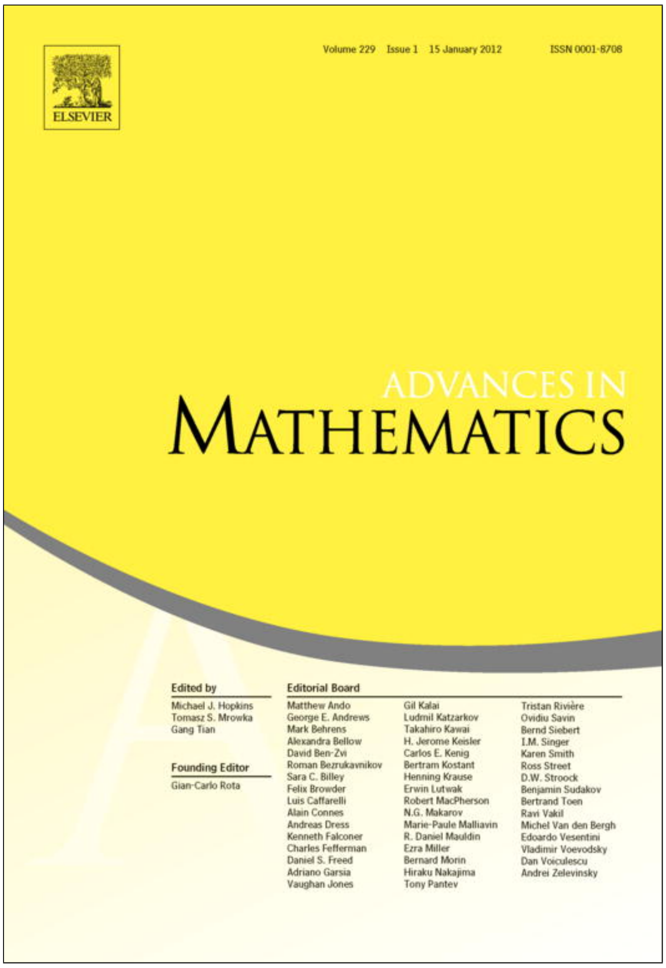
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Affine cellular algebras

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Abstract

Graham and Lehrer have defined cellular algebras and developed a theory that allows in particular to classify simple representations of finite dimensional cellular algebras. Many classes of finite dimensional algebras, including various Hecke algebras and diagram algebras, have been shown to be cellular, and the theory due to Graham and Lehrer successfully has been applied to these algebras.

We will extend the framework of cellular algebras to algebras that need not be finite dimensional over a field. Affine Hecke algebras of type A and infinite dimensional diagram algebras like the affine Temperley–Lieb algebras are shown to be examples of our definition. The isomorphism classes of simple representations of affine cellular algebras are shown to be parameterised by the complement of finitely many subvarieties in a finite disjoint union of affine varieties. In this way, representation theory of non-commutative algebras is linked with commutative algebra. Moreover, conditions on the cell chain are identified that force the algebra to have finite global cohomological dimension and its derived category to admit a stratification; these conditions are shown to be satisfied for the affine Hecke algebra of type A if the quantum parameter is not a root of the Poincaré polynomial.

MSC: 20C08; 16G30; 16G10; 16P40; 18G20; 16E10; 16D60; 81R50

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1. Introduction

Given an algebra or a group, a fundamental question is to find a parameter set of the simple (= irreducible) representations up to isomorphism. One of the most successful general techniques available in the case of finite dimensional algebras relies on cellular structures in the sense of Graham and Lehrer's definition of *cellular algebras* [7]. A (finite dimensional) cellular algebra has, by definition, a finite chain of cell ideals, and a subset of the index set of this chain is a natural parameter set of the isomorphism classes of simple representations. Once a cellular structure is known, it is a problem of linear algebra to identify this subset. Moreover, the problem to describe the simple modules, their dimensions or characters, also is reduced to a linear algebra problem, which, however, is hard in general. Cellular structures have been found and successfully applied for large classes of algebras, including group algebras of symmetric groups, their Hecke algebras, Temperley–Lieb algebras, Brauer algebras and other diagram algebras; other important classes of algebras such as generalised Schur algebras of reductive algebraic groups and blocks of the Bernstein–Gelfand–Gelfand category $\mathcal{O}$ have been studied by using properties such as quasi-heredity that are even stronger than cellular. Apart from providing a crucial method for finding parameter sets of simples, cellular algebras also have interesting homological features. In [19] we have given an explicit characterisation when a cellular algebra has finite global (= cohomological) dimension, and we also have shown the Cartan determinant conjecture to hold true for cellular algebras.

The aim of this article is to extend the theory of cellular algebras from finite dimensional algebras over a field to associative unitary algebras (not necessarily finitely generated) over a principal ideal domain. Our definition of affine cellular algebras A requires the existence of a finite cell chain. The subquotients (= cells) in this chain may be large over the given ground ring, but each subquotient is required to have a special form that makes it look similar, in a precise sense (Definition 2.1 and Section 2.2), to a matrix algebra over some commutative ring that depends on the given subquotient. Therefore, an affine cellular algebra has associated with it an asymptotic algebra (Definition 3.13) that is a finite direct sum of matrix rings, in general over varying commutative ground rings B_j. The problem of finding a parameter set of simple A-representations will be solved in complete generality (Theorem 3.12). The parameter set is contained, as the complement of the union of finitely many affine subvarieties, in the disjoint union $\bigcup_{j=1}^{n} \operatorname{MaxSpec}(B_j)$ of affine varieties, which form the parameter sets of the matrix ring summands of the asymptotic algebra, where $\operatorname{MaxSpec}(B_j)$ denotes the spectrum of B_j consisting of all maximal ideals in B_j. In particular, the parameter set is completely determined by the finitely many commutative algebras B_j and finitely many elements in each B_j. The methods we develop in this context are based on viewing subquotients in the cell chain as rings without units and comparing them with matrix rings. If the ground rings of the matrix rings are principal ideal domains, then we can in addition use linear algebra techniques and get a more precise description of the simple representations (Theorem 3.16).

Our definition applies to infinite dimensional diagram algebras like the affine Temperley–Lieb algebras (and presumably many other examples of this kind), but the most important example of affine cellular algebras to be discussed in this article is Lusztig's extended affine Hecke algebra of type A (Theorem 5.7). More precisely, our results will cover the extended affine Hecke algebras associated with general linear or special linear groups. Here, we are using Lusztig's cell theory for affine Hecke algebras and in particular N.H. Xi's proof [32] of Lusztig's conjecture about the so-called based rings.

Having shown that our definition of affine cellular algebra covers examples of interest in various contexts of mathematics and mathematical physics and having demonstrated its use for solving the problem of finding parameter sets of simple representations, we turn to homological aspects. In Theorem 4.1 we show that the parameter set of simple A-representations equals the full variety $\bigcup_{j=1}^{n} \mathrm{MaxSpec}(B_j)$ if and only if all the sections in the cell chain are idempotent. This is implied by the existence of a set of idempotents generating the cell ideals, providing us with a condition on cell chains that naturally extends the notion of heredity chains. Then we show that this condition implies finiteness of global (= cohomological) dimension (Theorem 4.4) – provided the global dimensions of the B_j are finite – and even a stratification of derived module categories. We show that the extended affine Hecke algebra of type A satisfies this condition if the quantum parameter is not a root of the Poincaré polynomial (Theorem 5.8); this includes the case of q not being a root of unity, which is the situation for which Kazhdan and Lusztig have proved the Deligne–Langlands classification [15].

The main results of this article are Theorem 3.12, which provides the classification of simple A-representations for any affine cellular algebra A, and the results of Section 4: Theorems 4.1, 4.3, 4.4 discussing the stronger 'quasi-hereditary' situation and its homological consequences – note that this is the 'generic' case for affine Hecke algebras.

The paper is organised as follows. In Section 2, we give the definition and examples of affine cellular algebras. In Section 3, we prove the main results Theorem 3.12 and Theorem 3.16. In Section 4, we investigate homological properties of affine cellular algebras and prove Theorems 4.1, 4.3 and 4.4. In Section 5, we verify that Lusztig's affine Hecke algebras of type A are affine cellular in the sense of this paper. Moreover, if the ground field has characteristic zero and the Poincaré polynomial does not vanish at the defining parameter, then the affine Hecke algebra of type A has finite global dimension.

Recently, further examples of affine cellular algebras are provided by Guilhot and Miemietz in [12]. Their main result is that any affine Hecke algebra of rank 2 is affine cellular in the sense of Definition 2.1 below.

2. Affine cellular algebras: definition, examples and comparison to existing concepts

We start this section by defining the central concept of this article, affine cellular algebras. Then we illustrate this concept by several examples. Next we interpret the notion of affine cell ideal in terms of what is called generalised matrix rings. Finally, we compare the new concept to existing ones.

First, we fix some notation to be used throughout this article. Let k be a noetherian domain; we will study k-algebras that are associative and have a unit, but on the way some algebras without unit will appear, too. We do not assume the algebras studied to be finitely generated or projective over k. For two k-modules V and W, we denote by τ the switch map: $V \otimes_k W \longrightarrow W \otimes_k V$, $v \otimes w \longmapsto w \otimes v$ for $v \in V$ and $w \in W$. A k-linear anti-automorphism i of a k-algebra A with $i^2 = id_A$ will be called a *k-involution* on A. A commutative k-algebra B is called *affine* if it is a quotient of a polynomial ring $k[x_1, \ldots, x_t]$ in finitely many variables $x_1, \ldots, x_t$ by an ideal I, that is, $B = k[x_1, \ldots, x_t]/I$. Affine commutative rings will come up as ground rings of certain matrix algebras that provide us with the affine spaces containing the parameter sets we are looking for.

2.1. *Definition and examples*

Definition 2.1. Let A be a unitary k-algebra with a k-involution i on A. A two-sided ideal J in A is called an *affine cell ideal* if and only if the following data are given and the following conditions are satisfied:

(1) The ideal J is fixed by i: $i(J) = J$.
(2) There exist a free k-module V of finite rank and an affine commutative k-algebra B with identity and with a k-involution σ such that $\Delta := V \otimes_k B$ is an A-B-bimodule, where the right B-module structure is induced by that of the right regular B-module B_B.
(3) There is an A-A-bimodule isomorphism $\alpha : J \longrightarrow \Delta \otimes_B \Delta'$, where $\Delta' = B \otimes_k V$ is a B-A-bimodule with the left B-structure induced by ${}_B B$ and with the right A-structure via i, that is, $(b \otimes v)a := \tau(i(a)(v \otimes b))$ for $a \in A$, $b \in B$ and $v \in V$, such that the following diagram is commutative:

$$\begin{array}{ccc} J & \xrightarrow{\alpha} & \Delta \otimes_B \Delta' \\ {\scriptstyle i}\downarrow & & \downarrow {\scriptstyle v_1 \otimes b_1 \otimes_B b_2 \otimes v_2 \longmapsto v_2 \otimes \sigma(b_2) \otimes_B \sigma(b_1) \otimes v_1} \\ J & \xrightarrow{\alpha} & \Delta \otimes_B \Delta'. \end{array}$$

The algebra A (with the involution i) is called *affine cellular* if and only if there is a k-module decomposition $A = J_1' \oplus J_2' \oplus \cdots \oplus J_n'$ (for some n) with $i(J_j') = J_j'$ for each j and such that setting $J_j = \bigoplus_{l=1}^{j} J_l'$ gives a chain of two-sided ideals of A: $0 = J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_n = A$ (each of them fixed by i) and for each j ($j = 1, \ldots, n$) the quotient $J_j' = J_j / J_{j-1}$ is an affine cell ideal of A/J_{j-1} (with respect to the involution induced by i on the quotient).

We call this chain a *cell chain* for the affine cellular algebra A. The module Δ will be called a *cell lattice* for the affine cell ideal J.

The subsections of a cell chain often will be called 'cells' or 'layers' of the affine cellular structure.

An affine cellular algebra need not be noetherian although each commutative algebra B appearing in Δ is noetherian (for an example, see Section 3.2 below).

Easy examples. A cellular algebra in the sense of [7] is a trivial example of an affine cellular algebra; here, all the rings B_j are equal to the ground ring k.

Another trivial example is a noetherian domain, which is an affine cellular algebra with respect to any involution.

The field $\mathbb{C}$ of complex numbers with the usual conjugation as an $\mathbb{R}$-involution is not a cellular $\mathbb{R}$-algebra, but it is an affine cellular algebra in the sense of Definition 2.1. In fact, every affine commutative k-algebra is an affine cellular k-algebra with respect to the identity map as a k-involution.

Suppose k is a field and A is a cellular k-algebra with respect to a k-involution i. Then the tensor product $k[x] \otimes_k A$ is an affine cellular k-algebra with respect to the involution $id \otimes_k i$. Suppose $J_1 \subset J_2 \subset \cdots \subset J_m = A$ is a cell chain of A. Then $k[x] \otimes_k J_1 \subset k[x] \otimes_k J_2 \subset \cdots \subset k[x] \otimes_k J_m = k[x] \otimes_k A$ is a cell chain for $k[x] \otimes_k A$. Using the language introduced in the next subsection, the forms ψ_j on the tensor product algebra are given by the forms defining the cellular structure on the algebra A, and all B_j are equal to $k[x]$. In this case, we see that each subquotient $(k[x] \otimes_k J_s)/(k[x] \otimes_k J_{s-1})$ is a free $k[x]$-module.

A non-trivial class of examples of affine cellular algebras will be discussed after the next subsection.

2.2. Ideals and generalised matrix algebras

Now we are collecting some basic properties of an affine cell ideal J in an algebra A. In particular, we derive the basic structure of J when viewed as an algebra (without unit) in itself.

Proposition 2.2. *Let J be an affine cell ideal in a k-algebra A with an involution i. We identify J with $\Delta \otimes_B \Delta' = V \otimes_k B \otimes_k V$. Then:*

(1) *$i(u \otimes b \otimes v) = v \otimes \sigma(b) \otimes u$ for all $u, v \in V$ and $b \in B$.*
(2) *There is a k-linear map $\psi : V \otimes_k V \longrightarrow B$ such that $\sigma(\psi(v, v')) = \psi(v', v)$, and*

$$(u \otimes b \otimes v)\big(u' \otimes b' \otimes v'\big) = u \otimes b\psi\big(v, u'\big)b' \otimes v'$$

for all $u, u', v, v' \in V$ and $b, b' \in B$.

(3) *For any element $a \in A$ and $u \otimes b \otimes v \in J$, we have*

$$a(u \otimes b \otimes v) \in V \otimes_k Bb \otimes_k v, \quad \text{and} \quad (u \otimes b \otimes v)a \in u \otimes_k bB \otimes_k V.$$

In particular, if I is an ideal in B and $u, v \in V$, then $V \otimes_k I \otimes_k v$ is a left ideal in A, and $u \otimes_k I \otimes_k V$ is a right ideal in A.

Proof. (1) is clear from the commutative diagram in Definition 2.1. (3) follows from the fact that α is an A-A-bimodule homomorphism since we can write $a(u \otimes b \otimes v) = a(u \otimes 1 \otimes_B b \otimes v) = (a(u \otimes 1)) \otimes_B (b \otimes v) = f(a, u)b \otimes_k v$, where $f(a, u) \in V \otimes_k B$ and $f(a, u)$ is independent of b and v. So, it remains to verify (2).

