda\otimes_{B^\sigma}B^\sigma\otimes_{A^\sigma}T^\bullet_\sigma$$

in $\mathscr{C}(\Lambda)$. By the proof of Theorem 3.1, the pullback algebra of $\varepsilon_2 p$ and i is derived equivalent to the pullback algebra A^α of $A^\sigma\xrightarrow{\pi}\Lambda\xleftarrow{\lambda}k^{\Delta_0^\sigma}$. Consequently, A'^α is derived equivalent to A^α. $\square$

Remark 4.6. (1) Note that two algebras A and B are derived equivalent if and only if their opposite algebras A^{op} and B^{op} are derived equivalent. So we can replace Δ by Δ^{op} and consider unifying arrows of Δ^{op}. This means that Theorem 4.5 also holds true for the subquiver Δ^{op}.

(2) The derived equivalence constructed in theorem 4.5 sends the simple A^α-modules corresponding to x and y again to simple A'^α-modules.

4.3 Derived equivalences from identifying socle elements

In this subsection, we introduce the third operation by identifying socle elements of algebras to get new derived equivalences.

Let A be a basic Artin algebra with the Jacobson radical r_A, and let $1_A=e_1+\cdots+e_n$ be a decomposition of 1_A into pairwise orthogonal primitive idempotents. Fix $i,j\in\{1,\cdots,n\}$. A *longest* (e_i,e_j)-*element* in A is a nonzero element $a\in e_ir_Ae_j$ such that $r_Aa=0=ar_A$, that is, $a\in\mathrm{soc}(r_Ae_j)\cap\mathrm{soc}(e_ir_A)$. In this case, the ideal $\langle a\rangle$ of A generated by a is 1-dimensional and contained in $\mathrm{soc}({}_AAe_j)\cap\mathrm{soc}(e_iA_A)$. A longest (e_i,e_i)-element is called a *complete* e_i-*cycle*.

For the rest of this subsection, we fix two algebras $A:=kQ/\langle\rho\rangle$ and $B:=k\Gamma/\langle\omega\rangle$ given by quivers with relations. Suppose that a is a longest (e_i,e_j)-element in A, and that b is a longest (e_s,e_t)-element in B, where $i,j\in Q_0$ and $s,t\in\Gamma_0$. We glue i and s into a new vertex u, and glue j and t into a new vertex v. Let σ be the corresponding partition of the set $\{i,j,s,t\}$. In case that $i=j$ or $s=t$, we actually glue all the vertices into one vertex, that is, $u=v$. Let $(A\times B)^\sigma$ be the σ-gluing algebra defined in Subsection 4.1. In case that $i=j$ and $s=t$, we simply write $A\,{}_{e_i}{\times}_{e_s}B$ for $(A\times B)^\sigma$. Now, it is easy to see that $a-b$ is a longest (e_u,e_v)-element in $(A\times B)^{\sigma}$ and the ideal $\langle a-b\rangle$ of $(A\times B)^{\sigma}$ generated by $a-b$ is 1-dimensional. So, we can define a new algebra

$$A\,{}_a\!\diamond_b B := (A\times B)^{\sigma}/\langle a-b\rangle.$$

It is called the algebra of *identifying socle elements* in A and B.

Suppose that $A' := kQ'/\langle \rho'\rangle$ is another algebra and there is a derived equivalence $F : \mathscr{D}^{\mathrm{b}}(A) \to \mathscr{D}^{\mathrm{b}}(A')$ such that $F(S_i) \simeq S_{i'}$ and $F(S_j) \simeq S_{j'}$ for some $i', j' \in Q'_0$. Let $T^{\bullet}$ be a basic, radical tilting complex associated to F. We may identify A' with $\mathrm{End}_{\mathscr{K}^{\mathrm{b}}(A)}(T^{\bullet})$ via the isomorphism $\mathrm{End}_{\mathscr{K}^{\mathrm{b}}(A)}(T^{\bullet}) \to A'$ induced by F. Further, by the proof of Lemma 2.4, both Ae_i and Ae_j only occur in degree zero with the multiplicity 1 in $T^{\bullet}$. For $x \in \{i, j\}$, let $T^{\bullet}_x$ be the indecomposable direct summand of $T^{\bullet}$ such that Ae_x is a direct summand of T^0_x, namely $T^0_x = Ae_x \oplus P_x$, and let $e_{x'}$ be the primitive idempotent element in A' corresponding to the summand $T^{\bullet}_x$. Let $m_a : T^{\bullet}_i \to T^{\bullet}_j$ be the following (well-defined) particular morphism

$$\begin{tikzcd}
\cdots \arrow[r] & T_i^{-1} \arrow[r] \arrow[d, "0"] & Ae_i\oplus P_i \arrow[r] \arrow[d, "\left[\begin{smallmatrix} \cdot a & 0 \\ 0 & 0\end{smallmatrix}\right]"] & T_i^{1} \arrow[r] \arrow[d, "0"] & \cdots \\
\cdots \arrow[r] & T_j^{-1} \arrow[r] & Ae_j\oplus P_j \arrow[r] & T_j^{1} \arrow[r] & \cdots
\end{tikzcd}$$

and let a' be the composite $T^{\bullet} \to T^{\bullet}_i \xrightarrow{m_a} T^{\bullet}_j \to T^{\bullet}$, where the first and last morphisms are the canonical projection and injection, respectively. This element a' has the following property.

