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Recollements induced from tilting modules over tame hereditary algebras

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Abstract. In this paper, we consider the endomorphism algebra of an infinitely generated tilting module of the form $R_{\mathcal{U}} \oplus R_{\mathcal{U}}/R$ over a tame hereditary k-algebra R with k an arbitrary field, where $R_{\mathcal{U}}$ is the universal localization of R at an arbitrary set $\mathcal{U}$ of simple regular R-modules. We show that the derived module category of this endomorphism algebra is a recollement of the derived module category $\mathscr{D}(R)$ of R and the derived module category $\mathscr{D}(\mathbb{A}_{\mathcal{U}})$ of the adèle ring $\mathbb{A}_{\mathcal{U}}$ associated with $\mathcal{U}$. When k is an algebraically closed field, the ring $\mathbb{A}_{\mathcal{U}}$ can be precisely described in terms of Laurent power series ring $k((x))$ over k. Moreover, if $\mathcal{U}$ is a union of finitely many cliques, we give two different stratifications of the derived category of this endomorphism algebra by derived categories of rings such that the two stratifications are of different finite lengths.

Keywords. Adèle ring, recollement, stratification, tame hereditary algebra, tilting module, universal localization.

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1 Introduction

Tilting modules over tame hereditary algebras have played a special role in the development of the representation theory of algebras: Finite-dimensional tilting modules provide a class of minimal representation-infinite algebras which can be used together with the covering techniques in [4] to decide whether an algebra is of finite representation type, while infinite-dimensional tilting modules involve the generic modules discovered by Ringel in [26], Prüfer modules and adic modules. Very recently, Angeleri-Hügel and Sánchez have classified all tilting modules over tame hereditary algebras up to equivalence in [3]. One of the main ingredients of their classification involves the universal localizations at simple regular modules, which were already studied by Crawley-Boevey in [14]. It is worth noting that

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Krause and Stovicek have recently shown in [20] that over hereditary rings universal localizations and ring epimorphisms coincide. For finite-dimensional tilting modules over tame hereditary algebras, their endomorphism algebras are well understood from the point of view of torsion theory and derived categories (see [7, 17, 18, 27] and others). For example, by Happel's general theorem, the given tame hereditary algebras and the endomorphism algebras of their tilting modules are derived equivalent. However, for infinite-dimensional tilting modules, one cannot get such derived equivalences anymore (see [5]). Nevertheless, if they are good tilting modules, then the derived module categories of their endomorphism algebras admit recollements by derived module categories of the given tame hereditary algebras themselves on the one side, and of certain universal localizations of their endomorphism algebras on the other side, as shown by a general result in [8]. Here, not much is known about the precise structures of these universal localizations as well as the derived composition factors of these recollements. In fact, it seems to be very difficult to describe them in general.

In the present paper, we will study these new recollements arising from a class of good tilting modules over tame hereditary algebras more explicitly. In this special situation, we can describe precisely the universal localizations appearing in the recollements in terms of adèle rings which occur often in algebraic number theory (see [23, Chapter V]), determine their derived composition factors, and provide two completely different stratifications of the derived module categories of the endomorphism algebras of these tilting modules.

Let R be an indecomposable finite-dimensional tame hereditary algebra over an arbitrary field k. Of our interest are simple regular R-modules. Now, we fix a complete set $\mathscr{S}$ of all non-isomorphic simple regular R-modules, and consider the equivalence relation $\sim$ on $\mathscr{S}$ generated by

$$L_1 \sim L_2 \quad \text{for } L_1, L_2 \in \mathscr{S} \text{ if } \mathrm{Ext}^1_R(L_1, L_2) \neq 0.$$

The equivalence classes of this relation are called *cliques* (see [14]). It is well known that all cliques are finite, and all but at most three cliques consist of only one simple regular module.

Let $\mathcal{C}$ be a clique of R and $V \in \mathcal{C}$. Then there is a unique Prüfer R-module, denoted by $V[\infty]$, such that its regular socle is equal to V (see [26]). Moreover, for any two non-isomorphic simple regular modules in $\mathcal{C}$, the endomorphism algebras of the Prüfer modules corresponding to them are isomorphic (see, for instance, Lemma 3.1 (3)). Hence we define $D(\mathcal{C})$ to be $\mathrm{End}_R(V[\infty])$ for an arbitrary but fixed module $V \in \mathcal{C}$. It is shown that this ring is a (not necessarily commutative) discrete valuation ring. Therefore, the so-called division ring $Q(\mathcal{C})$ of fractions of $D(\mathcal{C})$ exists, which is the "smallest" division ring containing $D(\mathcal{C})$ as a subring up to isomorphism.

Let $\mathcal{U} \subseteq \mathscr{S}$ be a set of simple regular modules, and let $R_{\mathcal{U}}$ stand for the universal localization of R at $\mathcal{U}$ in the sense of Schofield and Crawley-Boevey. Then it is proved in [2] that the R-module $T_{\mathcal{U}} := R_{\mathcal{U}} \oplus R_{\mathcal{U}}/R$ is a tilting module. Following [3, Example 1.3], if $\mathcal{U}$ is a union of cliques, the R-module $T_{\mathcal{U}}$ is called the *Reiten–Ringel tilting module* associated with $\mathcal{U}$. This class of modules was studied first in [26] and generalized then in [24]. As a main objective of the present paper, we will concentrate on the derived categories of the endomorphism algebras of tilting modules $T_{\mathcal{U}}$ for arbitrary subsets $\mathcal{U}$ of $\mathscr{S}$.

Let $k[[x]]$ and $k((x))$ be the algebras of formal and Laurent power series over k in one variable x, respectively. For an index set I, we define the I-adèle ring of $k((x))$ by

$$\mathbb{A}_I := \left\{ (f_i)_{i\in I} \in \prod_{i\in I} k((x)) \mid f_i \in k[[x]] \text{ for almost all } i \in I \right\},$$

where $\prod_{i\in I} k((x))$ stands for the direct product of I copies of $k((x))$. In particular, if I is a finite set, then $\mathbb{A}_I = k((x))^{|I|}$.

Our main result in this paper is the following theorem which provides us with a class of new recollements different from the one obtained by the structure of triangular matrix rings.

Theorem 1.1. *Let R be an indecomposable finite-dimensional tame hereditary algebra over an arbitrary field k. Let $\mathcal{U}$ be a nonempty set of simple regular R-modules with $\{\mathcal{C}_i\}_{i\in I}$, the set of all cliques contained in $\mathcal{U}$ where I is an index set, and let B be the endomorphism algebra of $R_{\mathcal{U}} \oplus R_{\mathcal{U}}/R$, where $R_{\mathcal{U}}$ stands for the universal localization of R at $\mathcal{U}$. Then there is the following recollement of derived module categories:*

$$\begin{tikzcd} \mathscr{D}(\mathbb{A}_{\mathcal{U}}) \arrow[r] & \mathscr{D}(B) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}(R) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

where $\mathbb{A}_{\mathcal{U}}$ is the I-adèle ring with respect to the rings $Q(\mathcal{C}_i)$ for $i \in I$, that is,

$$\mathbb{A}_{\mathcal{U}} := \left\{ (f_i)_{i\in I} \in \prod_{i\in I} Q(\mathcal{C}_i) \mid f_i \in D(\mathcal{C}_i) \textit{ for almost all } i \in I \right\}$$

and is Morita equivalent to a universal localization of B. In particular, if k is algebraically closed, then $\mathbb{A}_{\mathcal{U}}$ is isomorphic to the I-adèle ring $\mathbb{A}_I$ of the Laurent power series ring $k((x))$.

For more details on the six functors and relationship of the rings in the above recollement, we refer the reader to the explanation after Proposition 2.11.

As a consequence of Theorem 1.1, we obtain new stratifications of the derived categories of the endomorphism algebras of tilting modules arising from universal localizations at simple regular modules.

Corollary 1.2. *Let R be an indecomposable finite-dimensional tame hereditary algebra over an algebraically closed field k. Let r be the number of non-isomorphic simple R-modules. Suppose that $\mathcal{U}$ is a non-empty finite subset of $\mathscr{S}$ consisting of s cliques. Let B be the endomorphism algebra of the Reiten–Ringel tilting R-module associated with $\mathcal{U}$. Then $\mathscr{D}(B)$ admits two stratifications by derived module categories:*

- *one is of length $r + s$ with the following composition factors: r copies of the ring k and s copies of the ring $k((x))$,*
- *the other is of length $r + s - 1$ with the following composition factors: $r - 2$ copies of the ring k, s copies of the ring $k[[x]]$ and one copy of a Dedekind integral domain contained in the field $k(x)$ of fractions of the polynomial algebra $k[x]$.*

It follows from Corollary 1.2 that a derived module category may have two stratifications with different lengths and different sets of derived composition factors. This gives a negative answer to a general question whether Jordan–Hölder's theorem holds true for stratifications of derived module categories by derived module categories (see [1]).

Observe that if R is the Kronecker algebra and $\mathcal{U}$ consists of only one simple regular module, then we re-obtain the stratifications, shown in the example of [8, Section 8], from Corollary 1.2.

Now, let us describe the structure of this paper. In Section 2, we fix notation and recall some definitions and basic facts which will be used throughout the paper. In Section 3, we consider Prüfer modules and their endomorphism algebras. In Section 4, we make several preparations for proofs of our main results in the paper. First, we consider universal localizations at simple regular modules over tame hereditary algebras and establish a crucial result, Corollary 4.9. Second, we discuss the endomorphism algebras of tilting modules over arbitrary tame hereditary algebras. Finally, we reduce our consideration of universal localizations of arbitrary tame hereditary algebras to the special case of universal localizations of the Kronecker algebra at simple regular modules. In Section 5, we first apply the results in the previous sections to prove Theorem 1.1, and then, by using Theorem 1.1 together with a result in [19], determine the derived composition factors of the derived categories of the endomorphism algebras of Reiten–Ringel tilting modules, and therefore get a proof of Corollary 1.2. At the end of this section we mention a few questions related to the results in this paper.

2 Preliminaries

In this section, we first recall some standard notation which will be used throughout this paper, and then develop some properties on universal localizations and recollements. Finally, we collect some homological facts which are useful for our proofs.

2.1 Notation

All rings considered are assumed to be associative and with identity, all ring homomorphisms preserve identity, and all full subcategories $\mathcal{D}$ of a given category $\mathcal{C}$ are closed under isomorphic images, that is, if X and Y are objects in $\mathcal{C}$, then $Y \in \mathcal{D}$ whenever $Y \simeq X$ with $X \in \mathcal{D}$.

Let R be a ring.

We denote by R-Mod the category of all unitary left R-modules, and by R-mod the category of finitely generated unitary left R-modules. Unless stated otherwise, by an R-module we mean a left R-module. For an R-module M, we denote by $\mathrm{add}(M)$ (respectively, $\mathrm{Add}(M)$) the full subcategory of R-Mod consisting of all direct summands of finite (respectively, arbitrary) direct sums of copies of M. If I is an index set, we denote by $M^{(I)}$ the direct sum of I copies of M.

If $f : M \to N$ is a homomorphism of R-modules, then the image of $x \in M$ under f is denoted by $(x)f$ instead of $f(x)$. Also, for any R-module X, the induced morphisms

$$\mathrm{Hom}_R(X, f) : \mathrm{Hom}_R(X, M) \to \mathrm{Hom}_R(X, N)$$

and

$$\mathrm{Hom}_R(f, X) : \mathrm{Hom}_R(N, X) \to \mathrm{Hom}_R(M, X)$$

are denoted by f^* and f_*, respectively.

Given a class $\mathcal{U}$ of R-modules, we denote by $\mathscr{F}(\mathcal{U})$ the full subcategory of R-Mod consisting of all those R-modules M which have a finite filtration

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$$

such that M_i/M_{i-1} is isomorphic to a module in $\mathcal{U}$ for each i. We say that M is a *direct union of finite extensions* of modules in $\mathcal{U}$ if M is the direct limit of a direct system of submodules in $\mathscr{F}(\mathcal{U})$ of M (with respect to the inclusion ordering). Note that if M is the direct limit of a direct system of submodules $\{M_\alpha\}_{\alpha \in I}$ of M (with respect to the inclusion ordering), then $M = \bigcup_{\alpha \in I} M_\alpha$. In addition, if M is finitely generated, then $M = M_\alpha$ for some $\alpha \in I$.

Let $\mathscr{D}(R)$ be the (unbounded) derived category of R-Mod, which is the localization of the homotopy category of R-Mod at all quasi-isomorphisms. Furthermore, we always identify R-Mod with the subcategory of $\mathscr{D}(R)$ consisting of all stalk complexes concentrated on degree zero. It is well known that

$$\mathrm{Hom}_{\mathscr{D}(R)}(X, Y[n]) \simeq \mathrm{Ext}^n_R(X, Y)$$

for any $X, Y \in R$-Mod and $n \in \mathbb{N}$, where $[n]$ stands for the n-th shift functor of $\mathscr{D}(R)$, and that the triangulated category $\mathscr{D}(R)$ has small coproducts, that is, coproducts indexed by sets exist in $\mathscr{D}(R)$.

Let S be a ring and let $M^\bullet$ be a complex of R-S-bimodules. We shall denote by $M^\bullet \otimes^{\mathbb{L}}_S - : \mathscr{D}(S) \to \mathscr{D}(R)$ the total left-derived functor of $M^\bullet \otimes^\bullet_S -$, while $\mathbb{R}\mathrm{Hom}_R(M^\bullet, -) : \mathscr{D}(R) \to \mathscr{D}(S)$ is defined to be the total right-derived functor of $\mathrm{Hom}^\bullet_R(M^\bullet, -)$. Note that $(M^\bullet \otimes^{\mathbb{L}}_S -, \mathbb{R}\mathrm{Hom}_R(M^\bullet, -))$ is an adjoint pair of triangle functors.

If R is an Artin k-algebra over a commutative Artin ring k, we denote by D the usual duality, and by τ the Auslander–Reiten translation of R.

2.2 Ore localizations and universal localizations

In this subsection, we shall recall the definition of universal localizations, and mention two special cases of universal localizations: Ore localizations and universal localizations at a set of modules of projective dimension at most 1.

First of all, we have the following known result on universal localizations.

Lemma 2.1 ([28, Theorem 4.1]). *Let R be a ring and Σ be a set of homomorphisms between finitely generated projective R-modules. Then there is a ring R_Σ and a homomorphism $\lambda : R \to R_\Sigma$ of rings with the following properties:*

(1) *λ is Σ-inverting, that is, if $\alpha : P \to Q$ belongs to Σ, then*

$$R_\Sigma \otimes_R \alpha : R_\Sigma \otimes_R P \longrightarrow R_\Sigma \otimes_R Q$$

is an isomorphism of R_Σ-modules.

(2) *λ is universal Σ-inverting, that is, if S is a ring such that there exists a Σ-inverting homomorphism $\varphi : R \to S$, then there exists a unique homomorphism $\psi : R_\Sigma \to S$ of rings such that $\varphi = \lambda\psi$.*

(3) *The homomorphism $\lambda : R \to R_\Sigma$ is a ring epimorphism with*

$$\mathrm{Tor}^R_1(R_\Sigma, R_\Sigma) = 0.$$

We call $\lambda : R \to R_\Sigma$ in Lemma 2.1 the *universal localization* of R at Σ. If the R-module R_Σ has projective dimension at most 1, then λ is homological. Recall that a ring epimorphism $R \to S$ is said to be homological if $\mathrm{Tor}^R_n(S, S) = 0$ for every $n > 0$. Of our particular interest are the following two kinds of universal localizations.

The first one is associated with subsets of elements in rings.

Let Φ be a non-empty subset of R. Then we consider the universal localization of R at all homomorphisms ρ_r with $r \in \Phi$, where ρ_r is the right multiplication map $R \to R$ defined by $x \mapsto xr$ for $x \in R$. For simplicity, we write R_Φ for this universal localization, and say that R_Φ is the *universal localization of* R *at* Φ. Note that, by the property of universal localizations, R_Φ is also isomorphic to the "right" universal localization of R at all left multiplication maps $\sigma_r : R_R \to R_R$ defined by $x \mapsto rx$ for $x \in \Phi$, which are regarded as homomorphisms of right R-modules. Clearly, if $0 \in \Phi$, then $R_\Phi = 0$. If $0 \notin \Phi$, then we consider the smallest multiplicative subset Φ_1 of R containing Φ, and get $R_\Phi = R_{\Phi_1}$. Recall that a subset Φ of R is said to be *multiplicative* if $0 \notin \Phi$, $1 \in \Phi$, and it is closed under multiplication.

From now on, we assume that Φ is a multiplicative subset of R.

Under some extra assumptions on Φ, the ring R_Φ can be characterized by Ore localizations which generalizes the notion of localizations in commutative rings. To explain this point in detail, we first recall some relevant definitions about Ore localizations. For more details, we refer to [22, Chapter 4].

Definition 2.2. A subset Φ of R is called a *left denominator subset of* R if Φ satisfies the following two conditions:

(i) For any $a \in R$ and $s \in \Phi$, there holds $\Phi a \cap Rs \neq \emptyset$.

(ii) For any $r \in R$, if $rt = 0$ for some $t \in \Phi$, then there exists some $t' \in \Phi$ such that $t'r = 0$.

If Φ satisfies only the condition (i), then Φ is called a *left Ore subset of* R.

Similarly, we can define the notions of right denominator sets and right Ore sets, respectively. Clearly, if R is commutative, then every multiplicative subset of R is a left and right denominator set. Furthermore, if R is a domain, that is, R is a (not necessarily commutative) ring without left or right zero-divisors, then $R \setminus \{0\}$ is a left denominator set if and only if it is a left Ore set if and only if $Rr_1 \cap Rr_2 \neq \{0\}$ for any non-zero elements $r_1, r_2 \in R$. We say that R is a *left Ore domain* if R is a domain and $R \setminus \{0\}$ is a left denominator set.

The following lemma explains how left Ore localizations arise, and establishes a relationship between left Ore localizations and universal localizations.

Lemma 2.3 ([22, Theorem 10.6, Corollary 10.11]). *Let Φ be a left denominator subset of R and $\lambda : R \to R_\Phi$ be the universal localization of R at Φ. Then there is a ring, denoted by $\Phi^{-1}R$, and a ring homomorphism $\mu : R \to \Phi^{-1}R$ such that*

(1) *μ is Φ-invertible, that is, $(s)\mu$ is a unit in $\Phi^{-1}R$ for each $s \in \Phi$.*

(2) *Every element of $\Phi^{-1}R$ has the form $((t)\mu)^{-1}(r)\mu$ for some $t \in \Phi$ and some $r \in R$.*

(3) $\ker(\mu) = \{r \in R \mid sr = 0$ *for some* $s \in \Phi\}$.

(4) *There is a unique isomorphism*

$$\nu : \Phi^{-1}R \longrightarrow R_\Phi$$

of rings such that $\lambda = \mu\nu$.

The ring $\Phi^{-1}R$ in Lemma 2.3 is called a *left ring of fractions of* R (with respect to $\Phi \subseteq R$), or alternatively, a *left Ore localization* of R at Φ. Clearly, for commutative rings, Ore localizations and the usual localizations at multiplicative subsets coincide.

Similarly, when Φ is a right denominator subset of R, we can define a right ring $R\Phi^{-1}$ of fractions of R. If Φ is a left and right denominator subset of R, then $\Phi^{-1}R$ is called the *ring of fractions* of R, or the *Ore localization* of R at Φ. Actually, in this case, both $\Phi^{-1}R$ and $R\Phi^{-1}$ are isomorphic to R_Φ. Furthermore, if R is a left and right Ore domain, then the ring of fractions of R with respect to $R \setminus \{0\}$ is usually denoted by $Q(R)$. Notice that, up to isomorphism, $Q(R)$ is the smallest division ring containing R as a subring. So we call $Q(R)$ the *division ring of fractions* of R.

Now, we introduce a class of Ore domains, that is, discrete valuation rings.

Definition 2.4. A ring R is called a *discrete valuation ring* (which may not be commutative) if the following conditions hold true:

(1) R is a local ring, that is, R has a unique maximal left ideal $\mathfrak{m}$,

(2) $\bigcap_{i\geq 1} \mathfrak{m}^i = 0$,

(3) $\mathfrak{m} = pR = Rp$, where p is some non-nilpotent element of R.

We remark that an equivalent definition of discrete valuation rings is the following: A non-division ring R is called a *discrete valuation ring* if it is a local domain with $\mathfrak{m}$ the unique maximal ideal of R such that the only left ideals and the only right ideals of R are of the form $\mathfrak{m}^i$ for $i \in \mathbb{N}$.