Suppose $\{v_j \mid j = 1, 2, \ldots, m\}$ is a k-basis of V. Let us compute $(v_p \otimes_k b \otimes_k v_q)(v_s \otimes_k c \otimes_k v_t)$ where b and c are in B. On the one hand, since $\Delta = V \otimes_k B$ is a left A-module, we have

$$(v_p \otimes_k b \otimes_k v_q)(v_s \otimes_k 1 \otimes_B c \otimes_k v_t) = \sum_j v_j \otimes_k f_j(v_p, v_q, b, v_s)c \otimes_k v_t,$$

where $f_j(v_p, v_q, b, v_s)$ is a function with values in B. On the other hand, since Δ' is a right A-module, we have

$$(v_p \otimes_k b \otimes_B 1 \otimes_k v_q)(v_s \otimes_k c \otimes_k v_t) = \sum_j v_p \otimes_k bg_j(v_q, v_s, c, v_t) \otimes_k v_j,$$

where $g_j(v_p, v_s, c, v_t)$ is a function with values in B. By observing linear independence (over k) of the summands and then comparing the coefficients, we find that

$$\begin{aligned}(v_p \otimes_k b \otimes_k v_q)(v_s \otimes_k c \otimes_k v_t) &= v_p \otimes_k f_p(v_p, v_q, b, v_s)c \otimes_k v_t \\ &= v_p \otimes_k bg_t(v_q, v_s, c, v_t) \otimes_k v_t.\end{aligned}$$

Thus $f_p(v_p, v_q, b, v_s)c = bg_t(v_q, v_s, c, v_t)$ for all $b, c \in B$ and $p, q, s, t \in \{1, 2, \dots, m\}$. In particular, if we take $b = 1$, then $f_p(v_p, v_q, 1, v_s)c = g_t(v_q, v_s, c, v_t)$. This shows that $g_t(v_q, v_s, c, v_t)$ is independent of v_t, thus we may write $g_t(v_q, v_s, c, v_t) = g_t(v_q, v_s, c)$. Similarly, $f_p(v_p, v_q, b, v_s)$ is independent of v_p, and we can write $f_p(v_p, v_q, b, v_s) = f_p(v_q, b, v_s)$. This shows that

$$(v_p \otimes_k b \otimes_k v_q)(v_s \otimes_k c \otimes_k v_t) = v_p \otimes_k bf_p(v_q, 1, v_s)c \otimes_k v_t.$$

If we define $\psi(v_q, v_s) := f_p(v_q, 1, v_s)$, then $\psi(v_q, v_s)$ is a k-bilinear form because the multiplication in J is k-bilinear. Finally, the condition $\sigma(\psi(v, v')) = \psi(v', v)$, relating ψ and σ follows from (1). $\square$

Therefore, the action of A on Δ is given as follows:

$$a(v \otimes_k b) = f(a, v)b, \quad a \in A,\ v \in V,\ b \in B,$$

where $f(a, v) \in V \otimes_k B$ is independent of b. In particular, if $a = w \otimes_k b' \otimes_k u \in J$, then

$$a(v \otimes b) = w \otimes b'\psi(u, v)b.$$

Also, Proposition 2.2 provides a general recipe of constructing affine cell ideals: Given a free k-module V, a (commutative) k-algebra B and a k-bilinear form $\psi : V \otimes_k V \longrightarrow B$, we may define an associative algebra $\mathcal{A}(V, B, \psi)$ as follows: As a k-module, $\mathcal{A}(V, B, \psi) := V \otimes_k B \otimes_k V$. The multiplication on the module is defined by

$$(u \otimes_k b \otimes_k v)\big(u' \otimes_k b' \otimes_k v'\big) = u \otimes b\psi\big(v, u'\big)b' \otimes_k v'$$

for all $u, u', v, v' \in V$, and $b, b' \in B$. Moreover, if B admits a k-involution i such that $i\psi(v, v') = \psi(v', v)$, then $\mathcal{A}(V, B, \psi)$ admits a k-involution j which sends $v \otimes b \otimes w$ to $w \otimes i(b) \otimes v$ for all $v, w \in V$ and $b \in B$.

Apart from the additional requirement concerning the involution i, this construction is exactly that of a *generalised matrix algebra* over the ground ring B. Suppose that B is a k-algebra and fix a natural number n, set $V = k^n$ and choose a k-bilinear form $\psi : V \otimes_k V \longrightarrow B$. Then the generalised matrix algebra $(M_n(B), \psi)$ over B with respect to ψ as a k-space equals the ordinary matrix algebra $M_n(B)$ of $n \times n$ matrices over B, but multiplication is deformed in the following way. We write matrices in $(M_n(B), \psi)$ as sums of 'matrix units' $a \otimes x \otimes b$ with $a, b \in V$, $x \in B$ and put $(a \otimes x \otimes b) \cdot (c \otimes y \otimes d) := a \otimes x\psi(b, c)y \otimes d$. Linearly extending this setting defines an associative and k-linear multiplication on $(M_n(B), \psi)$, which thus becomes a k-algebra, in general without unit. Note that $\mathcal{A}(V, B, \psi) \simeq (M_n(B), \psi)$ and there may not be a B-algebra structure present. This notion goes back to W.P. Brown [3] who used it when studying Brauer algebras (see also [20] for a further use of this concept, using finite dimensional non-commutative algebras as B_j).

Summarising the above discussion, we have the following description of affine cellular algebras.

Proposition 2.3. *Let k be a noetherian domain, and let A be a k-algebra (with a k-involution i). Suppose there is a decomposition:*

$$A = J_1' \oplus J_2' \oplus \cdots \oplus J_n'$$

of k-modules, such that

(1) *each J'_t is invariant under i;*
(2) *the chain $J_0 = 0 \subset J_1 \subset \cdots \subset J_n = A$ with $J_t = \bigoplus_{j=1}^{t} J'_j$ is a chain of ideals in A; and*
(3) *for each $1 \leqslant s \leqslant n$, there is a free k-module V_s of finite rank, an affine commutative k-algebra B_s, a k-bilinear form $\psi_s : V_s \otimes_k V_s \longrightarrow B_s$, and a k-linear involution i_s on B_s, such that the subquotient J_s/J_{s-1} is isomorphic to $\mathcal{A}(V_s, B_s, \psi_s) \simeq \Delta_s \otimes_B \Delta'_s$ as an algebra (not necessarily with identity) and as an A-A-bimodule, and that the induced involution of i on J_s/J_{s-1} is given by $i(v_1 \otimes_k b \otimes_k v_2) = v_2 \otimes_k i_s(b) \otimes_k v_1$ for all $v_1, v_2 \in V$, $b \in B_s$. Then A is an affine cellular algebra.*

Example. Let $A = \begin{pmatrix} k[x] & k[x] \\ xk[x] & k[x] \end{pmatrix}$, where k is a field. Then we define an involution on A by sending $\begin{pmatrix} a & c \\ xb & d \end{pmatrix}$ to $\begin{pmatrix} a & b \\ xc & d \end{pmatrix}$, where a, b, c and d are elements in $k[x]$. Let $J'_1 = \begin{pmatrix} k[x] & k[x] \\ xk[x] & xk[x] \end{pmatrix}$, and $J'_2 = \begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix}$. Then A is a direct sum of $k[x]$-modules J'_1 and J'_2 each of which is invariant under the involution. Let V_1 be a k-space of dimension 2 with basis $\{v_1, v_2\}$. We define a k-bilinear form $\psi_1 : V_1 \otimes_k V_1 \longrightarrow k[x]$ by the matrix $\begin{pmatrix} 1 & 1 \\ x & x \end{pmatrix}$. This defines a generalised matrix ring $\mathcal{A}(V_1, k[x], \psi_1)$. Let $V_2 = k$ and $\psi_2 = (1)$. Then A is an affine cellular algebra. The same kind of example exists over any Dedekind domain; A always is an example of a hereditary order.

Lemma 2.4. *Suppose K is another noetherian domain and $\phi : k \longrightarrow K$ is a homomorphism of rings with identity. If A is an affine cellular k-algebra with an involution i, then $K \otimes_k A$ is an affine cellular K-algebra with respect to the involution $id_K \otimes i$.*

Proof. Let A be an affine cellular algebra. We define $I'_j = J'_j \otimes_k K$, $I_j = \bigoplus_{l=1}^{j} I'_l = J_j \otimes_k K$, $W_j = V_j \otimes_k K$ and $C_j = K \otimes_k B_j$. The $(K \otimes_k A)$-C_j-bimodule structure on $W_j \otimes_K C_j$ is induced from the natural module structure on $V_j \otimes_k K \otimes_k B_j$ which is identified with $W_j \otimes_K C_j$. We define $\beta_j : I_j \longrightarrow W_j \otimes_K C_j \otimes_K W_j$ by $\lambda \otimes_k x \longmapsto \lambda\alpha(x)$ for $x \in J_j$ and $\lambda \in K$. Note that we identify the K-module $V_j \otimes_k (K \otimes_k B_j) \otimes_k V_j$ with the K-module $W_j \otimes_K C_j \otimes_K W_j$. Then the I_j, $1 \leqslant j \leqslant n$ form a cell chain of $K \otimes_k A$. Thus $K \otimes_k A$ is an affine cellular algebra over K. □

Remark. While the definition of affine cellular algebra is in many respects similar to that of cellular algebra, there is one fundamental difference; the new ingredients in the affine case are the algebras B_j, and the main problem is that these algebras are given as kind of external data, which are not a priori related to the algebra A. For instance, modifying the above example into $A = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & \mathbb{Z} \end{pmatrix}$ we get an affine cellular algebra, which has $B_2 = \mathbb{Z}/p\mathbb{Z}$ associated with the top cell (which actually equals B_2). In this case, and in general, the algebras B_j need not be subalgebras of the centre of A. There is no natural action of an algebra B_j on A. Indeed, although the cell ideal J is a generalised matrix ring over B, there is no natural ring homomorphism from B to J with the image contained in the centre of J. Moreover, while the bimodule structure of ${}_A\Delta_B$ induces a ring homomorphism $B \longrightarrow \mathrm{End}_A(\Delta)$, the image of this map may in general not be contained in the centre of $\mathrm{End}_A(\Delta)$. All of these problems force us to give up on the methods used in the theory of finite dimensional cellular algebras.

2.3. Affine Temperley–Lieb algebras

Infinite dimensional diagram algebras that have arisen in mathematical physics or in knot theory are natural sources of examples of affine cellular algebras. Here, we consider affine

Temperley–Lieb algebras and show that these are affine cellular algebras. First, let us recall the definition of the affine Temperley–Lieb algebra $TL_n^a(\delta)$ given in [8].

We fix a vertical cylinder with n marked points on the top circle of its boundary and n marked points on the bottom circle. An affine diagram of type (n,n) is obtained by joining these points pairwise by arcs on the surface of the cylinder without intersection. We can also add to an affine diagram a finite number of circles which circumnavigate the cylinder if there is no intersection with arcs. We denote by $D(n)$ the set of isotopy classes of such affine diagrams of type (n,n). If α and β are two elements in $D(n)$, then we may glue the bottom boundary circle of α to the top boundary circle of β so that the corresponding marked points coincide, and thus we get a cylinder again; there may be some loops on its surface. We denote by $m(\alpha,\beta)$ the number of loops, and by $\alpha\circ\beta$ the affine diagram obtained by removing all loops. Then $\alpha\circ\beta$ is an element in $D(n)$.

Let k be a field and δ an element in k. The affine Temperley–Lieb algebra $TL_n^a(\delta)$ is an associative k-algebra spanned over k by affine diagrams in $D(n)$ with multiplication

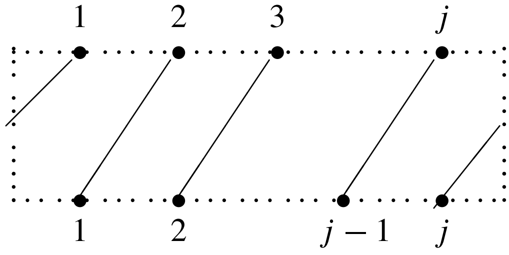
$$\alpha\beta = \delta^{m(\alpha,\beta)}\alpha\circ\beta$$

for all $\alpha,\beta\in D(n)$. This is an associative algebra with identity. Note that there is an involution $*$ on $TL_n^a(\delta)$, which is given by turning around the cylinder, thus turning the bottom points into the top points and vice versa.

An arc in an affine diagram is called a through arc if it connects a bottom point and a top point. Let j be an integer with $0\leqslant j\leqslant n$ and $j\equiv n \pmod 2$. We denote by J'_j the k-space spanned by the set of affine diagrams of $D(n)$ with exactly j through arcs. Clearly, J'_j is invariant under the involution $*$.

We first consider the case $j=0$, which only occurs if n is even. Then J'_0 is an infinite dimensional k-space. Each affine diagram D in J'_0 can be expressed as x^rD' where x is a variable and r is the number of circles in D, and D' is obtained from D by deleting all circles. So, we may define x acting on D by adding one circle. If we denote by I the k-space spanned by all such affine diagrams D', then J'_0 is isomorphic to $k[x]\otimes_k I$ as $k[x]$-modules. The involution on $k[x]$ is defined to be the identity map. Note that each diagram in I splits up uniquely into two half-diagrams D_t, the top part of D, and D_b, the bottom part of D. Let V_0 be the vector space spanned by all these D_t. Now we define a bilinear form $\psi_0: V_0\otimes_k V_0 \longrightarrow k[x]$ which controls the multiplication inside J'_0. We will identify J'_0 with $V_0\otimes_k k[x]\otimes_k V_0$. For this, we first compose D_t with D_b by identifying all corresponding vertices, and then count the number $n(D_t,D_b)$ of circles and the number $m(D_t,D_b)$ of loops in the resulting diagram, and finally, we define $\psi_0(D_t,D_b)=\delta^{m(D_t,D_b)}x^{n(D_t,D_b)}$. Thus J'_0 is an affine cell ideal. In the terminology of the previous subsection we have shown that $J'_0=\mathcal{A}(V_0,k[x],\psi_0)$.

If $j\neq 0$, then there is a twist τ_j:



and we may consider J'_j as a left module over $k[x,x^{-1}]:=k[x,y]/(xy-1)$, where x acts on an affine diagram D by putting τ_j on top of D, but only identifying those marked points of D which are end points of through arcs.

We note that every affine diagram D in J'_j is of the form $\tau_j^r D'$, where r is an integer, and where D' is a standard diagram, namely, D' has no circumnavigating circles and it can be drawn without intersections of through arcs with a straight line segment from the top vertex numbered 1 to the bottom vertex also numbered 1.

Let V_j be the k-space spanned by all half-diagrams of standard diagrams D in J'_j. Here, a half-diagram of type (n, j) is a diagram with n marked points on a circle, such that there are $n - j$ arcs each of which has two end points on the circle, and j rays each of which has its starting point on the circle, and such that there is no intersections between arcs, nor between arcs and rays. Then we may identify J'_j with $V_j \otimes_k k[x, x^{-1}] \otimes_k V_j$. Now we define a k-bilinear map $\psi_j : V_j \otimes_k V_j \longrightarrow k[x, x^{-1}]$. As in the case $j = 0$, we compose two half-diagrams D_t and D_b, and count the number $m(D_t, D_b)$ of loops. Note that for $j \neq 0$, there are no circumnavigating circles in the resulting diagram. We put $\psi_j(D_t, D_b) = \delta^{m(D_t, D_b)} \in k[x, x^{-1}]$. Then the subquotient J_j/J_{j-1} is an affine cell; it is isomorphic to $\mathcal{A}(V_j, k[x, x^{-1}], \psi_j)$. The involution i on $k[x, x^{-1}]$ is defined by $x \longmapsto x^{-1}$. Thus an affine Temperley–Lieb algebra is an affine cellular algebra and we have shown:

Proposition 2.5. *For any choice of parameters n and δ, the affine Temperley–Lieb algebra $TL_n^a(\delta)$ is an affine cellular algebra.*

2.4. Comparison with existing concepts

Obviously, finite dimensional cellular algebras over fields, as studied extensively ever since Graham and Lehrer's ground-breaking article [7], are special examples of affine cellular algebras. We note, however, that our definition is more general even in the finite dimensional case, since it allows to vary the ground rings B_j of the cells. Thus field extensions as the complex numbers viewed as an algebra over the real numbers do become examples. In the context of finite groups, work of Roggenkamp [28,29] implies that group rings of dihedral groups are affine cellular without, in general, being cellular.

When passing from cellular to affine cellular algebras, the following major problems do arise. While for cellular algebras, each layer of the cell chain contributes one or no simple module, we will see that each layer of a cell chain of an affine cellular algebra may (and in general does) contribute infinitely many simple representations, which have to be parameterised. Moreover, while for a cellular algebra each layer either has trivial multiplication modulo lower layers or it is a heredity ideal modulo lower layers [17], in the affine case we will have to find the infinite dimensional analogue of a heredity ideal in order to get started with a reasonable homological theory.

Graham and Lehrer formulated their definition of cellular algebras in terms of k-bases having special properties. An equivalent reformulation in structural terms has been given in [17], which we follow here. It is possible to rephrase our definition of affine cell ideal in terms of bases, too, but this uses a finite B-basis, not a k-basis. Thus defining affine cellularity in terms of bases means working with bases over different ground rings for different layers. We refrain from giving the obvious, but tedious details.