Lemma 4.7. *The element a' just defined is a longest $(e_{i'}, e_{j'})$-element in A'.*

Proof. Since $a \in e_i r_A e_j$ is nilpotent, the element a' is nilpotent and lies in $e_{i'} r_{A'} e_{j'}$. It remains to show $r_{A'} a' = 0 = a' r_{A'}$.

Let $g^{\bullet} : T^{\bullet} \to T^{\bullet}$ be in $r_{A'}$. Then $g^{\bullet}$ is nilpotent, that is, $(g^{\bullet})^m$ is null-homotopic for some integer $m \geq 1$. Particularly, $(g^0)^m = h^0 d^{-1} + d^0 h^1$ for some homomorphisms $h^0 : T^0 \to T^{-1}$ and $h^1 : T^1 \to T^0$ of A-modules. Since the differential maps of $T^{\bullet}$ are radical by assumption, the map $(g^0)^m$ is radical, and so is g^0. It follows that the composite $g^0\pi^0 : T^0 \to T^0_i$ is also a radical map, where $\pi^{\bullet}$ is a canonical projection $T^{\bullet} \to T^{\bullet}_i$. Now, the fact $r_A a = 0$ indicates that the composite $Ae_l \xrightarrow{r} Ae_i \xrightarrow{\cdot a} Ae_j$ is zero for all $l \in Q_0$ and all radical maps r. Hence the chain map $g^{\bullet}\pi^{\bullet} m_a$ is zero in all degrees, and consequently $g^{\bullet} a' = 0$. This shows $r_{A'} a' = 0$. Similarly, using $a r_A = 0$, we can prove $a' r_{A'} = 0$. $\square$

The following theorem shows that we can extend the derived equivalence between A and A' by identifying socle elements.

Theorem 4.8. *The algebras $A\,{}_a\!\diamond_b B$ and $A'\,{}_{a'}\!\diamond_b B$ are derived equivalent.*

Proof. For simplicity, we write Λ for $(A\times B)^{\sigma}$. As explained in Subsection 4.1, Λ is the pullback algebra of the canonical surjective homomorphisms $B \to k^{\sigma}$ and $A \to k^{\sigma}$. Let $\sigma' = \{i', s\} \cup \{j', t\}$ be the corresponding partition of $\{i', j', s, t\}$. By the proof of Theorem 3.1, the complex $\tilde{T}^{\bullet} := M(T^{\bullet}, B, 1)$ is a tilting complex over Λ with the endomorphism algebra isomorphic to $(A'\times B)^{\sigma'}$. By definition, $\tilde{T}^{\bullet}_i := M(T^{\bullet}_i, Be_s, 1)$ and $\tilde{T}^{\bullet}_j := M(T^{\bullet}_j, B_t, 1)$ are indecomposable direct summands of $\tilde{T}^{\bullet}$. Note that all other indecomposable direct summands of $\tilde{T}^{\bullet}$ are of the form $M(P^{\bullet}, 0, 0)$ or $M(0, Q, 0)$, where $P^{\bullet}$ is an indecomposable direct summand of $T^{\bullet}$ and Q is an indecomposable projective B-module. Moreover, the indecomposable projective Λ-modules Λe_u and Λe_v, which are isomorphic to $M(Ae_i, Be_s, 1)$ and $M(Ae_j, Be_t, 1)$, respectively, only occur in degree zero with the multiplicity 1 in $\tilde{T}^{\bullet}$. Thus $\tilde{T}^{\bullet}$ is a basic, radical complex over Λ.

Set $I := \langle a-b\rangle$. Then $Ie_v = I = e_u I$ and $IX = 0$ for all indecomposable projective Λ-modules X not isomorphic to Λe_v. It follows that $I\tilde{T}^{\bullet} = I\tilde{T}^0 \simeq {}_{\Lambda}I$. Note that ${}_{\Lambda}I$ is a simple Λ-module with $e_u I \neq 0$. Hence $\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(\Lambda)}(\tilde{T}^{\bullet}, I\tilde{T}^{\bullet}[l]) \simeq \mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A)}(\tilde{T}^{\bullet}, I[l]) = 0$ for all $l \neq 0$. Now, the short exact sequence $0 \to I\tilde{T}^{\bullet} \to \tilde{T}^{\bullet} \to \tilde{T}^{\bullet}/I\tilde{T}^{\bullet} \to 0$ in the category of complexes over Λ gives raise to a triangle

$$I\tilde{T}^{\bullet} \longrightarrow \tilde{T}^{\bullet} \longrightarrow \tilde{T}^{\bullet}/I\tilde{T}^{\bullet} \longrightarrow I\tilde{T}^{\bullet}[1]$$

in $\mathscr{D}^{\mathrm{b}}(\Lambda)$. Applying $\mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(\Lambda)}(\tilde{T}^{\bullet}, -)$ to this triangle, we get an exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(\Lambda)}(\tilde{T}^{\bullet}, \tilde{T}^{\bullet}/I\tilde{T}^{\bullet}[-1]) \longrightarrow \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(\Lambda)}(\tilde{T}^{\bullet}, I\tilde{T}^{\bullet}) \longrightarrow \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(\Lambda)}(\tilde{T}^{\bullet}, \tilde{T}^{\bullet}),$$

which is isomorphic to

$$(\sharp)\qquad 0 \longrightarrow \mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(\Lambda)}(\tilde{T}^\bullet, \tilde{T}^\bullet/I\tilde{T}^\bullet[-1]) \longrightarrow \mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(\Lambda)}(\tilde{T}^\bullet, I\tilde{T}^\bullet) \xrightarrow{\theta} \mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(\Lambda)}(\tilde{T}^\bullet, \tilde{T}^\bullet).$$