The element p in the above condition (3) is called a *prime element* of R. Clearly, for each invertible element v of R, both vp and pv are prime elements. A discrete valuation ring is said to be *complete* if the canonical map

$$R \longrightarrow \varprojlim_i R/\mathfrak{m}^i$$

is an isomorphism. Note that every discrete valuation ring can be embedded into a complete discrete valuation ring.

The following lemma collects some basic properties of discrete valuation rings, which will be frequently used in our proofs.

Lemma 2.5 ([21, Chapter 1], [22]). *Let R be a discrete valuation ring, $\mathfrak{m}$ be the unique maximal ideal of R, and p be a prime element of R. Then the following statements are true:*

(1) *The ideals $\mathfrak{m}^i$, with $i \in \mathbb{N}$, are the only left ideals and the only right ideals of R.*

(2) *For any non-zero element $x \in R$, there is a unique subset $\{x_1, x_2\} \subseteq R \setminus \mathfrak{m}$ such that $x = x_1 p^n = p^n x_2$ for some $n \in \mathbb{N}$.*

(3) *R is a left and right Ore domain. In particular, the division ring $Q(R)$ of fractions of R exists.*

(4) *$Q(R)$ is isomorphic to the universal localization of R at the map $\rho_p : R \to R$ defined by $r \mapsto rp$ for each $r \in R$.*

The other kind of universal localizations is provided by universal localizations at injective homomorphisms between finitely generated projective modules, and therefore related to finitely presented modules of projective dimension at most 1.

Suppose that $\mathcal{U}$ is a set of finitely presented R-modules of projective dimension at most 1. For each $U \in \mathcal{U}$, there is an exact sequence of R-modules

$$0 \longrightarrow P_1 \xrightarrow{f_U} P_0 \longrightarrow U \longrightarrow 0,$$

such that P_1 and P_0 are finitely generated and projective. Set

$$\Sigma := \{ f_U \mid U \in \mathcal{U}\},$$

and let $R_{\mathcal{U}}$ be the universal localization of R at Σ. If $f'_U : Q_1 \to Q_0$ is another such a sequence of U, then the universal localization of R at $\Sigma' := \{f'_U \mid U \in \mathcal{U}\}$ is isomorphic to $R_{\mathcal{U}}$. Hence $R_{\mathcal{U}}$ does not depend on the choices of the injective homomorphisms f_U, and we may say that $R_{\mathcal{U}}$ is *the universal localization of R at $\mathcal{U}$*.

Clearly, we have $\mathrm{Tor}_i^R(R_{\mathcal{U}}, U) = 0$ for all $i \geq 0$ and $U \in \mathcal{U}$, and therefore $\mathrm{Tor}_i^R(R_{\mathcal{U}}, X) = 0$ for all $i \geq 0$ and $X \in \mathcal{F}(\mathcal{U})$.

Now, we recall the following property of universal localizations, which states that iterated universal localizations are again universal localizations.

Lemma 2.6 ([28, Theorem 4.6]). *Let Σ and Γ be sets of homomorphisms between finitely generated projective R-modules. Set $\overline{\Gamma} := \{R_\Sigma \otimes_R f \mid f \in \Gamma\}$. Then the universal localization of R at $\Sigma \cup \Gamma$ is isomorphic to the universal localization of R_Σ at $\overline{\Gamma}$, that is, $R_{\Sigma\cup\Gamma} \simeq (R_\Sigma)_{\overline{\Gamma}}$ as rings.*

Finally, we point out a special case of universal localizations which arise from ring epimorphisms.

Lemma 2.7. *Let $\lambda : R \to S$ be a ring epimorphism. If the R-module S is finitely generated and projective, then λ is the universal localization of R at the homomorphism λ.*

Proof. Suppose that ${}_RS$ is finitely generated and projective. Then $\lambda : R \to S$ is a homomorphism of finitely generated projective R-modules. Let $\Sigma := \{\lambda\}$. To show that λ is the universal localization of R at Σ, we shall check the conditions (1) and (2) in Lemma 2.1. Actually, since λ is a ring epimorphism, we know that $S \otimes \lambda : S \otimes_R R \to S \otimes_R S$ is an isomorphism of S-modules. Thus λ is Σ-inverting and verifies Lemma 2.1 (1). Suppose that $\varphi : R \to T$ is a ring homomorphism such that

$$T \otimes \lambda : T \otimes_R R \longrightarrow T \otimes_R S$$

is an isomorphism of T-modules. Clearly, the homomorphism $\mu : T \to T \otimes_R R$, given by $t \mapsto t \otimes 1$ for $t \in T$, is an isomorphism of T-R-bimodules. Now, we define

$$\psi : S \longrightarrow T, \quad s \longmapsto (1 \otimes s)\big(\mu(T \otimes \lambda)\big)^{-1}$$

for each $s \in S$. Clearly, ψ is well defined and can be illustrated by the following commutative diagram of homomorphisms of T-R-bimodules:

$$\begin{tikzcd}
T \arrow[r, "\mu"] \arrow[d, "\cdot(s)\psi"'] & T \otimes_R R \arrow[r, "T\otimes\lambda"] & T \otimes_R S \arrow[d, "T\otimes(\cdot s)"] \\
T \arrow[r, "\mu"] & T \otimes_R R \arrow[r, "T\otimes\lambda"] & T \otimes_R S
\end{tikzcd}$$

where $\cdot(s)\psi : T \to T$ and $\cdot s : S \to S$ stand for the right multiplication maps by $(s)\psi$ and s, respectively. From this diagram, we see that ψ is a ring homomorphism such that $\varphi = \lambda\psi$. Further, since λ is a ring epimorphism, we know that if there exists another ring homomorphism $\psi' : S \to T$ such that $\varphi = \lambda\psi'$, then $\psi' = \psi$. Consequently, λ is universal Σ-inverting and satisfies Lemma 2.1 (2). Thus λ is the universal localization of R at Σ. □

2.3 Recollements induced by tilting modules

Now, let us recall the definition of recollements of triangulated categories. This notion was first introduced by Beilinson, Bernstein and Deligne in [6] to study the triangulated categories of perverse sheaves over singular spaces, and later was used by Cline, Parshall and Scott in [11] to stratify the derived categories of quasi-hereditary algebras arising from the representation theory of semisimple Lie algebras and algebraic groups.

Let $\mathcal{D}$ be a triangulated category. We denote the shift functor of $\mathcal{D}$ by [1].

Definition 2.8 ([6]). Let $\mathcal{D}'$ and $\mathcal{D}''$ be triangulated categories. We say that $\mathcal{D}$ is a *recollement* of $\mathcal{D}'$ and $\mathcal{D}''$ if there are six triangle functors $i_*, i^*, i^!, j^!, j_*$ and $j_!$ as in the diagram

$$\begin{tikzcd}
\mathcal{D}'' \arrow[r, "i_*=i_!"] & \mathcal{D} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^!=j^*"] & \mathcal{D}' \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"]
\end{tikzcd}$$

such that

(1) $(i^*, i_*), (i_!, i^!), (j_!, j^!)$ and (j^*, j_*) are adjoint pairs,

(2) i_*, j_* and $j_!$ are fully faithful,

(3) $i^! j_* = 0$ (and thus also $j^! i_! = 0$ and $i^* j_! = 0$),

(4) for each object $C \in \mathcal{D}$, there are two triangles in $\mathcal{D}$:

$$i_! i^!(C) \longrightarrow C \longrightarrow j_* j^*(C) \longrightarrow i_! i^!(C)[1]$$

and

$$j_! j^!(C) \longrightarrow C \longrightarrow i_* i^*(C) \longrightarrow j_! j^!(C)[1],$$

where $i_! i^!(C) \to C$ and $j_! j^!(C) \to C$ are counit adjunction morphisms, and where $C \to j_* j^*(C)$ and $C \to i_* i^*(C)$ are unit adjunction morphisms.

In the following, if $\mathcal{D}$ is a recollement of $\mathcal{D}'$ and $\mathcal{D}''$, we also say that there is a recollement among $\mathcal{D}'$, $\mathcal{D}$ and $\mathcal{D}''$, or very briefly, that $\mathcal{D}$ admits a recollement.

A well-known example of recollements of derived categories of rings is given by triangular matrix rings: Suppose that A, B are rings and M is an A-B-bimodule. Let $R = \left(\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right)$, the triangular matrix ring associated with A, B and M. Then there is a recollement of derived categories:

$$\begin{tikzcd}
\mathscr{D}(A) \arrow[r] & \mathscr{D}(R) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}(B). \arrow[l, bend right] \arrow[l, bend left]
\end{tikzcd}$$

A generalization of this situation is the so-called stratifying ideals defined by Cline, Parshall and Scott, and can be found in [11].

Another type of examples of recollements of derived categories of rings appears in the tilting theory of infinitely generated tilting modules over arbitrary rings (see [8]). Before we state this kind of examples, we first recall the definition of tilting modules over arbitrary rings from [13], and then construct tilting modules from universal localizations.

Definition 2.9. An R-module T is called a *tilting module* (of projective dimension at most 1) if the following conditions are satisfied:

(T1) The projective dimension of T is at most 1, that is, there exists an exact sequence
$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow T \longrightarrow 0$$
with P_i projective for $i = 0, 1$.

(T2) $\mathrm{Ext}^i_R(T, T^{(\alpha)}) = 0$ for each $i \geq 1$ and each index set α.

(T3) There exists an exact sequence
$$0 \longrightarrow {}_RR \longrightarrow T_0 \longrightarrow T_1 \longrightarrow 0$$
of R-modules such that $T_i \in \mathrm{Add}(T)$ for $i = 0, 1$.

A tilting R-module T is called *good* if T_0 and T_1 in (T3) lie in $\mathrm{add}(T)$, and *classical* if T is good and finitely presented.

A special kind of good tilting modules can be constructed from certain universal localizations.

Lemma 2.10 ([2, Theorem 3.5, Theorem 2.6]). *Let R be a ring and Σ be a set of homomorphisms between finitely generated projective R-modules. If the universal localization $\lambda : R \to R_\Sigma$ is injective and the R-module R_Σ has projective dimension at most* 1*, then $R_\Sigma \oplus R_\Sigma/R$ is a tilting R-module with*
$$\mathrm{Hom}_R(R_\Sigma/R, R_\Sigma) = 0.$$

Now, we state the promised example of recollements as a proposition which is a consequence of [8, Lemma 6.2, Corollary 6.6]. It is worth noticing that the recollement in this proposition is, in general, different from the one obtained from the structure of triangular matrix rings.

Proposition 2.11. *Let R be a ring and let $\mathcal{U}$ denote a set of finitely presented R-modules of projective dimension at most* 1*. Suppose that the universal localization $\lambda : R \to R_\mathcal{U}$ of R at $\mathcal{U}$ is injective and that the R-module $R_\mathcal{U}$ has projective dimension at most* 1*. Set $B := \mathrm{End}_R(R_\mathcal{U} \oplus R_\mathcal{U}/R)$, $S := \mathrm{End}_R(R_\mathcal{U}/R)$ and $\Sigma := \{S \otimes_R f_U \mid U \in \mathcal{U}\}$. Then there is a recollement of derived module categories*
$$\begin{tikzcd} \mathscr{D}(S_\Sigma) \arrow[r] & \mathscr{D}(B) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}(R) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$
where S_Σ is the universal localization of S at Σ, and where the right R-module structure on S is induced by the ring homomorphism sending $r \in R$ to the right multiplication map $\cdot r : R_\mathcal{U}/R \to R_\mathcal{U}/R$.

Let us give an explicit description of the six functors appearing in the above recollement.

Let $\mu : R \to S$ be the right multiplication map defined by $r \mapsto (y \mapsto yr)$ for $r \in R$ and $y \in R_{\mathcal{U}}/R$. Clearly, this is a ring homomorphism and endows S with a natural R-R-bimodule structure. Further, let $\phi : S \to S_\Sigma$ be the universal localization of S at Σ. Then, by Lemma 2.1 (2), there exists a unique ring homomorphism $\rho : R_{\mathcal{U}} \to S_\Sigma$ such that the following diagram of ring homomorphisms is commutative:

$$\begin{tikzcd} R \arrow[r, "\lambda"] \arrow[d, "\mu"'] & R_{\mathcal{U}} \arrow[d, "\rho"] \\ S \arrow[r, "\phi"] & S_\Sigma. \end{tikzcd}$$

Moreover, by [8, Lemma 6.4 (2) and Lemma 6.5 (2)], the ring B can be identified with the triangular matrix ring $\left(\begin{smallmatrix} R_{\mathcal{U}} & R_{\mathcal{U}}\otimes_R S \\ 0 & S \end{smallmatrix}\right)$ up to isomorphism. Now, we define a homomorphism φ between finitely generated projective B-modules

$$\varphi : \begin{pmatrix} R_{\mathcal{U}} \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} R_{\mathcal{U}} \otimes_R S \\ S \end{pmatrix}, \quad \begin{pmatrix} x \\ 0 \end{pmatrix} \longmapsto \begin{pmatrix} x \otimes 1 \\ 0 \end{pmatrix} \quad \text{for } x \in R_{\mathcal{U}}.$$

Since $R_{\mathcal{U}}$ and S are R-R-bimodules via λ and μ, respectively, the map φ can be regarded as a homomorphism of B-R-bimodule. This implies that the mapping cone $P^\bullet$ of φ between stalk complexes is actually a complex of B-R-bimodule. Moreover, $\mathrm{Hom}_B(P^\bullet[-1], B)$ is quasi-isomorphic to $T := R_{\mathcal{U}} \oplus R_{\mathcal{U}}/R$ as complexes of R-B-bimodules. By [8, Lemmas 6.1 and 6.2], the universal localization of B at φ is given by

$$\theta : B = \begin{pmatrix} R_{\mathcal{U}} & R_{\mathcal{U}} \otimes_R S \\ 0 & S \end{pmatrix} \longrightarrow C := \begin{pmatrix} S_\Sigma & S_\Sigma \\ S_\Sigma & S_\Sigma \end{pmatrix},$$
$$\begin{pmatrix} x_1 & x_2 \otimes s_2 \\ 0 & s_1 \end{pmatrix} \longmapsto \begin{pmatrix} (x_1)\rho & (x_2)\rho(s_2)\phi \\ 0 & (s_1)\phi \end{pmatrix}$$

for $x_i \in R_{\mathcal{U}}$ and $s_i \in S$ with $i = 1, 2$. Furthermore, by [8, Theorem 1.1], the map θ is homological, and induces a recollement of derived module categories

$$\begin{tikzcd} \mathscr{D}(C) \arrow[r, "i_*"] & \mathscr{D}(B) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^!"] & \mathscr{D}(R) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

where

$$i^* := C \otimes^{\mathbb{L}}_B -, \qquad i_* := C \otimes^{\mathbb{L}}_C -, \quad i^! := \mathbb{R}\mathrm{Hom}_B(C, -),$$
$$j_! := P^\bullet[-1] \otimes^{\mathbb{L}}_R -, \quad j^! := T \otimes^{\mathbb{L}}_B -, \quad j_* := \mathbb{R}\mathrm{Hom}_R(T, -).$$

Observe that C is Morita equivalent to S_Σ. So, we can replace $\mathscr{D}(C)$ by $\mathscr{D}(S_\Sigma)$ in the recollement, and obtain a recollement of derived module categories in Proposition 2.11.

For a systematic investigation on the recollements induced from pairs of ring homomorphisms, we refer the reader to the recent preprint [9].

2.4 Homological facts

Finally, we prepare several homological results for our later proofs.

Lemma 2.12. *Let R be a ring and let*

$$0 \longrightarrow X \xrightarrow{(f,g)} Y \oplus Z \xrightarrow{h} W \longrightarrow 0$$

be an exact sequence of R-modules. Assume that $f : X \to Y$ is injective and that there is a homomorphism $\tilde{g} : Y \to Z$ with

$$g = f\tilde{g} : X \to Z.$$

Then there exists an automorphism γ of the module $Y \oplus Z$ and an isomorphism $\psi : W \to \mathrm{Coker}(f) \oplus Z$ such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "{(f,g)}"] \arrow[d, equal] & Y \oplus Z \arrow[r, "h"] \arrow[d, "\gamma"] & W \arrow[r] \arrow[d, "\psi"] & 0 \\
0 \arrow[r] & X \arrow[r, "{(f,0)}"] & Y \oplus Z \arrow[r, "{\left(\begin{smallmatrix} \pi & 0 \\ 0 & 1 \end{smallmatrix}\right)}"] & \mathrm{Coker}(f) \oplus Z \arrow[r] & 0
\end{tikzcd}$$

where $\pi : Y \to \mathrm{Coker}(f)$ stands for the canonical surjection.

Proof. Set $\gamma := \left(\begin{smallmatrix} 1 & -\tilde{g} \\ 0 & 1 \end{smallmatrix}\right)$. Then γ is an automorphism of the module $Y \oplus Z$. Since $g = f\tilde{g}$, we have

$$(f, g)\gamma = (f, 0).$$

Thus there exists a unique homomorphism $\psi : W \to \mathrm{Coker}(f) \oplus Z$ such that the above diagram is commutative. Clearly, ψ is an isomorphism. This completes the proof. □

The following homological facts can be found in the literature. For example, see [16, Lemma 3.1.6, Lemma 3.3.4] for proofs of (1, ii) and (1, iii).

Lemma 2.13. *Let R be a ring.*

(1) *If $\{X_\alpha\}_{\alpha\in I}$ is a direct system of R-modules, then:*

(i) *We have*

$$\mathrm{Hom}_R(\varinjlim_\alpha X_\alpha, M) \simeq \varprojlim_\alpha \mathrm{Hom}_R(X_\alpha, M)$$

for any R-module M.

(ii) *Let $n \geq 0$. If M is an R-module with a projective resolution*

$$\cdots \longrightarrow P_{n+1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

such that all P_j, with $0 \leq j \leq n+1$, are finitely generated, then

$$\mathrm{Ext}^i_R(M, \varinjlim_\alpha X_\alpha) \simeq \varinjlim_\alpha \mathrm{Ext}^i_R(M, X_\alpha)$$

for all $i \leq n$. In particular, if M is a finitely presented R-module, then

$$\mathrm{Hom}_R(M, \varinjlim_\alpha X_\alpha) \simeq \varinjlim_\alpha \mathrm{Hom}_R(M, X_\alpha).$$

(iii) *If M is a pure-injective R-module (for example, M is of finite length over its endomorphism ring), then*

$$\mathrm{Ext}^i_R(\varinjlim_\alpha X_\alpha, M) \simeq \varprojlim_\alpha \mathrm{Ext}^i_R(X_\alpha, M)$$

for all $i \geq 0$. Conversely, if this isomorphism is true for $i = 1$ and for every directed system X_α, then M is pure-injective.

(2) *If $\{Y_\alpha\}_{\alpha\in I}$ is an inverse system of R-modules, then, for any R-module M,*

$$\mathrm{Hom}_R(M, \varprojlim_\alpha Y_\alpha) \simeq \varprojlim_\alpha \mathrm{Hom}_R(M, Y_\alpha).$$

Remark. (1) The statement (1, iii) is due to Maurice Auslander.

(2) The class of all pure-injective R-modules is closed under products, direct summands and finite direct sums. In general, it is not closed under extensions.

Lemma 2.14. *Let A be a finite-dimensional k-algebra over a field k, M be a finite-dimensional A-module and N be an arbitrary A-module.*

(1) *If* proj.dim$(M) \leq 1$, *then*

$$D\mathrm{Ext}^1_A(M, N) \simeq \mathrm{Hom}_A(N, \tau M),$$

where proj.dim(M) *stands for the projective dimension of M.*

(2) *If* inj.dim$(M) \leq 1$, *then*

$$\operatorname{Ext}_A^1(N, M) \simeq D\operatorname{Hom}_A(\tau^{-1}M, N),$$

where inj.dim(M) *stands for the injective dimension of* M.

Proof. It is known that every A-module N is a direct limit of finitely presented A-modules $\{X_\alpha\}_{\alpha \in I}$ (see [16, Lemma 1.2.3]) and that (1) and (2) hold true for finitely generated modules N. Then, it follows from Lemma 2.13 that

$$\begin{aligned} D\operatorname{Ext}_A^1(M, N) &\simeq D\operatorname{Ext}_A^1(M, \varinjlim_\alpha X_\alpha) \simeq D \varinjlim_\alpha \operatorname{Ext}_A^1(M, X_\alpha) \\ &\simeq \varprojlim_\alpha D\operatorname{Ext}_A^1(M, X_\alpha) \simeq \varprojlim_\alpha \operatorname{Hom}_A(X_\alpha, \tau M) \\ &\simeq \operatorname{Hom}_A(\varinjlim_\alpha X_\alpha, \tau M) = \operatorname{Hom}_A(N, \tau M). \end{aligned}$$

This proves (1). The statement (2) can be shown similarly. □

3 Prüfer modules and their endomorphism algebras

In this section, we shall consider the endomorphism algebra of the direct sum of all Prüfer modules obtained from a given tube. This ring was calculated already in [26]. For convenience of the reader and also for the later proofs of our main results, we include here some details of this calculation.