A tempting way of generalising Graham and Lehrer's definition of cellular algebras is to keep the definition of cell ideals, but allowing the cell chain to be infinite (with finite dimensional layers). As the example of the polynomial ring $k[x]$ in one variable shows, such infinite 'cell chains' sometimes do exist, but do not give any information on simple representations of the algebra studied. Since localisations of the polynomial ring have the same kind of infinite cell chain, it is indeed unlikely that such cell chains give any reasonable information on the algebra.

There are two existing concepts dealing with infinite dimensional generalisations of cellular algebras, both due to R.M. Green. The first one, procellular algebras [9], drops the finiteness of the cell chain and studies completions, that is projective limits, of finite dimensional cellular algebras. This works well for certain quantum groups such as Lusztig's algebra $\dot{U}$, which is known to have infinitely many cells. An easy example of such a procellular algebra is a power series ring in one variable, while the polynomial ring and most of its localisations are excluded. Although the concept of procellular algebras does cover interesting examples, it seems to lead to a theory rather disjoint from the one to be developed in this article and it does not cover examples such as affine Hecke algebras. The parameter set of finite dimensional simple modules in some sense is built into the definition of procellular algebras, in the form of a 'cell datum of profinite type', and thus needs to be found already when checking that an algebra is procellular; so in that theory, the problem of finding a parameter set of simples is rather different in its nature than in our setting. Homological properties of procellular algebras have not been studied yet.

Another generalisation of cellular algebras, also due to Green, is the concept of tabular algebras [10]. This exhibits similar structures in important classes of examples; indeed, our proof that extended affine Hecke algebras of type A are affine cellular uses the same results of Lusztig's cell theory and N.H. Xi's work on Lusztig's conjecture as Green's proof [11] that these algebras are tabular. So, there is much overlap in examples, although tabular algebras in general are not affine cellular, and affine cellular algebras are in general not tabular (since we do not require the positivity of the structure constants of B_j; for example, $B_j = \mathbb{Q}[x, y]/(x^2 - xy, xy + y^2)$ is not a hypergroup). As Green remarks, there is no result known how to parameterise simple representations of tabular algebras or reasonable classes thereof, while we solve the corresponding problem for affine cellular algebras. Since our methods are quite general, they potentially can be adapted to other algebras, which are not affine cellular, too. Also, our homological results might indicate on how to proceed with a similar theory for tabular algebras.

Much of the motivation for our definition of affine cellular algebras comes from our earlier work on cellular algebras, in particular our work on inflations and on Brauer algebras [17,18,20]. There we have rephrased the definition of cellular algebras in terms of generalised matrix rings (which Brown had already used in a special case half a century ago) and we have shown the feasibility of this concept. Here, and in particular in the case of Brauer algebras and many other diagram algebras, generalised matrix rings with entries in a non-commutative ring B have turned out to be very useful; for instance, B may be the group algebra of a symmetric group. This also has led to a stronger homological theory [14]. We refrain from discussing affine versions of these results here.

Much of the theory to be developed below works also for a more general definition of cellularity, where the algebras B_l are allowed to be arbitrary (not necessarily affine) commutative rings. In such generality, however, one cannot expect to achieve a classification of simple A-modules, since a similar classification is missing for the algebras B_l. In other words, there is an analogue of Theorem 3.10, but not of Theorem 3.12, and thus the main result on classifying simples would only reduce the problem to the same problem for all B_l, without solving it. Therefore, we do not work in this generality. We remark, however, that there may be meaningful extensions of the theory below for choices of B_l that are not affine, but still have a known representation theory in some sense. We refrain from discussing details here in order to avoid overloading this article with technicalities and also, because the approach given here covers all the examples we are interested in.

This article has been motivated by and represents our solution to a problem posed by Gus Lehrer in survey lectures on cellular algebras during a workshop at Oxford in 2005.

3. Classification of simple representations of affine cellular algebras

This section contains the first main result of this article, a classification of simple modules of affine cellular algebras in full generality. The parameter set will turn out to be subset of the finite disjoint union of the affine spaces parameterising the simple B_l-modules. The complement of the parameter set will be shown to be a disjoint union of (possibly empty) subvarieties, one for each l, of these affine spaces. In order to prove this classification, we first set up a general ring theoretic technique that in the special case of an affine cellular algebra A turns the set of simple A-modules into a disjoint union of the sets of simple modules over the rings without unit J_l/J_{l-1}, that is, over generalised matrix rings over the commutative algebras B_l. In the second step, again based on representation theory of rings without unit, we construct injective maps from the parameter sets of simple J_l/J_{l-1}-modules to affine spaces over B_l. In this step we also get coarse upper bounds, but not precise formulae, for the dimensions of simple A-modules; this involves a kind of highest weight theory for our algebras. At this point, the classification Theorem 3.12 can be stated and proved. Under the additional assumption that all the algebras B_l are principal ideal domains, we are able to go further and to use linear algebra methods to give more precise descriptions of simple A-modules and better bounds for their dimensions. Here, it turns out to be an advantage to work with rings without unit; indeed, a normal form for such a ring can be obtained simply by Gauss elimination, since there is no unit to be preserved by isomorphisms.

3.1. Simple modules, ideals and algebras without unit

In this subsection we shall carry out the first step in the above programme. We will work in a quite general situation, comparing modules over an algebra with modules over an ideal in that algebra, viewed as a non-unitary algebra. This material is elementary, but a suitable reference seems to be missing, and therefore we provide the details.

Let Λ be a k-algebra which may not have an identity element; by a k-algebra we then just mean a (not necessarily free) k-module structure on the ring Λ such that multiplication is k-bilinear. By a Λ-module M we mean a (not necessarily free) k-module together with the usual Λ-module structure requirements except for the unitary condition. A Λ-module M is called *simple* if $\Lambda M \neq 0$ and there are no submodules different from 0 and M; note that a module with zero Λ-action cannot be simple, by definition. If the algebra Λ has a unit then we require a Λ-module to be unitary.

From the definition of simple modules we have the following easy properties, where Λ^j denotes the j-fold product $\Lambda \cdot \Lambda \cdot \cdots \cdot \Lambda$, that is, $\Lambda^j = \{\sum_{i=1}^m a_{i1} \cdots a_{ij} \mid m \in \mathbb{N},\ a_{ip} \in \Lambda,\ 1 \leqslant p \leqslant j\}$: If M is a simple Λ-module, then (1) $M \neq 0$, (2) $\Lambda^j M = M$ for all positive integers j, (3) for any element x in M and any left ideal I in Λ, if $Ix \neq 0$, then $M = Ix$, (4) for any nonzero element m in M, we have $\Lambda m = M$. (Otherwise, we consider the proper k-submodule km of M, which contains $m \neq 0$, but which is contained in the submodule $\Lambda(km) = k(\Lambda m) = 0$.) In particular, if $\Lambda^n = 0$ for some positive integer n, then there is no simple Λ-module. Conversely, suppose M is a Λ-module. Then M is simple if (1) $\Lambda M \neq 0$, and (2) for any $0 \neq m \in M$ we have $\Lambda m = M$.

A construction of simple Λ-modules using ideals of Λ goes as follows. Let I be a non-zero left ideal of Λ, and suppose I' is a left ideal of Λ, which is properly contained in I. If I' is maximal in I with the property that $\Lambda I \nsubseteq I'$, then I/I' is a simple Λ-module.

Now let J be an ideal (= two-sided ideal) in Λ. Suppose L is a simple Λ-module such that $Ju \neq 0$ for some element $u \in L$. Then $JL = L = Ju$, and $L = J^n u$ for all positive integers n. Moreover, there is an exact sequence of Λ-modules:

$$0 \longrightarrow \mathrm{Ann}_J(u) \longrightarrow J \xrightarrow{\mu} Ju = L \longrightarrow 0,$$

where μ is given by $x \longmapsto xu$ for all $x \in J$, and where $\mathrm{Ann}_J(u)$ denotes the annihilator of the element u in J. Clearly, $\mathrm{Ann}_J(u)$ has the following properties:

(a) $\mathrm{Ann}_J(u)$ is a left ideal of Λ; and it is maximal with respect to being a left ideal of Λ properly contained in J, that is, $\mathrm{Ann}_J(u) \neq J$, and if I is a left ideal in Λ such that $\mathrm{Ann}_J(u) \subsetneq I \subseteq J$ then $I = J$.

(b) $J^2 \nsubseteq \mathrm{Ann}_J(u)$, thus $J^2 + \mathrm{Ann}_J(u) = J$ by (a).

Conversely, if a left ideal I of Λ is contained in J with the above two properties, then J/I is a simple Λ-module.

Using property (a) we can derive the following property.

(c) $\mathrm{Ann}_J(u)$ is a maximal left ideal in J (which is seen as an algebra in itself). In fact, if I is a left ideal of J and properly contains $\mathrm{Ann}_J(u)$, then $I + \Lambda I$ is a left ideal of Λ and is contained in J. By (a), we must have $I + \Lambda I = J$. Note that $(\Lambda I)u$ is a Λ-submodule of L. Thus either $\Lambda I u = 0$ or $\Lambda I u = L$. In the first case, we have $\Lambda I \subseteq \mathrm{Ann}_J(u)$, and therefore, $I + \Lambda I \subseteq I + \mathrm{Ann}_J(u) = I$, so I is a left ideal in Λ, contradicting (a). Thus we must have $\Lambda I u = L$. This implies that $L = JL = J\Lambda I u \subseteq Iu$ and hence $Iu = L$ is a Λ-module. Note that $\mathrm{Ann}_I(u) = I \cap \mathrm{Ann}_J(u) = \mathrm{Ann}_J(u)$. Now, we consider the following exact commutative diagram of J-modules:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Ann}_I(u) & \longrightarrow & I & \longrightarrow & Iu = L & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Ann}_J(u) & \longrightarrow & J & \longrightarrow & Ju = L & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & J/I & & 0 & &
\end{array}$$

The snake lemma shows that $J/I = 0$, that is, $J = I$. Thus $\mathrm{Ann}_J(u)$ is a maximal left ideal of J.

As a consequence of (c), we see that $J/\mathrm{Ann}_J(u) \simeq L$ is simple as a J-module, too. The converse is also true:

Lemma 3.1. *Let J be an ideal in Λ. Then every simple J-module L is a simple Λ-module.*

Proof. If L is a simple J-module, then there exists a maximal left ideal J' of J such that $J^2 \nsubseteq J'$ and $L \simeq J/J'$. Let J_1 be the left ideal of Λ generated by J'. Then $J_1 = J' + \Lambda J' \subseteq J$. Thus $J_1^2 \subseteq J_1 J' + J_1 \Lambda J' \subseteq J J' + J J' \subseteq J'$. Now we consider the following exact commutative diagram of J-modules:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J' & \longrightarrow & J & \longrightarrow & L & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & J_1 & \longrightarrow & J & \longrightarrow & L' & \longrightarrow & 0. \end{array}$$

This provides an exact sequence

$$0 \longrightarrow J_1/J' \longrightarrow L \longrightarrow L' \longrightarrow 0.$$

Since $J^2 \nsubseteq J'$ and $J_1^2 \subseteq J'$, we see that $J_1 \neq J$ and hence $L' \neq 0$. Note that L is a simple J-module. Therefore, $J' = J_1$, and it is a left ideal in Λ. Thus $L = L'$, as a quotient of left ideals in Λ, is a Λ-module and simple, too. □

As a direct consequence of Lemma 3.1, we get the following general statement, which is what we need when continuing our programme of classifying simple modules. For A being affine cellular, we reduce in this way the problem of classifying simple A-modules to the problem of classifying the simple modules for each generalised matrix ring $\tilde{\Lambda} = J_l/J_{l-1}$.

Corollary 3.2. *Let $J = J_0 \subseteq J_1 \subseteq \cdots \subseteq J_n = \Lambda$ be a chain of k-submodules in Λ. If J_{l-1} is an ideal in J_l for all $1 \leqslant l \leqslant n$, then every simple J-module L is a simple Λ-module. Conversely, if a simple Λ-module L satisfies $JL \neq 0$, then L is a simple J-module.*

3.2. Swich algebras and classification of simple representations

In this subsection we will carry out the second step of our programme, culminating in two results. The first one, Theorem 3.10 provides a very general procedure to compare simple representations of algebras and of what we call swich algebras (see below for an explanation of the terminology); this includes the situation we are interested in, namely to compare simple representations of generalised matrix algebras and of ordinary matrix algebras. Applying this result to an affine cellular algebra, we then get Theorem 3.12, which is our main result in this context; it describes the parameter set as a finite disjoint unions of subsets in affine varieties, contained in the affine spaces $\mathrm{MaxSpec}(B_j)$, the spectrum of all maximal ideals in B_j.

Again, we will work in a quite general setup, in order to clearly exhibit the crucial structures. We keep the noetherian domain k as a ground ring, and we require all unitary algebras Λ to be finitely generated over their centres. These assumptions are valid in the application to affine cellular algebras since, in the affine cellular situation, the algebras called Λ in the current subsection are matrix algebras of finite size over the affine commutative algebras B_l associated with subquotients in the cell chain, and each of them is noetherian over its centre.

Definition 3.3. Let Λ be a k-algebra and a_0 a fixed element in Λ. We define a new k-algebra $\tilde{\Lambda} = S(\Lambda, a_0)$, called the *swich algebra* of Λ with respect to a_0, where as a set $\tilde{\Lambda} = \{\tilde{a} \mid a \in \Lambda\}$, and the algebra structure on $\tilde{\Lambda}$ is given by

$$\tilde{a} + \tilde{b} = \widetilde{a+b}, \quad a, b \in \Lambda,$$
$$\tilde{a} \cdot \tilde{b} = \widetilde{aa_0b}, \quad a, b \in \Lambda,$$
$$\lambda\tilde{a} = \widetilde{\lambda a}, \quad \lambda \in k,\ a \in \Lambda.$$

The algebra $\tilde{\Lambda}$ is an associative algebra, not necessarily with identity. The term 'swich' is a composition of 'sandwich' indicating the definition of multiplication by putting a_0 in between the two elements to be multiplied, and of 'switch' indicating the main use we will make of these algebras by switching from cells to matrix algebras.

A typical example of an algebra of the form $\tilde{\Lambda}$ is an affine cell ideal J, when viewed as a generalised matrix algebra, see Proposition 2.2 and the discussion following it. Using the notation in Section 2.2, the multiplication in J is given by $xy = x\Psi y$ for all $x, y \in M_n(B)$, where Ψ is the matrix describing the bilinear form ψ with respect to some basis of V. In this case, we have $\Lambda = M_n(B)$, $a_0 = \Psi$ and $\tilde{\Lambda} = (M_n(B), \Psi)$ with the multiplication in $\tilde{\Lambda}$ being given by Ψ. By Corollary 3.2, our task is to classify the simple $(M_n(B), \Psi)$-representations, where $\tilde{\Lambda}$ is a subquotient (= layer) in a cell chain. Thus, we are going to classify, in the general setup, simple $\tilde{\Lambda}$-representations in terms of simple Λ-representations. In the application to affine cellular algebras, Λ is an ordinary matrix algebra over some B_l, hence it is Morita equivalent to B_l and its simple representations are parameterised by MaxSpec(B_l).

Note that $\tilde{\Lambda}$ may not be a noetherian algebra even if Λ is a noetherian algebra. A small example is $\Lambda = k[x]$ with k a field and $a_0 = 0$. Using this $\tilde{\Lambda}$, we get an affine cellular algebra $A = k \oplus \tilde{\Lambda}$ with cell chain $0 \subset \tilde{\Lambda} \subset A$ such that $\tilde{\Lambda}$ is not noetherian, since it is just a vector space with trivial multiplication.

In the following, we always assume that the algebra Λ, which for now is any k-algebra, has an identity. The main purpose of this subsection is to establish a general correspondence between the simple modules over Λ and the simple modules over $\tilde{\Lambda}$, and thus to parameterise simple $\tilde{\Lambda}$-modules via simple Λ-modules. Let us first construct an algebra homomorphism from $\tilde{\Lambda}$ to Λ.