Note that the map $\cdot(a-b) : \Lambda e_u \to {}_\Lambda I$ induces a morphism $g^\bullet$ in $\mathrm{End}_{\mathscr{K}^{\mathrm{b}}(\Lambda)}(\tilde{T}_i^\bullet)$:

$$\begin{tikzcd}
\cdots \arrow[r] & T_i^{-1} \arrow[r, "d"] \arrow[d] & \Lambda e_u \oplus P_i \arrow[r] \arrow[d, "{\left[\begin{smallmatrix} \cdot(a-b) \\ 0 \end{smallmatrix}\right]}"] & T_i^1 \arrow[r] \arrow[d] & 0 \\
& 0 \arrow[r] \arrow[d] & I \arrow[r] \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & T_j^{-1} \arrow[r, "d"] & \Lambda e_v \oplus P_j \arrow[r] & T_j^1 \arrow[r] & 0.
\end{tikzcd}$$

The image of g^0 is $I\Lambda e_v = I$. It follows that $g^\bullet$ cannot be null-homotopic, since the image of any morphism from T_j^{-1} or T_i^1 to Λe_v has image contained in Ae_j which intersects I trivially. Hence $g^\bullet \neq 0$, and therefore

$$\tilde{g}^\bullet := \begin{bmatrix} g^\bullet & 0 \\ 0 & 0 \end{bmatrix}$$

is a nonzero endomorphism of $\tilde{T}^\bullet$ and lies in $\mathrm{Im}(\theta)$ (see the sequence $(\sharp)$). Note that

$$\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(\Lambda)}(\tilde{T}^\bullet, I\tilde{T}^\bullet) \simeq \mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(\Lambda)}(\tilde{T}^\bullet, {}_\Lambda I) \simeq \mathrm{Hom}_\Lambda(\Lambda e_u, {}_\Lambda I) \simeq e_u I = I$$

and I is 1-dimensional. Hence θ is an injective map and $\mathrm{Im}(\theta)$ is a 1-dimensional k-space with $\tilde{g}$ as a basis. It follows from $(\sharp)$ that $\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(\Lambda)}(\tilde{T}^\bullet, \tilde{T}^\bullet/I\tilde{T}^\bullet[-1]) = 0$. Thus $\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(\Lambda)}(\tilde{T}^\bullet/I\tilde{T}^\bullet, \tilde{T}^\bullet/I\tilde{T}^\bullet[-1]) \simeq \mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(\Lambda)}(\tilde{T}^\bullet, \tilde{T}^\bullet/I\tilde{T}^\bullet[-1]) = 0$. Now, by [7, Theorem 4.2], the algebras Λ/I and $\mathrm{End}_{\mathscr{K}^{\mathrm{b}}(\Lambda)}(\tilde{T}^\bullet)/\mathrm{Im}(\theta)$ are derived equivalent. It is easy to check that the isomorphism $\mathrm{End}_{\mathscr{K}^{\mathrm{b}}(\Lambda)}(\tilde{T}^\bullet) \simeq (A' \times B)^{\sigma'}$, which is induced by the projections $\Lambda \to A$ and $\Lambda \to B$, sends the element $\tilde{g}^\bullet$ in $\mathrm{End}_{\mathscr{K}^{\mathrm{b}}(\Lambda)}(\tilde{T}^\bullet)$ to $a' - b$ in $(A' \times B)^{\sigma'}$. As a result, $\mathrm{End}_{\mathscr{K}^{\mathrm{b}}(\Lambda)}(\tilde{T}^\bullet)/\mathrm{Im}(\theta)$ is isomorphic to $A'\,{}_{a'}\!\diamond_b B$. Note that the algebra Λ/I is just $A\,{}_a\!\diamond_b B$. Hence $A\,{}_a\!\diamond_b B$ is derived equivalent to $A'\,{}_{a'}\!\diamond_b B$. This finishes the proof. $\square$

Remark that the derived equivalence in Theorem 4.8 sends the simple modules over $A\,{}_a\!\diamond_b B$ corresponding to u and v also to simple modules over $A'\,{}_{a'}\!\diamond_b B$ corresponding to u' and v'.

A special case of Theorem 4.8 is that we take complete cycles with the same starting and ending vertices.

Corollary 4.9. *Suppose that e and f are primitive idempotent elements in A and B, respectively, and that $a \in A$ is a complete e-cycle and $b \in B$ is a complete f-cycle. Let $T^\bullet$ be a basic, radical tilting complex over A with $[T^\bullet : Ae] = 1$, and let $A' = \mathrm{End}_{\mathscr{K}^{\mathrm{b}}(A)}(T^\bullet)$. Then $A\,{}_a\!\diamond_b B$ and $A'\,{}_{a'}\!\diamond_b B$ are derived equivalent.*

5 Derived equivalences and Frobenius type

As an application of our constructions in Section 4, we consider, in this section, whether Frobenius type of algebras is invariant under derived equivalences. Solutions to this question are presented in Proposition 5.1, Corollary 5.5 and Example 5.6.

Throughout this section, all algebras are finite-dimensional over a field.