Unless stated otherwise, we assume from now on that R is an indecomposable finite-dimensional tame hereditary k-algebra over an arbitrary but fixed field k.

Let $\mathscr{S} := \mathscr{S}(R)$ be a fixed complete set of isomorphism classes of all simple regular R-modules. For each $U \in \mathscr{S}$ and $n > 0$, we denote by $U[n]$ the R-module of regular length n on the ray

$$U = U[1] \subset U[2] \subset \cdots \subset U[n] \subset U[n+1] \subset \cdots,$$

and define the Prüfer module corresponding to U as

$$U[\infty] := \varinjlim_n U[n].$$

Note that $U[\infty]$ has a unique regular submodule $U[n]$ of regular length n, and therefore admits a unique chain of regular submodules, and that each endomorphism of $U[\infty]$ restricts to an endomorphism of $U[n]$ for any $n > 0$. For further information on regular modules and Prüfer modules over tame hereditary algebras, we refer the reader to [26, Sections 4 and 5] and [15].

Recall that we have defined an equivalence relation $\sim$ on $\mathscr{S}$ in Section 1. It is known that two simple regular modules lie in the same clique if and only if they lie in the same tube. Thus a clique is just the set of all simple regular modules belonging to a fixed tube.

Let $U \in \mathscr{S}$ and $\mathcal{U} \subseteq \mathscr{S}$. We denote by $\mathscr{C}(U)$ the clique containing U, and by $c(U)$ the cardinality of $\mathscr{C}(U)$. Similarly, we denote by $\mathscr{C}(\mathcal{U})$ the union of all cliques $\mathscr{C}(U)$ with $U \in \mathcal{U}$, and by $c(\mathcal{U})$ the cardinality of $\mathscr{C}(\mathcal{U})$. As mentioned before, $c(U)$ is always finite, and furthermore, $c(U) = 1$ for almost all $U \in \mathscr{S}$. In fact, there are at most three cliques consisting of more than one element. Also, we know that R has only two isomorphism classes of simple modules if and only if every clique of R consists of one simple regular R-module. If the field k is algebraically closed, this is equivalent to the statement that R is Morita equivalent to the Kronecker algebra.

Throughout this section, let $\mathcal{C}$ be a clique of R, U be an element in $\mathcal{C}$, and $\mathbf{t}$ be the tube of rank $m \geq 1$ containing $\mathcal{C}$. Set $U_i := \tau^{-(i-1)}U$ for $i \in \mathbb{Z}$. Then $\tau^{-m}U \simeq U$ and $\mathcal{C} = \{U_1, U_2, \ldots, U_{m-1}, U_m\}$ which is a complete set of non-isomorphic simple regular modules in $\mathbf{t}$. Since $U_j \simeq U_{j+m}$ for any $j \in \mathbb{Z}$, the subscript of U_j is always modulo m in our discussion below. It is well known that $\mathrm{End}_R(U_i)$ is a division algebra and $\mathrm{Hom}_R(U_i, U_j) = 0$ for $1 \leq i \neq j \leq m$, and that $D\mathrm{Ext}^1_R(U_i, U_j) \simeq \mathrm{End}_R(U_i)$ if $j = i - 1$, and zero otherwise. Furthermore, $\mathbf{t}$ is an exact abelian subcategory of R-mod, and every indecomposable module in $\mathbf{t}$ is serial, that is, it has a unique regular composition series in $\mathbf{t}$. For example, for any $i \in \mathbb{Z}$ and $j > 0$, the module $U_i[j]$ admits successive regular composition factors $U_i, U_{i+1}, \ldots, U_{i+j-1}$ with U_i as its unique regular socle and with U_{i+j-1} as its unique regular top. For details, see [27, Section 3.1].

Now, we collect some properties of Prüfer modules.

Lemma 3.1. *The following statements hold true for the tube* $\mathbf{t}$.

(1) *For any* $1 \leq i \leq m$ *and for any regular module* X *in* $\mathbf{t}$, *we have*

$$\mathrm{Hom}_R(U_i[\infty], X) = 0 = \mathrm{Ext}^1_R(X, U_i[\infty]).$$

Further, if $1 \leq i < j \leq m$, *then* $\mathrm{Hom}_R(U_i[n], U_j[\infty]) = 0$ *for* $1 \leq n \leq j-i$, *and* $\mathrm{Hom}_R(U_j[n], U_i[\infty]) = 0$ *for* $1 \leq n \leq m - j + i$.

(2) *Let* $i, j \in \mathbb{N}$ *with* $1 \leq i < j$. *Then, for any* $n > j - i$, *there is a canonical exact sequence of* R-*modules*

$$0 \longrightarrow U_i[j-i] \longrightarrow U_i[n] \xrightarrow{\varepsilon_{i,j}[n]} U_j[n-(j-i)] \longrightarrow 0.$$

In particular, we get a canonical exact sequence

$$0 \longrightarrow U_i[j-i] \longrightarrow U_i[\infty] \xrightarrow{\varepsilon_{i,j}} U_j[\infty] \longrightarrow 0,$$

where $\varepsilon_{i,j} := \varinjlim_n \varepsilon_{i,j}[n]$. *Moreover, we have*

$$\varepsilon_{i,j} = \varepsilon_{i+m,j+m} \quad \text{and} \quad \varepsilon_{i,j}\varepsilon_{j,p} = \varepsilon_{i,p} \quad \text{for any } p > j.$$

(3) *If $i, j \in \mathbb{N}$ with $1 \le j - i < m$, then $\varepsilon_{i,j}$ induces an isomorphism of left* $\mathrm{End}_R(U_i[\infty])$*-modules*

$$(\varepsilon_{i,j})^* : \mathrm{End}_R(U_i[\infty]) \xrightarrow{\sim} \mathrm{Hom}_R(U_i[\infty], U_j[\infty]),$$

and an isomorphism of right $\mathrm{End}_R(U_j[\infty])$*-modules*

$$(\varepsilon_{i,j})_* : \mathrm{End}_R(U_j[\infty]) \xrightarrow{\sim} \mathrm{Hom}_R(U_i[\infty], U_j[\infty]).$$

In particular, we get a ring isomorphism

$$\varphi_{i,j} : \mathrm{End}_R(U_i[\infty]) \longrightarrow \mathrm{End}_R(U_j[\infty]), \quad f \longmapsto f'$$

for $f \in \mathrm{End}_R(U_i[\infty])$ and $f' \in \mathrm{End}_R(U_j[\infty])$, with $f\varepsilon_{i,j} = \varepsilon_{i,j} f'$.

(4) *Suppose $1 \le r, s, t \le m$. Set*

$$\Delta_{r,s} := \begin{cases} 0 & \text{if } r < s, \\ 1 & \text{if } r \ge s, \end{cases}$$

and define

$$\pi_{r,s} := \varepsilon_{r,s+\Delta_{r,s}m} \in \mathrm{Hom}_R(U_r[\infty], U_{s+\Delta_{r,s}m}[\infty]).$$

Then

$$\pi_{r,s}\,\pi_{s,t} = \begin{cases} \pi_{r,t} & \text{if } \Delta_{r,s} + \Delta_{s,t} = \Delta_{r,t}, \\ \pi_{r,r}\pi_{r,t} & \text{otherwise.} \end{cases}$$

In particular, we have $(\pi_{i,i})\varphi_{i,j} = \pi_{j,j}$ for any $1 \le i < j \le m$.

(5) *The ring $\mathrm{End}_R(U_i[\infty])$ is a complete discrete valuation ring with $\pi_{i,i}$ as a prime element. If k is an algebraically closed field, then there is a ring isomorphism $\varphi_i : \mathrm{End}_R(U_i[\infty]) \to k[[x]]$ which sends $\pi_{i,i}$ to x.*

Proof. (1) Note that we have $D\mathrm{Ext}^1_R(X, U_i[\infty]) \simeq \mathrm{Hom}_R(U_i[\infty], \tau X)$ for any $X \in \mathbf{t}$ by Lemma 2.14 (1), and that every indecomposable module in $\mathbf{t}$ is serial. Thus, to prove the first statement in (1), it suffices to show $\mathrm{Hom}_R(U_i[\infty], U_j) = 0$ for all $1 \le j \le m$. In fact, since the inclusion map $U_i[n] \to U_i[n+1]$ induces a zero map from $\mathrm{Hom}_R(U_i[n+1], U_j)$ to $\mathrm{Hom}_R(U_i[n], U_j)$ for all n. This implies that

$$\mathrm{Hom}_R(U_i[\infty], U_j) = \mathrm{Hom}_R(\varinjlim_n U_i[n], U_j) \simeq \varprojlim_n \mathrm{Hom}_R(U_i[n], U_j) = 0.$$

The last statement in (1) follows from the fact that the abelian category $\mathbf{t}$ is serial.

(2) For any $n > j - i$, we can easily see from the structure of the tube $\mathbf{t}$ that there is an exact commutative diagram of R-modules

$$\begin{tikzcd}
0 \arrow[r] & U_i[j-i] \arrow[r] \arrow[d, equal] & U_i[n] \arrow[r, "\varepsilon_{i,j}[n]"] \arrow[d, hook] & U_j[n-(j-i)] \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & U_i[j-i] \arrow[r] & U_i[n+1] \arrow[r, "\varepsilon_{i,j}[n+1]"] & U_j[n-(j-i)+1] \arrow[r] & 0
\end{tikzcd}$$

where the map $\varepsilon_{i,j}[n]$ is induced by the canonical inclusion $U_i[j-i] \hookrightarrow U_i[n]$. Thus, by taking the direct limit of the above diagram, we obtain the canonical exact sequence

$$0 \longrightarrow U_i[j-i] \longrightarrow U_i[\infty] \xrightarrow{\varepsilon_{i,j}} U_j[\infty] \longrightarrow 0 \qquad (*)$$

where $\varepsilon_{i,j} := \varinjlim_n \varepsilon_{i,j}[n]$. This is the first assertion in (2). In the following, we shall show that $\varepsilon_{i,j} = \varepsilon_{i+m,j+m}$ and $\varepsilon_{i,j}\varepsilon_{j,p} = \varepsilon_{i,p}$ for any $p > j$. In fact, the former clearly follows from $\varepsilon_{i,j}[n] = \varepsilon_{i+m,j+m}[n]$ for any $n > j - i$ since we have $U_i = U_{i+m}$ and $U_j = U_{j+m}$ by our convention. As for the latter, it follows that, for any $u > p - i$, the composite of

$$\varepsilon_{i,j}[u] : U_i[u] \longrightarrow U_j[u-(j-i)],$$

and

$$\varepsilon_{j,p}[u-(j-i)] : U_j[u-(j-i)] \longrightarrow U_p[u-(p-i)]$$

coincides with $\varepsilon_{i,p}[u] : U_i[u] \longrightarrow U_p[u-(p-i)]$. So

$$\varepsilon_{i,j}[u]\varepsilon_{j,p}[u-(j-i)] = \varepsilon_{i,p}[u].$$

By taking the direct limit of the two sides of this equality, we have $\varepsilon_{i,j}\varepsilon_{j,p} = \varepsilon_{i,p}$ for any $p > j$. This completes the proof of (2).

(3) If we apply $\mathrm{Hom}_R(U_i[\infty], -)$ to the sequence $(*)$ in the proof of (2), then we get the following exact sequence:

$$\begin{aligned}
0 \longrightarrow \mathrm{Hom}_R(U_i[\infty], U_i[j-i]) &\longrightarrow \mathrm{Hom}_R(U_i[\infty], U_i[\infty]) \\
&\xrightarrow{(\varepsilon_{i,j})^*} \mathrm{Hom}_R(U_i[\infty], U_j[\infty]) \\
&\longrightarrow \mathrm{Ext}^1_R(U_i[\infty], U_i[j-i]).
\end{aligned}$$

Note that $\mathrm{Hom}_R(U_i[\infty], U_i[j-i]) = 0$ by (1). Thus, to prove that $(\varepsilon_{i,j})^*$ is an isomorphism, it suffices to show $\mathrm{Ext}^1_R(U_i[\infty], U_i[j-i]) = 0$. In fact, this follows from

$$\begin{aligned}
\mathrm{Ext}^1_R(U_i[\infty], U_i[j-i]) &\simeq D\mathrm{Hom}_R(\tau^-(U_i[j-i]), U_i[\infty]) \\
&\simeq D\mathrm{Hom}_R(U_{i+1}[j-i], U_i[\infty]) = 0,
\end{aligned}$$

where the last equality holds for $1 \le j - i < m$ by (1).

Next, by applying $\mathrm{Hom}_R(-, U_j[\infty])$ to the sequence $(*)$, we get the following exact sequence:

$$0 \longrightarrow \mathrm{End}_R(U_j[\infty]) \xrightarrow{(\varepsilon_{i,j})_*} \mathrm{Hom}_R(U_i[\infty], U_i[\infty]) \longrightarrow \mathrm{Hom}_R(U_i[j-i], U_j[\infty]).$$

Since $1 \leq j - i < m$, we have $\mathrm{Hom}_R(U_i[j-i], U_j[\infty]) = 0$ by (1), and therefore $(\varepsilon_{i,j})_*$ is an isomorphism.

Now, it follows from the isomorphisms $(\varepsilon_{i,j})^*$ and $(\varepsilon_{i,j})_*$ that the map

$$\varphi_{i,j} : \mathrm{End}_R(U_i[\infty]) \longrightarrow \mathrm{End}_R(U_j[\infty])$$

in (3) is well defined and thus a ring isomorphism.

(4) By definition, for $1 \leq r, s, t \leq m$, one can check

$$\begin{aligned}\pi_{r,s}\pi_{s,t} &= \varepsilon_{r,s+\Delta_{r,s}m}\varepsilon_{s,t+\Delta_{s,t}m}\\ &= \varepsilon_{r,s+\Delta_{r,s}m}\varepsilon_{s+\Delta_{r,s}m,t+(\Delta_{s,t}+\Delta_{r,s})m}\\ &= \varepsilon_{r,t+(\Delta_{r,s}+\Delta_{s,t})m}.\end{aligned}$$

On the one hand, for any $p > r$ and $q > r$, we infer from (2) that $\varepsilon_{r,p} = \varepsilon_{r,q}$ if and only if $p = q$. On the other hand, we always have $\Delta_{r,s} + \Delta_{s,t} - \Delta_{r,t} \in \{0, 1\}$. Consequently, the first statement in (4) follows. In particular, this implies that

$$\pi_{i,j}\pi_{j,j} = \pi_{i,i}\pi_{i,j}$$

for $1 \leq i < j \leq m$. By the definition of $\varphi_{i,j}$ in (3), the second statement in (4) follows.

(5) Set $D_i := \mathrm{End}_R(U_i[\infty])$. It follows from [26, Section 4.4] that D_i is a complete discrete valuation ring. Let $\mathfrak{m}$ be the unique maximal ideal of D_i. We shall prove that $\pi_{i,i}$ is a prime element of D_i, that is, $\mathfrak{m} = \pi_{i,i}D_i = D_i\pi_{i,i}$. Indeed, by applying $\mathrm{Hom}_R(-, U_i[\infty])$ to the exact sequence

$$0 \longrightarrow U_i[m] \longrightarrow U_i[\infty] \xrightarrow{\pi_{i,i}} U_i[\infty] \longrightarrow 0,$$

we obtain another exact sequence of right D_i-modules:

$$0 \longrightarrow D_i \xrightarrow{(\pi_{i,i})_*} D_i \longrightarrow \mathrm{Hom}_R(U_i[m], U_i[\infty]) \longrightarrow 0,$$

due to $\mathrm{Ext}^1_R(U_i[\infty], U_i[\infty]) = 0$, which follows from [26, Section 4.5]. To show $\mathfrak{m} = \pi_{i,i}D_i$, we first claim that

$$\mathrm{Hom}_R(U_i[m], U_i[\infty]) \simeq \mathrm{Hom}_R(U_i, U_i[\infty]) \simeq D_i/\mathfrak{m}$$

as right D_i-modules.

Let

$$0 \longrightarrow U_i \longrightarrow U_i[m] \xrightarrow{\varepsilon_{i,i+1}[m]} U_{i+1}[m-1] \longrightarrow 0$$

be the exact sequence defined in (2). Then we get the following exact sequence of k-modules:

$$\begin{aligned}\mathrm{Hom}_R(U_{i+1}[m-1], U_i[\infty]) &\longrightarrow \mathrm{Hom}_R(U_i[m], U_i[\infty])\\ &\longrightarrow \mathrm{Hom}_R(U_i, U_i[\infty])\\ &\longrightarrow \mathrm{Ext}^1_R(U_{i+1}[m-1], U_i[\infty]).\end{aligned}$$

Since $\mathrm{Hom}_R(U_{i+1}[m-1], U_i[\infty]) = 0 = \mathrm{Ext}^1_R(U_{i+1}[m-1], U_i[\infty])$ by (1), we have $\mathrm{Hom}_R(U_i[m], U_i[\infty]) \simeq \mathrm{Hom}_R(U_i, U_i[\infty])$ as right D_i-modules.

It remains to show $\mathrm{Hom}_R(U_i, U_i[\infty]) \simeq D_i/\mathfrak{m}$ as right D_i-modules. Let

$$0 \longrightarrow U_i \xrightarrow{\zeta} U_i[\infty] \xrightarrow{\varepsilon_{i,i+1}} U_{i+1}[\infty] \longrightarrow 0$$

be the exact sequence defined in (2) with ζ the canonical inclusion. Since

$$\mathrm{Ext}^1_R(U_{i+1}[\infty], U_i[\infty]) = 0$$

by [26, Section 4.5], we infer that, for any $f : U_i \to U_i[\infty]$, there is a $g \in D_i$ such that $f = \zeta g$. This means $\mathrm{Hom}_R(U_i, U_i[\infty]) = \zeta D_i$. Clearly, $\zeta D_i \simeq D_i/N$ as right D_i-modules, where $N := \{h \in D_i \mid \zeta h = 0\}$. As the canonical ring homomorphism from D_i to $\mathrm{End}_R(U_i)$ via the map ζ induces a ring isomorphism from $D_i/\mathfrak{m}$ to $\mathrm{End}_R(U_i)$, we have $\zeta\mathfrak{m} = 0$, that is, $\mathfrak{m} \subseteq N$. Since D_i is a local ring and $N \subsetneq D_i$, we get $N = \mathfrak{m}$, and therefore $\mathrm{Hom}_R(U_i, U_i[\infty]) \simeq D_i/\mathfrak{m}$ as right D_i-modules. This finishes the claim.

From the above claim, we conclude that $\mathfrak{m}$ coincides with the image of $(\pi_{i,i})_*$, that is, $\mathfrak{m} = \pi_{i,i} D_i$. Similarly, we have $\mathfrak{m} = D_i \pi_{i,i}$. This means that $\pi_{i,i}$ is a prime element of D_i. As for the second statement in (5), we note that, for any $p \in \mathbb{N}$ and $1 \le q < m$, the canonical inclusion map

$$U_i[pm+q] \longrightarrow U_i[pm+q+1]$$

induces an isomorphism

$$\mathrm{Hom}_R(U_i[pm+q+1], U_i[\infty]) \xrightarrow{\simeq} \mathrm{Hom}_R(U_i[pm+q], U_i[\infty]).$$

Consequently, we have the following isomorphisms of abelian groups:

$$\begin{aligned} D_i &= \mathrm{Hom}_R(\varinjlim_n U_i[n], U_i[\infty])\\ &\simeq \varprojlim_n \mathrm{Hom}_R(U_i[n], U_i[\infty])\\ &\simeq \varprojlim_n \mathrm{Hom}_R(U_i[(n-1)m+1], U_i[\infty])\\ &\simeq \varprojlim_n k[x]/(x^n) \simeq k[[x]].\end{aligned}$$

Here we need the assumption that k is an algebraically closed field. Now, one can check directly that the composite of the above isomorphisms yields a ring isomorphism $\varphi_i : D_i \to k[[x]]$, which sends $\pi_{i,i}$ to x. This finishes the proof. $\square$

By Lemma 3.1 (3), the rings $\mathrm{End}_R(U_i[\infty])$, with $1 \leq i \leq m$, are all isomorphic. From now on, we always identify these rings, and simply denote them by $D(\mathcal{C})$. Further, we write $\mathfrak{m}(\mathcal{C})$ and $Q(\mathcal{C})$ for the maximal ideal of $D(\mathcal{C})$ and the division ring of fractions of $D(\mathcal{C})$, respectively. In particular,

$$\mathfrak{m}(\mathcal{C}) = \pi_{i,i} D(\mathcal{C}) = D(\mathcal{C})\pi_{i,i}.$$

Suppose that C is a $\mathbb{Z}$-module and $c \in C$. For $1 \leq i, j \leq m$, we denote by $E_{i,j}(c)$ the $m \times m$ matrix which has the (i, j)-entry c, and the other entries 0. For simplicity, we write $E_{i,j}$ for $E_{i,j}(1)$ if C is a ring with the identity 1. Moreover, let $\pi_{i,j}$ be the homomorphisms defined in Lemma 3.1 (4).