Lemma 3.4. *Let Λ be a k-algebra with identity and $\tilde{\Lambda}$ the swich algebra defined by a_0. Then there is an algebra homomorphism $\varphi : \tilde{\Lambda} \longrightarrow \Lambda$ defined by $\tilde{a} \longmapsto \varphi(\tilde{a}) = aa_0$. Similarly, the map $\varphi' : \tilde{\Lambda} \longrightarrow \Lambda$ defined by $\tilde{a} \longmapsto \varphi'(\tilde{a}) = a_0a$ also is an algebra homomorphism. The map φ is injective if and only if a_0 is not a right zero-divisor, and it is surjective if and only if a_0 is a right unit in Λ (that is, there is $b \in \Lambda$ such that $ba_0 = 1$).*

Proof. This is a straightforward computation. □

Thus, via φ, each Λ-module M will become a $\tilde{\Lambda}$-module, which is denoted by M_φ.

Lemma 3.5. *There is a Λ-$\tilde{\Lambda}$-bimodule structure on Λ: The left Λ-module structure on Λ is the regular one, and the right $\tilde{\Lambda}$-module structure is defined by*

$$x \cdot \tilde{a} = xa_0a, \quad a, x \in \Lambda.$$

Similarly, there is a left $\tilde{\Lambda}$-module structure on Λ. Moreover, we have ${}_{\tilde{\Lambda}}\Lambda \simeq {}_{\tilde{\Lambda}}\tilde{\Lambda}$.

Proof. Clearly, the left module structure is well defined. It remains to verify that the right structure is well defined and the associative law is fulfilled by the two module structures: Suppose $x \in \Lambda$, and $\tilde{a}, \tilde{b} \in \tilde{\Lambda}$ with $a, b \in \Lambda$. Then

$$x \cdot (\tilde{a}\tilde{b}) = xa_0(aa_0b) = (x \cdot \tilde{a}) \cdot \tilde{b},$$
$$(ax) \cdot \tilde{b} = axa_0b = a(xa_0b) = a(x \cdot \tilde{b}).$$

Thus Λ is a Λ-$\tilde{\Lambda}$-bimodule. The second half of the lemma can be proved similarly. □

Remark. Using Lemma 3.5, the algebra homomorphism φ in Lemma 3.4 can be described as $\tilde{a} \longmapsto a \cdot \tilde{1} = a(1 \cdot \tilde{1})$, where 1 is the identity in Λ. The right $\tilde{\Lambda}$-module structure on Λ is induced from φ', not from φ, namely $x \cdot \tilde{a} = x\varphi'(\tilde{a})$ for all $x, a \in \Lambda$.

The algebra homomorphisms in Lemma 3.4 provide us with functors Λ-mod $\longrightarrow$ $\tilde{\Lambda}$-mod. In order to get functors backwards, we need some kind of induction procedure, involving a tensor product over the non-unitary algebra $\tilde{\Lambda}$. Following a definition by Lusztig [22] in the context of affine Hecke algebras, we define such a tensor product as follows. Given a Λ-$\tilde{\Lambda}$-bimodule X and a $\tilde{\Lambda}$-module Y, we define $X \otimes_{\tilde{\Lambda}} Y$ as the quotient of $X \otimes_k Y$ modulo the k-submodule generated by $\{x\tilde{a} \otimes y - x \otimes \tilde{a}y \mid x \in X,\ y \in Y,\ a \in \Lambda\}$. Then $X \otimes_{\tilde{\Lambda}} Y$ is a Λ-module in the usual sense.

Using the Λ-$\tilde{\Lambda}$-bimodule structure on Λ, we can now compare the modules over Λ with those over $\tilde{\Lambda}$, as the following lemma shows.

Lemma 3.6.

(1) *Let X be a Λ-module and M a $\tilde{\Lambda}$-submodule of X_φ. Then there is a homomorphism of Λ-modules:*

$$\theta : {}_\Lambda\Lambda \otimes_{\tilde{\Lambda}} M \longrightarrow X, \qquad a \otimes m \longmapsto \varphi(\tilde{a})m, \quad m \in M,\ a \in \Lambda$$

such that $\varphi(\tilde{\Lambda})\operatorname{Ker}(\theta) = 0$.

(2) *Let Y be a $\tilde{\Lambda}$-module. Then there is a $\tilde{\Lambda}$-module homomorphism:*

$$p : (\Lambda \otimes_{\tilde{\Lambda}} Y)_\varphi \longrightarrow Y, \qquad a \otimes y \longmapsto \tilde{a}y, \quad a \in \Lambda,\ y \in Y$$

such that $\tilde{\Lambda}\operatorname{Ker}(p) = 0$.

Proof. (1) It follows from $\varphi(\tilde{a} \cdot \tilde{b})m = (\varphi(\tilde{a})\varphi(\tilde{b}))m = \varphi(\tilde{a})(\varphi(\tilde{b})m) = \varphi(\tilde{a})(\tilde{b}m)$ that the images of $a \otimes \tilde{b}m$ and $a \cdot \tilde{b} \otimes m$ under θ coincide with each other. Thus θ is well defined. Now we check that θ is a Λ-homomorphism. Let $a \in \Lambda$ and $b \otimes m \in \Lambda \otimes_{\tilde{\Lambda}} M$. Then the image under θ of $a(b \otimes m) = ab \otimes m$ is $\varphi(\widetilde{ab})m = aba_0m$. This is equal to $a(\varphi(\tilde{b})m) = a(ba_0m)$. Thus θ preserves the left action of elements $a \in \Lambda$, and it is a Λ-module homomorphism.

Let $\sum_i a_i \otimes m_i$ be in the kernel of θ with $a_i \in \Lambda$ and $m_i \in M$. Then $\sum_i \varphi(\widetilde{a_i})m_i = 0$. Pick an $\tilde{a}$ in $\tilde{\Lambda}$. We get

$$\begin{aligned}\varphi(\tilde{a})\Big(\sum_i a_i \otimes m_i\Big) &= \sum_i \varphi(\tilde{a})a_i \otimes m_i = \sum_i a a_0 a_i \otimes m_i \\ &= \sum_i a \cdot \widetilde{a_i} \otimes m_i = \sum_i a \otimes \widetilde{a_i} m_i \\ &= a \otimes \sum_i \widetilde{a_i} m_i = a \otimes \sum_i \varphi(\widetilde{a_i}) m_i = 0.\end{aligned}$$

This finishes the proof of (1).

(2) The proof of (2) is similar to that of (1). □

This has a useful consequence for our programme of classifying $\tilde{\Lambda}$-modules; every simple $\tilde{\Lambda}$-module is a quotient of the restriction to $\tilde{\Lambda}$ of some Λ-module.

Corollary 3.7. *Let S be a simple $\tilde{\Lambda}$-module. Then S is a quotient of $(\Lambda \otimes_{\tilde{\Lambda}} S)_\varphi$.*

Proof. Since S is simple and $\tilde{\Lambda} S \neq 0$, the map $p : (\Lambda \otimes_{\tilde{\Lambda}} S)_\varphi \longrightarrow S$ is a surjective $\tilde{\Lambda}$-module homomorphism by Lemma 3.6(2). □

This result provides the first step towards a kind of highest weight theory for swich algebras. Simple modules over $\tilde{\Lambda}$ are to be constructed as the unique simple quotients of the restrictions to $\tilde{\Lambda}$ of simple Λ-modules. The next proposition shows that the restriction to $\tilde{\Lambda}$ of a simple Λ-module either does not contribute any $\tilde{\Lambda}$-simple or precisely one. Before stating the proposition we comment on the term "composition factor" to be used now. A composition factor of a $\tilde{\Lambda}$-module M by definition is a simple subquotient L of M. According to our definition, the simple module L is required to carry a non-zero $\tilde{\Lambda}$-action. A module with trivial action does not qualify. Therefore, it may happen that the restriction to $\tilde{\Lambda}$ of some Λ-module, for instance a simple one, say E, has no composition factor at all. It is exactly in this case that E does not contribute to the classification of simple $\tilde{\Lambda}$-modules. In the case of finite dimensional cellular algebras, this is precisely the case when a cell module equals the radical of its associated bilinear form.

Proposition 3.8. *Let E be a simple Λ-module.*

(1) *The $\tilde{\Lambda}$-module E_φ has a composition factor if and only if $\varphi(\tilde{\Lambda})E \neq 0$.*
(2) *If a simple $\tilde{\Lambda}$-module S is a composition factor of the $\tilde{\Lambda}$-module E_φ, then S is a quotient of E_φ.*
(3) *If the $\tilde{\Lambda}$-module E_φ has a composition factor, then it has a unique simple quotient S. The kernel of the epimorphism $E_\varphi \longrightarrow S$ has no composition factor.*

Proof. First, we observe that $\varphi(\tilde{\Lambda})E$ by definition equals $\Lambda a_0 E$, which is a Λ-submodule of E; thus it is either zero or equals E. Suppose it is non-zero. Then $E = \Lambda a_0 E$. The subset $M = \{m \in E \mid \Lambda a_0 m = 0\}$ is a $\tilde{\Lambda}$-submodule of E_φ. Moreover, we claim that M is a maximal submodule of E_φ. Suppose M' is a $\tilde{\Lambda}$-submodule of E_φ such that M' properly contains M. Pick an element $m' \in M' \setminus M$. Then $\Lambda a_0 m' \neq 0$. Since E is simple, we have $E = \Lambda a_0 m'$. This implies that $E_\varphi = E = \Lambda a_0 m' = \tilde{\Lambda} \cdot m' \subseteq M'$. Hence, as claimed, M is a maximal submodule of E_φ. Since $E = \Lambda a_0 E$, we see that M is properly contained in E and that $\tilde{\Lambda}$ acts non-trivially on the quotient E_φ / M. Thus E_φ / M is a simple $\tilde{\Lambda}$-module, which proves one implication of (1).

Suppose that the $\tilde{\Lambda}$-module E_φ has a composition factor, say S. By definition, there is a chain $X_2 \subseteq X_1 \subseteq E_\varphi$ of submodules of E_φ such that $X_1/X_2 \simeq S$. Since $\tilde{\Lambda} S \neq 0$, we see $\varphi(\tilde{\Lambda})X_1 \neq 0$. This means that there are elements $\tilde{a} \in \tilde{\Lambda}$ and $x \in X_1$ such that $\tilde{a}x \neq 0$. Hence the Λ-homomorphism

$$\theta : \Lambda \otimes_{\tilde{\Lambda}} X_1 \longrightarrow E, \qquad a \otimes x \longmapsto \varphi(\tilde{a})x, \quad a \in \Lambda,\ x \in X_1$$

is non-zero, and therefore it is surjective since E is simple. It follows that the image $\mathrm{Im}(\theta)$ of θ is equal to E, and $\mathrm{Im}(\theta)$ also equals $\varphi(\tilde{\Lambda})X_1$ which is contained in the k-submodule X_1. Thus $E = X_1 = E_\varphi$, which implies that S is a quotient of E_φ. This proves (2) and also the other implication of (1), since the $\tilde{\Lambda}$-action on the simple quotient S being non-zero implies the action on E_φ also to be non-zero.

Note that the previous argument proves a statement stronger than (2). If X is any $\tilde{\Lambda}$-submodule of E_φ with non-trivial $\tilde{\Lambda}$-action, then $X = E$, since the map θ has image $\mathrm{Im}(\theta)$ equal to E and $\mathrm{Im}(\theta)$ also is contained in the $\tilde{\Lambda}$-module X. In particular, the above submodule X_2 is a proper submodule, since X_2 has already been shown to be the kernel of an epimorphism $X_1 = E_\varphi \longrightarrow S \neq 0$. Thus X_2 must be acted upon trivially by $\tilde{\Lambda}$. Therefore, $X_2 \subseteq \{x \in E_\varphi \mid \tilde{\Lambda}x = 0\} =: Z$. Note that Z is a submodule of E_φ. Since Z/X_2 is a proper submodule of the simple $\tilde{\Lambda}$-module E_φ/X_2, we must have $X_2 = Z$. This implies (3). □

In general, if E is a simple Λ-module such that its restriction E_φ has simple quotient S over $\tilde{\Lambda}$, the three Λ-modules E, $\Lambda \otimes_{\tilde{\Lambda}} S$ and $\Lambda \otimes_{\tilde{\Lambda}} E_\varphi$ may all be different. In a special situation, however, which will be investigated in more detail in the subsequent section, there are isomorphisms between these three modules.

Proposition 3.9. *Let Λ be a noetherian k-algebra. Suppose $\varphi(\tilde{\Lambda})E' \neq 0$ for each simple Λ-module E'. Let S be a simple $\tilde{\Lambda}$-module, and let E be a simple Λ-module.*

If S is a composition factor of E_φ, then $\Lambda \otimes_{\tilde{\Lambda}} E_\varphi \simeq E \simeq \Lambda \otimes_{\tilde{\Lambda}} S$ as Λ-modules.

Proof. Suppose S is a composition factor of E_φ. Then $\varphi(\tilde{\Lambda})E \neq 0$. This implies that the map $\theta : \Lambda \otimes_{\tilde{\Lambda}} E_\varphi \longrightarrow E$ is surjective since E is simple. Now we show that $\mathrm{Ker}(\theta) = 0$. If x is an element in $\mathrm{Ker}(\theta)$, then Λx is a finitely generated Λ-module. Since we are assuming that Λ is a noetherian k-algebra, the module Λx is a noetherian Λ-module. Thus, if $x \neq 0$, then there is a maximal submodule K of Λx such that $F := \Lambda x/K$ is a simple Λ-module, which by assumption carries a non-trivial $\tilde{\Lambda}$-action. However, $\varphi(\tilde{\Lambda})(F) \subset \varphi(\tilde{\Lambda})(\mathrm{Ker}(\theta)/K) = 0$ by Lemma 3.6, which is a contradiction. Thus $\mathrm{Ker}(\theta) = 0$ and θ is injective, and $E \simeq \Lambda \otimes_{\tilde{\Lambda}} E_\varphi$. Since S is a quotient of E_φ by Lemma 3.8(1), we have a surjective Λ-homomorphism from $\Lambda \otimes_{\tilde{\Lambda}} E_\varphi$ to $\Lambda \otimes_{\tilde{\Lambda}} S \neq 0$. Note that $\Lambda \otimes_{\tilde{\Lambda}} E_\varphi \simeq E$ is simple. So, $\Lambda \otimes_{\tilde{\Lambda}} S \simeq \Lambda \otimes_{\tilde{\Lambda}} E_\varphi$. □

The following result establishes a relationship between the set of all simple modules over Λ and that over a swich algebra $\tilde{\Lambda}$.

Theorem 3.10. *Let Λ be a k-algebra with identity such that Λ is finitely generated over its centre Z. Let $\tilde{\Lambda} = S(\Lambda, a_0)$ be the swich algebra of Λ with respect to a_0 in Λ. Then there is a bijection ω between the set of non-isomorphic simple Λ-modules E with $\varphi(\tilde{\Lambda})E \neq 0$, and the set of all non-isomorphic simple $\tilde{\Lambda}$-modules, which is given by $E \longmapsto E_\varphi/\{x \in E_\varphi \mid \tilde{\Lambda}x = 0\}$.*

Moreover, all simple $\tilde{\Lambda}$-modules are modules over Z and as such they are semisimple and artinian.

Proof. By Proposition 3.8, the map ω is well defined. We first show that ω is surjective. Let S be a simple $\tilde{\Lambda}$-module. We shall show that there is a simple Λ-module E such that S is a (hence the unique) quotient of E_φ.

Since S is a simple $\tilde{\Lambda}$-module, there is an element $s_0 \in S$ such that $S = \tilde{\Lambda}s_0$. Hence we have a surjective homomorphism $q : \Lambda \longrightarrow S$ of $\tilde{\Lambda}$-modules, which is defined by $\lambda \longmapsto \tilde{\lambda}s_0$ for $\lambda \in \Lambda$. Let M be the kernel of q. Note that Λ is a left $\tilde{\Lambda}$-Z-bimodule. In order to make the bimodule structure more visible, we will write the morphism q on the right of its argument.