Frobenius parts of algebras have played an important role in several aspects of the representation theory of algebras. For instance, concerning the Auslander-Reiten conjecture (or Alperin-Auslander conjecture referred in [17]) which states that stable equivalent algebras should have the same number of non-isomorphic, non-projective simple modules, Martínez-Villa reduced the validity of this conjecture for algebras without nodes to that for Frobenius parts (see [13]). In [9], the problem of lifting stable equivalences of Morita type to derived equivalences for arbitrary algebras is reduced to the one for their Frobenius parts. Moreover, there are close connections between dominant dimensions and Frobenius parts of algebras (see [3]).

Let A be an algebra. We may suppose A-stp $= \mathrm{add}(Ae)$ for e an idempotent element in A and that Ae is a basic A-module. Following [9], the algebra eAe is called the *Frobenius part* of A. It is a self-injective algebra introduced first in [13] (see also [9, Lemma 2.5]) and uniquely determined by A up to Morita equivalence.

We say that A is *Frobenius-finite (-tame or -wild)* if its Frobenius part is representation-finite (-tame or -wild). By Frobenius type we mean the representation type of the Frobenius part. If the Frobenius part of A is zero, we say that A is *Frobenius-free*.

Frobenius-finite algebras include representation-finite algebras, Auslander-algebras, cluster-tilted algebras, and can be produced from triangular matrix algebras, Auslander-Yoneda algebras and Frobenius extensions (see [9, Section 5] for details). For Frobenius-finite algebras over an algebraically closed field, every stable equivalence of Morita type lifts to a derived equivalence (see [9, Theorem 1.1]). Thus this large class of algebras shares many stable and derived invariants (see [16, 15, 11, 13, 7])

Now, we consider behaviors of Frobenius type under stable and derived equivalences.

From [9, Lemma 3.3] it follows that, for indecomposable algebras with separable semisimple quotients by their Jacobson radicals, Frobenius type is preserved by stable equivalences of Morita type. Actually, a more general statement is true, namely stable equivalences preserve Frobenius type. This follows from a simple observation (see Proposition 5.1(1) below). Recall that a simple A-module S is called a *node* in [12] if it is neither projective, nor injective, and the almost split sequence $0\to S\to P\to \mathrm{Tr}D(S)\to 0$ has a projective middle term P. For the definition of almost split sequences, we refer the reader to [1].

Proposition 5.1. (1) *Let A and B be algebras over an algebraically closed field and without nodes. If A and B are stably equivalent (that is, the stable categories A-$\underline{\mathrm{mod}}$ and B-$\underline{\mathrm{mod}}$ are equivalent), then they have the same Frobenius type.*

(2) *Let A be an algebra over an arbitrary field and ${}_AT$ be a tilting A-module with $B := \mathrm{End}_A(T)$. Then the Frobenius parts of A and B are isomorphic.*

Proof. (1) Under the assumptions of the proposition, we know from [13] that a stable equivalence between A and B induces a stable equivalence between their Frobenius parts. Since stable equivalences preserve representation type by [11], we see that the Frobenius parts of A and B have the same representation type, and therefore A and B have the same Frobenius type.

(2) This follows from [3, Lemma 4.3]. $\square$

As is known, derived equivalences between self-injective algebras over a field preserve representation type. Also, by Proposition 5.1(2), derived equivalences induced by tilting modules over arbitrary algebras preserve Frobenius type. Furthermore, almost ν-stable derived equivalences also preserve Frobenius type (see [9, Proposition 3.3]). So, based on these phenomena, one may naturally ask the following question:

Question. Does a derived equivalence always preserve Frobenius type of algebras?

In the following, we shall answer the question negatively.

Let A and B be basic algebras, and let e and f be primitive idempotents in A and B, respectively. Suppose that $a\in A$ is a complete e-cycle and that $b\in B$ is a complete f-cycle. Set $\Lambda := A\,{}_e\!\times_f B$, and $\Gamma := A\,{}_a\!\diamond_b B$. Recall that Γ is the quotient algebra of Λ modulo the one-dimensional ideal $I := \langle a-b\rangle$. For $x\in\Lambda$, we write $\bar{x} = x+I$ in Γ. As before, let $1_A = e+e_2+\cdots+e_n$ and $1_B = f+f_2+\cdots+f_m$ be decompositions of identities into pairwise orthogonal primitive idempotents. Then $1_\Gamma = \overline{e+f}+\bar{e_2}+\cdots+\bar{e_n}+\bar{f_2}+\cdots+\bar{f_m}$ is a decompositions of 1_Γ into pairwise orthogonal primitive idempotents.

In the following, we describe the Frobenius part of the algebra Γ.

Lemma 5.2. *Let A be an algebra, and let e_1, e_2 be primitive idempotents in A. Then the following are equivalent.*

(1) $\nu_A(Ae_1)\simeq Ae_2$.

(2) $e_1\,\mathrm{soc}(Ae_2)\neq 0$ *and, for each* $0\neq u\in e_1\,\mathrm{soc}(Ae_2)$, *the following two conditions are satisfied:*

(i) *For each* $0\neq x\in Ae_2$, *there is an element* $y\in e_1A$ *such that* $yx=u$.

(ii) *For each* $0\neq y\in e_1A$, *there is an element* $x\in Ae_2$ *such that* $yx=u$.

(3) *There is a nonzero element* $u\in e_1Ae_2$, *satisfying the conditions* (i) *and* (ii) *in* (2).