Lemma 3.2. *There exists a ring isomorphism*

$$\rho : \mathrm{End}_R\Big(\bigoplus_{i=1}^{m} U_i[\infty]\Big) \longrightarrow \Gamma(\mathcal{C}) := \begin{pmatrix} D(\mathcal{C}) & D(\mathcal{C}) & \cdots & D(\mathcal{C}) \\ \mathfrak{m}(\mathcal{C}) & D(\mathcal{C}) & \ddots & \vdots \\ \vdots & \ddots & \ddots & D(\mathcal{C}) \\ \mathfrak{m}(\mathcal{C}) & \cdots & \mathfrak{m}(\mathcal{C}) & D(\mathcal{C}) \end{pmatrix}_{m\times m}$$

which sends the matrix $E_{m,1}(\pi_{m,1})$ to $E_{m,1}(\pi_{m,m})$ and the matrix $E_{r,r+1}(\pi_{r,r+1})$ to $E_{r,r+1}$ for $1 \leq r < m$, where the maximal ideal $\mathfrak{m}(\mathcal{C})$ of the ring $D(\mathcal{C})$ is generated by the element $\pi_{m,m}$.

Proof. For any $1 \leq i < m$, by Lemmas 3.1 (2) and (4), we have the following exact sequence of R-modules:

$$0 \longrightarrow U_i[m-i] \longrightarrow U_i[\infty] \xrightarrow{\pi_{i,m}} U_m[\infty] \longrightarrow 0.$$

Summing up these sequences, we can get the following exact sequence:

$$0 \longrightarrow \bigoplus_{i=1}^{m-1} U_i[m-i] \longrightarrow \bigoplus_{j=1}^{m} U_j[\infty] \xrightarrow{\xi} U_m[\infty]^{(m)} \longrightarrow 0,$$

where $\xi := \mathrm{diag}(\pi_{1,m}, \pi_{2,m}, \dots, \pi_{m-1,m}, 1)$ is the $m \times m$ diagonal matrix with $\pi_{i,m}$ in the (i,i)-position for $1 \leq i < m$, and with 1 in the (m,m)-position.

Let $D := \mathrm{End}_R(U_m[\infty])$, and let $\mathfrak{m}$ be the unique maximal ideal of D. Set

$$\Lambda := \mathrm{End}_R\Big(\bigoplus_{j=1}^{m} U_j[\infty]\Big).$$

Since $\mathrm{Hom}_R(U_i[m-i], U_m[\infty]) = 0$ for $1 \le i < m$, we see that, for any $g \in \Lambda$, there exists a unique homomorphism f and a unique homomorphism h such that the following diagram is commutative:

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{i=1}^{m-1} U_i[m-i] \arrow[r] \arrow[d, dashed, "f"'] & \bigoplus_{j=1}^{m} U_j[\infty] \arrow[r, "\xi"] \arrow[d, "g"'] & U_m[\infty]^{(m)} \arrow[r] \arrow[d, dashed, "h"'] & 0 \\
0 \arrow[r] & \bigoplus_{i=1}^{m-1} U_i[m-i] \arrow[r] & \bigoplus_{j=1}^{m} U_j[\infty] \arrow[r, "\xi"] & U_m[\infty]^{(m)} \arrow[r] & 0.
\end{tikzcd}$$

This yields a ring homomorphism $\rho : \Lambda \to M_m(D)$ defined by $g \mapsto h$. More precisely, if $g = (g_{u,v})_{1\le u,v\le m} \in \Lambda$ with $g_{u,v} \in \mathrm{Hom}_R(U_u[\infty], U_v[\infty])$, then we have $h = (h_{u,v})_{1\le u,v\le m} \in M_m(D)$ with $h_{u,v} \in D$ satisfying

(a) $g_{u,v}\pi_{v,m} = \pi_{u,m}h_{u,v}$ if $u < m$ and $v < m$,

(b) $h_{m,v} = g_{m,v}\pi_{v,m}$ if $u = m$ and $v < m$,

(c) $g_{u,m} = \pi_{u,m}h_{u,m}$ if $u < m$ and $v = m$,

(d) $h_{m,m} = g_{m,m}$.

In particular, the map ρ sends $E_{u,u}$ in Λ to $E_{u,u}$ in $M_m(D)$. In this sense, we may write $\rho = (\rho_{u,v})_{1\le u,v\le m}$, where

$$\rho_{u,v} : \mathrm{Hom}_R(U_u[\infty], U_v[\infty]) \longrightarrow D$$

is defined by $g_{u,v} \mapsto h_{u,v}$.

Clearly, ρ is injective since

$$\mathrm{Hom}_R(U_j[\infty], U_i[m-i]) = 0$$

for $1 \le j \le m$ and $1 \le i < m$ by Lemma 3.1 (1). In the following, we shall determine the image of ρ, which is clearly a subring of $M_m(D)$.

On the one hand, for any $a \in \mathrm{End}_R(U_u[\infty])$, $b \in \mathrm{Hom}_R(U_u[\infty], U_v[\infty])$ and $c \in \mathrm{End}_R(U_v[\infty])$, we have

$$(abc)\rho_{u,v} = (a)\rho_{u,u}(b)\rho_{u,v}(c)\rho_{v,v}.$$

On the other hand, it follows from Lemma 3.1 (3) that $\rho_{u,u}$ is always a ring isomorphism, and the left $\mathrm{End}_R(U_u[\infty])$-module $\mathrm{Hom}_R(U_u[\infty], U_v[\infty])$ is freely generated by $\pi_{u,v}$ for $1 \le u \ne v \le m$. This implies that the image of ρ coincides with the $m \times m$ matrix ring having $D(\pi_{u,v})\rho_{u,v}$ in the (u, v)-position if $1 \le u \ne v \le m$, and D otherwise. By Lemma 3.1 (3) and (4), if $1 \le s < t < m$

and $1 \leq w < m$, we can form the following commutative diagrams:

$$\begin{tikzcd} U_s[\infty] \arrow[r, "\pi_{s,m}"] \arrow[d, "\pi_{s,t}"] & U_m[\infty] \arrow[d, equal] \\ U_t[\infty] \arrow[r, "\pi_{t,m}"] & U_m[\infty], \end{tikzcd} \qquad \begin{tikzcd} U_t[\infty] \arrow[r, "\pi_{t,m}"] \arrow[d, "\pi_{t,s}"] & U_m[\infty] \arrow[d, "\pi_{m,m}"] \\ U_s[\infty] \arrow[r, "\pi_{s,m}"] & U_m[\infty], \end{tikzcd}$$

$$\begin{tikzcd} U_m[\infty] \arrow[r, equal] \arrow[d, "\pi_{m,w}"] & U_m[\infty] \arrow[d, "\pi_{m,m}"] \\ U_w[\infty] \arrow[r, "\pi_{w,m}"] & U_m[\infty], \end{tikzcd} \qquad \begin{tikzcd} U_w[\infty] \arrow[r, "\pi_{w,m}"] \arrow[d, "\pi_{w,m}"] & U_m[\infty] \arrow[d, equal] \\ U_m[\infty] \arrow[r, equal] & U_m[\infty]. \end{tikzcd}$$

In other words, we have

$$(\pi_{s,t})\rho_{s,t} = 1 = (\pi_{w,m})\rho_{w,m} \quad \text{and} \quad (\pi_{t,s})\rho_{t,s} = \pi_{m,m} = (\pi_{m,w})\rho_{m,w}.$$

Thus, the image of ρ is equal to the $m \times m$ matrix ring having $D\pi_{m,m}$ as the (p,q)-entry for $1 \leq q < p \leq m$, and D as the other entries. By Lemma 3.1 (5), we know that $\mathfrak{m} = D\pi_{m,m}$. Now, by identifying D with $D(\mathscr{C})$ and $\mathfrak{m}$ with $\mathfrak{m}(\mathscr{C})$, we infer that the image of ρ coincides with the ring $\Gamma(\mathscr{C})$ defined in Lemma 3.2. Therefore, we conclude that $\rho : \Lambda \to \Gamma(\mathscr{C})$ is a ring isomorphism which sends $E_{m,1}(\pi_{m,1})$ to $E_{m,1}(\pi_{m,m})$ and $E_{r,r+1}(\pi_{r,r+1})$ to $E_{r,r+1}$ for $1 \leq r < m$. This completes the proof. □

Combining Lemma 3.2 with Lemma 3.1 (5), we then obtain the following result which will be used for the calculation of stratifications of derived module categories in the last section.

Corollary 3.3. *Assume that k is an algebraically closed field. Then there exists a ring isomorphism*

$$\sigma : \operatorname{End}_R\Big(\bigoplus_{i=1}^{m} U_i[\infty]\Big) \longrightarrow \Gamma(m) := \begin{pmatrix} k[[x]] & k[[x]] & \cdots & k[[x]] \\ (x) & k[[x]] & \ddots & \vdots \\ \vdots & \ddots & \ddots & k[[x]] \\ (x) & \cdots & (x) & k[[x]] \end{pmatrix}_{m\times m},$$

which sends $E_{m,1}(\pi_{m,1})$ to $E_{m,1}(x)$ and $E_{r,r+1}(\pi_{r,r+1})$ to $E_{r,r+1}$ for $1 \leq r < m$.

As another consequence of Lemma 3.2, we have the following description of the universal localization of the endomorphism algebra of the direct sum of all Prüfer modules from a given tube.

Lemma 3.4. *Define* $M := \bigoplus_{i=1}^{m} U_i[\infty]$, $\Lambda := \mathrm{End}_R(M)$ *and*

$$\Pi := \{\mathrm{Hom}_R(M, \pi_{r,r+1}) \mid 1 \le r < m\} \cup \{\mathrm{Hom}_R(M, \pi_{m,1}\}.$$

Then the universal localization Λ_Π *of* Λ *at* Π *is isomorphic to* $M_m(Q(\mathcal{C}))$, *the* $m \times m$ *matrix ring over* $Q(\mathcal{C})$.

Proof. By Lemma 3.2, there is a ring isomorphism $\rho : \Lambda \to \Gamma := \Gamma(\mathcal{C})$, which sends $E_{m,1}(\pi_{m,1})$ to $E_{m,1}(\pi_{m,m})$ and $E_{r,r+1}(\pi_{r,r+1})$ to $E_{r,r+1}$ for $1 \le r < m$. Let $\varphi_m : \Gamma E_{m,m} \to \Gamma E_{1,1}$ and $\varphi_r : \Gamma E_{r,r} \to \Gamma E_{r+1,r+1}$ be the canonical homomorphisms induced by multiplying on the right by $E_{m,1}(\pi_{m,m})$ and $E_{r,r+1}$, respectively, and define $\Theta := \{\varphi_r \mid 1 \le r < m\} \cup \{\varphi_m\}$. Then, under the isomorphism ρ, we see that $\Lambda_\Pi \simeq \Gamma_\Theta$ as rings. It remains to prove $\Gamma_\Theta \simeq M_m(Q(\mathcal{C}))$.

We first claim that the inclusion $f : \Gamma \hookrightarrow \widetilde{\Gamma} := M_m(D(\mathcal{C}))$ is the universal localization of Γ at the set $\Sigma_0 := \{\varphi_r \mid 1 \le r < m\}$.

Indeed, let $\varphi_{\ge r} := \varphi_r \varphi_{r+1} \cdots \varphi_{m-1}$, and let $\psi_r : \Gamma E_{r,r} \to \widetilde{\Gamma} E_{r,r}$ be the composite of $\varphi_{\ge r}$ with the right multiplication map $\cdot E_{m,r} : \Gamma E_{m,m} \to \widetilde{\Gamma} E_{r,r}$. We define

$$\Sigma_1 := \{\varphi_{\ge r} \mid 1 \le r < m\} \quad \text{and} \quad \Sigma_2 := \{\psi_r \mid 1 \le r < m\}.$$

Clearly, $\varphi_{\ge r} : \Gamma E_{r,r} \to \Gamma E_{m,m}$ is the right multiplication map by $E_{r,m}$, and we have

$$\Gamma_{\Sigma_0} = \Gamma_{\Sigma_1} \quad \text{and} \quad \Gamma E_{m,m} = \widetilde{\Gamma} E_{m,m}.$$

Thus ${}_\Gamma\widetilde{\Gamma}$ is isomorphic to the direct sum of m copies of $\Gamma E_{m,m}$. This implies that ${}_\Gamma\widetilde{\Gamma}$ is a finitely generated projective Γ-module and the multiplication map $\widetilde{\Gamma} \otimes_\Gamma \widetilde{\Gamma} \to \widetilde{\Gamma}$ is an isomorphism, that is, the inclusion map f is a ring epimorphism. By Lemma 2.7, f is the universal localization of Γ at f. Moreover, due to the isomorphism $\widetilde{\Gamma} E_{i,j} \simeq \widetilde{\Gamma} E_{r,s}$ of $\widetilde{\Gamma}$-modules, we have

$$\Gamma_{\Sigma_1} = \Gamma_{\Sigma_2}.$$

Since $E_{r,m} E_{m,r} = E_{r,r}$, it is easy to see that ψ_r is the right multiplication map by $E_{r,r}$ and coincides with the inclusion $\Gamma E_{r,r} \hookrightarrow \widetilde{\Gamma} E_{r,r}$. Hence Γ_{Σ_2} is the same as the universal localization Γ_f of Γ at f, and consequently

$$\Gamma_{\Sigma_0} = \Gamma_f.$$

This completes the proof of the claim.

By Lemma 2.5 (4), the universal localization $D(\mathcal{C})_{\pi_{m,m}}$ of $D(\mathcal{C})$ at $\pi_{m,m}$ is equal to $Q(\mathcal{C})$. Let $\varphi'_m : \widetilde{\Gamma} E_{m,m} \to \widetilde{\Gamma} E_{1,1}$ be the right multiplication map by $E_{m,1}(\pi_{m,m})$. Now, combining Lemma 2.6 with Corollary [8, Corollary 3.5], we have

$$\Gamma_\Theta \simeq \widetilde{\Gamma}_{\varphi'_m} \simeq M_m(D(\mathcal{C})_{\pi_{m,m}}) = M_m(Q(\mathcal{C})).$$

Thus $\Lambda_\Pi \simeq \Gamma_\Theta \simeq M_m(Q(\mathcal{C}))$ as rings. □

4 Universal localizations of tame hereditary algebras

In this section, we shall consider universal localizations of tame hereditary algebras at simple regular modules, and the endomorphism algebras of tilting modules produced by these localizations.

Throughout this section, we fix a nonempty subset $\mathcal{U}$ of $\mathscr{S}$, where $\mathscr{S}$ is a complete set of isomorphism classes of all simple regular R-modules. Recall that $\lambda : R \to R_{\mathcal{U}}$ is the universal localization of R at $\mathcal{U}$. It follows from [28, Theorems 4.9, 5.1 and 5.3] that λ is injective and $R_{\mathcal{U}}$ is hereditary. By Lemma 2.10, the R-module

$$T_{\mathcal{U}} := R_{\mathcal{U}} \oplus R_{\mathcal{U}}/R$$

is a tilting module with $\mathrm{Hom}_R(R_{\mathcal{U}}/R, R_{\mathcal{U}}) = 0$. Note that we always have an exact sequence

$$0 \longrightarrow R \xrightarrow{\lambda} R_{\mathcal{U}} \xrightarrow{\pi} R_{\mathcal{U}}/R \longrightarrow 0$$

of R-modules with π the canonical surjection.

Set $B := \mathrm{End}_R(T_{\mathcal{U}})$ and $S := \mathrm{End}_R(R_{\mathcal{U}}/R)$. Recall that the right multiplication map $\mu : R \to S$, defined by $r \mapsto (y \mapsto yr)$ for $r \in R$ and $y \in R_{\mathcal{U}}/R$, is a ring homomorphism, which endows S with a natural R-R-bimodule structure. Further, for each $U \in \mathcal{U}$, we choose a finitely generated projective resolution of R-modules

$$0 \longrightarrow P_1 \xrightarrow{f_U} P_0 \longrightarrow U \longrightarrow 0$$

and define $\Sigma := \{S \otimes_R f_U \mid U \in \mathcal{U}\}$. Then, by Proposition 2.11, we obtain a recollement of derived module categories

$$\begin{tikzcd} \mathscr{D}(S_\Sigma) \arrow[r] & \mathscr{D}(B) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}(R) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

where S_Σ is the universal localization of S at Σ.

From now on, we always assume in this section that $\mathcal{U} = \mathcal{U}_0 \,\dot{\cup}\, \mathcal{U}_1 \subseteq \mathscr{S}$ such that $\mathcal{U}_0$ contains no cliques and $\mathcal{U}_1$ is a union of cliques.

4.1 Universal localizations at simple regular modules

In this subsection, we shall calculate S_Σ concretely and show Corollary 4.9. This will play an important role in proving our main results.

Let $\mathcal{U}^+$ be the full subcategory of R-Mod, defined by

$$\mathcal{U}^+ := \{X \in R\text{-Mod} \mid \mathrm{Ext}^i_R(U, X) = 0 \text{ for all } U \in \mathcal{U} \text{ and all } i \in \mathbb{N}\}.$$

By Lemma 3.1 (1), $\mathcal{U}^+$ contains the Prüfer module $V[\infty]$ for $V \in \mathscr{S} \setminus \mathcal{U}$. Further, we have $\mathcal{U}_0 \subseteq \mathcal{U}_1^+$ and $\mathcal{U}_1 \subseteq \mathcal{U}_0^+$. This follows from the fact that if $U \in \mathcal{U}_0$ and $V \in \mathcal{U}_1$, then they belong to different tubes.

The subcategory $\mathcal{U}^+$ has the following property, due to [2, Proposition 4.8].

Lemma 4.1. *The subcategory $\mathcal{U}^+$ coincides with the image of the restriction functor $\lambda_* : R_{\mathcal{U}}$-Mod $\to R$-Mod. In particular, for any $Y \in \mathcal{U}^+$, the unit adjunction $\eta_Y : Y \to R_{\mathcal{U}} \otimes_R Y$, defined by $y \mapsto 1 \otimes y$ for $y \in Y$, is an isomorphism of R-modules.*

Thus, for an R-module $Y \in \mathcal{U}^+$, we may endow it with an $R_{\mathcal{U}}$-module structure via the isomorphism η_Y, and in this way, we consider the R-module Y as an $R_{\mathcal{U}}$-module. Note that this $R_{\mathcal{U}}$-module structure on Y extending the R-module structure of Y is unique.

Concerning the universal localization $R_{\mathcal{U}}$ of R at $\mathcal{U}$, there are the following facts (see [3, Proposition 1.11], [28] and [14]).

Lemma 4.2. *The following statements hold:*

(1) *Suppose that $\mathcal{U}$ contains no cliques. Then $R_{\mathcal{U}}$ is a finite-dimensional tame hereditary k-algebra. In particular, the tilting R-module $T_{\mathcal{U}}$ is classical. Moreover, $\{R_{\mathcal{U}} \otimes_R V \mid V \in \mathscr{S} \backslash \mathcal{U}\}$ is a complete set of non-isomorphic simple regular $R_{\mathcal{U}}$-modules, and $(R_{\mathcal{U}} \otimes_R V)[\infty] \simeq V[\infty]$ as $R_{\mathcal{U}}$-modules for each $V \in \mathscr{S} \backslash \mathcal{U}$.*

(2) *Suppose that $\mathcal{U}$ contains cliques. Then $R_{\mathcal{U}}$ is a hereditary order. Moreover, $\{R_{\mathcal{U}} \otimes_R V \mid V \in \mathscr{S} \backslash \mathcal{U}\}$ is a complete set of non-isomorphic simple $R_{\mathcal{U}}$-modules, and the injective envelope of the $R_{\mathcal{U}}$-module $R_{\mathcal{U}} \otimes_R V$ is isomorphic to $V[\infty]$ for each $V \in \mathscr{S} \backslash \mathcal{U}$.*

(3) *Suppose $\mathcal{V} \subseteq \mathscr{S} \backslash \mathcal{U}$. Then*

$$R_{\mathcal{U} \cup \mathcal{V}} = (R_{\mathcal{U}})_{\overline{\mathcal{V}}},$$

where $\overline{\mathcal{V}} := \{R_{\mathcal{U}} \otimes_R V \mid V \in \mathcal{V}\}$. In particular, there are injective ring epimorphisms $R_{\mathcal{U}} \to R_{\mathcal{U} \cup \mathcal{V}}$ and $R_{\mathcal{U} \cup \mathcal{V}} \to R_{\mathscr{S}}$.