Claim 1. *$M = ZM$. In particular, M is a Z-module, and S is a $\tilde{\Lambda}$-Z-bimodule.*

In fact, it follows from $(\tilde{\Lambda} \cdot \Lambda)q = \tilde{\Lambda}(\Lambda)q = \tilde{\Lambda}S = S$ that $\tilde{\Lambda} \cdot \Lambda \nsubseteq M$. Clearly, ZM is a $\tilde{\Lambda}$-submodule of ${}_{\tilde{\Lambda}}\Lambda$. To show that $ZM = M$, it is sufficient to prove that $ZM \subsetneqq \Lambda$ since M is a maximal $\tilde{\Lambda}$-submodule of Λ. If $ZM = \Lambda$, then, for any $\lambda, \mu \in \Lambda$, we write $\mu = \sum_i z_i m_i$ with $z_i \in Z$ and $m_i \in M$, and get

$$\tilde{\lambda} \cdot \mu = \tilde{\lambda} \cdot \left(\sum_i z_i m_i\right) = \sum_i \lambda a_0 z_i m_i = \sum_i \widetilde{\lambda z_i} \cdot m_i \in M$$

since M is a $\tilde{\Lambda}$-module. This contradicts the fact that $\tilde{\Lambda} \cdot \Lambda \nsubseteq M$. Thus we have shown that $M = ZM$.

Claim 2. *There is a maximal ideal $\mathfrak{m}$ of Z such that $\mathfrak{m}S = 0$.*

Indeed, if for each maximal ideal $\mathfrak{m}$ in Z we have $\mathfrak{m}S = S$, then, by localisation at $\mathfrak{m}$, we have $S_\mathfrak{m} = \mathfrak{m}_\mathfrak{m} S_\mathfrak{m}$, where the subindex means the localisation at $\mathfrak{m}$. Since $Z_\mathfrak{m}$ is a local ring with maximal ideal $\mathfrak{m}_\mathfrak{m}$ and since S is a finitely generated Z-module, Nakayama's Lemma implies that $S_\mathfrak{m} = 0$. Thus, by a well-known fact in commutative algebra (see [2, Proposition 3.8]), we have $S = 0$, a contradiction. This shows Claim 2.

Claim 3. *Let $\mathfrak{m}$ be a maximal ideal in Z such that $\mathfrak{m}S = 0$. Then the map q induces a surjective Z-homomorphism $q' : \Lambda/\Lambda\mathfrak{m} \longrightarrow S$. Moreover, the Λ-module $\Lambda/\Lambda\mathfrak{m}$ is both artinian and noetherian.*

In fact, the first statement in Claim 3 follows from $(\mathfrak{m}\Lambda)q = \mathfrak{m}(\Lambda)q = \mathfrak{m}S = 0$ since q is also a Z-homomorphism (by Claim 1).

Since Λ is finitely generated as a Z-module, we have a surjective Z-homomorphism $f : Z^n \longrightarrow \Lambda$ for some positive integer n. The quotient of Z^n modulo the Z-submodule $(\mathfrak{m}Z)^n$ is isomorphic to $(Z/\mathfrak{m})^n$, which is a direct sum of n copies of the field $Z/\mathfrak{m}$. Thus it is a semisimple, artinian and noetherian Z-module. Note that $\Lambda\mathfrak{m} = \mathfrak{m}\Lambda$. Since the image of the Z-submodule $(\mathfrak{m}Z)^n$ of Z^n under f is $\mathfrak{m}\Lambda$, it follows that f induces a surjective Z-homomorphism from $(Z/\mathfrak{m})^n$ to $\Lambda/\Lambda\mathfrak{m}$. Thus $\Lambda/\Lambda\mathfrak{m}$ is a semisimple Z-module which is both artinian and noetherian. Since Λ is a Z-algebra, this implies that $\Lambda/\Lambda\mathfrak{m}$ is also artinian and noetherian as a Λ-module. This proves Claim 3.

Thus we have an exact sequence of $\tilde{\Lambda}$-modules:

$$0 \longrightarrow \operatorname{Ker}(q') \longrightarrow (\Lambda/\Lambda\mathfrak{m})_\varphi \xrightarrow{q'} S \longrightarrow 0.$$

Since $\tilde{\Lambda}(\Lambda/\Lambda\mathfrak{m})_\varphi \neq 0$ and since $\Lambda/\Lambda\mathfrak{m}$ is both artinian and noetherian, there is a composition series of the Λ-module $\Lambda/\Lambda\mathfrak{m}$:

$$0 \subset X_1 \subset X_2 \subset \cdots \subset X_{s-1} \subset X_s \subset \cdots \subset X_n = \Lambda/\Lambda\mathfrak{m}$$

such that $(X_{s-1})q' = 0 \neq (X_s)q'$ and $E = X_s/X_{s-1}$ is a simple Λ-module. Note that X_{s-1} is contained in $\operatorname{Ker}(q')$. So we have the following commutative diagram with exact rows in $\tilde{\Lambda}$-mod:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (X_{s-1})_\varphi & \longrightarrow & (X_s)_\varphi & \longrightarrow & E_\varphi & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\mu'} & & \\
0 & \longrightarrow & \operatorname{Ker}(q') & \longrightarrow & (\Lambda/\Lambda\mathfrak{m})_\varphi & \xrightarrow{q'} & S & \longrightarrow & 0,
\end{array}$$

where μ is the canonical inclusion, and μ' is the induced map. Suppose $\mu' = 0$. Then X_s would be included in $\operatorname{Ker}(q')$, thus $(X_s)q' = 0$, a contradiction. So the map μ' is non-zero. This shows that S is a (in fact, the unique) quotient of E_φ with E simple and $\varphi(\tilde{\Lambda})E \neq 0$.

Finally, we show that the map ω is injective. Suppose that E and F are two simple Λ-modules such that E_φ and F_φ have the same simple quotient S. We have to show that E and F are isomorphic.

Indeed, let π' be the projection from E_φ to S. Then we have a surjective Λ-homomorphism $\pi := 1 \otimes \pi' : \Lambda \otimes_{\tilde{\Lambda}} E_\varphi \longrightarrow \Lambda \otimes_{\tilde{\Lambda}} S$. By Lemma 3.6, there is a Λ-homomorphism $\theta : \Lambda \otimes_{\tilde{\Lambda}} E_\varphi \longrightarrow E$, which is surjective. So, we get the following commutative exact diagram of Λ-modules:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Ker}(\theta) & \xrightarrow{\alpha} & \Lambda \otimes_{\tilde{\Lambda}} E_\varphi & \longrightarrow & E & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\nu} & & \\
0 & \longrightarrow & \operatorname{Im}(\alpha\pi) & \xrightarrow{\gamma} & \Lambda \otimes_{\tilde{\Lambda}} S & \longrightarrow & \operatorname{Cok}(\gamma) & \longrightarrow & 0.
\end{array}$$

Since E is simple and ν is surjective, we have that $\operatorname{Cok}(\gamma) \simeq E$ or $\operatorname{Cok}(\gamma) = 0$. By Lemma 3.6, $\varphi(\tilde{\Lambda}) \operatorname{Ker}(\theta) = 0$. This implies that the second case, $\operatorname{Cok}(\gamma) = 0$, cannot happen. Moreover, β is surjective, and $\varphi(\tilde{\Lambda}) \operatorname{Im}(\alpha\pi) = 0$. As a consequence, the exact sequence $0 \longrightarrow \operatorname{Im}(\alpha\pi) \longrightarrow \Lambda \otimes_{\tilde{\Lambda}} S \longrightarrow E \longrightarrow 0$ shows that E is the unique quotient of $\Lambda \otimes_{\tilde{\Lambda}} S$ such that $\varphi(\tilde{\Lambda})E \neq 0$. Similarly, F satisfies the same conditions. Hence E and F must be isomorphic. $\square$

Theorem 3.10 describes how to parameterise the simple $\tilde{\Lambda}$-modules by using the simple Λ-modules in a way strongly reminiscent of highest weight classifications in Lie theory.

Specialising to the situation of an affine cell ideal, that is, to $\tilde{\Lambda}$ being a generalised matrix algebra and Λ an ordinary matrix algebra, we get the following step in our programme.

Proposition 3.11. *Suppose B is an affine commutative k-algebra with identity and $\Psi = (\psi_{mn})$ is an $n \times n$ matrix over B. We denote by $\Lambda = M_n(B)$ the $n \times n$ matrix k-algebra over B, and by $\tilde{\Lambda}$ the algebra $(M_n(B), \Psi)$. Then there is a bijection between the set of isomorphism classes of simple $\tilde{\Lambda}$-modules and the set of maximal ideals m in B such that m does not contain the ideal of B generated by all the entries ψ_{mn} of Ψ, and this set is in bijection with the set of all simple B-modules S such that there is some ψ_{st} with $\psi_{st} S \neq 0$.*

So, in order to determine the parameter space of $\tilde{\Lambda}$-simples it is enough to know the finitely many entries of the matrix Ψ.

Proof of Propsosition 3.11. Since simple B-modules are parameterised by maximal ideals in B and since Λ is Morita equivalent to B, it is sufficient (by Theorem 3.10) to decide for which simple Λ-module E we can have $(\Lambda\Psi)E \neq 0$. Given a simple B-module S, there is a unique maximal ideal m in B such that $\mathsf{m}S = 0$. The corresponding simple Λ-module E associated with S (under the Morita equivalence) satisfies that $M_n(\mathsf{m})E = 0$. Note that $\Lambda\Psi E = 0$ if and only if $\Psi E = 0$ if and only if $\Psi \in M_n(\mathsf{m})$ since $M_n(\mathsf{m})$ is a maximal ideal in Λ. The last condition is equivalent to $\psi_{mn} \in \mathsf{m}$ for all m, n. $\square$

Summarising the above discussion, we get the main result, classifying the simple modules of an affine cellular algebra. For free, we also get a criterion on block decompositions of affine cellular algebras. Recall that by $\mathrm{MaxSpec}(B)$ we mean the maximal spectrum of B consisting of all maximal ideals in B.

Theorem 3.12. *Let A be an affine cellular algebra with a cell chain*

$$0 = J_0 \subset J_1 \subset \cdots \subset J_n = A$$

such that $J_j/J_{j-1} \simeq \mathcal{A}(V_j, B_j, \psi_j)$. Let $(\psi_{st}^{(j)})$ be the matrix corresponding to the bilinear form ψ_j. Then:

(1) *There is a bijection between the set of isomorphism classes of simple A-modules and the set*

$$\left\{(j, \mathsf{m}) \;\middle|\; 1 \leqslant j \leqslant n,\ \mathsf{m} \text{ is a maximal ideal of } B_j \text{ such that there is some } \psi_{st}^{(j)} \notin \mathsf{m}\right\}.$$

In particular, the parameter set of simple A-modules is the disjoint union of sets $\mathcal{V}_j = \{\mathsf{m} \subset B \mid \mathsf{m}$ is a maximal ideal in B_j such that there is some $\psi_{st}^{(j)} \notin \mathsf{m}\}$, and each set $\mathcal{V}_j$ is contained in an affine variety $\mathrm{MaxSpec}(B_j)$; the complement of $\mathcal{V}_j$ in $\mathrm{MaxSpec}(B_j)$ is an affine subvariety of $\mathrm{MaxSpec}(B_j)$.
Each simple A-module S is finite dimensional over some field B_j/m (which is a k-module), and its dimension (over this field B_j/m) is bounded above by the dimension (over the same field) of the corresponding simple module E of the matrix ring $M_{n_j}(B_j)$, which is a quotient of the cell lattice Δ_j.

(2) *Let $1 \leqslant j \leqslant n$. Then ψ_j is an isomorphism if and only if the determinant $\det(\psi_{st}^{(j)})$ of ψ_j is a unit in B_j. In particular, if all ψ_j are isomorphisms, then A is isomorphic, as an affine cellular k-algebra, to $\bigoplus_{j=1}^{n} M_{n_j}(B_j)$, where n_j is the k-rank of V_j.*

Here, by an isomorphism of affine cellular algebras we mean a k-algebra isomorphism that carries the cell chain of the first algebra into that of the second one, thereby inducing isomorphisms of the associated affine algebras B_j and of the corresponding generalised matrix algebra structures. The term 'affine variety' is used in a generalised sense, denoting the maximal spectrum of an 'affine k-algebra' in the general sense used above.

Proof of Theorem 3.12. By assumption, each algebra B_j is affine, hence a quotient of a polynomial algebra P over k, with finitely many variables. Therefore, the maximal spectrum $\mathrm{MaxSpec}(B_j)$ is an affine variety contained in the affine space corresponding to the polynomial ring P. The points of $\mathrm{MaxSpec}(B_j)$ not contained in $\mathcal{V}_j$ correspond to the maximal ideals M of B_j such that B_j/M is annihilated by J_j/J_{j-1}. These maximal ideals M are exactly the maximal ideals containing the ideal generated by the entries of the matrix Ψ_j of the corresponding bilinear form ψ_j. That is, the M to be excluded are the points of an affine subvariety of $\mathrm{MaxSpec}(B_j)$.

The other statements in (1) follow inductively from Proposition 3.11. Claim (2) follows from the proof of [20, Lemma 7.1], which carries over to the affine situation without any change, and induction on the length n of a cell chain. □

We remark that the existence of a cell chain and the inductive use of Proposition 3.11 also adds to the interpretation of the above 'highest weight' theory for A-simples in the following way. Any simple A-module S is a simple module of a unique layer in the cell chain. This layer is a cell ideal J of an affine cellular quotient A' of A. Thus S is the unique simple J-quotient of some simple $M_n(B)$-module E of the matrix algebra $M(B)$ belonging to the generalised matrix algebra J. The ideal J acts as zero on the kernel $K(S)$ of the projection $E \longrightarrow S$. As an A-module, E is a quotient of the cell lattice Δ, and the kernel $K(S)$ is an A'/J-module, and S is the only composition factor that is not an A'/J-module, hence S is the highest composition factor when defining an order on the layers in the usual way of highest weight categories or quasi-hereditary algebras.

Due to the close relation of parameter sets, the sum of the ordinary matrix rings deserves to get a name.

Definition 3.13. Let A be an affine cellular algebra with a cell chain

$$0 = J_0 \subset J_1 \subset \cdots \subset J_n = A$$

such that $J_j/J_{j-1} \simeq \mathcal{A}(V_j, B_j, \psi_j)$. Then the algebra $\bigoplus_{j=1}^{n} M_{n_j}(B_j)$ is called the *asymptotic algebra* of the affine cellular algebra A; here n_j is the rank of the free k-module V_j.

When specialising A to be an extended affine Hecke algebra of type A and studying its affine cellular structure we will see that the asymptotic algebra in this case coincides with an algebra Lusztig defined in a completely different way as the asymptotic algebra of the extended affine Hecke algebra.

If all B_j in Theorem 3.12 are principal ideal domains, we can describe the simple modules over A in more detail by using linear algebra methods. This will be done in the next subsection.

3.3. Simple representations of affine cellular algebras over a principal ideal domain

Our general results on parameter sets of simple representations of an affine cellular algebra are based on ring theoretic considerations, in particular on our theory of swich algebras. Once the simple representations have been classified for a given class of affine cellular algebras, the next step is to try describing them in more detail, in particular by working out dimensions or characters. While a general answer cannot be expected – after all, the same question is already wide open in the finite dimensional case, as the example of symmetric groups shows – there are two possible approaches to this second step. One approach will be indicated in some of the proofs in the next section; for a given affine cell ideal, associated with a commutative ring B, a simple module has as parameter a maximal ideal of B. As we will show, it is possible to factor out this maximal ideal and to get a finite dimensional cell ideal, even a heredity ideal over a quotient algebra, such that the same simple module is the unique simple quotient of the standard module associated with this heredity ideal. In this way, one can study individual simple modules by a reduction to finite dimensional situations.

Another approach, to be worked out in this subsection, works in case the ring B is a principal ideal domain. Then linear algebra over B is available (see for instance [4, 16.6]) and we are able to derive a normal form of the affine cell ideal. Consequently we will get a much more detailed description of simple modules. One of our main results here is an exact formula for the dimension of simple modules depending only on knowing the finitely many entries of the swich element Ψ. In contrast to the first approach, this one simultaneously deals with all simple modules in a given cell. Of course, there is also a hybrid approach possible, using the first one to reduce not to a field, but to some principal ideal domain, and then using the methods of the current section to study families of simple modules.

Our setup in this subsection is as follows. Suppose A is an affine cellular algebra as in Definition 2.1, and the cell chain of A is

$$J_0 = 0 \subset J_1 \subset \cdots \subset J_n = A.$$

Throughout this section we assume that all k-algebras B_j appearing in the quotients $J_j/J_{j-1} \simeq V_j \otimes_k B_j \otimes_k V_j$ are principal ideal domains.