Proof. (1) $\Rightarrow$ (2) Suppose $\nu_A(Ae_1)\simeq Ae_2$. Then $\mathrm{soc}(Ae_2)$ is isomorphic to the top of Ae_1. Hence $e_1\,\mathrm{soc}(Ae_2)\neq 0$. Let u be a nonzero element in $e_1\,\mathrm{soc}(Ae_2)$. We claim that $u\in\mathrm{soc}(e_1A)$. For $r\in r_A$, let $\phi_r : Ae_2\to A, z\mapsto zr$. Then ϕ_r is a homomorphism of left A-modules. Since $\mathrm{Im}(\phi_r) = Ae_2r$ which is a nilpotent left ideal in A and since $Ae_2\simeq \nu_A(Ae_1)$ which is indecomposable and injective, ϕ_r is not injective. Otherwise, we would have $Ae_2r\simeq Ae_2$ and $A = Ae_2r\oplus L$ for some left ideal L of A since the module Ae_2r is injective, and consequently Ae_2r would contain an nonzero idempotent element and therefore not be nilpotent, a contradiction. Thus $(\mathrm{soc}(Ae_2))\phi_r = 0$, and therefore $ur=0$ and $u\in\mathrm{soc}(e_1A)$. Now, for each $0\neq x\in Ae_2$,

Ax is a nonzero submodule of Ae_2. Hence $\operatorname{soc}(Ae_2) \subseteq Ax$ and $u \in Ax$. Thus there is some $a \in A$ such that $u = ax$. Let $y = e_1 a$. Then $u = e_1 u = e_1 ax = yx$. That is, u satisfies the condition (i). Similarly, one can prove that u satisfies the condition (ii) by the fact that $u \in \operatorname{soc}(e_1 A)$ and $e_1 A \simeq D(e_2 A)$ which is an indecomposable, injective right A-module.

(2) $\Rightarrow$ (3) This is trivial.

(3) $\Rightarrow$ (1) Suppose that $u \in e_1 \operatorname{soc}(Ae_2)$ is a nonzero element satisfying the conditions (i) and (ii). Let α be a linear map in $D(e_1 A)$ such that $(u)\alpha = 1$. Define $\phi : Ae_2 \longrightarrow D(e_1 A), \quad z \mapsto (z\cdot)\alpha$. Then ϕ is a homomorphism of A-modules. For $0 \neq x \in Ae_2$, there is an element $y \in e_1 A$ such that $yx = u$ by the condition (i). It follows that $(x)\phi$ sends y to $(yx)\alpha = (u)\alpha = 1$. Thus $(x)\phi \neq 0$. This implies that ϕ is injective. Similarly, let β be a linear map in $D(Ae_2)$ such that $(u)\beta = 1$. Using the condition (ii), one can prove that the map

$$e_1 A \longrightarrow D(Ae_2), \quad y \mapsto \beta(\cdot\, y)$$

is an injective homomorphism of right A-modules. Then $\dim_k D(e_1 A) = \dim_k e_1 A \leq \dim_k D(Ae_2) = \dim_k Ae_2 \leq \dim_k D(e_1 A)$. It follows that these dimensions are equal, and consequently ϕ is an isomorphism. $\square$

To describe ν-stably projective Γ-modules, we also need the following lemma.

Lemma 5.3. *The assignment $(e)\theta = \overline{e+f}$, $(e_i)\theta = \bar{e}_i$ for all $i \geq 2$, and $(r)\theta = \bar{r}$ for all $r \in r_A$, defines an injective k-linear map $\theta : A \to \Gamma$ such that $(xy)\theta = (x)\theta(y)\theta$ for all $x, y \in A$.*

Proof. Since $A = ke \oplus ke_2 \oplus \cdots \oplus ke_n \oplus r_A$, the assignment $(e)\phi := e + f$, $(e_i)\phi := e_i$ for all $i \geq 2$ and $(r)\phi := r$ for all $r \in r_A$ defines a k-linear map $\phi : A \to \Lambda$. Now, it is rather straightforward to check from definition that ϕ is injective and satisfies $(xy)\phi = (x)\phi(y)\phi$ for all $x, y \in A$. Clearly, θ is the composite $\phi\pi$, where π is the canonical surjective algebra homomorphism from Λ to Γ. Note that θ is injective since $A \cap I = \{0\}$, and satisfies the other conditions of the lemma. $\square$

Proposition 5.4. *Keep the above notation. We have the following statements:*

(1) *For each $i \geq 2$, $\Gamma\bar{e}_i$ is ν-stably projective if and only if so is Ae_i.*

(2) *For each $i \geq 2$, $\Gamma\bar{f}_i$ is ν-stably projective if and only if so is Bf_i.*

(3) *$\Gamma(\overline{e+f})$ is ν-stably projective if and only if $\nu_A(Ae) \simeq Ae$ and $\nu_B(Bf) \simeq Bf$.*

Proof. We shall frequently use the injective map $\theta : A \to \Gamma$ in Lemma 5.3.