As remarked in [14, Section 4], in the case of Lemma 4.2 (1), the set of simple regular $R_{\mathcal{U}}$-modules in a clique is of the form

$$\{R_{\mathcal{U}} \otimes_R V \mid V \in \mathcal{C},\ V \notin \mathcal{U}\},$$

where $\mathcal{C}$ is a clique of R. Further, by Lemma 4.2 (1), for each $V \in \mathcal{C} \backslash \mathcal{U}$, the Prüfer modules corresponding to $R_{\mathcal{U}} \otimes_R V$ and to V are isomorphic. In particular, they have the isomorphic endomorphism algebra.

Thus, if $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_s$ are all cliques from non-homogeneous tubes of R and if $\mathcal{U}$ is a union of $c(\mathcal{C}_i) - 1$ simple regular R-modules from each $\mathcal{C}_i$, then each clique of $R_{\mathcal{U}}$ consists of only one single element. This implies that $R_{\mathcal{U}}$ has only two isomorphism classes of simple modules. If, in addition, the field k is algebraically closed, then $R_{\mathcal{U}}$ is Morita equivalent to the Kronecker algebra. In this case, since the set of cliques of the Kronecker algebra are parameterized by $\mathbb{P}^1(k)$, we see that the set of cliques of an arbitrary tame hereditary k-algebra can be indexed by $\mathbb{P}^1(k)$.

A description of the structure of the module $R_{\mathcal{U}}/R$ was first given in [29], and a further substantial discussion has been carried out recently in [3]. Especially, the following lemma is proved in [3, Propositions 1.7 (6) and 1.10].

Lemma 4.3. *The following statements hold:*

(1) *The R-module $R_{\mathcal{U}}/R$ is a direct union of finite extensions of modules in $\mathcal{U}$.*

(2) *Let $\mathbf{t} \subset R\text{-mod}$ be a tube of rank $m > 1$, and let $\mathcal{U} = \{U_1, U_2, \dots, U_{m-1}\}$ be a set of $m-1$ simple regular modules in $\mathbf{t}$ such that $U_{i+1} = \tau^- U_i$ for all $1 \le i \le m-1$. Then*

$$R_{\mathcal{U}}/R \simeq U_1[m-1]^{(\delta_{U_1})} \oplus U_2[m-2]^{(\delta_{U_2})} \oplus \cdots \oplus U_{m-1}[1]^{(\delta_{U_{m-1}})},$$

with $\delta_{U_j} := \dim_{\mathrm{End}_R(U_j)} \mathrm{Ext}^1_R(U_j, R)$ for $1 \le j \le m-1$. Moreover,

$$R_{\mathcal{U}} \otimes_R U_m \simeq U_m[m]$$

as $R_{\mathcal{U}}$-modules.

(3) *If $\mathcal{U}$ is a union of cliques, then, for any finitely generated projective R-module P,*

$${}_R(R_{\mathcal{U}}/R) \otimes_R P \simeq \bigoplus_{U \in \mathcal{U}} U[\infty]^{(\delta_{U,P})},$$

where $\delta_{U,P} := \dim_{\mathrm{End}_R(U)} \mathrm{Ext}^1_R(U, P)$.

Next, we shall show that $R_{\mathcal{U}}$ and $\mathrm{End}_R(R_{\mathcal{U}}/R)$ can be interpreted as the tensor product and direct sum of some rings, respectively.

Lemma 4.4. *Let $\mathcal{V} = \mathcal{V}_0 \,\dot\cup\, \mathcal{V}_1 \subseteq \mathscr{S}$ such that $\mathcal{V}_0$ contains no cliques and such that $\mathcal{V}_1 \subseteq \mathcal{V}_0^+$. Then the following statements are true:*

(1) *We have*

$$R_{\mathcal{V}} \simeq R_{\mathcal{V}_1} \otimes_R R_{\mathcal{V}_0}$$

as $R_{\mathcal{V}_1}$-$R_{\mathcal{V}_0}$-bimodules, and

$$R_{\mathcal{V}}/R_{\mathcal{V}_1} \simeq R_{\mathcal{V}_1} \otimes_R (R_{\mathcal{V}_0}/R)$$

as $R_{\mathcal{V}_1}$-R-bimodules.

(2) *If* $\mathcal{V}_0 \subseteq \mathcal{V}_1^+$, *then* $R_{\mathcal{V}}/R \simeq R_{\mathcal{V}_0}/R \oplus R_{\mathcal{V}}/R_{\mathcal{V}_0}$ *as* R*-modules and there is a ring isomorphism*

$$\mathrm{End}_R(R_{\mathcal{V}}/R) \longrightarrow \mathrm{End}_R(R_{\mathcal{V}_0}/R) \times \mathrm{End}_{R_{\mathcal{V}_0}}(R_{\mathcal{V}}/R_{\mathcal{V}_0}).$$

Proof. (1) By assumption, we have $\mathcal{V}_1 \subseteq \mathcal{V}_0^+$. It follows from Lemma 4.1 that the unit adjunction $\eta_V : V \to R_{\mathcal{V}_0} \otimes_R V$ is an isomorphism of R-modules for any $V \in \mathcal{V}_1$. This implies that every module in $\mathcal{V}_1$ can be endowed with a unique $R_{\mathcal{V}_0}$-module structure that preserves the given R-module structure via the universal localization $\lambda_0 : R \to R_{\mathcal{V}_0}$. Consequently, $R_{\mathcal{V}} = (R_{\mathcal{V}_0})_{\mathcal{V}_1}$ by Lemma 4.2 (3). Now, we construct the following exact commutative diagram of R-modules:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & R \arrow[r, "\lambda_0"] \arrow[d, equal] & R_{\mathcal{V}_0} \arrow[r, "\pi_0"] \arrow[d, "\lambda_1"] & R_{\mathcal{V}_0}/R \arrow[r] \arrow[d, "\lambda_2"] & 0 \\
0 \arrow[r] & R \arrow[r] & R_{\mathcal{V}} \arrow[r] \arrow[d, "\pi_1"] & R_{\mathcal{V}}/R \arrow[r] \arrow[d, "\pi_2"] & 0 \\
 & & R_{\mathcal{V}}/R_{\mathcal{V}_0} \arrow[r, equal] \arrow[d] & R_{\mathcal{V}}/R_{\mathcal{V}_0} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd} \qquad (*)$$

where λ_1 is the universal localization of $R_{\mathcal{V}_0}$ at $\mathcal{V}_1$, and λ_2 is the canonical injection induced by λ_1, and where π_0, π_1 and π_2 are canonical surjections.

Clearly, $R_{\mathcal{V}_0}$ is a finite-dimensional tame hereditary algebra by Lemma 4.2 (1). From $R_{\mathcal{V}} = (R_{\mathcal{V}_0})_{\mathcal{V}_1}$ we see that $R_{\mathcal{V}}/R_{\mathcal{V}_0}$ is a direct union of finite extensions of modules in $\mathcal{V}_1$ by Lemma 4.3 (1). Since $R_{\mathcal{V}_1}$ is the universal localization of R at $\mathcal{V}_1$, we have $\mathrm{Tor}_i^R(R_{\mathcal{V}_1}, V) = 0$ for any $i \geq 0$ and $V \in \mathcal{V}_1$. Note that the i-th left derived functor $\mathrm{Tor}_i^R(R_{\mathcal{V}_1}, -) : R\text{-Mod} \to \mathbb{Z}\text{-Mod}$ commutes with direct limits. Thus

$$\mathrm{Tor}_i^R(R_{\mathcal{V}_1}, R_{\mathcal{V}}/R_{\mathcal{V}_0}) = 0$$

for any $i \geq 0$. This implies that the homomorphisms $R_{\mathcal{V}_1} \otimes_R \lambda_1$ and $R_{\mathcal{V}_1} \otimes_R \lambda_2$ are isomorphisms. Moreover, by Lemma 4.2 (3), we have $R_{\mathcal{V}} = (R_{\mathcal{V}_1})_{\overline{\mathcal{V}}_0}$ with $\overline{\mathcal{V}}_0 := \{R_{\mathcal{V}_1} \otimes_R V \mid V \in \mathcal{V}_0\}$, and therefore $R_{\mathcal{V}}$ can be regarded as an $R_{\mathcal{V}_1}$-module. Since we have a ring epimorphism $R \to R_{\mathcal{V}_1}$, the canonical multiplication map $\nu_2 : R_{\mathcal{V}_1} \otimes_R R_{\mathcal{V}} \to R_{\mathcal{V}}$ is an isomorphism.

Now we apply the tensor functor $R_{\mathcal{V}_1} \otimes_R -$ to the diagram $(*)$ and get the following exact commutative diagram of $R_{\mathcal{V}_1}$-R-bimodules:

$$\begin{tikzcd}
 & R_{\mathcal{V}_1} \otimes_R R \arrow[r, "R_{\mathcal{V}_1}\otimes_R \lambda_0"] \arrow[d, equal] & R_{\mathcal{V}_1} \otimes_R R_{\mathcal{V}_0} \arrow[r, "R_{\mathcal{V}_1}\otimes_R \pi_0"] \arrow[d, "\simeq"', "R_{\mathcal{V}_1}\otimes_R \lambda_1"] & R_{\mathcal{V}_1} \otimes_R (R_{\mathcal{V}_0}/R) \arrow[r] \arrow[d, "\simeq"', "R_{\mathcal{V}_1}\otimes_R \lambda_2"] & 0 \\
 & R_{\mathcal{V}_1} \otimes_R R \arrow[r] \arrow[d, "\simeq"', "\nu_1"] & R_{\mathcal{V}_1} \otimes_R R_{\mathcal{V}} \arrow[r] \arrow[d, "\simeq"', "\nu_2"] & R_{\mathcal{V}_1} \otimes_R (R_{\mathcal{V}}/R) \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & R_{\mathcal{V}_1} \arrow[r] & R_{\mathcal{V}} \arrow[r] & R_{\mathcal{V}}/R_{\mathcal{V}_1} \arrow[r] & 0
\end{tikzcd}$$

where ν_1 is the multiplication map, and where the exactness of the last row follows from Lemma 4.2 (3). Thus $R_{\mathcal{V}} \simeq R_{\mathcal{V}_1} \otimes_R R_{\mathcal{V}_0}$ as $R_{\mathcal{V}_1}$-$R_{\mathcal{V}_0}$-bimodules and $R_{\mathcal{V}}/R_{\mathcal{V}_1} \simeq R_{\mathcal{V}_1} \otimes_R (R_{\mathcal{V}_0}/R)$ as $R_{\mathcal{V}_1}$-R-bimodules.

(2) Since $R_{\mathcal{V}_0}/R$ is a finite-dimensional k-space and a direct union of finite extensions of modules in $\mathcal{V}_0$ by Lemma 4.3 (1), we have $R_{\mathcal{V}_0}/R \in \mathcal{F}(\mathcal{V}_0)$. Thanks to Lemmas 4.2 (1) and 4.3 (1), we can write

$$R_{\mathcal{V}}/R_{\mathcal{V}_0} = \varinjlim_{\alpha} X_\alpha$$

with $X_\alpha \in \mathcal{F}(\mathcal{V}_1)$. Then, by Lemma 2.13 (1, ii) and the inclusion $\mathcal{V}_1 \subseteq \mathcal{V}_0^+$, we have

$$\mathrm{Ext}^j_R(R_{\mathcal{V}_0}/R, R_{\mathcal{V}}/R_{\mathcal{V}_0}) \simeq \varinjlim_{\alpha} \mathrm{Ext}^j_R(R_{\mathcal{V}_0}/R, X_\alpha) = 0 \quad \text{for } j \geq 0. \qquad \text{(a)}$$

Similarly, it follows from Lemma 2.13 (1, iii) and the inclusion $\mathcal{V}_0 \subseteq \mathcal{V}_1^+$ that

$$\mathrm{Ext}^j_R(R_{\mathcal{V}}/R_{\mathcal{V}_0}, R_{\mathcal{V}_0}/R) \simeq \varprojlim_{\alpha} \mathrm{Ext}^j_R(X_\alpha, R_{\mathcal{V}_0}/R) = 0 \quad \text{for } j \geq 0. \qquad \text{(b)}$$

Particularly, the canonical exact sequence

$$0 \longrightarrow R_{\mathcal{V}_0}/R \xrightarrow{\lambda_2} R_{\mathcal{V}}/R \xrightarrow{\pi_2} R_{\mathcal{V}}/R_{\mathcal{V}_0} \longrightarrow 0 \qquad \text{(c)}$$

splits in R-Mod, that is,

$$R_{\mathcal{V}}/R \simeq R_{\mathcal{V}_0}/R \oplus R_{\mathcal{V}}/R_{\mathcal{V}_0}$$

as R-modules. Since $R \to R_{\mathcal{V}_0}$ is a ring epimorphism, we have

$$\mathrm{End}_R(R_{\mathcal{V}}/R_{\mathcal{V}_0}) = \mathrm{End}_{R_{\mathcal{V}_0}}(R_{\mathcal{V}}/R_{\mathcal{V}_0}).$$

It follows from (a) and (b) for $j = 0$ that

$$\mathrm{End}_R(R_{\mathcal{V}}/R) \simeq \mathrm{End}_R(R_{\mathcal{V}_0}/R) \times \mathrm{End}_{R_{\mathcal{V}_0}}(R_{\mathcal{V}}/R_{\mathcal{V}_0}).$$

This isomorphism can be described as follows: For $f \in \mathrm{End}_R(R_{\mathcal{V}}/R)$, it follows from (c) that there is a unique endomorphism $f_1 \in \mathrm{End}_R(R_{\mathcal{V}_0}/R)$ and a unique endomorphism $f_2 \in \mathrm{End}_R(R_{\mathcal{V}}/R_{\mathcal{V}_0})$ such that $\lambda_2 f = f_1\lambda_2$ and $\pi_2 f_2 = f\pi_2$, and therefore the map

$$\varphi : \mathrm{End}_R(R_{\mathcal{V}}/R) \longrightarrow \mathrm{End}_R(R_{\mathcal{V}_0}/R) \times \mathrm{End}_R(R_{\mathcal{V}}/R_{\mathcal{V}_0}), \quad f \longmapsto (f_1, f_2)$$

is the desired isomorphism of rings. This completes the proof of (2). □

As an obvious consequence of Lemma 4.4, we have the following result.

Corollary 4.5. *The following statements are true for $\mathcal{U}$:*

(1) *We have*

$$R_{\mathcal{U}} \simeq R_{\mathcal{U}_1} \otimes_R R_{\mathcal{U}_0}$$

as $R_{\mathcal{U}_1}$-$R_{\mathcal{U}_0}$-bimodules, and

$$R_{\mathcal{U}}/R_{\mathcal{U}_1} \simeq R_{\mathcal{U}_1} \otimes_R (R_{\mathcal{U}_0}/R)$$

as $R_{\mathcal{U}_1}$-R-bimodules.

(2) *There is a ring isomorphism*

$$\mathrm{End}_R(R_{\mathcal{U}}/R) \longrightarrow \mathrm{End}_R(R_{\mathcal{U}_0}/R) \times \mathrm{End}_{R_{\mathcal{U}_0}}(R_{\mathcal{U}}/R_{\mathcal{U}_0}).$$

Remark. We should point out that $\mathcal{U}_1$ can be regarded as a set of simple regular $R_{\mathcal{U}_0}$-modules, and is a union of cliques of $R_{\mathcal{U}_0}$. In fact, it follows from $\mathcal{U}_1 \subseteq \mathcal{U}_0^+$ and Lemma 4.1 that $R_{\mathcal{U}_0} \otimes_R V \simeq V$ as R-modules for $V \in \mathcal{U}_1$, and therefore each V in $\mathcal{U}_1$ can be viewed as an $R_{\mathcal{U}_0}$-module. Hence, by the statements pointed out after Lemma 4.2, we infer that $\mathcal{U}_1$ is a union of cliques of $R_{\mathcal{U}_0}$.

The following result reduces the calculation of S_Σ to the consideration of the cliques contained in $\mathcal{U}$.

Lemma 4.6. *Define*

$$\Lambda := \mathrm{End}_{R_{\mathcal{U}_0}}(R_{\mathcal{U}}/R_{\mathcal{U}_0})$$

and

$$\Theta := \{\Lambda \otimes_{R_{\mathcal{U}_0}} (R_{\mathcal{U}_0} \otimes_R f_V) \mid V \in \mathcal{U}_1\}.$$

Then S_Σ is isomorphic to the universal localization Λ_Θ of Λ at Θ.

Proof. Note that the $R_{\mathcal{U}_0}$-module structure on Λ is given by the ring homomorphism $R_{\mathcal{U}_0} \to \Lambda$, which is defined by the right multiplication map.

By Lemma 4.2 (1), we know that $R_{\mathcal{U}_0}$ is a finite-dimensional tame hereditary k-algebra. Moreover, from the remark following Corollary 4.5, we know that $\mathcal{U}_1$ can be seen as a set of simple regular $R_{\mathcal{U}_0}$-modules. Thus $R_{\mathcal{U}} = (R_{\mathcal{U}_0})_{\mathcal{U}_1}$ by Lemma 4.2 (3). More precisely, for each $V \in \mathcal{U}_1$, we fix a minimal projective presentation

$$0 \longrightarrow P_1 \xrightarrow{f_V} P_0 \longrightarrow V \longrightarrow 0$$

of V in R-mod, and get a projective presentation of V in $R_{\mathcal{U}_0}$-mod

$$0 \longrightarrow R_{\mathcal{U}_0} \otimes_R P_1 \xrightarrow{R_{\mathcal{U}_0} \otimes_R f_V} R_{\mathcal{U}_0} \otimes_R P_0 \longrightarrow V \longrightarrow 0.$$

This is due to the fact that

$$\mathrm{Tor}_1^R(R_{\mathcal{U}_0}, V) \simeq \mathrm{Tor}_1^R(R_{\mathcal{U}_0}, R_{\mathcal{U}_0} \otimes_R V) \simeq \mathrm{Tor}_1^{R_{\mathcal{U}_0}}(R_{\mathcal{U}_0}, R_{\mathcal{U}_0} \otimes_R V) = 0.$$

Therefore, $R_{\mathcal{U}}$ is the universal localization of $R_{\mathcal{U}_0}$ at $\{R_{\mathcal{U}_0} \otimes_R f_V \mid V \in \mathcal{U}_1\}$.

Recall that

$$\Lambda := \mathrm{End}_{R_{\mathcal{U}_0}}(R_{\mathcal{U}}/R_{\mathcal{U}_0}) \quad \text{and} \quad \Theta := \{\Lambda \otimes_{R_{\mathcal{U}_0}} (R_{\mathcal{U}_0} \otimes_R f_V) \mid V \in \mathcal{U}_1\}.$$

In the following, we shall show that S_Σ is isomorphic to Λ_Θ.

Let $\Gamma := \mathrm{End}_R(R_{\mathcal{U}_0}/R)$ and $\varphi = (\varphi_0, \varphi_1) : S \to \Gamma \times \Lambda$, where $\varphi_0 : S \to \Gamma$ and $\varphi_1 : S \to \Lambda$ are the ring homomorphisms given in the proof of Lemma 4.4 (2). Recall that $\mu : R \to S$ is the right multiplication map. Set $\mu_0 = \mu\varphi_0 : R \to \Gamma$ and $\mu_1 = \mu\varphi_1 : R \to \Lambda$. Clearly, both μ_0 and μ_1 are ring homomorphisms, through which both Λ and Γ have a right R-module structure. Now, we write

$$\Sigma := \{S \otimes_R f_U \mid U \in \mathcal{U}\}$$

as $\Sigma = \Phi \times \Psi$ with

$$\Phi := \{\Gamma \otimes_R f_U \mid U \in \mathcal{U}\} \quad \text{and} \quad \Psi := \{\Lambda \otimes_R f_U \mid U \in \mathcal{U}\}.$$

Consequently, the ring isomorphism φ implies that $S_\Sigma \simeq \Gamma_\Phi \times \Lambda_\Psi$. To finish the proof, it suffices to prove that $\Gamma_\Phi = 0$ and $\Lambda_\Psi \simeq \Lambda_\Theta$.