Let L be an arbitrary simple A-module. Then there is a minimal number t with $1 \leqslant t \leqslant n$ such that $J_t L \neq 0$. Since t is minimal, we must have $J_{t-1}L = 0$. Hence the simple A-module L is, in fact, a simple A/J_{t-1}-module, where A/J_{t-1} is again an affine cellular algebra. So we may and will assume that $t = 1$, and we fix $J = J_1$, $B = B_1$ and $m = \operatorname{rank}_k(V_1)$. Then the multiplication of the algebra J, viewed as a generalised matrix algebra, can be described by a bilinear form $\psi : V \otimes_k V \longrightarrow B$. As before, we may regard J as a generalised matrix algebra, that is, as a k-module, J is the set of all $m \times m$ matrices over B, and the multiplication $\cdot$ in J is given by a matrix Ψ (over B), that is a swich element, corresponding to ψ:

$$X \cdot Y = X\Psi Y$$

for all $X, Y \in J$. As before we denote this algebra by $\widetilde{J} = (J, \Psi)$, or simply by $\widetilde{J}$ if the meaning of Ψ is clear. The case of B being a field is already contained in Brown's work [3]. In this case, if Ψ is not zero, then J has exactly one simple module whose k-dimension equals the rank of the matrix Ψ.

Note that if Φ is another matrix such that $\Phi = P\Psi Q$ with P and Q invertible matrices over B, then the algebras (J, Φ) and (J, Ψ) are isomorphic. That is, isomorphism of generalised matrix algebras (as algebras without unit) is in terms of linear algebra given by Gauss elimination.

Since B is a principal ideal domain, there are invertible matrices P and Q such that $P\Psi Q$ is a diagonal matrix $\operatorname{diag}\{\sigma_1, \ldots, \sigma_r, 0, \ldots, 0\}$ with $0 \neq \sigma_j \mid \sigma_{j+1}$ for $j = 1, \ldots, r-1$. The scalars σ_j are called the *invariant factors* or *invariant divisors* of Ψ. The σ_j's are uniquely (up to units) determined by Ψ. If $r \neq 0$, then we call σ_1 the minimal invariant divisor of Ψ. By the above discussion, we may now assume that Ψ is a diagonal matrix $\mathrm{diag}\{\sigma_1, \dots, \sigma_r, 0, \dots, 0\}$ with $0 \neq \sigma_j$ for all j and $\sigma_j \mid \sigma_{j+1}$ for $j = 1, \dots, r-1$.

Since the matrices $X = (X_{ij})$ with $X_{ij} = 0$ for all $1 \leqslant i, j \leqslant r$ form a nilpotent ideal in J, these matrices belong to the radical of A, see Section 3 of [20]. Thus these matrices annihilate the simple module L. So, in order to describe the simple module L, from now on *we assume that* $r = m$. But we have to take into account that $r = 0$ is possible and this will produce a special case in the classification to be obtained.

We shall use the notation $E_{ij}(a)$ for the matrix with entry a at the (i, j)-position, and zero elsewhere. As usual, I_m stands for the $m \times m$ identity matrix.

Let u be an element in L such that $Ju \neq 0$. Then $L = Ju$, and there is a matrix $X \in J$ such that $Xu \neq 0$. This implies that there is also a matrix $X_0 = E_{i_0, j_0}(a)$ with $a \in B$ such that $X_0 u \neq 0$. Thus $L = J(X_0 u)$, and there is an exact sequence of A-modules:

$$0 \longrightarrow \mathrm{Ann}_J(X_0 u) \longrightarrow J \xrightarrow{\mu} J(X_0 u) = L \longrightarrow 0,$$

where μ is given by $X \longmapsto X(X_0 u) = (X\Psi X_0)u = X \cdot (X_0 u)$, and where $\mathrm{Ann}_J(v)$ denotes the annihilator of v in J; $\mathrm{Ann}_J(v)$ is a left ideal of A.

Since $L = J(X_0 u) = (J \cdot X_0)u = (J\Psi X_0)u$, we have $\Psi X_0 \neq 0$, that is, $\Psi X_0 = E_{i_0, j_0}(\sigma_{i_0} a) \neq 0$, where σ_{i_0} is a diagonal entry of Ψ. Let $c = \sigma_{i_0} a$. Thus

$$L = \overset{\hphantom{0 \cdots 0\ }i_0\hphantom{\ 0 \cdots 0}}{\begin{pmatrix} 0 & \cdots & 0 & Bc & 0 & \cdots & 0 \\ 0 & \cdots & 0 & Bc & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & Bc & 0 & \cdots & 0 \end{pmatrix}} u,$$

where the matrix is $J\Psi X_0$.

Let $V_m(B)$ denote the set of all column vectors of size m with entries in B. It has a natural J-module structure by $ax := a\Psi x$ for all $a \in J$ and $x \in V_m(B)$. Then L can be described by the following exact sequence of J-modules:

$$0 \longrightarrow M \longrightarrow V_m(B) \xrightarrow{\mu} L \longrightarrow 0.$$

Here the J-module homomorphism μ is given by sending the column vector $(b_1, \dots, b_m)^T$ in $V_m(B)$ to the element $\sum_{j=1}^{m} E_{j,i_0}(b_j c)u$ in $L = (J\Psi X_0)u$. Moreover, M is the kernel of μ, which is a maximal submodule of $V_m(B)$ such that $JV(B_m) \nsubseteq M$. We are going to determine all such M in $V_m(B)$.

We define a map $\pi_j : M \longrightarrow \widetilde{B} = \mathcal{A}(k, B, \sigma_j)$ to be the j-th projection, where $\widetilde{B}$ is the generalised matrix algebra defined by the 1×1 matrix (σ_j), that is, for $a, b \in B$, the multiplication of a and b in $\widetilde{B}$ is $a\sigma_j b$.

Claim (i). *The image* $\mathrm{Im}(\pi_j)$ *of* π_j *is an ideal in* $\widetilde{B}$.

In fact, if we pick $m \in \mathrm{Im}(\pi_j)$ and $b \in B$, then there is a column vector $x \in M$ with $\pi_j(x) = m$. Since M is a J-module, $E_{j,j}(b)x$ belongs to M and its j-th component is $b\sigma_j m = b \cdot m$. Thus $b \cdot m \in \mathrm{Im}(\pi_j)$, hence $\mathrm{Im}(\pi_j)$ is an ideal in $\widetilde{B}$.

Claim (ii). *$M + JV_m(B) = V_m(B)$. In particular, if M' is another submodule of $V_m(B)$ such that $M + M'$ is properly contained in $V_m(B)$, then $JV_m(B)$ is not contained in $M + M'$.*

Indeed, the first statement follows from exactness of the sequence $0 \longrightarrow M \longrightarrow V_m(B) \longrightarrow L = JL \longrightarrow 0$ by observing that $JV_m(B)$ maps onto $L = JL$ and that $JV_m(B) + M$ contains M and is contained in $V_m(B)$. To prove the second statement, assume that $JV_m(B)$ is contained in $M + M'$. Then $V_m(B) = M + JV_m(B) \subseteq M + M' \subseteq V_m(B)$. Thus $V_m(B) = M + M'$, a contradiction.

Now, we divide our further considerations into two cases:

Case (1). *There are units among the diagonal elements of Ψ.* Then we can write $\Psi = \mathrm{diag}\{1, \ldots, 1, \sigma_{r+1}, \ldots, \sigma_m\}$, where σ_{r+1} is not a unit (this implies that all the σ_j are non-units).

We will describe the modules M in terms of irreducible elements in B. In order to include the case of 0 being a maximal ideal, that is of B not having any irreducible element, we formally define in this situation $p = 0$ to be an irreducible element.

Lemma 3.14. *Suppose we are in Case* (1). *If M is a maximal submodule of the J-module $V_m(B)$ such that $JV_m(B) \nsubseteq M$, then there exists an irreducible element p in B such that*

$$M = \begin{pmatrix} pB \\ \vdots \\ pB \\ B \\ \vdots \\ B \end{pmatrix},$$

where pB appears r times. Conversely, if M is of this form, then M is a maximal submodule of the J-module $V_m(B)$ such that $JV_m(B) \nsubseteq M$.

Proof. By Claim (i) above, the image $\mathrm{Im}(\pi_i)$ is an ideal in B for $i = 1, \ldots, r$.

To prove that $\mathrm{Im}(\pi_j)$ is a maximal ideal in B for $1 \leqslant j \leqslant r$, we first show that $\mathrm{Im}(\pi_j) \neq B$. If not, then for each $b \in B$ there is an $x \in M$ such that $\pi_j(x) = b$. Since M is a J-module, we have $E_{l,j}(1)x = E_l(\sigma_j b) = E_l(b) \in M$ for all l. (Here, and later on, $E_l(b)$ denotes the column vector in M that has entry b at l-th place and 0 everywhere else.) Thus $V_m(B) \subseteq M$, a contradiction. We have shown that $\mathrm{Im}(\pi_j)$ is a proper ideal.

Secondly, we show that $\mathrm{Im}(\pi_j)$ is a maximal ideal in B. If not, then there exists an ideal I' of B with $I \subsetneq I' \subsetneq B$. Since $V_m(I')$ is a submodule of $V_m(B)$, the j-th component of $M + V_m(I')$ is contained in I'. It follows that $M + V_m(I')$ is a proper submodule of $V_m(B)$. Note that $M \subsetneq M + V_m(I') \subsetneq V_m(B)$. This contradicts the maximality of M by Claim (ii) above (note that we are using the particular kind of maximality as used in the statement of (ii)).

So we have shown the claim that $\mathrm{Im}(\pi_j)$ is a maximal ideal in B for all $1 \leqslant j \leqslant r$. Thus $\mathrm{Im}(\pi_j) = Bp_j$ for an irreducible element p_j in B since B is a principal ideal domain.

Next, we show that all $\mathrm{Im}(\pi_j)$ are equal for $1 \leqslant j \leqslant r$. If not, this means that $p_j \notin Bp_l$ and $p_l \notin Bp_j$, for some $j \neq l$. Since both Bp_j and Bp_l are maximal ideals in B, we have

$Bp_j + Bp_l = B$. In this case, there are two elements u and v in B such that $p_j u + p_l v = 1$ since B is a principal ideal domain. Using this fact and $JM \subseteq M$ together with $E_{lj}(w)x = E_l(w\pi_j(x))$ for $w \in B$ and $x \in V_m(B)$, we can conclude that $V_m(B) \subseteq M$, a contradiction. Thus all p_j are equal and we denote their common value by p.

Finally, we show that M is of the desired form. Let

$$M' = \begin{pmatrix} pB \\ \vdots \\ pB \\ B \\ \vdots \\ B \end{pmatrix},$$

where pB appears r times. Then M' is a J-submodule of $V_m(B)$ and $M \subseteq M' \subsetneq V_m(B)$. By Claim (ii) and the maximality of M, we have that $M = M'$.

Conversely, if M is of this form, then $V_m(B)/M$ is a simple J-module. Indeed, this follows by verifying (1) $JM \neq 0$ and (2) for each non-zero element $\bar{x} \in V_m(B)/M$ and an arbitrary element $\bar{b} \in V_m(B)/M$ there is a matrix $X \in J$ such that $X\bar{x} = \bar{b}$. □

Now we proceed to the second case.

Case (2). *There are no units among the diagonal elements of* Ψ. So, $\Psi = \operatorname{diag}\{\sigma_1, \dots, \sigma_m\}$, where all σ_j are non-units in B.

Lemma 3.15. *Suppose we are in Case* (2). *If M is a maximal submodule of the J-module $V_m(B)$ such that $JV_m(B) \nsubseteq M$, then there is an irreducible element p in B such that p does not divide σ_1, and that*

$$M = \begin{pmatrix} pB \\ \vdots \\ pB \\ B \\ \vdots \\ B \end{pmatrix},$$

where pB appears r times with r being the maximal number such that p does not divide σ_r. Conversely, if M is of this form, then M is a maximal submodule of the J-module $V_m(B)$ such that $JV_m(B) \nsubseteq M$.

Proof. We start by showing that $\operatorname{Im}(\pi_1)$ is a maximal ideal in B.

In fact, by (i), $\operatorname{Im}(\pi_1)$ is an ideal in $\widetilde{B}$. Note that an ideal in $\widetilde{B}_1$ is not automatically an ideal in B. Let $B\operatorname{Im}(\pi_1)$ be the ideal of B generated by $\operatorname{Im}(\pi_1)$ in B.

Claim 1. $B\operatorname{Im}(\pi_1) \neq B$.

Otherwise, we may write $1 = \sum_{j=1}^{s} b_j x_j$ with $x_j \in \operatorname{Im}(\pi_1)$ and $b_j \in B$. Then, for an arbitrary $b \in B$, we have $b = \sum_{j=1}^{s} bb_j x_j$ and $b\sigma_1 = \sum_{j=1}^{s} bb_j\sigma_1 x_j$. Suppose that $x_l = \pi_1(m_l)$ for $m_l \in M$. This implies that

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ b\sigma_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = E_{j,1}(bb_1)m_1 + E_{j,1}(bb_2)m_2 + \cdots + E_{j,1}(bb_s)m_s$$

lies in M for all j. Since σ_1 divides all the other invariant factors, we get $JV_m(B) \subseteq M$, a contradiction. So we have shown Claim 1, that is, $B\,\mathrm{Im}(\pi_1) \neq B$.

Claim 2. *$B\,\mathrm{Im}(\pi_1)$ is a maximal ideal in B.*

Let I be a maximal ideal of B containing $B\,\mathrm{Im}(\pi_1)$. Then $V_m(I)$ and $M + V_m(I)$ both are J-submodules of $V_m(B)$. Note that the set of all first components of elements in $M + V_m(I)$ is contained in I. This shows that $M + V_m(I) \subsetneq V_m(B)$. By property (ii), $JV_m(B)$ is not contained in $M + V_m(I)$. Hence we must have $M + V_m(I) = M$ by the maximality of M. In particular, $\mathrm{Im}(\pi_1) = I$. This shows Claim 2, that is, $\mathrm{Im}(\pi_1)$ is a maximal ideal in B.

Since B is a principal ideal domain, there exists an irreducible element p in B such that $\mathrm{Im}(\pi_1) = Bp$.

Claim 3. *p does not divide σ_1.*

Assume it does so, then $JV_m(B) \subseteq V_m(B\sigma_1) \subseteq V_m(Bp)$. Moreover, $V_m(B) = M + JV_m(B) \subseteq M + V_m(Bp) \subsetneq V_m(B)$, where properness of the last inclusion follows by checking the first component of $M + V_m(Bp)$; indeed, the entries are multiples of p. This is a contradiction. So we have proved Claim 3, that p does not divide σ_1.

Let r be the maximal number of those j such that p does not divide σ_j. Then we define

$$M' = \begin{pmatrix} pB \\ \vdots \\ pB \\ B \\ \vdots \\ B \end{pmatrix},$$

where pB appears r times.

Claim 4. *M' is a maximal J-submodule of $V_m(B)$.*

Clearly, this module is not equal to $V_m(B)$. To see it is a maximal submodule of $V_m(B)$, we consider the quotient $V_m(B)/M'$. As in the proof of Case (1), we can show that the quotient is a simple J-module, thus proving Claim 4.

Now it follows from the maximality of M and M' that $M = M'$ since $M + M' \subsetneq V_m(B)$ by $\pi_1(M + M') = Bp$.

The converse also follows as in the proof of Case (1). □

Thus, all simple J-modules, and therefore all simple A-modules which are not annihilated by J, are completely determined by Ψ, that is, by a finite set of data.

Summarising the above discussions, we get the following result on simple modules of affine cellular algebras if all B_j are principal ideal domains.