(1) Since $a \in A$ is a complete e-cycle and contained in $e \cdot \operatorname{soc}(Ae)$ by definition, the socle of Ae is isomorphic to the top of Ae. So, it cannot happen that $\nu_A(Ae_i) \simeq Ae$ for any $i \geq 2$. Thus, if Ae_i, with $i \geq 2$, is ν-stably projective, then $\nu_A^t(Ae_i)$ is isomorphic to some module in $\{Ae_2, \cdots, Ae_n\}$ for all $t \geq 1$. Similarly, $\bar{a}$ is a complete $\overline{e+f}$-cycle in Γ, and it cannot happen that $\nu_\Gamma(\Gamma\bar{e}_i) \simeq \Gamma(\overline{e+f})$. It is also impossible that $\nu_\Gamma(\Gamma\bar{e}_i) \simeq \Gamma\bar{f}_l$ for any $l \geq 2$, since $\bar{e}_i\Gamma\bar{f}_l = 0$. We shall show, for $i, j \geq 2$, that $\nu_A(Ae_i) \simeq Ae_j$ if and only if $\nu_\Gamma(\Gamma\bar{e}_i) \simeq \Gamma\bar{e}_j$. Note that θ induces isomorphisms of vector spaces $e_i A \to \bar{e}_i\Gamma$ and $Ae_j \to \Gamma\bar{e}_j$. Then it is easy to check that a nonzero element $u \in e_i Ae_j$ satisfies both (i) and (ii) in Lemma 5.2 if and only if $(u)\theta$, which is a nonzero element in $\bar{e}_i\Gamma\bar{e}_j$, satisfies the same conditions. By Lemma 5.2, $\nu_A(Ae_i) \simeq Ae_j$ if and only if $\nu_\Gamma(\Gamma\bar{e}_i) \simeq \Gamma\bar{e}_j$. Repeating this process, we see that $\Gamma\bar{e}_i$ is ν-stably projective if and only if Ae_i is ν-stably projective. This proves (1).

(2) This can be shown similarly.

(3) We assume that $\Gamma(\overline{e+f})$ is ν-stably projective. Then $\Gamma(\overline{e+f})$ is indecomposable and projective-injective, and has a 1-dimensional simple socle. The element $\bar{a} = \bar{b}$ is in the socle of $\Gamma(\overline{e+f})$ and $(\overline{e+f})\bar{a} = \bar{a}$. It follows that the socle of $\Gamma(\overline{e+f})$ is isomorphic to the simple Γ-module corresponding to the primitive idempotent $\overline{e+f}$. Hence $\nu_\Gamma\Gamma(\overline{e+f}) \simeq \Gamma(\overline{e+f})$. By Lemma 5.2, the element $\bar{a} \in (\overline{e+f})\Gamma(\overline{e+f})$ satisfies the condition (i) and (ii) in Lemma 5.2(3). We shall prove that $a \in eAe$ satisfies the conditions (i) and (ii) in Lemma 5.2(3). Let $x \in Ae$ be a nonzero element. Suppose $x = \lambda e + r$ for some $\lambda \in k$ and $r \in r_A e$. If $\lambda \neq 0$, then $\frac{1}{\lambda}ax = a + \frac{1}{\lambda}ar = a$. Now, we assume that $\lambda = 0$ and $x = r \in r_A e$. In this case, $(x)\theta = \bar{r}$ is a non-zero element in $\Gamma(\overline{e+f})$. By Lemma 5.2, there is an element $w = \mu(\overline{e+f}) + \bar{r}_1 + \bar{r}_2$ in $(\overline{e+f})\Gamma$, where $\mu \in k$, $r_1 \in er_A$ and $r_2 \in fr_B$, such that $w \cdot (x)\theta = \bar{a}$. Note that $\bar{r}_2 \cdot (x)\theta = \bar{r}_2\bar{r} = 0$. Hence $\bar{a} = w \cdot (x)\theta = (\mu(\overline{e+f}) + \bar{r}_1) \cdot (x)\theta = (\mu e + r_1)\theta \cdot (x)\theta$. Let $y = \mu e + r_1$. Then $y \in eA$ and $(yx)\theta = \bar{a}$. Since $a \in r_A$, we have $\bar{a} = (a)\theta$. It follows that $(yx)\theta = (a)\theta$, and consequently $yx = a$. This shows that a satisfies the condition (i) in Lemma 5.2. Similarly, a also satisfies the condition (ii) in Lemma 5.2. Consequently, $\nu_A(Ae) \simeq Ae$. Similarly, $\nu_B(Bf) \simeq Bf$.

Conversely, we assume that $\nu_A(Ae) \simeq Ae$ and $\nu_B(Bf) \simeq Bf$. By Lemma 5.2, $a \in eAe$ (respectively, $b \in fBf$) satisfies the conditions (i) and (ii) in Lemma 5.2. We claim that $\bar{a} \in (\overline{e+f})\Gamma(\overline{e+f})$ also satisfies the conditions (i) and (ii) in Lemma 5.2. Let $w = \lambda(\overline{e+f}) + \bar{r}_1 + \bar{r}_2 \in \Gamma(\overline{e+f})$ be a nonzero element with

$\lambda \in k$, $r_1 \in r_A e$ and $r_2 \in r_B f$. If $\lambda \neq 0$, $\frac{1}{\lambda}\bar{a} \in (\overline{e+f})\Gamma$, and $\frac{1}{\lambda}\bar{a}w = \bar{a}$. Next we assume $\lambda = 0$. Then $w = \bar{r}_1 + \bar{r}_2$. We can assume that w is not a multiple of $\bar{a} = \bar{b}$ (Otherwise, w clearly satisfies the condition (i) in Lemma 5.2). Then either $\bar{r}_1$ or $\bar{r}_2$ is not a multiple of $\bar{a}$. Without loss of generality, we assume that $\bar{r}_1 \neq 0$ is not a multiple of $\bar{a}$. This is equivalent to $Ar_1 \neq ka$. Since ka is the simple socle of Ae and Ar_1 is a nonzero submodule of Ae, ka is a proper submodule of Ar_1, and there is some $z \in r_A$ such that $zr_1 = a$. Let $y = ez$. Then $yr_1 = ezr_1 = ea = a$. Hence $\bar{y} \cdot w = \bar{y}\bar{r}_1 + \bar{y}\bar{r}_2 = \bar{a}$. Altogether, $\bar{a}$ satisfies the condition (i) in Lemma 5.2. Similarly, $\bar{a}$ also satisfies the condition (ii) in Lemma 5.2. As a result, $\nu_\Gamma\big(\Gamma(\overline{e+f})\big) \simeq \Gamma(\overline{e+f})$. $\square$