Indeed, we write $\Phi = \Phi_0 \cup \Phi_1$ with

$$\Phi_0 := \{\Gamma \otimes_R f_U \mid U \in \mathcal{U}_0\} \text{ and } \Phi_1 := \{\Gamma \otimes_R f_U \mid U \in \mathcal{U}_1\}.$$

Then, by Lemma 2.6, we have

$$\Gamma_\Phi \simeq (\Gamma_{\Phi_0})_{\overline{\Phi}_1},$$

where

$$\overline{\Phi}_1 := \{\Gamma_{\Phi_0} \otimes_R f_U \mid U \in \mathcal{U}_1\}.$$

To prove $\Gamma_\Phi = 0$, it suffices to prove $\Gamma_{\Phi_0} = 0$. Consider the canonical exact sequence of R-modules

$$0 \longrightarrow R \xrightarrow{\lambda_0} R_{\mathcal{U}_0} \xrightarrow{\pi_0} R_{\mathcal{U}_0}/R \longrightarrow 0.$$

By Lemma 4.2 (1), the module $T_{\mathcal{U}_0} := R_{\mathcal{U}_0} \oplus R_{\mathcal{U}_0}/R$ is a classical tilting R-module, and therefore $\mathscr{D}(R)$ is triangle equivalent to $\mathscr{D}(\mathrm{End}_R(T_{\mathcal{U}_0}))$ in the recollement of $\mathscr{D}(R)$, $\mathscr{D}(\mathrm{End}_R(T_{\mathcal{U}_0}))$ and $\mathscr{D}(\Gamma_{\Phi_0})$ by Proposition 2.11. Thus $\Gamma_{\Phi_0} = 0$ and $\Gamma_\Phi = 0$.

It remains to show $\Lambda_\Psi \simeq \Lambda_\Theta$. Let $\mu_2 : R_{\mathcal{U}_0} \to \Lambda$ denote the right multiplication map defined by $r \mapsto (x \mapsto xr)$ for $r \in R_{\mathcal{U}_0}$ and $x \in R_{\mathcal{U}}/R_{\mathcal{U}_0}$. Then, along the diagram $(*)$ in the proof of Lemma 4.4, one can check that the following diagram of ring homomorphisms commutes:

$$\begin{tikzcd} R \arrow[r, "\lambda_0"] \arrow[d, "\mu"] & R_{\mathcal{U}_0} \arrow[d, "\mu_2"] \\ S \arrow[r, "\varphi_1"] & \Lambda. \end{tikzcd}$$

Now, we write $\Psi = \Psi_0 \cup \Psi_1$ with

$$\Psi_0 := \{\Lambda \otimes_R f_U \mid U \in \mathcal{U}_0\} \quad \text{and} \quad \Psi_1 := \{\Lambda \otimes_R f_V \mid V \in \mathcal{U}_1\},$$

and claim $\Lambda_{\Psi_0} = \Lambda$. It suffices to show that $\Lambda \otimes_R f_U$ is an isomorphism for any $U \in \mathcal{U}_0$. However, this follows from $\Lambda \otimes_R f_U \simeq \Lambda \otimes_{R_{\mathcal{U}_0}} (R_{\mathcal{U}_0} \otimes_R f_U)$ and $R_{\mathcal{U}_0} \otimes_R f_U$ being an isomorphism by the definition of universal localizations. Hence $\Lambda_{\Psi_0} = \Lambda$.

Now, we have $\overline{\Psi}_1 := \{\Lambda_{\Psi_0} \otimes_\Lambda h \mid h \in \Psi_1\} = \Psi_1$. It follows from Lemma 2.6 that $\Lambda_\Psi \simeq (\Lambda_{\Psi_0})_{\overline{\Psi}_1} \simeq \Lambda_{\Psi_1}$. Further, $\Lambda \otimes_R f_V \simeq \Lambda \otimes_{R_{\mathcal{U}_0}} (R_{\mathcal{U}_0} \otimes_R f_V)$ for any $V \in \mathcal{U}_1$. By comparing the elements in Θ with the ones in Ψ_1, one knows immediately that $\Lambda_\Psi \simeq \Lambda_\Theta$, and therefore $S_\Sigma \simeq \Lambda_\Theta$, finishing the proof. □

To proceed with our discussion, let us now introduce some notation.

Let $\mathcal{C}$ be a clique of R. Recall that $D(\mathcal{C})$ stands for the endomorphism algebra of a Prüfer module $V[\infty]$ with $V \in \mathcal{C}$. Then $D(\mathcal{C})$ is a discrete valuation ring with the division ring $Q(\mathcal{C})$ of fractions of $D(\mathcal{C})$. Clearly, $D(\mathcal{C})$ is a subring of $Q(\mathcal{C})$.

For $U \in \mathcal{C}$, let

$$0 \longrightarrow U \xrightarrow{\zeta_U} U[\infty] \xrightarrow{\pi_U} (\tau^- U)[\infty] \longrightarrow 0$$

be the canonical exact sequence defined in Lemma 3.1(2), where ζ_U is the canonical inclusion.

We write $\mathcal{C} = \{U_1, U_2, \ldots, U_{m-1}, U_m\}$ with $m \geq 1$ such that $U_{i+1} = \tau^- U_i$ for $1 \leq i \leq m$, where the subscript of U_i is modulo m. If we have $U = U_j$ for some $1 \leq j \leq m$, then π_U is defined to be $\pi_{j,j+1} : U_j[\infty] \longrightarrow U_{j+1}[\infty]$ in Lemma 3.1 (4), where $\pi_{m,m+1} := \pi_{m,1}$ by our convention.

To calculate S_Σ efficiently, we first simplify the homomorphisms appearing in Σ.

Lemma 4.7. *If $\mathcal{U}$ is a union of cliques, then, for each $U \in \mathcal{U}$, there exists an exact commutative diagram of R-modules*

$$\begin{tikzcd}
0 \arrow[r] & U \arrow[r] \arrow[d, equal] & (R_{\mathcal{U}}/R) \otimes_R P_1 \arrow[r, "(R_{\mathcal{U}}/R)\otimes_R f_U"] \arrow[d, "\simeq"] & (R_{\mathcal{U}}/R) \otimes_R P_0 \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & U \arrow[r, "(\zeta_U, 0)"] & U[\infty] \oplus E \arrow[r, "\begin{pmatrix} \pi_U & 0 \\ 0 & 1 \end{pmatrix}"] & (\tau^- U)[\infty] \oplus E \arrow[r] & 0
\end{tikzcd}$$

where E is an R-module.

Proof. Recall that we have a projective resolution of U,

$$0 \longrightarrow P_1 \xrightarrow{f_U} P_0 \longrightarrow U \longrightarrow 0$$

where P_1 and P_0 are finitely generated projective R-modules. As $\lambda : R \to R_{\mathcal{U}}$ is the universal localization of R at $\mathcal{U}$, the homomorphism

$$R_{\mathcal{U}} \otimes_R f_U : R_{\mathcal{U}} \otimes_R P_1 \longrightarrow R_{\mathcal{U}} \otimes_R P_0$$

is an isomorphism. This yields the exact commutative diagram of R-modules:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[d] & & U \arrow[d, "\psi"] & \\
0 \arrow[r] & P_1 \arrow[r, "\lambda\otimes_R P_1"] \arrow[d, "f_U"'] & R_{\mathcal{U}} \otimes_R P_1 \arrow[r, "\pi\otimes_R P_1"] \arrow[d, "R_{\mathcal{U}}\otimes_R f_U", "\simeq"'] & (R_{\mathcal{U}}/R) \otimes_R P_1 \arrow[r] \arrow[d, "(R_{\mathcal{U}}/R)\otimes_R f_U"] & 0 \\
0 \arrow[r] & P_0 \arrow[r, "\lambda\otimes_R P_0"] \arrow[d] & R_{\mathcal{U}} \otimes_R P_0 \arrow[r, "\pi\otimes_R P_0"] & (R_{\mathcal{U}}/R) \otimes_R P_0 \arrow[r] \arrow[d] & 0 \\
 & U \arrow[d] & & 0 & \\
 & 0 & & &
\end{tikzcd}$$

which provides the following short exact sequence of R-modules:

$$0 \longrightarrow U \xrightarrow{\psi} (R_{\mathcal{U}}/R) \otimes_R P_1 \xrightarrow{(R_{\mathcal{U}}/R)\otimes_R f_U} (R_{\mathcal{U}}/R) \otimes_R P_0 \longrightarrow 0. \quad \text{(a)}$$

Suppose that $\mathcal{U}$ is a union of cliques, say $\mathcal{U} = \dot{\bigcup}_{i\in I} \mathcal{C}_i$ with I an index set. By Lemma 4.3 (3), we have

$$(R_{\mathcal{U}}/R) \otimes_R P_1 \simeq \bigoplus_{i\in I} \bigoplus_{V\in\mathcal{C}_i} V[\infty]^{(n_V)} \quad \text{for some } n_V \in \mathbb{N},$$

where n_U is non-zero since U can be embedded into $(R_{\mathcal{U}}/R) \otimes_R P_1$ and since $\mathrm{Hom}_R(U, W[\infty]) = 0$ for $W \in \mathcal{U}$ with $W \not\simeq U$. So we may write

$$(R_{\mathcal{U}}/R) \otimes_R P_1 = U[\infty] \oplus E$$

with E an R-module and

$$\psi = (\psi_1, g) : U \longrightarrow U[\infty] \oplus E$$

with $0 \neq \psi_1 \in \mathrm{Hom}_R(U, U[\infty])$ and $g \in \mathrm{Hom}_R(U, E)$. Let $D := \mathrm{End}_R(U[\infty])$. Then D is a local ring with a maximal ideal $\mathfrak{m}$. Moreover, it follows from the proof of Lemma 3.1 (5) that there is an exact sequence

$$0 \longrightarrow \mathfrak{m} \longrightarrow D \xrightarrow{(\zeta_U)^*} \mathrm{Hom}_R(U, U[\infty]) \longrightarrow 0.$$

This means that, for any $\alpha : U \to U[\infty]$, there is a homomorphism $\beta \in D$ such that $\alpha = \zeta_U \beta$, and that if the homomorphism α is non-zero, then β must be an automorphism. In particular, there is an automorphism $\beta \in D$ such that $\psi_1 = \zeta_U \beta$. Thus we can form the following commutative diagram:

$$\begin{tikzcd} U \arrow[r, "\psi"] \arrow[d, equal] & (R_{\mathcal{U}}/R) \otimes_R P_1 \arrow[d, "\simeq"'] \\ U \arrow[r, "(\zeta_U, g)"] & U[\infty] \oplus E. \end{tikzcd} \quad \text{(b)}$$

Note that $g : U \to E$ factorizes through ζ_U. Then, by applying Lemma 2.12 to (a) with the property (b), we obtain the following exact commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & U \arrow[r, "\psi"] \arrow[d, equal] & (R_{\mathcal{U}}/R) \otimes_R P_1 \arrow[r, "(R_{\mathcal{U}}/R)\otimes_R f_U"] \arrow[d, "\simeq"] & (R_{\mathcal{U}}/R) \otimes_R P_0 \arrow[r] \arrow[d, "\simeq"] & 0 \\ 0 \arrow[r] & U \arrow[r, "(\zeta_U, 0)"] & U[\infty] \oplus E \arrow[r, "\begin{pmatrix} \pi_U & 0 \\ 0 & 1 \end{pmatrix}"] & (\tau^- U)[\infty] \oplus E \arrow[r] & 0. \end{tikzcd}$$

This finishes the proof. □

Next, we show that the universal localizations in Lemma 4.6, which are of interest for us, take actually the form of adèle rings in the algebraic number theory (see [23, Chapter V, Section 1]).

Lemma 4.8. *If $\mathcal{U} \subseteq \mathscr{S}$ is a union of cliques, say $\mathcal{U} = \dot{\bigcup}_{i\in I} \mathcal{C}_i$ with I an index set, then the following statements are true:*

(1) *The ring S is Morita equivalent to $\prod_{i\in I} \Gamma(\mathcal{C}_i)$, where the ring $\Gamma(\mathcal{C})$ is defined in Lemma 3.2 for each clique $\mathcal{C}$ of R.*

(2) *The ring S_Σ is Morita equivalent to the adèle ring*

$$\mathbb{A}_{\mathcal{U}} := \left\{ (f_i)_{i\in I} \in \prod_{i\in I} Q(\mathcal{C}_i) \mid f_i \in D(\mathcal{C}_i) \text{ for almost all } i \in I \right\}.$$

Proof. (1) By Lemma 4.3 (3), we have

$$R_{\mathcal{U}}/R \simeq \bigoplus_{i\in I} \bigoplus_{V\in\mathcal{C}_i} V[\infty]^{(\delta_V)}$$

as R-modules, where

$$\delta_V := \dim_{\mathrm{End}_R(V)} \mathrm{Ext}^1_R(V, R) = \dim_{\mathrm{End}_R(V)^{\mathrm{op}}}(\tau V) \neq 0.$$

We claim that there is a natural number d such that $\delta_V \leq d$ for all $V \in \mathcal{U}$.

In fact, let $\{S_j \mid 1 \leq j \leq r\}$ be a complete set of isomorphism classes of simple R-modules with r a natural number. For each $X \in R$-mod, denote by $\underline{\dim}\, X \in \mathbb{N}^r$ the dimension vector of X. Now, let $\langle -,- \rangle : \mathbb{N}^r \times \mathbb{N}^r \to \mathbb{Z}$ be the Euler form of the tame hereditary k-algebra R, that is,

$$\langle \underline{\dim}\, Y, \underline{\dim}\, Z \rangle := \dim_k \mathrm{Hom}_R(Y, Z) - \dim_k \mathrm{Ext}^1_R(Y, Z)$$

with $Y, Z \in R$-mod, and further, let $q : \mathbb{N}^r \to \mathbb{Z}$ be the quadratic form of R, that is, $q(\underline{\dim}\, Y) := \langle \underline{\dim}\, Y, \underline{\dim}\, Y \rangle$, and let $h = (h_i)_{1\leq i\leq r}$ be the minimal positive radical vector of q. It is known that h is equal to the sum of the dimension vectors of all simple regular R-modules in $\mathbf{t}'$ for an arbitrary tube $\mathbf{t}'$ of R. Therefore, we have

$$\delta_U \leq \dim_k(\tau U) \leq \Big(\sum_i h_i\Big)\Big(\sum_j \dim_k S_j\Big) < \infty$$

for $U \in \mathscr{S}$. In particular, if we take

$$d = \Big(\sum_i h_i\Big)\Big(\sum_j \dim_k S_j\Big),$$

then $\delta_V \leq d$ for all $V \in \mathcal{U}$, as claimed.

Set

$$N := \bigoplus_{i \in I} \bigoplus_{V \in \mathcal{C}_i} V[\infty] \quad \text{and} \quad \Gamma := \mathrm{End}_R(N).$$

The above claim implies that $\mathrm{Hom}_R(R_{\mathcal{U}}/R, N)$ is a finitely generated, projective generator for S-Mod, and therefore S is Morita equivalent to Γ.

Note that if $i, j \in I$ with $i \neq j$, then $\mathrm{Hom}_R(U[\infty], V[\infty]) = 0$ for all $U \in \mathcal{C}_i$ and $V \in \mathcal{C}_j$. Thus, by Lemma 3.2, we get the following isomorphisms:

$$\Gamma \simeq \prod_{i \in I} \mathrm{End}_R\Big(\bigoplus_{V \in \mathcal{C}_i} V[\infty]\Big) \simeq \prod_{i \in I} \Gamma(\mathcal{C}_i).$$

Thus S is Morita equivalent to $\prod_{i \in I} \Gamma(\mathcal{C}_i)$. This finishes the proof of (1).

(2) For any finitely generated projective R-module P, we have

$$S \otimes_R P = \mathrm{Hom}_R(R_{\mathcal{U}}/R, R_{\mathcal{U}}/R) \otimes_R P \simeq \mathrm{Hom}_R(R_{\mathcal{U}}/R, (R_{\mathcal{U}}/R) \otimes_R P)$$

as S-modules. So, we can rewrite

$$\Sigma = \{\mathrm{Hom}_R(R_{\mathcal{U}}/R, (R_{\mathcal{U}}/R) \otimes_R f_V) \mid V \in \mathcal{U}\}.$$

It follows from Lemma 4.7 that S_Σ is the same as the universal localization of S at $\Sigma' := \{\mathrm{Hom}_R(R_{\mathcal{U}}/R, \pi_V) \mid V \in \mathcal{U}\}$. Since Morita equivalences preserve universal localizations by [8, Corollary 3.5], we know that $S_{\Sigma'}$ (and also S_Σ) is Morita equivalent to Γ_Φ with

$$\Phi := \{\mathrm{Hom}_R(N, \pi_V) \mid V \in \mathcal{U}\}.$$

Let $\mathcal{U} = \mathcal{L} \dot\cup \mathcal{W}$ be a decomposition such that $\mathcal{L}$ is a union of cliques $\mathcal{C}_i$ with i in an index set I_0 and that $\mathcal{W}$ is a union of cliques $\mathcal{C}_j$ with j in an index set I_1. Since $I = I_0 \dot\cup I_1$, we obtain the following isomorphisms of rings:

$$\Gamma \simeq \prod_{i \in I} \mathrm{End}_R\Big(\bigoplus_{V \in \mathcal{C}_i} V[\infty]\Big) \simeq \prod_{i \in I} \Gamma(\mathcal{C}_i) \simeq \prod_{i \in I_0} \Gamma(\mathcal{C}_i) \times \prod_{i \in I_1} \Gamma(\mathcal{C}_i). \qquad (*)$$

First of all, we define $\Gamma_0 := \prod_{i \in I_0} \Gamma(\mathcal{C}_i)$ and $\Gamma_1 := \prod_{i \in I_1} \Gamma(\mathcal{C}_i)$, and decompose $\Phi = \Phi_0 \cup \Phi_1$ where

$$\Phi_0 := \{\mathrm{Hom}_R(N, \pi_V) \mid V \in \mathcal{L}\} \quad \text{and} \quad \Phi_1 := \{\mathrm{Hom}_R(N, \pi_W) \mid W \in \mathcal{W}\}.$$

Then $\Gamma \simeq \Gamma_0 \times \Gamma_1$ as rings. Note that if two Prüfer modules belong to different tubes of R, then there are no nonzero homomorphisms between them. So, under these isomorphisms $(*)$, we can regard Φ_0 (respectively, Φ_1) as the set of homomorphisms between finitely generated projective Γ_0-modules (respectively, Γ_1-modules), and therefore the calculation of Γ_Φ can be done along the blocks Γ_0 and Γ_1 of the ring Γ. In other words, $\Gamma_\Phi \simeq (\Gamma_0)_{\Phi_0} \times (\Gamma_1)_{\Phi_1}$ as rings.

Next, we assume that each clique in $\mathcal{W}$ is of rank 1, and each clique L in $\mathcal{L}$ is of rank greater than 1. It is known that $\mathcal{L}$ is a finite set. Thus the calculation of $(\Gamma_0)_{\Phi_0}$ is reduced to each block $\Gamma(\mathcal{C}_i)$ of Γ_0. It follows from Lemma 3.4 that

$$(\Gamma_0)_{\Phi_0} \simeq \prod_{i \in I_0} M_{c(\mathcal{C}_i)}(Q(\mathcal{C}_i)).$$

Clearly, $\Gamma_1 = \prod_{i \in I_1} D(\mathcal{C}_i)$. Since I_1 is not necessarily a finite set, we cannot express $(\Gamma_1)_{\Phi_1}$ as a direct product of corresponding universal localization of each block of Γ_1. Nevertheless, we claim $(\Gamma_1)_{\Phi_1} \simeq \mathbb{A}_{\mathcal{W}}$ as rings, where

$$\mathbb{A}_{\mathcal{W}} := \left\{ (f_i)_{i \in I_1} \in \prod_{i \in I_1} Q(\mathcal{C}_i) \mid f_i \in D(\mathcal{C}_i) \text{ for almost all } i \in I_1 \right\}.$$

Actually, for each $i \in I_1$, the clique $\mathcal{C}_i$ consists of only one simple regular module. Hence we write $D(\mathcal{C}_i) = \operatorname{End}_R(\mathcal{C}_i)$, which is a discrete valuation ring with a unique maximal ideal generated by π_i.

Define $e_i := (\beta_j)_{j \in I_1} \in \Gamma_1$ by $\beta_i = 1$ and $\beta_j = 0$ if $j \neq i$, and let

$$\varphi_i : \Gamma_1 e_i \longrightarrow \Gamma_1 e_i$$

be the right multiplication map defined by $g \mapsto g\pi_i$ for every $g \in D(\mathcal{C}_i)$. Under those isomorphisms $(*)$, we can identify Φ_1 with $\{\varphi_j \mid j \in I_1\}$. Further, we define $\varepsilon_i := (\theta_j)_{j \in I_1} \in \Gamma_1$ by $\theta_i = \pi_i$ and $\theta_j = 1$ if $j \neq i$. Then, the right multiplication map $\bar{\varepsilon}_i$ defined by ε_i has the following form:

$$\bar{\varepsilon}_i = \begin{pmatrix} \varphi_i & 0 \\ 0 & 1 \end{pmatrix} : \Gamma_1 e_i \oplus \Gamma_1(1 - e_i) \longrightarrow \Gamma_1 e_i \oplus \Gamma_1(1 - e_i).$$

Consequently, we have $(\Gamma_1)_{\Phi_1} \simeq (\Gamma_1)_\Psi$ with $\Psi := \{\bar{\varepsilon}_j \mid j \in I_1\}$.