Theorem 3.16. *Let A be an affine cellular k-algebra with cell chain $J_0 = 0 \subset J_1 \subset \cdots \subset J_n = A$ such that J_j/J_{j-1} is a generalised matrix algebra $\mathcal{A}(B_j, \Psi_j)$ with B_j a principal ideal domain. Then:*

(1) *The isomorphism classes of simple modules are parameterised by $\{(j, p) \mid 1 \leqslant j \leqslant n,$ $\Psi_j \neq 0,$ p is an irreducible element (representing an equivalence class up to multiplication by units) of B_j such that p does not divide the minimal invariant divisor of $\Psi_j\}$.*
Here we define $p = 0$, if there is no irreducible element in B_j.
(2) *If $L(j, p)$ is the simple A-module corresponding to (j, p), then*

$$\dim_{B_j/(p)} L(j,p) = \begin{cases} r, & \text{if some invariant divisors of } \Psi_j \text{ are units, and} \\ & r \text{ is the number of such unit divisors}, \\ r, & \text{if no invariant divisor of } \Psi_j \text{ is a unit, and} \\ & r \text{ is the number of invariant divisors not divisible by } p. \end{cases}$$

(3) *Suppose k is a field. If $L(j, p)$ is the simple A-module corresponding to (j, p), then*

$$\dim_k L(j,p) = \begin{cases} r \cdot \dim_k(B_j/(p)), & \text{if some invariant divisors of } \Psi_j \text{ are units, and} \\ & r \text{ is the number of such unit divisors}, \\ r \cdot \dim_k(B_j/(p)), & \text{if no invariant divisor of } \Psi_j \text{ is a unit, and} \\ & r \text{ is the number of invariant divisors not divisible by } p. \end{cases}$$

Proof. (1) Let L be a simple A-module. Then there is a minimal j such that $J_j L \neq 0 = J_{j-1} L$. Thus L is a simple A/J_{j-1}-module, and also a simple J_j/J_{j-1}-module. So, $J_j^2 \nsubseteq J_{j-1}$, that is, $\Psi_j \neq 0$. If B_j is a field, then there is only one simple J_j/J_{j-1}-module (up to isomorphism). Thus L can be labelled as $L(j, 0)$. If B_j is not a field, then, by Lemmas 3.14 and 3.15, there is a maximal ideal of B_j generated by an irreducible element p (unique up to units) such that p does not divide the minimal invariant factor of Ψ_j, and L is isomorphic to a unique simple J_j/J_{j-1}-module associated with p. Thus L can be labelled as $L(j, p)$. On the other hand, each simple J_j/J_{j-1} is a simple A-module by Lemma 3.1. Note that for different l and j, the simple A-modules $L(l, q)$ and $L(j, p)$ are not isomorphic. Also, $L(j, p)$ and $L(j, q)$ are not isomorphic if there is no unit u in B_j such that $q = pu$. This shows the statement (1).

Statements (2) and (3) follow from the proofs of Lemmas 3.14 and 3.15. □

Remarks. (1) If all B_j in Proposition 2.3 are isomorphic to k with k a field, then we get back the notion of cellular algebra (see [7]). In this way Theorem 3.16 generalises a main result of the representation theory of cellular algebras over a field. Moreover, Theorem 3.16 extends also some results in the representation theory of linear semigroups in [24, pp. 111–120].

(2) In Theorem 3.16(3) the dimension $\dim_k(B_j/(p))$ of a simple module $B_j/(p)$ may be infinite. However, if B_j is a polynomial algebra $k[X]$ or a Laurent polynomial algebra $k[X_1^{\pm}, \ldots, X_n^{\pm}]$ over the ground field k, then all simple A-modules are finite dimensional over k.

4. Idempotent ideals and cohomology

Having classified the simple modules of an affine cellular algebra we proceed now to develop a structure theory of affine cellular algebras. Let us first recall relevant features of finite dimensional cellular algebras. Given a cell ideal J in a finite dimensional algebra A over a field, there are two cases (see [17]). Either J is nilpotent and then $J^2 = 0$ or J is what is called a *heredity* ideal. Recall that an ideal J in A is called a heredity ideal if (a) J is idempotent, that is, $J^2 = J$ (and hence generated by an idempotent), (b) it is projective as a left and right A-module, (c) its endomorphism ring $\mathrm{End}_A(J)$ is semisimple and (d) the multiplication map $Ae \otimes_{eAe} eA \longrightarrow J$ is always an A-bimodule isomorphism. Note that in the cellular case $\mathrm{End}_A(J)$ is even a simple ring. It is exactly in this second case when J is contributing a simple module to the classification of simple A-modules. In [19] we have shown that a cellular algebra A has finite global (= cohomological) dimension if and only if in one, and then any, of its cell chains each subquotient is of the second kind, that is a heredity ideal in the respective quotient. An even stronger property is true in this case; the (bounded or unbounded) derived category of A-modules admits a stratification by derived categories of simple algebras.

Now let A be an affine cellular k-algebra and let J be an affine cell ideal in its cell chain. Our aim in this section is to determine precisely the connection between the properties of J to be idempotent, or (not equivalent in general) to be generated by an idempotent, to be left or right projective over A, the property of the endomorphism ring of Δ to be B, the property of the parameter set of this layer to be the full affine space, and finiteness of the global dimension of A. In particular, we will get a sufficient criterion for finiteness of the global dimension of A and for the existence of a stratification of its derived module categories. In the application to affine Hecke algebras this criterion will be satisfied in a very natural situation. Thus we get meaningful and very general extensions of many results that have been fundamental in the finite dimensional situation and which are in particular crucial in the theory of quasi-hereditary algebras and highest weight categories. It has to be noted, however, that all the proofs developed in this section are completely different from all the proofs known in the finite dimensional situation, and technical statements behind the facts just mentioned are unexpected and reveal new features of this theory.

We mention at this point an *open problem*; we do not know a precise characterisation of finiteness of global dimension that would be a rather far reaching generalisation of the main result of [19]. In particular, we do not know what datum should play the role of the Cartan determinant for affine cellular algebras.

We start by discussing the following question of independent interest. How to characterise when the embedding of the affine parameter set of the cell layer, seen as a generalised matrix algebra, into the affine parameter space of the corresponding ordinary matrix algebra, is the identity? In the language of the previous section, this means, when does each module E_φ have a composition factor? There is a natural answer in terms of cell ideals being idempotent.

Theorem 4.1. *Suppose $J = V \otimes_k B \otimes_k V$ is an affine cell ideal in a k-algebra A with a bilinear form $\psi : V \otimes_k V \longrightarrow B$ which defines multiplication inside J.*

(1) *The ideal J is idempotent if and only if for each maximal ideal m of B, the cell ideal $J/J(\mathsf{m})$ (of $\bar{A}$) is a heredity ideal in the quotient algebra $\bar{A} := A/J(\mathsf{m})$ if and only if each simple module of $M_n(B)$ has a composition factor when viewed as a J-module if and only if the embedding of the parameter set of J-simples into the parameter set of $M_n(B)$-simples is the identity map. Here we denote by $J(\mathsf{m})$ the set $V \otimes_k \mathsf{m} \otimes_k V$ in J.*

(2) *Suppose J is idempotent and m is a maximal ideal in B. Then the A-module $W(\mathsf{m}) := V \otimes_k (B/\mathsf{m})$ has endomorphism algebra isomorphic to B/m. In this case $\bar{J} := J/J(\mathsf{m})$ contains a primitive idempotent e such that $\bar{A}e\bar{A} = \bar{J}$; any primitive idempotent f in $\bar{J}$ is equivalent to e, that is, $\bar{A}f \simeq \bar{A}e$.*

Note that by J being an idempotent ideal we mean $J^2 = J$. This does not imply that J is generated by an idempotent element; the existence of such an idempotent element will be discussed later.

Proof of Theorem 4.1. First we need to explain some terminology used in statement (2) and a way to produce finite dimensional ideals from the given one. Suppose A is an affine cellular algebra over a principal ideal domain k, and $J \neq 0$ is a cell ideal in A. So J is a swich algebra $\tilde{\Lambda}$ of a matrix algebra $\Lambda = M_n(B)$ with the swich element $a_0 = \Psi$ in Λ. Note that Λ is a noetherian ring with identity. Suppose now that m is a maximal ideal in B, hence B/m is a field, denoted by K. By the multiplication rule in J, we see that $J(\mathsf{m}) := V \otimes_k \mathsf{m} \otimes_k V$ is an ideal in $J = V \otimes_k B \otimes_k V$. Moreover, it is also an ideal in the given affine cellular algebra A by Proposition 2.2(3). If m is fixed by the involution i of A then $A/J(\mathsf{m})$ actually is an affine cellular algebra with respect to the involution $\bar{i}$ induced from i on A since i fixes the ideal $J(\mathsf{m})$; this is obvious from the isomorphism $(A/J(\mathsf{m}))/(J/J(\mathsf{m})) \simeq A/J$ and the description of affine cell ideals in terms of generalised matrix algebras. In general this may not be the case, but it still makes sense to ask whether $J/J(\mathsf{m})$ is a heredity ideal in $A/J(\mathsf{m})$; note that the definition of heredity ideal does not refer to an involution i.

To prove the equivalences in (1) we first suppose that for each simple Λ-module L we have $\tilde{\Lambda}L \neq 0$, that is, $a_0 L \neq 0$. We claim that J is an idempotent ideal in A, that is, $J^2 = J$. Using the definition of the algebra $\tilde{\Lambda}$, J being idempotent is equivalent to the equality $\tilde{\Lambda} \cdot \tilde{\Lambda} = \tilde{\Lambda}$ that in turn is equivalent to $\Lambda a_0 \Lambda = \Lambda$. To prove the claim, we observe first that each simple Λ-module L is the quotient of Λ by some maximal left ideal in Λ. Therefore, the set $a_0\Lambda$ is not contained in any maximal left ideal in Λ. So the ideal $\Lambda a_0 \Lambda$ is not contained in any maximal left ideal of Λ. On the other hand, Λ is a noetherian algebra and every proper left ideal of Λ is contained in a maximal left ideal of Λ. This implies $\Lambda a_0 \Lambda = \Lambda$ and hence the claim.

Conversely, if J is an idempotent affine cell ideal in A, then $\Lambda a_0 \Lambda = \Lambda$, which implies that $a_0 L \neq 0$ for all simple Λ-modules L. Thus we have shown that J is idempotent if and only if each simple B-module provides a simple A-module if and only if for each maximal ideal m in B there are elements $u, v \in V$ such that $\psi(u, v)$ does not lie in m.

Now we assume that J is an idempotent affine cell ideal in a k-algebra A: We claim that $J/J(\mathsf{m})$ is a heredity ideal in $A/J(\mathsf{m})$. To prove the claim, we denote $J/J(\mathsf{m})$ by $\bar{J}$ and $A/J(\mathsf{m})$ by $\bar{A}$. Clearly, $\bar{J} = V \otimes_k K \otimes_k V$. Since J is idempotent in A, also $\bar{J}$ is idempotent in $\bar{A}$. Moreover, the ideal J being idempotent implies also that $J^2 \nsubseteq J(\mathsf{m})$; Otherwise, the ideal of B generated by all $\psi(u, v)$ for $u, v \in V$ would be contained in m. So we choose elements $u, v \in V$ such that $\psi(u, v) \notin \mathsf{m}$, and we calculate the product $(v \otimes_k 1 \otimes_k u)^2 = v \otimes_k \psi(u, v) \otimes_k u$. Since $K = B/\mathsf{m}$ is a field, we have an element $\lambda \in B$ such that $\lambda\psi(u, v) \equiv 1 \pmod{\mathsf{m}}$. This means that the element $v \otimes_k \bar{\lambda} \otimes_k u$ is an idempotent element in $\bar{J}$. Let $e = v \otimes_k \bar{\lambda} \otimes_k u$. Then $\bar{A}e = \bar{J}e = V \otimes_k K \otimes_k u$, $J(\mathsf{m})\bar{J}e = 0$ and $e\bar{J}e \simeq K$. This implies that $e\bar{A}e \simeq K$. Moreover, since $\bar{A}e = \bar{J}e$ we get $\bar{J} = (V \otimes_k K) \otimes_K (K \otimes_k V) \simeq \bar{J}e \otimes_K e\bar{J} = \bar{A}e \otimes_{e\bar{A}e} e\bar{A}$. Thus $\bar{J}$ is a heredity ideal in $\bar{A}$. Note that $\bar{J}$ is a finite dimensional K-module, and thus the known results on finite dimensional heredity ideals can be applied and all assertions in (2) follow readily as well as the remaining equivalence in (1). □

We note that the proof of (2) goes through for a particular ideal m in B provided the image of ψ is not contained in m. That is, a simple A-module L corresponding to a simple module of the affine cell ideal J, also is a simple module over some quotient algebra $A/J(\mathsf{m})$ and then it corresponds to a heredity ideal. This provides the reduction to a finite dimensional situation mentioned above in Section 3.3. In order to study the simple modules associated with the cell ideal, one may use all the linear algebra methods used in the theory of finite dimensional cellular algebras. The fact, that the quotient algebra as a whole still may have infinite dimension over k (when k is a field) does not matter in this context.

By the known properties of heredity ideals it follows that the endomorphism ring of L is B/m. By induction we get:

Corollary 4.2. *Let A be an affine cellular algebra. Then the endomorphism ring of the simple A-module $L = L(j, \mathsf{m})$ corresponding to the index (j, m) is isomorphic to the field B_j/m.*

In this way, $L = L(j, \mathsf{m})$ becomes a module over the field B_j/m and as such it is finite dimensional.

We reiterate that, a priori, a module over a generalised matrix algebra over B is not a B-module, since there is no natural B-action on the generalised matrix algebra. In Theorem 3.12 we have already shown that $L = L(j, \mathsf{m})$ actually is a B-module and as such it is semisimple, artinian and noetherian. The corollary makes this statement more precise.

Before we proceed, let us recall the finite dimensional situation. Then, each layer in a cell chain contributes either none or exactly one simple module to the classification, and the second case appears if and only if the subquotient is a heredity ideal. In the language of Theorem 4.1 this means that when $k = B$ is a field, the parameter set of simples corresponding to the particular cell equals the full affine space $\mathrm{MaxSpec}(k)$ if and only if the ideal is idempotent. In the finite dimensional situation it is then automatic that J is generated by an idempotent, that it is projective over A on either side and that its endomorphism ring is simple. In our very general situation these consequences may fail. In particular, idempotents cannot, in general, be lifted from the finite dimensional situation over B/m to the situation over B. To show the existence of idempotents for examples, it will be necessary to use special features of these examples, as in the case of affine Hecke algebras to be discussed below. Concerning projectivity and endomorphism ring, we do, however, prove natural analogues of the finite dimensional statements under realistic assumptions on the ring B associated with J, for example if B is a (Laurent) polynomial ring (as in all our applications).

The affine algebra B occurring in our setup is, by definition, a quotient of a polynomial ring in finitely many variables. Let m be a maximal ideal in B. Then $\bigcap_{j\in\mathbb{N}} \mathsf{m}^j = 0$, since the same holds true for the preimage of m in the polynomial ring. This property will be used in the proof of the first statement of the following theorem. The second statement needs an additional assumption, which excludes, for instance, non-trivial finite dimensional quotients of polynomial rings. This condition is satisfied if B is a polynomial algebra or a Laurent polynomial algebra in finitely many variables, which is the case in all our examples.

Theorem 4.3. *Let $J = V \otimes_k B \otimes_k V$ be an idempotent affine cell ideal in a k-algebra A with the cell lattice $\Delta = V \otimes_k B$.*

(1) *If there is a non-zero idempotent e in J, then ${}_AJ$ is a projective A-module and $J = AeA$.*
(2) *Suppose that $\bigcap_{\mathsf{m}} \mathsf{m} = 0$, where m runs over all maximal ideals in B, that is,* $\mathrm{rad}(B) = 0$. *Then* $\mathrm{End}({}_A\Delta) \simeq B$.

Proof. Let $J = V \otimes_k B \otimes_k V$ be an idempotent affine cell ideal in a k-algebra A, equipped with a bilinear form $\psi : V \otimes_k V \longrightarrow B$ defining the multiplication inside J. We are first going to prove (1). Suppose that $0 \neq e$ is an idempotent element in J. Let m be a maximal ideal in B; quotients indicated by overlining are to be taken modulo m. Since J is idempotent, the cell ideal $\bar{J} := J/J(\mathsf{m})$ is heredity for each maximal ideal m of B by Theorem 4.1. Thus there is a primitive idempotent element f in $\bar{J}$ such that $\bar{A} f \bar{A} = \bar{J}$.

Claim 1. *The idempotent e is not contained in* $J(\mathsf{m})$, *hence* $\bar{e} \neq 0 \in \bar{J}$.

Proof. Assume to the contrary that $e \in J(\mathsf{m})$, that is, it is represented in $J = V \otimes_k B \otimes_k V$ by a matrix with all entries in m. Using the definition of multiplication in J as matrix multiplication involving the sandwich element ψ, we get $e^j \in (J(\mathsf{m}))^j$ for all j. This means that $e \in \bigcap_j J(\mathsf{m}^j) = J(\bigcap_j \mathsf{m}^j) = 0$ and $e = 0$, a contradiction. Thus Claim 1 is true. □

Claim 2. $AeA + J(\mathsf{m}) = J$ *for each maximal ideal* m *of* B.