Note that if both A and B are symmetric algebras, then Proposition 5.4 tells that $A\,{}_a\!\diamond_b B$ is a self-injective algebra. Further, we have the following corollary of Proposition 5.4 and Corollary 4.9. For notation, see Section 4.3.

Corollary 5.5. *Let A be a Frobenius-free k-algebra given by a quiver with relations. Suppose that $e \in A$ is a primitive idempotent and that $a \in A$ is a complete e-cycle element. Let $T^\bullet$ be a basic, radical tilting complex over A such that $[T^\bullet : Ae] = 1$, that $A' := \mathrm{End}_{\mathscr{K}^{\mathrm{b}}(A)}(T^\bullet)$ is Frobenius-finite, and that the only ν-stably projective, indecomposable A'-module is $A'\tilde{e}$ with $\tilde{e}^2 = \tilde{e} \in A'$, where $\tilde{e}$ is the idempotent element in A' corresponding to the indecomposable direct summand of $T^\bullet$ in which Ae appears. Let B be a basic, symmetric k-algebra which is splitting over k and has no nonzero semisimple direct summands, and let f be a primitive idempotent in B and $0 \neq b \in \mathrm{soc}({}_B Bf)$. Then $A\,{}_a\!\diamond_b B$ and $A'\,{}_{a'}\!\diamond_b B$ are derived equivalent, and their Frobenius parts are $(1_B - f)B(1_B - f)$ and B, respectively.*

Finally, we employ Corollary 5.5 to construct a series of examples, showing that Frobenius type of algebras may change under derived equivalences.

Example 5.6. Let A and A' be k-algebras given by the following quivers with relations, respectively:

$$\begin{tikzcd} \underset{1}{\bullet} \arrow[r, shift left, "\alpha"] & \underset{2}{\bullet} \arrow[l, shift left, "\delta"] \arrow[r, shift left, "\beta"] & \underset{3}{\bullet} \arrow[l, shift left, "\gamma"] \end{tikzcd} \qquad \begin{tikzcd} \overset{1}{\bullet} \arrow[r, "\alpha'"] & \overset{2}{\bullet} \arrow[dl, "\beta'"] \\ \underset{3}{\bullet} \arrow[u, "\gamma'"] & \end{tikzcd}$$

$$\alpha\delta\alpha, \gamma\delta, \delta\alpha - \beta\gamma\,; \qquad \alpha'\beta'\gamma'\alpha', \gamma'\alpha'\beta'\gamma'.$$

Then $\dim_k(A) = 12$ and $\dim_k(A') = 13$. We denote by e_i the primitive idempotent element of A corresponding to the vertex i. Let $e = e_1$. Then there is a tilting complex $T^\bullet = T_1^\bullet \oplus Ae_2[1] \oplus Ae_3[1]$ over A, where $T_1^\bullet$ is the complex

$$0 \longrightarrow Ae_2 \xrightarrow{\cdot\delta} Ae \longrightarrow 0$$

with Ae in degree zero. Then the assignment

$$\alpha' \ \mapsto \ \begin{tikzcd} 0 \arrow[r] & Ae_2 \arrow[r, "\cdot\delta"] \arrow[d, equal] & Ae \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & Ae_2 \arrow[r] & 0 & \end{tikzcd} \qquad \beta' \ \mapsto \ \begin{tikzcd} 0 \arrow[r] & Ae_2 \arrow[r] \arrow[d, "\cdot\beta"] & 0 \\ 0 \arrow[r] & Ae_3 \arrow[r] & 0 \end{tikzcd}$$

$$\gamma' \ \mapsto \ \begin{tikzcd} 0 \arrow[r] & Ae_3 \arrow[r] \arrow[d, "\cdot(-\gamma)"] & 0 \arrow[d] & \\ 0 \arrow[r] & Ae_2 \arrow[r, "\cdot\delta"] & Ae \arrow[r] & 0 \end{tikzcd}$$

induces an isomorphism between A' and $\widetilde{A} := \mathrm{End}_{\mathscr{K}^{\mathrm{b}}(A)}(T^\bullet)$. In the following, we identify A' with $\mathrm{End}_{\mathscr{K}^{\mathrm{b}}(A)}(T^\bullet)$, that is, $A' = \widetilde{A}$. The element $a := \alpha\delta$ is a complete e-cycle in A. Let a' be the particular element in $\widetilde{A}$:

$$\begin{tikzcd} 0 \arrow[r] & Ae_2 \arrow[r, "\cdot\delta"] \arrow[d, "0"] & Ae \arrow[r] \arrow[d, "\cdot\alpha\delta"] & 0 \\ 0 \arrow[r] & Ae_2 \arrow[r, "\cdot\delta"] & Ae \arrow[r] & 0. \end{tikzcd}$$

Due to the relation $\delta\alpha - \beta\gamma$ in A, we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & Ae_2 \arrow[r, "\cdot\delta"] \arrow[d, "\cdot\beta\gamma"'] & Ae \arrow[r] \arrow[d, "\cdot\alpha\delta"] \arrow[dl, dashed, "\cdot\alpha"] & 0 \\
0 \arrow[r] & Ae_2 \arrow[r, "\cdot\delta"'] & Ae \arrow[r] & 0.
\end{tikzcd}$$

This implies $a' = \alpha'\beta'\gamma'$.