Now, let Υ be the minimal multiplicative subset of Γ_1 containing all ε_j for $j \in I_1$. It follows that $(\Gamma_1)_\Psi$ is also the universal localization $(\Gamma_1)_\Upsilon$ of Γ_1 at the set Υ, that is, the universal localization of Γ_1 at the set of all right multiplication maps induced by the elements of Υ (see Section 2.2). Moreover,

$$\Upsilon = \left\{ (f_i)_{i \in I_1} \in \prod_{i \in I_1} \{(\pi_i)^n \mid n \in \mathbb{N}\} \mid f_i = 1 \text{ for almost all } i \in I_1 \right\} \subseteq \Gamma_1,$$

where $(\pi_i)^0 := 1$. Next, we show that Υ is a left and right denominator subset of Γ_1 (see Definition 2.2).

Indeed, let $a = (a_i)_{i \in I_1} \in \Gamma_1$ and $s = (\pi_i^{n_i})_{i \in I_1} \in \Upsilon$ with $n_i \in \mathbb{N}$. As $D(\mathcal{C}_i)$ is a discrete valuation ring for each $i \in I_1$, we have $D(\mathcal{C}_i)\pi_i^{n_i} = \pi_i^{n_i} D(\mathcal{C}_i)$, and therefore

$$\Gamma_1 s = \prod_{i \in I_1} D(\mathcal{C}_i)\pi_i^{n_i} = \prod_{i \in I_1} \pi_i^{n_i} D(\mathcal{C}_i).$$

This means $sa \in \Upsilon a \cap \Gamma_1 s \neq \emptyset$, which verifies condition (i) in Definition 2.2. Moreover, if $as = 0$, then $a_i \pi_i^{n_i} = 0$ for $i \in I_1$. Since $\pi_i^{n_i} \neq 0$ and $D(\mathcal{C}_i)$ is a domain for $i \in I_1$, we have $a_i = 0$, and so $a = 0$, which verifies condition (ii) in Definition 2.2. Thus, Υ is a left denominator subset of Γ_1. Similarly, we can prove that Υ is also a right denominator subset of Γ_1.

By Lemma 2.3, the Ore localization $\Upsilon^{-1}\Gamma_1$ of Γ_1 at Υ does exist and is isomorphic to $(\Gamma_1)_\Upsilon$. Thus

$$(\Gamma_1)_{\Phi_1} \simeq (\Gamma_1)_\Psi \simeq (\Gamma_1)_\Upsilon \simeq \Upsilon^{-1}\Gamma_1$$

as rings. Hence, to prove $(\Gamma_1)_{\Phi_1} \simeq \mathbb{A}_{\mathcal{W}}$, it is sufficient to prove $\Upsilon^{-1}\Gamma_1 \simeq \mathbb{A}_{\mathcal{W}}$ as rings. However, by Lemma 2.3, it is enough to show that the canonical inclusion $\mu : \Gamma_1 \to \mathbb{A}_{\mathcal{W}}$ is an Ore localization of Γ_1 at Υ.

Recall that $Q(\mathcal{C}_j)$ denotes the division ring of fractions of the domain $D(\mathcal{C}_j)$ for $j \in I_1$. This implies that μ satisfies both Lemma 2.3 (1) and Lemma 2.3 (3). Now, suppose $f := (f_j)_{j \in I_1} \in \mathbb{A}_{\mathcal{W}}$. By definition, there is a finite subset Δ of I_1 such that $f_j \in Q(\mathcal{C}_j)$ if $j \in \Delta$, and that $f_j \in D(\mathcal{C}_j)$ if $j \notin \Delta$. Note that $Q(\mathcal{C}_j)$ is the Ore localization of $D(\mathcal{C}_j)$ at the subset $S_j := \{(\pi_j)^n \mid n \in \mathbb{N}\}$, due to Lemma 2.5. It follows from Lemma 2.3 (2) that each $x \in Q(\mathcal{C}_j)$ has the form t/s with $t \in D(\mathcal{C}_j)$ and $s \in S_j$. So, if $j \in \Delta$, then we can write $f_j = t_j/s_j$ with $t_j \in D(\mathcal{C}_j)$ and $s_j \in S_j$. Define $g := (g_j)_{j \in I_1}$ by

$$g_j = \begin{cases} t_j & \text{if } j \in \Delta, \\ f_j & \text{if } j \notin \Delta, \end{cases}$$

and $h := (h_j)_{j \in I_1}$ by

$$h_j = \begin{cases} s_j & \text{if } j \in \Delta, \\ 1 & \text{if } j \notin \Delta. \end{cases}$$

Then $g \in \Gamma_1$, $h \in \Upsilon$ and $f = g/h \in \mathbb{A}_{\mathcal{W}}$. Thus μ fulfills Lemma 2.3 (2), and therefore is the Ore localization of Γ_1 at Υ. This shows

$$(\Gamma_1)_{\Phi_1} \simeq \Upsilon^{-1}\Gamma_1 \simeq \mathbb{A}_{\mathcal{W}}.$$

Summing up what we have proved, we obtain

$$\Gamma_\Phi \simeq (\Gamma_0)_{\Phi_0} \times (\Gamma_1)_{\Phi_1} \simeq \left(\prod_{i \in I_0} M_{c(\mathcal{C}_i)}(Q(\mathcal{C}_i))\right) \times \mathbb{A}_{\mathcal{W}},$$

where the last ring is Morita equivalent to $\mathbb{A}_{\mathcal{U}}$. As S_Σ is Morita equivalent to Γ_Φ, we see that S_Σ is Morita equivalent to $\mathbb{A}_{\mathcal{U}}$. This completes the proof of (2). □

Finally, we give a description of S_Σ up to Morita equivalence for an arbitrary $\mathcal{U}$. This will be used for the proof of Theorem 1.1.

Corollary 4.9. *Let $\{\mathcal{C}_i\}_{i\in I}$ be the set of all cliques contained in $\mathcal{U}$, where I is an index set. Then S_Σ is Morita equivalent to the adèle ring*

$$\mathbb{A}_{\mathcal{U}} := \left\{(f_i)_{i\in I} \in \prod_{i\in I} Q(\mathcal{C}_i) \mid f_i \in D(\mathcal{C}_i) \text{ for almost all } i \in I\right\}.$$

Proof. We write $\mathcal{U} = \mathcal{U}_0 \cup \mathcal{U}_1 \subseteq \mathscr{S}$ such that $\mathcal{U}_0$ contains no cliques and $\mathcal{U}_1$ is a union of cliques $\mathcal{C}_i$ with $i \in I$. It follows from Lemma 4.6 that S_Σ is isomorphic to the universal localization Λ_Θ of Λ at Θ with

$$\Lambda := \mathrm{End}_{R_{\mathcal{U}_0}}(R_{\mathcal{U}}/R_{\mathcal{U}_0}) \quad \text{and} \quad \Theta := \{\Lambda \otimes_{R_{\mathcal{U}_0}} (R_{\mathcal{U}_0} \otimes_R f_V) \mid V \in \mathcal{U}_1\}.$$

By Lemma 4.2 (1), $R_{\mathcal{U}_0}$ is a finite-dimensional tame hereditary k-algebra, and the endomorphism algebra of the Prüfer module corresponding to a simple regular module in $\mathcal{U}_1$ is preserved (up to isomorphism). Furthermore, by the remark following Corollary 4.5, we can regard $\mathcal{U}_1$ as a set of simple regular $R_{\mathcal{U}_0}$-modules. In this case, $\mathcal{U}_1$ is a union of cliques of $R_{\mathcal{U}_0}$, and each $V \in \mathcal{U}_1$ admits a projective presentation

$$0 \longrightarrow R_{\mathcal{U}_0} \otimes_R P_1 \xrightarrow{R_{\mathcal{U}_0} \otimes_R f_V} R_{\mathcal{U}_0} \otimes_R P_0 \longrightarrow V \longrightarrow 0$$

in $R_{\mathcal{U}_0}$-mod (see the proof of Lemma 4.6). Now, we can pass from R to $R_{\mathcal{U}_0}$ and apply Lemma 4.8 (2) to $R_{\mathcal{U}_0}$ and $\mathcal{U}_1$, and deduce that Λ_Θ is Morita equivalent to $\mathbb{A}_{\mathcal{U}}$. Hence, S_Σ is Morita equivalent to $\mathbb{A}_{\mathcal{U}}$. □

4.2 Endomorphism algebras of tilting modules

In this subsection, we shall discuss the endomorphism algebras of tilting modules obtained by universal localizations of tame hereditary algebras at simple regular modules. The consideration here will serve as a part of preparations for the proof of Corollary 1.2.

First of all, we mention a relationship between universal localizations of an arbitrary tame hereditary algebra and the ones of the Kronecker algebra.

Lemma 4.10. *For the given $\mathcal{U} \subset \mathscr{S}$, there exists a $\mathcal{V} \subseteq \mathscr{S}$ with $\mathcal{U} \cap \mathcal{V} = \emptyset$ such that, for $\mathcal{W} := \mathcal{U} \cup \mathcal{V}$, the following statements are true:*

(1) *There is a finite-dimensional tame hereditary k-algebra Λ with only two non-isomorphic simple modules, and a set $\mathcal{S}$ of simple regular Λ-modules such that $R_{\mathcal{W}}$ coincides with the universal localization $\Lambda_{\mathcal{S}}$ of Λ at $\mathcal{S}$.*

(2) *The $R_{\mathcal{U}}$-module $T := R_{\mathcal{W}} \oplus R_{\mathcal{W}}/R_{\mathcal{U}}$ is a classical tilting module. In particular, $R_{\mathcal{U}}$ and $\mathrm{End}_{R_{\mathcal{U}}}(T)$ are derived equivalent.*

Proof. Write $\mathcal{U} = \mathcal{U}_0 \,\dot\cup\, \mathcal{U}_1 \subseteq \mathscr{S}$ such that $\mathcal{U}_0$ contains no cliques and $\mathcal{U}_1$ is a union of cliques. Observe that we may assume $\mathcal{U}_0 = \emptyset$. In fact, if $\mathcal{U}_0$ is not empty, we can replace R by $R_{\mathcal{U}_0}$ and $\mathcal{U}$ by $\mathcal{U}_1$ since $R_{\mathcal{U}_0}$ is a tame hereditary algebra and $\mathcal{U}_1$ can be seen as a set of simple regular $R_{\mathcal{U}_0}$-modules.

Now, we suppose $\mathcal{U}_0 = \emptyset$, that is, $\mathcal{U}$ is a union of cliques. Let $\mathcal{V}$ be a maximal subset of $\mathscr{S}$ with respect to the following properties: $\mathcal{V} \cap \mathcal{U} = \emptyset$ and $\mathcal{V}$ contains no cliques. In other words, from each clique $\mathcal{C}$ not contained in $\mathcal{U}$, we choose $c(\mathcal{C}) - 1$ elements, and let $\mathcal{V}$ be the union of all these elements. Clearly, the choice of $\mathcal{V}$ is not unique in general.

Let $\mathcal{W} := \mathcal{U} \,\dot\cup\, \mathcal{V}$, and let $\mathcal{U}_{>1}$ be the union of all cliques $\mathcal{C}_{i\in I}$ in $\mathcal{U}$ of rank greater than one, where I is a finite set. We choose $c(\mathcal{C}_i) - 1$ elements from each $\mathcal{C}_i$ for $i \in I$, and let $\mathcal{V}'$ be the set consisting of all of these elements. Now, we define $\mathcal{L} := \mathcal{V} \cup \mathcal{V}'$ and write $\mathcal{W} = \mathcal{L} \,\dot\cup\, \mathcal{M}$.

We claim that statement (1) holds true. Indeed, it follows from Lemma 4.2 (1) that $R_{\mathcal{L}}$ is a tame hereditary algebra such that all cliques of $R_{\mathcal{L}}$ consist of only one simple regular module. This means that $R_{\mathcal{L}}$ has exactly two isomorphism classes of simple modules. By Lemma 4.2 (3), we have

$$R_{\mathcal{W}} = (R_{\mathcal{L}})_{\overline{\mathcal{M}}}$$

with $\overline{\mathcal{M}} := \{R_{\mathcal{L}} \otimes_R L \mid L \in \mathcal{M}\}$. Thus, setting $\Lambda := R_{\mathcal{L}}$ and $\mathcal{S} := \overline{\mathcal{M}}$, we get statement (1).

In the following, we shall show statement (2). Note that $\mathcal{V}$ contains no cliques. Thus, it follows from Lemma 4.2 (1) that $R_{\mathcal{V}}$ is a finite-dimensional tame hereditary k-algebra and $R_{\mathcal{V}}/R$ is a finitely presented R-module. By Corollary 4.5 (1), $R_{\mathcal{W}}/R_{\mathcal{U}} \simeq R_{\mathcal{U}} \otimes_R (R_{\mathcal{V}}/R)$ as $R_{\mathcal{U}}$-R-bimodules. This implies that $R_{\mathcal{W}}/R_{\mathcal{U}}$ is a finitely presented $R_{\mathcal{U}}$-module, and therefore so are the $R_{\mathcal{U}}$-modules $R_{\mathcal{W}}$ and T. Hence T is a classical $R_{\mathcal{U}}$-module. □

As a consequence of Lemma 4.10, we obtain the following result which describes $R_{\mathcal{U}}$ (up to derived equivalence) by a triangular matrix ring with the rings in the diagonal being relatively simple.

Corollary 4.11. *Suppose that $\mathcal{U} \subseteq \mathscr{S}$ is a union of cliques $\mathcal{C}_{i\in I}$ with I an index set. Let $\mathcal{V}$ be a maximal subset of $\mathscr{S}$ such that $\mathcal{V} \cap \mathcal{U} = \emptyset$ and $\mathcal{V}$ contains no cliques, and let $\mathscr{C}(\mathcal{V}) = \dot\bigcup_{j\in J} \mathcal{C}_j$ with J an index set. Define $\mathcal{W} := \mathcal{U} \cup \mathcal{V}$ and $T_{\mathcal{U}} := R_{\mathcal{U}} \oplus R_{\mathcal{U}}/R$. Then the following statements hold true:*

(1) *There is a canonical ring isomorphism:*

$$\mathrm{End}_R(T_{\mathcal{U}}) \simeq \begin{pmatrix} R_{\mathcal{U}} & \mathrm{Hom}_R(R_{\mathcal{U}}, R_{\mathcal{U}}/R) \\ 0 & \mathrm{End}_R(R_{\mathcal{U}}/R) \end{pmatrix}.$$

(2) *$R_{\mathcal{U}}$ is derived equivalent to the triangular matrix ring*

$$\mathrm{End}_{R_{\mathcal{U}}}(R_{\mathcal{W}} \oplus R_{\mathcal{W}}/R_{\mathcal{U}}) = \begin{pmatrix} R_{\mathcal{W}} & \mathrm{Hom}_{R_{\mathcal{U}}}(R_{\mathcal{W}}, R_{\mathcal{W}}/R_{\mathcal{U}}) \\ 0 & \mathrm{End}_{R_{\mathcal{U}}}(R_{\mathcal{W}}/R_{\mathcal{U}}) \end{pmatrix}$$

such that:

(a) *$R_{\mathcal{W}}$ is the universal localization $\Lambda_{\mathcal{S}}$ of a finite-dimensional tame hereditary k-algebra Λ, which has two isomorphism classes of simple modules, at a set $\mathcal{S}$ of simple regular Λ-modules,*

(b) *$\mathrm{End}_{R_{\mathcal{U}}}(R_{\mathcal{W}}/R_{\mathcal{U}})$ is Morita equivalent to $\prod_{j\in J} T_{c(\mathcal{C}_j)-1}(\mathrm{End}_R(V_j))$, where $V_j \in \mathcal{C}_j$ is a fixed element for each $j \in J$, and $T_n(A)$ stands for the $n \times n$ upper triangular matrix ring over a ring A.*

Proof. Clearly, (1) follows from $\lambda : R \to R_{\mathcal{U}}$ being a ring epimorphism and from $\mathrm{Hom}_R(R_{\mathcal{U}}/R, R_{\mathcal{U}}) = 0$ (see Lemma 2.10). As to (2), we first show statement (b). In fact, by the proof of Lemma 4.10, we know $R_{\mathcal{W}}/R_{\mathcal{U}} \simeq R_{\mathcal{U}} \otimes_R (R_{\mathcal{V}}/R)$ as $R_{\mathcal{U}}$-R-bimodules. Since $\mathcal{V} \subseteq \mathcal{U}^+$, we have $R_{\mathcal{U}} \otimes_R (R_{\mathcal{V}}/R) \simeq R_{\mathcal{V}}/R$ as R-modules by Lemma 4.1, and therefore $R_{\mathcal{W}}/R_{\mathcal{U}} \simeq R_{\mathcal{V}}/R$ as R-modules. This implies that

$$\mathrm{End}_{R_{\mathcal{U}}}(R_{\mathcal{W}}/R_{\mathcal{U}}) \simeq \mathrm{End}_R(R_{\mathcal{W}}/R_{\mathcal{U}}) \simeq \mathrm{End}_R(R_{\mathcal{V}}/R).$$

Now, we define $m_j := c(\mathcal{C}_j)$ for each $j \in J$. Then it follows from Lemmas 4.4 (2) and 4.3 (2) that

$$R_{\mathcal{V}}/R \simeq \bigoplus_{j\in J} \bigoplus_{i=1}^{m_j-1} U_{i,j}[m_j - i]^{(\delta_{i,j})},$$

where $\delta_{i,j} > 0$ and $\mathcal{V} \cap \mathcal{C}_j = \{U_{i,j} \mid 1 \le i < m_j\}$ such that $U_{i+1,j} = \tau^- U_{i,j}$ for all $1 \le i < m_j - 1$. Further, for a fixed $j \in J$, we have an exact sequence

$$0 \longrightarrow U_{i,j} \longrightarrow U_{i,j}\,[m_j - i] \xrightarrow{\gamma} U_{i+1,j}[m_j - i - 1] \longrightarrow 0$$

of R-modules with $1 \le i < m_j - 1$. Since

$$\mathrm{Hom}_R(U_{i,j}[m_j - i], U_{i,j}) = \mathrm{Ext}^1_R(U_{i,j}[m_j - i], U_{i,j}) = 0$$

and since

$$\mathrm{Hom}_R(U_{i,j}, U_{i+1,j}[m_j - i - 1]) = 0,$$

we see that γ induces isomorphisms

$$\begin{aligned} \mathrm{End}_R(U_{i,j}[m_j - i]) &\simeq \mathrm{Hom}_R(U_{i,j}[m_j - i], U_{i+1,j}[m_j - i - 1]) \\ &\simeq \mathrm{End}_R(U_{i+1,j}[m_j - i - 1)]). \end{aligned}$$

Moreover,

$$\mathrm{Hom}_R(U_{r,j}[m_j - r], U_{s,j}[m_j - s]) = 0$$

for $1 \leq s < r \leq m_j - 1$. Hence

$$\mathrm{End}_R\left(\bigoplus_{i=1}^{m_j-1} U_{i,j}[m_j - i]\right) \simeq T_{m_j-1}(\mathrm{End}_R(V_j)),$$

where V_j is a fixed element of $\mathcal{C}_j$ with $j \in J$. Note that, up to isomorphism, $\mathrm{End}_R(V_j)$ is independent of the choice of elements of $\mathcal{C}_j$. Thus $\mathrm{End}_{R_{\mathcal{U}}}(R_{\mathcal{W}}/R_{\mathcal{U}})$ is Morita equivalent to $\prod_{j\in J} T_{m_j-1}(\mathrm{End}_R(V_j))$ since there is no non-trivial homomorphism between two different tubes.

Note that the other conclusions in (2) are consequences of Lemma 4.10 and properties of injective ring epimorphisms (see also [8, Lemma 6.4 (2)]). This completes the proof. □

Thus, by Corollary 4.11 (2), the consideration of the derived category $\mathscr{D}(R_{\mathcal{U}})$ needs first to understand universal localizations of tame hereditary algebras with two isomorphism classes of simple modules, at simple regular modules. If k is an algebraically closed field, then each tame hereditary algebra with two isomorphism classes of simple modules is Morita equivalent to the Kronecker algebra. So, in the next subsection, we shall focus our attention on the universal localizations of the Kronecker algebra.

4.3 Kronecker algebra

In this subsection, we shall consider a particular tame hereditary algebra, the Kronecker algebra. The results obtained here will serve again as a preparation for the discussion of stratifications of derived module categories in the next section.