Let $\tilde{e}$ be the primitive idempotent in $\widetilde{A}$ corresponding to the direct summand $T_1^{\bullet}$. Under the identification of $\tilde{A}$ with A', $\tilde{e}$ corresponds to the vertex 1 in A'. Note that A is Frobenius-free, the Frobenius part of A' is isomorphic to $k[x]/(x^2)$, and the only ν-stably projective, indecomposable A'-module is $A'\tilde{e}$, that is A'-stp $= \mathrm{add}(A'\tilde{e})$.

Let B be a basic, symmetric algebra without semisimple direct summands, and let f be a primitive idempotent in B. Then any nonzero element in the socle of Bf is a complete f-cycle. Let b be such an element. Then, by Corollary 5.5, the algebras $A\,{}_a\!\diamond_b B$ and $A'\,{}_{a'}\!\diamond_b B$ are derived equivalent, while the Frobenius part of $A\,{}_a\!\diamond_b B$ is $(1_B - f)B(1_B - f)$ and the Frobenius part of $A'\,{}_{a'}\!\diamond_b B$ is B.

Thus, if we choose a basic, symmetric algebra B and a primitive idempotent $f \in B$ such that B is wild (or tame) and $(1_B - f)B(1_B - f)$ is tame or representation-finite, then $A'\,{}_{a'}\!\diamond_b B$ is Frobenius-wild (or tame), while $A\,{}_a\!\diamond_b B$ is Frobenius-tame or Frobenius-finite. This means that Frobenius type may change under derived equivalences in general.

Note that under the derived equivalence defined by $T^{\bullet}$ in Example 5.6, the simple A-module S_1 corresponding to the vertex 1 is sent to the simple A'-module S_1' corresponding to the vertex 1 by Lemma 2.4. Thus, by Theorem 4.1, if we glue an arbitrary algebra B at the vertex 1 in A and A', respectively, then the resulting extension algebras of A and A' are also derived equivalent.

References

[1] M. Auslander, I. Reiten and S. O. Smalø, *Representation theory of Artin algebras*, Cambridge Studies in Advanced Mathematics **36**, Cambridge University Press, Cambridge, 1995.

[2] I. Burban and Y. Drozd, Derived categories of nodal algebras, *J. Algebra* **272** (2004), 46-94.

[3] H. X. Chen and C. C. Xi, Dominant dimensions, derived equivalences and tilting modules, *Israel J. Math.* **215** (2016), 349-395.

[4] D. Happel, *Triangulated categories in the representation theory of finite dimensional algebras*, Cambridge University Press, 1988.

[5] D. Herbara and P. Prihoda, Infinitely generated projective modules over pullbacks of rings, *Trans. Amer. Math. Soc.* **366** (2014), 1433-1454.

[6] W. Hu, On iterated almost ν-stable derived equivalences. *Comm. Algebra* **40** (2012), 3920-3932.

[7] W. Hu and C. C. Xi, Derived equivalences for Φ-Auslander-Yoneda algebras, *Trans. Amer. Math. Soc.* **365** (2013), 5681-5711.

[8] W. Hu and C. C. Xi, Derived equivalences and stable equivalences of Morita type, I, *Nagoya Math. J.* **200** (2010), 107–152.

[9] W. Hu and C. C. Xi, Derived equivalences and stable equivalences of Morita type, II, *Rev. Mat. Iberoam.* (to appear). Preprint is available at: `http://math0.bnu.edu.cn/~ccxi/`.

[10] E. Kirkman and J. Kuzmanovich, On the global dimension of fibre products. *Pacific J. Math.* **134** (1988), 121-132.

[11] H. Krause, Representation type and stable equivalence of Morita type for finite dimensional algebras, *Math. Z.* **229** (1998), 601-606.

[12] R. Martínez-Villa, Algebras stably equivalent to l-hereditary, in: Representation theory, II (Proc. Second Internat. Conf., Carleton Univ., Ottawa, Ont., 1979), *Lecture Notes in Math.* **832**, Springer, Berlin, 1980, pp. 396-431.

[13] R. MARTÍNEZ-VILLA, Properties that are left invariant under stable equivalence, *Comm. Algebra* **18** (1990), 4141-4169.

[14] J. W. MILNOR, *Introduction to algebraic K-theory*, Annals of Mathematics Studies, No. 72, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1971.

[15] J. RICKARD, Derived categories and stable equivalence, *J. Pure Appl. Algebra* **61** (1989), 303-317.

[16] J. RICKARD, Morita theory for derived categories, *J. London Math. Soc.* **39** (1989), 436-456.

[17] R. ROUQUIER, Derived equivalences and finite dimensional algebras. In: *International Congress of Mathematicians* Vol. II. 191-221, Eur. Math. Soc., Zürich, 2006.

[18] C. C. XI, On the finitistic dimension conjecture, II: related to finite global dimension, *Adv. Math.* **201** (2006), 116-142.

[19] C. C. XI, Constructions of derived equivalences for algebras and rings. *Front. Math. China* **12** (2017), 1-18.

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