Throughout this subsection, k is a field and R is the Kronecker algebra $\left(\begin{smallmatrix} k & k^2 \\ 0 & k \end{smallmatrix}\right)$, where the k-k-bimodule structure of k^2 is given by

$$a(b,c)d = (abd, acd)$$

with $a, b, c, d \in k$. It is known that R is isomorphic to the path algebra of the quiver

$$Q : 2 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 1,$$

and that R-Mod (respectively, R-mod) is equivalent to the category of representations (respectively, finite-dimensional representations) of Q over k.

In this subsection, we denote by V the representation

$$k \underset{1}{\overset{0}{\rightrightarrows}} k.$$

By [8, Section 8], we have $R_V = M_2(k[x])$, and the universal localization map $\lambda : R \to R_V$ is given by

$$\begin{pmatrix} a & (c,d) \\ 0 & b \end{pmatrix} \longmapsto \begin{pmatrix} a & c+dx \\ 0 & b \end{pmatrix}$$

for $a, b, c, d \in k$. In particular, the restriction functor $\lambda_* : R_V\text{-Mod} \to R\text{-Mod}$ induced by λ is fully faithful. Let

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in R_V.$$

Clearly, the tensor functor $R_V e \otimes_{k[x]} - : k[x]\text{-Mod} \to R_V\text{-Mod}$ is an equivalence. Now, we define $F : k[x]\text{-Mod} \to R\text{-Mod}$ to be the composition of the functors $R_V\, e \otimes_{k[x]} -$ and λ_*. Then F is a fully faithful exact functor, and sends each $k[x]$-module M to the representation

$$M \overset{1}{\underset{x}{\rightrightarrows}} M.$$

Moreover, we have the following result.

Lemma 4.12 ([25, Theorem 4]). *The functor F induces an equivalence between the category of finite-dimensional $k[x]$-modules and the category of finite-dimensional regular R-modules with regular composition factors not isomorphic to V.*

Let $\mathcal{P}$ be the set of all monic irreducible polynomials in $k[x]$. For each polynomial $p(x) \in \mathcal{P}$, we denote by $k_{p(x)}$ the extension field $k[x]/(p(x))$ of k, and by $V_{p(x)}$ the representation

$$k_{p(x)} \overset{1}{\underset{x}{\rightrightarrows}} k_{p(x)},$$

which is the image of $k_{p(x)}$ under F. Since simple $k[x]$-modules are parameterized by monic irreducible polynomials, it follows from Lemma 4.12 that

$$\mathcal{S} := \{V\} \cup \{V_{p(x)} \mid p(x) \in \mathcal{P}\}$$

is a complete set of isomorphism classes of simple regular R-modules. If k is algebraically closed, then $\mathcal{P} = \{x - a \mid a \in k\}$, and therefore $\mathcal{S}$ can be identified with the projective line $\mathbb{P}^1(k)$.

The following corollary describes the endomorphism algebras of Prüfer modules over the Kronecker algebra.

Corollary 4.13. *Let t be a variable and $p(x) \in \mathcal{P}$. Then there are isomorphisms of rings:*

$$\mathrm{End}_R(V[\infty]) \simeq k[[t]] \quad \textit{and} \quad \mathrm{End}_R(V_{p(x)}[\infty]) \simeq k_{p(x)}[[t]].$$

Proof. Recall that, for any simple regular R-module U, we have

$$\operatorname{End}_R(U[\infty]) \simeq \varprojlim_n \operatorname{End}_R(U[n])$$

as rings. If $U = V$, then $\operatorname{End}_R(U[n]) \simeq k[t]/(t^n)$ for any $n > 0$, and therefore

$$\operatorname{End}_R\big(U[\infty]\big) \simeq \varprojlim_n k[t]/(t^n) \simeq k[[t]].$$

Suppose $U = V_{p(x)}$. It follows from Lemma 4.12 that $U[n] \simeq F(k[x]/(p(x)^n))$ as R-modules and that $\operatorname{End}_R(U[n]) \simeq \operatorname{End}_{k[x]}(k[x]/(p(x)^n)) \simeq k[x]/(p(x)^n)$ for any $n > 0$. Thus

$$\operatorname{End}_R(U[\infty]) \simeq \varprojlim_n k[x]/(p(x)^n).$$

This implies that $\operatorname{End}_R(U[\infty])$ is a complete commutative discrete valuation ring (see Lemma 3.1 (5)), and therefore it is a regular ring of Krull dimension 1. Recall that a regular ring is by definition a commutative noetherian ring of finite global dimension. For regular rings, the global dimension agrees with the Krull dimension.

It remains to prove

$$\varprojlim_n k[x]/(p(x)^n) \simeq k_{p(x)}[[t]].$$

Actually, this is a straightforward consequences of the following classical result (see [12, Theorem 15] for details):

Let S be a complete regular local ring of Krull dimension m with the residue class field K. If S contains a field, then S is isomorphic to the formal power series ring $K[[t_1, \dots, t_m]]$ over K in variables $t_1, \dots, t_m$.

Hence

$$\operatorname{End}_R(U[\infty]) \simeq \varprojlim_n k[x]/(p(x)^n) \simeq k_{p(x)}[[t]],$$

which finishes the proof. □

Finally, we prove the following lemma as the last preparation for the proof of Corollary 1.2.

Lemma 4.14. *Let Δ be a subset of $\mathcal{P}$, and let $\mathcal{U} := \{V\} \cup \{V_{p(x)} \mid p(x) \in \Delta\}$. Suppose that D is the smallest subring of the fraction field $k(x)$ of $k[x]$ containing both $k[x]$ and $\frac{1}{p(x)}$ with all $p(x) \in \Delta$. Then $R_{\mathcal{U}} \simeq M_2(D)$, the 2×2 matrix ring over D. In particular, $R_{\mathcal{U}}$ is Morita equivalent to the Dedekind integral domain D.*

Proof. Define $\mathcal{W} := \{R_V \otimes_R V_{p(x)} \mid p(x) \in \Delta\}$. Then $R_{\mathcal{U}} = (R_V)_{\mathcal{W}}$ by Lemma 4.2 (3). Recall that $R_V = M_2(k[x])$ and $\lambda : R \to R_V$ is the universal localization of R at V. On the one hand, for each $p(x) \in \Delta$, it follows from

$$V_{p(x)} = F(k_{p(x)}) = \lambda_*(R_V e \otimes_{k[x]} k_{p(x)})$$

that

$$R_V \otimes_R V_{p(x)} \simeq V_{p(x)} = R_V e \otimes_{k[x]} k_{p(x)} = \begin{pmatrix} k_{p(x)} \\ k_{p(x)} \end{pmatrix}$$

as R_V-modules. On the other hand, by [8, Corollary 3.5], Morita equivalences preserve universal localizations. Consequently, we have

$$R_{\mathcal{U}} = (M_2(k[x]))_{\mathcal{W}} \simeq M_2(k[x]_\Theta)$$

with $\Theta := \{k_{p(x)} \mid p(x) \in \Delta\} \subseteq k[x]$-Mod. Now, one may readily see that $k[x]_\Theta$ coincides with the localization of $k[x]$ at the smallest multiplicative subset of $k[x]$ containing $\{p(x) \mid p(x) \in \Delta\}$, which is exactly the ring D defined in Lemma 4.14. Since $k[x]$ is a Dedekind integral domain and since localizations of Dedekind integral domains are again Dedekind integral domains, we see that D is a Dedekind integral domain. As a result, we have $R_{\mathcal{U}} \simeq M_2(D)$. This completes the proof. □

Remark. (1) If k is an algebraically closed field, then, for any simple regular R-module U, we can choose an automorphism $\sigma : R \to R$ such that the induced functor $\sigma_* : R$-Mod $\to R$-Mod by σ is an equivalence with $\sigma_*(U) \simeq V$. This implies that, up to isomorphism, Lemma 4.14 provides a complete description of $R_{\mathcal{V}}$ for any subset $\mathcal{V}$ of $\mathscr{S}$. In particular, $R_{\mathcal{V}}$ is Morita equivalent to a Dedekind integral domain.

(2) If we localize R at all non-isomorphic simple regular modules $\mathscr{S}$ which is indexed by all monic irreducible polynomials, then, by Lemma 4.14, we have $R_{\mathscr{S}} \simeq M_2(k(x))$ since the smallest subring containing the inverses of all irreducible polynomials $p(x)$ is just $k(x)$.

5 Proof of the main results

In this section, we prove our main results, Theorem 1.1 and Corollary 1.2, in this paper.

5.1 Proof of Theorem 1.1

Recall that

$$B := \mathrm{End}_R(R_{\mathcal{U}} \oplus R_{\mathcal{U}}/R)$$

and

$$S := \mathrm{End}_R(R_{\mathcal{U}}/R).$$

By Proposition 2.11, there is a recollement of derived module categories

$$\begin{tikzcd} \mathscr{D}(S_\Sigma) \arrow[r] & \mathscr{D}(B) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}(R) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd} \tag{$\dagger$}$$

where S_Σ is the universal localization of S at $\Sigma := \{S \otimes_R f_U \mid U \in \mathcal{U}\}$.

Note that I is an index set such that $\{\mathcal{C}_i\}_{i \in I}$ is the set of all cliques contained in $\mathcal{U}$. It follows from Corollary 4.9 that S_Σ is Morita equivalent to the adèle ring $\mathbb{A}_\mathcal{U}$ in Theorem 1.1. So, if we substitute $\mathscr{D}(S_\Sigma)$ by $\mathscr{D}(\mathbb{A}_\mathcal{U})$ in ($\dagger$), then we obtain the desired recollement of derived module categories in Theorem 1.1:

$$\begin{tikzcd} \mathscr{D}(\mathbb{A}_\mathcal{U}) \arrow[r] & \mathscr{D}(B) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}(R). \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

This completes the proof of the first part of Theorem 1.1.

As for the second part, we note that if k is algebraically closed, then, for each clique $\mathcal{C}$ of R, the rings $D(\mathcal{C})$ and $Q(\mathcal{C})$ are isomorphic to $k[[x]]$ and $k((x))$ by Lemma 3.1 (5), respectively. Now, combining this fact with the first part of Theorem 1.1, we know that $\mathbb{A}_\mathcal{U}$ is isomorphic to $\mathbb{A}_I$. This finishes the proof. □

In the following, we give two consequences of Theorem 1.1.

If we take $\mathcal{U} = \mathscr{S}$, then the module $T_\mathscr{S} := R_\mathscr{S} \oplus R_\mathscr{S}/R$ is a Reiten–Ringel tilting R-module (see [26] and [3, Example 1.3]). Actually, this module is of the form

$$G^{(n)} \oplus \bigoplus_{U \in \mathscr{S}} U[\infty]^{(\delta_U)},$$

where G is the unique generic R-module, and where

$$n = \dim G_{\mathrm{End}_R(G)} \quad \text{and} \quad \delta_U = \dim_{\mathrm{End}_R(U)} \mathrm{Ext}^1_R(U, R)$$

for $U \in \mathscr{S}$ (see [3, Proposition 1.10]). Recall that $\mathscr{S}$ is parameterized by the projective line $\mathbb{P}^1(k)$ if k is algebraically closed. As a direct consequence of Theorem 1.1, we have the following corollary.

Corollary 5.1. *If k is an algebraically closed field and T is the Reiten–Ringel tilting R-module $T_\mathscr{S}$, then there is a recollement*

$$\begin{tikzcd} \mathscr{D}(\mathbb{A}_{\mathbb{P}^1(k)}) \arrow[r] & \mathscr{D}(\mathrm{End}_R(T)) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}(R). \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

Now, let Δ be a subset of $\mathscr{P}$, the set of all monic irreducible polynomials in $k[x]$, and let $\mathcal{U} := \{V\} \cup \{V_{p(x)} \mid p(x) \in \Delta\}$ (see Section 4.3 for notation). We define the Δ-adèle ring of $k[x]$ as follows:

$$\mathbb{A}(\Delta) := k((t)) \times \Big\{(\theta_{p(x)})_{p(x)\in\Delta} \in \prod_{p(x)\in\Delta} k_{p(x)}((t)) \mid \theta_{p(x)} \in k_{p(x)}[[t]] \text{ for almost all } p(x) \in \Delta\Big\}.$$

Combining Theorem 1.1 with Corollary 4.13, we get the following result.

Corollary 5.2. *Suppose that R is the Kronecker algebra. Let B be the endomorphism algebra of the tilting R-module $R_{\mathcal{U}} \oplus R_{\mathcal{U}}/R$. Then there is a recollement of derived categories:*

$$\begin{tikzcd} \mathscr{D}(\mathbb{A}(\Delta)) \arrow[r] & \mathscr{D}(B) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}(R). \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

5.2 Proof of Corollary 1.2

We first recall the definition of stratifications of derived categories of rings.

Following [1, Sections 4 and 5], the derived module category $\mathscr{D}(A)$ of a ring A is called *derived simple* if it is not a non-trivial recollement of any derived categories of rings. A *stratification* of $\mathscr{D}(A)$ of a ring A by derived categories of rings is defined to be a sequence of iterated recollements of the following form: a recollement of A, if it is not derived simple,

$$\begin{tikzcd} \mathscr{D}(A_1) \arrow[r] & \mathscr{D}(A) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}(A_2), \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

a recollement of the ring A_1, if it is not derived simple,

$$\begin{tikzcd} \mathscr{D}(A_{11}) \arrow[r] & \mathscr{D}(A_1) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}(A_{12}), \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

and a recollement of the ring A_2, if it is not derived simple,

$$\begin{tikzcd} \mathscr{D}(A_{21}) \arrow[r] & \mathscr{D}(A_2) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}(A_{22}), \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

and recollements of the rings A_{ij} with $1 \le i, j \le 2$, if they are not derived simple, and so on, until one arrives at derived simple rings at all positions, or continues to infinitum. All the derived simple rings appearing in this procedure are called *composition factors* of the stratification. The cardinality of the set of all composition factors (counting the multiplicity) is called the *length* of the stratification. If the length of a stratification is finite, we say that this stratification is *finite* or *of finite length*.

Proof of Corollary 1.2. Under the assumption that k is an algebraically closed field, the following two facts are known:

(a) For any simple regular R-module U, the algebras $\mathrm{End}_R(U)$ and $\mathrm{End}_R(U[\infty])$ are isomorphic to k and $k[[x]]$, respectively. This is due to Lemma 3.1 (5).

(b) Each tame hereditary algebra with two isomorphism classes of simple modules is Morita equivalent to the Kronecker algebra.

One the one hand, it follows from Theorem 1.1 that $\mathscr{D}(B)$ is stratified by $\mathscr{D}(R)$ and $\mathscr{D}(\mathbb{A}_I)$, where $I = \{1, 2, \dots, s\}$ is an index set of the cliques contained in $\mathcal{U}$, and the ring $\mathbb{A}_I$ is defined in the Introduction. Since $\mathcal{U}$ is a union of finitely many cliques of $\mathscr{S}$, we know that $\mathbb{A}_I$ is equal to $k((x))^s$, the direct product of s copies of $k((x))$. Thus $\mathscr{D}(\mathbb{A}_I)$ has a stratification by derived module categories with s copies of the composition factor $k((x))$. Note that $\mathscr{D}(R)$ has a stratification by derived module categories with r copies of the composition factor k, where r is the number of non-isomorphic simple R-modules. Thus $\mathscr{D}(B)$ has a stratification of length $r + s$ with the composition factor k of multiplicity r, and the composition factor $k((x))$ of multiplicity s.

On the other hand, by Corollary 4.11, we know that $\mathscr{D}(B)$ can be stratified by $\mathscr{D}(R_{\mathcal{W}})$, $\mathscr{D}(\mathrm{End}_{R_{\mathcal{U}}}(R_{\mathcal{W}}/R_{\mathcal{U}}))$ and $\mathscr{D}(\mathrm{End}_R(R_{\mathcal{U}}/R))$, where $\mathcal{W}$ is defined in Corollary 4.11. Here, we have used the known fact that every 2×2 triangular matrix ring yields a recollement of derived module categories of the rings in the diagonal. In the following, we shall calculate composition factors of $\mathscr{D}(B)$.

First, it follows from Corollary 4.11 (2) and Lemma 4.14 that $R_{\mathcal{W}}$ is Morita equivalent to a Dedekind integral domain and that $\mathrm{End}_{R_{\mathcal{U}}}(R_{\mathcal{W}}/R_{\mathcal{U}})$ is Morita equivalent to $\prod_{j \in J} T_{c(\mathcal{C}_j)-1}(k)$. It is known from [1, Proposition 4.11 (3)] that every Dedekind domain is derived simple. Thus $R_{\mathcal{W}}$ contributes one composition factor to $\mathscr{D}(B)$. It is easy to see that $\mathscr{D}(T_{c(\mathcal{C}_j)-1}(k))$ has a stratification with $c(\mathcal{C}_j) - 1$ copies of the composition factor k. Thus $\mathscr{D}(\mathrm{End}_{R_{\mathcal{U}}}(R_{\mathcal{W}}/R_{\mathcal{U}}))$ admits a stratification with $\sum_{j \in J}(c(\mathcal{C}_j) - 1)$ copies of k.

Second, combining Lemma 4.8 (1) with Corollary 3.3, we can conclude that $\mathrm{End}_R(R_{\mathcal{U}}/R)$ is Morita equivalent to $\prod_{i=1}^{s} \Gamma(c(\mathcal{C}_i))$, where $\mathcal{U}$ is assumed to be a union of s cliques $\mathcal{C}_i$ with $1 \le i \le s$, and where $\Gamma(m)$ is defined in Corollary 3.3 for each positive integer m. Note that the canonical inclusion f of $\Gamma(m)$ into $M_m(k[[x]])$ is a ring epimorphism and that $M_m(k[[x]])$ is finitely generated and projective as a left $\Gamma(m)$-module. Let $E_{m,m}$ be the diagonal matrix with 1 in the (m, m)-entry, and 0 in other entries. Then the sequence

$$0 \longrightarrow \Gamma(m) \xrightarrow{f} M_m(k[[x]]) \longrightarrow \mathrm{Coker}(f) \longrightarrow 0$$

is an $\mathrm{add}(\Gamma(m)E_{m,m})$-split sequence in the category of all left $\Gamma(m)$-modules (see [30, Lemma 3.1]), and therefore we see that $\mathrm{End}_{\Gamma(m)}(\Gamma(m) \oplus M_m(k[[x]]))$ and

$\mathrm{End}_{\Gamma(m)}(M_m(k[[x]]) \oplus \mathrm{Coker}(f))$ are derived equivalent by [19, Theorem 1.1]. Clearly, the former ring is Morita equivalent to $\Gamma(m)$ and the latter is Morita equivalent to $\mathrm{End}_{\Gamma(m)}(M_m(k[[x]])E_{m,m} \oplus \mathrm{Coker}(f))$. Hence $\Gamma(m)$ is derived equivalent to $\mathrm{End}_{\Gamma(m)}(M_m(k[[x]])E_{m,m} \oplus \mathrm{Coker}(f))$ which is just the following matrix ring:

$$\begin{pmatrix} k[[x]] & k & \cdots & k \\ 0 & k & \ddots & \vdots \\ \vdots & \ddots & \ddots & k \\ 0 & \cdots & 0 & k \end{pmatrix}_{m\times m}.$$

For a general consideration of derived equivalences between subrings of matrix rings, we refer to [10]. Thus, we know that $\mathscr{D}(\Gamma(m))$ has a stratification with the composition factor $k[[x]]$ of multiplicity 1, and the composition factor k of multiplicity $m-1$. Therefore, $\mathscr{D}(\mathrm{End}_R(R_{\mathcal{U}}/R))$ admits a stratification with the following composition factors: s copies of $k[[x]]$ and $\sum_{i=1}^{s}(c(\mathcal{C}_i)-1)$ copies of k.

Finally, by summarizing up the above discussions, we conclude that $\mathscr{D}(B)$ has a stratification of length $r+s-1$ with the following composition factors: $r-2$ copies of k, s copies of $k[[x]]$ and one copy of a fixed Dedekind domain. Here, we use the well-known fact

$$\sum_{\mathcal{C}}(c(\mathcal{C})-1) = r-2,$$

where $\mathcal{C}$ runs over all of the cliques of R. Thus the proof is completed. □

Let us end this section by mentioning the following questions suggested by our results.

(1) For tilting modules of the form $R_{\mathcal{U}} \oplus R_{\mathcal{U}}/R$, we have provided a recollement of the derived categories of their endomorphism algebras. It would be interesting to have a similar result for tilting modules of other types described in [3].

(2) In Corollary 1.2, it would be nice to know that $\mathscr{D}(B)$ has no other composition factors (up to derived equivalence) except the ones displayed there.

(3) It would be interesting to generalize the results in this paper to hereditary orders.

(4) Suppose that the derived category $\mathscr{D}(A)$ of a ring A admits a stratification of finite length by derived categories of rings. Does $\mathscr{D}(A)$ then have only finitely many derived composition factors (up to derived equivalence)?

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