Proof. Since $\bar{J}$ is finite dimensional over $B/(\mathsf{m})$, the idempotent $\bar{e} \in \bar{J}$ can be written as a finite sum of pairwise orthogonal primitive idempotent elements in $\bar{J}$. Theorem 4.1 implies that $\bar{A}\bar{e}\bar{A} = \bar{A} f \bar{A} = \bar{J}$. Thus Claim 2 is true. □

Claim 3. AeA *is an* A-B-*submodule of* J.

Proof. The left A-structure is clear; we have to show that AeA is a right B-module. Note that ${}_A J_B \simeq {}_A\Delta_B^n$ with n being the rank of V. So, we may identify the right B-module structure on J with the action of the image of $\beta : B \longrightarrow \mathrm{End}({}_A J)$, where β is the right multiplication of B-elements on J. With this identification, the ideal AeA is invariant under $\mathrm{End}({}_A J)$ by the following computation. If $g : J \longrightarrow J$ is an A-endomorphism and $aea' \in AeA$ with $a, a' \in A$, then $(aea')g = (ae \cdot ea')g = ae(ea')g \in AeA$. This shows that AeA is a right B-submodule of J, and Claim 3 is true. □

Claim 4. $AeA = J$.

Proof. Combining Claims 2 and 3, we have $AeA + J(\mathsf{m}) = J$ as right B-modules. Passing to the localisation at m, we get an equality:

$$(AeA)_{\mathsf{m}} + J(\mathsf{m})_{\mathsf{m}} = J_{\mathsf{m}}.$$

The right-hand side is a direct sum of copies of the local ring B_{m}. The Jacobson radical of J_{m} is $J(\mathsf{m})_{\mathsf{m}}$. Since J_{m} is a finitely generated B_{m}-module, we may apply Nakayama's Lemma to get $(AeA)_{\mathsf{m}} = J_{\mathsf{m}}$ for each maximal ideal m in B. Hence $AeA = J$. Claim 4 has been shown. □

From now on we are going to use the assumption $J^2 = J$.

Claim 5. *Each element* $f \in \mathrm{End}({}_A\Delta)$ *lies in* $\mathrm{End}(\Delta_B)$, *that is, A-endomorphisms of* Δ *are automatically B-morphisms as well. Multiplication by* $b \in B$ *on the right provides an endomorphism in* $\mathrm{End}({}_A\Delta)$ *that lies in the centre of* $\mathrm{End}({}_A\Delta)$.

Proof. We identify J with $(M_n(B),\Psi)$, as before. Then the bimodule ${}_A\Delta_B$ gets identified with the set $V_n(B)$ of all $n\times 1$ matrices over B. The swich algebra J acts on Δ via the matrix Ψ, which is the swich element in this case, and B acts on the right by multiplying with scalars in B, which is the same as multiplying $x\in V_n(B)$ on the left with the diagonal matrix $d_b=\operatorname{diag}\{b,b,\dots,b\}$ for $b\in B$. In other words, we have $xb=d_bx$. Thus we have an embedding β of B into $\operatorname{End}({}_A\Delta)$ by sending b to $d_b\in\operatorname{End}({}_A\Delta)$.

Now we shall show that d_b lies in the centre of $\operatorname{End}({}_A\Delta)$. Let $f\in\operatorname{End}({}_A\Delta)$, $x\in V_n(B)=\Delta$ and let $b\in B$. Then we have the following chain of equalities, where Ψ is the swich element and the sign $\cdot$ indicates products in the generalised matrix ring J: $\Psi(f(x)b)=\Psi(d_bf(x))=(\Psi d_b)f(x)=d_b\Psi f(x)=d_b\cdot f(x)=f(d_b\cdot x)=f(d_b\Psi x)=f(\Psi d_bx)=f(\Psi(xb))=f(I_n\cdot(xb))=I_n\cdot f(xb)=\Psi f(xb)$. Thus $\Psi(f(x)b-f(xb))=\Psi(d_bf-fd_b)(x)=0$ for all $x\in V_n(B)$. This means that $\Psi(d_bf-fd_b)(V_n(B))=0$. Note that $g:=d_bf-fd_b\in\operatorname{End}({}_A\Delta)$. Since $J^2=J$, we have $J\Delta=\Delta$, that is, $M_n(B)\Psi V_n(B)=V_n(B)$. Thus $g(\Delta)=g(J\Delta)=Jg(\Delta)=M_n(B)\Psi g(V_n(B))=0$ and $g=0$. This shows that d_b lies in the centre of $\operatorname{End}({}_A\Delta)$ and $f\in\operatorname{End}(\Delta_B)$. Thus Claim 5 is true. □

Claim 6. *There is an equality* $\operatorname{add}({}_AAe)=\operatorname{add}({}_A\Delta)$.

Proof. We first show that $Ae\in\operatorname{add}({}_A\Delta)$. Multiplying by e gives a surjective homomorphism $AeA\longrightarrow Ae=AeAe$. This implies that Ae is a direct summand of AeA. Hence $Ae\in\operatorname{add}({}_A\Delta)$.

In the following, we shall show $\Delta\in\operatorname{add}(Ae)$. We first observe that if X is an A-submodule of Δ such that $X=X\mathsf{m}:=\sum_{x\in X}x\mathsf{m}$ for a maximal ideal m in B, then $X=0$. In fact, the definition of the bimodule ${}_A\Delta_B$ implies that $(V\otimes_k B)\mathsf{m}^j=V\otimes_k\mathsf{m}^j$ for all $j\geqslant 1$. Therefore, $X=X\mathsf{m}^j\subseteq(V\otimes_k B)\mathsf{m}^j\subseteq V\otimes_k\mathsf{m}^j$. However, $\bigcap_{\mathrm{j}}(V\otimes_k\mathsf{m}^j)=0$. Thus we have proved the observation.

Now we may assume that $\Delta=X_1\oplus X_2$ is a decomposition of Δ into A-submodules, where $0\neq X_1\in\operatorname{add}({}_AAe)$, and where $X_2\notin\operatorname{add}({}_AAe)$ is a complement, which we have to show to be equal to zero. Let f_i be the idempotent in $\operatorname{End}({}_A\Delta)$, which is the projection of Δ onto X_i. Since f_i commutes with elements in B, we see that $X_i=\operatorname{Im}(f_i)\supseteq f_i(\Delta\mathsf{m})=f_i(\Delta)\mathsf{m}=X_i\mathsf{m}$. By passing to the quotient module with respect to a maximal ideal m of B, we get that $W(\mathsf{m}):=\Delta/(\Delta\mathsf{m})=(V\otimes_k B)/(V\otimes_k\mathsf{m})=V\otimes_k(B/\mathsf{m})=\bar{X}_1\oplus\bar{X}_2$. Since $W(\mathsf{m})$ is indecomposable by Theorem 4.1(1), one of the $\bar{X}_1$ and $\bar{X}_2$ must be zero. Note that X_1 contains a non-zero direct summand of Ae and $\bar{e}\neq 0$. So $\bar{X}_2=0$. This is true for an arbitrary maximal ideal m in B. By our observation, we get $X_2=0$. Hence $\Delta\in\operatorname{add}(Ae)$, as desired. This finishes the proof of Claim 6. □

This finishes the proof of statement (1) and we proceed to prove statement (2).

As before, we identify J with $(M_n(B),\Psi)$. Again, the bimodule ${}_A\Delta_B$ gets identified with the set $V_n(B)$ of all $n\times 1$ matrices over B, where J acts on Δ via the matrix Ψ and B acts from the right-hand side by multiplying with scalars in B. By Claim 5, the B-action is in the centre of $\operatorname{End}({}_A\Delta)$. Thus we may write an element f in $\operatorname{End}({}_A\Delta)$ as an $n\times n$ matrix (f_{jl}) in $M_n(B)=\operatorname{End}(\Delta_B)$, where we identify Δ_B with $V_n(B)_B$. Let m be a maximal ideal in B; then we can identify $\Delta\mathsf{m}$ with $V_n(\mathsf{m})$. Since each f in $\operatorname{End}({}_A\Delta)$ induces a homomorphism on $\Delta\mathsf{m}$ by restriction, f induces a homomorphism $\bar{f}:\Delta/(\Delta\mathsf{m})\longrightarrow\Delta/(\Delta\mathsf{m})$ of A-modules. Identifying $\Delta/(\Delta\mathsf{m})$ with $V_n(B/\mathsf{m})$, we get that $\bar{f}=\overline{(f_{jl})}=(\overline{f_{jl}})$, where $\overline{f_{jl}}\in B/\mathsf{m}$. By Theorem 4.1(1), $(\overline{f_{jl}})$ is a scalar matrix over B/m. Hence $f_{jl}\in\mathsf{m}$ for $j\neq l$, and $f_{jj}\equiv f_{ll}\pmod{\mathsf{m}}$. It follows from $\bigcap_{\mathsf{m}}\mathsf{m}=0$ that $f_{jl}=0$ for $j\neq l$, and $f_{jj}=f_{ll}$. Hence $f\in B$, and $\operatorname{End}({}_A\Delta)=B$. □

Using the structure exhibited by the previous results, we can now state a strong homological consequence. Here $\mathrm{gldim}(A)$ denotes the left global dimension of A, that is the maximum degree in which cohomology does not vanish on the category of all left modules. We refer to [16] for the concepts of stratifications and recollements of derived categories used in the following result. Note that in case of finite global dimension a stratification of the unbounded derived category implies one on the bounded level, see [16].

Theorem 4.4. *Let A be an affine cellular algebra with a cell chain $J_0 = 0 \subset J_1 \subset \cdots \subset J_n = A$ such that $J_j/J_{j-1} = V_j \otimes_k B_j \otimes_k V_j$ as in Definition* 2.1. *Suppose that each B_j satisfies* $\mathrm{rad}(B_j) = 0$. *Suppose moreover that each J_j/J_{j-1} is idempotent and contains a non-zero idempotent element in A/J_{j-1}. Then:*

(a) *The unbounded derived category $D(A\text{-Mod})$ of A admits a stratification, that is an iterated recollement whose strata are the derived categories of the various algebras B_j.*
(b) *The global dimension* $\mathrm{gldim}(A)$ *is finite if and only if* $\mathrm{gldim}(B_j)$ *is finite for all j.*

Proof. (a) The assertion on the stratification follows by induction from a standard fact (see [16]): Let J be a projective ideal in an algebra Λ; then the surjective ring homomorphism $\Lambda \longrightarrow \Lambda/J$ is a homological epimorphism. Therefore there exists a recollement of unbounded derived categories relating the rings Λ/J, Λ and $e\Lambda e$.

Note that the derived category of an algebra of finite global dimension in general need not have a stratification. The existence of a stratification implies further statements, in particular on vanishing of cohomology.

(b) By induction on the length of the cell chain, it is sufficient to consider the following situation: Let $J = J_1$ be an idempotent affine cell ideal in A and the affine cellular algebra A/J has finite global dimension. By Theorem 4.3, the left module ${}_AJ$ is projective and $\mathrm{add}(Ae) = \mathrm{add}({}_A\Delta)$ for some non-zero idempotent e. Also, the endomorphism algebra of Δ is Morita equivalent to eAe, thus Morita equivalent to B_1 by Theorem 4.3. Summing up, it is sufficient to prove the following general statement.

Lemma 4.5. *Let Λ be a ring, $e^2 = e \in \Lambda$, and ${}_\Lambda J := \Lambda e\Lambda$. If ${}_\Lambda J$ is projective, then* $\mathrm{gldim}(\Lambda)$ *is finite if and only if both* $\mathrm{gldim}(\Lambda/J)$ *and* $\mathrm{gldim}(e\Lambda e)$ *are finite.*

Proof. We denote by $\mathrm{pd}({}_\Lambda X)$ the projective dimension of a module X over Λ. Clearly, $\mathrm{pd}({}_\Lambda Y) \leqslant \mathrm{pd}({}_{\Lambda/J}Y) + \mathrm{pd}({}_\Lambda \Lambda/J) \leqslant \mathrm{pd}({}_{\Lambda/J}Y) + 1$ for $Y \in \Lambda/J\text{-Mod}$. Observe that $\Lambda e \otimes_{e\Lambda e} e\Lambda \simeq \Lambda e\Lambda$ since ${}_\Lambda \Lambda e\Lambda$ is projective (see [5]).

Suppose $m := \mathrm{gldim}(\Lambda) < \infty$. Then, it is shown in [5] that $\mathrm{gldim}(\Lambda/J) \leqslant \mathrm{gldim}(\Lambda) = m$. To see $\mathrm{gldim}(e\Lambda e) < \infty$, we pick an $e\Lambda e$-module X, and consider the Λ-module $\Lambda \otimes_{e\Lambda e} X$. Since $\mathrm{gldim}(\Lambda) \leqslant m$, there is a projective resolution of $\Lambda e \otimes_{e\Lambda e} X$, say

$$0 \longrightarrow P_m \longrightarrow \cdots \longrightarrow P_0 \longrightarrow \Lambda e \otimes_{e\Lambda e} X \longrightarrow 0.$$

This gives rise to an exact sequence

$$0 \longrightarrow eP_m \longrightarrow \cdots \longrightarrow eP_0 \longrightarrow X \longrightarrow 0$$

in $e\Lambda e$-Mod. If we can show that each eP_j is a projective $e\Lambda e$-module, then $\mathrm{pd}({}_{e\Lambda e}X) \leqslant m$ and $\mathrm{gldim}(e\Lambda e) \leqslant m$. Since ${}_\Lambda \Lambda e\Lambda$ is projective, the map $\bigoplus_{x \in e\Lambda} \Lambda e \longrightarrow \Lambda e\Lambda$ given by

$(a_x)_{x\in e\Lambda} \longmapsto \sum_{x\in e\Lambda} a_x x$ is surjective, and therefore $\Lambda e\Lambda$ is a direct summand of $\bigoplus_{x\in e\Lambda} \Lambda e$. This means that $e\Lambda e\Lambda$ is a direct summand of $e(\bigoplus_{x\in e\Lambda} \Lambda e)$, which is a free $e\Lambda e$-module. The inclusions $e\Lambda \subset \Lambda e\Lambda \subset \Lambda$ imply $e\Lambda e\Lambda \subset e\Lambda \subset e\Lambda e\Lambda$. Therefore $e\Lambda = e\Lambda e\Lambda$ is a projective $e\Lambda e$-module, and direct summands of copies of $e\Lambda$ are projective $e\Lambda e$-modules. This implies that each eP_j is a projective $e\Lambda e$-module.

Conversely, we assume that both $a := \mathrm{gldim}(\Lambda/J)$ and $b := \mathrm{gldim}(e\Lambda e)$ are finite. Let X be a Λ-module. Then we have two canonical exact sequences of Λ-modules:

$$0 \longrightarrow JX \longrightarrow X \longrightarrow X/JX \longrightarrow 0, \qquad 0 \longrightarrow K_X \longrightarrow J \otimes_\Lambda X \longrightarrow JX \longrightarrow 0.$$

Here K_X is the kernel of the multiplication map $J \otimes_\Lambda X \longrightarrow JX$. Now we show that $\mathrm{pd}({}_\Lambda JX) \leqslant a + b + 3$. Indeed, it follows from $\Lambda e \otimes_{e\Lambda e} e\Lambda \simeq \Lambda e\Lambda$ that $J \otimes_\Lambda X \simeq \Lambda e \otimes_{e\Lambda e} eX$ and $eJ \otimes_\Lambda X \simeq eJX = eX$. This implies $eK_X = 0$. Thus K_X is a Λ/J-module, and $\mathrm{pd}_\Lambda(K_X) \leqslant a + 1$. Let

$$0 \longrightarrow Q_b \longrightarrow \cdots \longrightarrow Q_0 \longrightarrow eX \longrightarrow 0$$

be a projective resolution of eX. By tensoring this sequence by Λe, we get the following complex in Λ-Mod:

$$0 \longrightarrow \Lambda e \otimes_{e\Lambda e} Q_b \longrightarrow \cdots \longrightarrow \Lambda e \otimes_{e\Lambda e} Q_0 \longrightarrow \Lambda e \otimes_{e\Lambda e} eX \longrightarrow 0$$

with homologies $\mathrm{Tor}_i^{e\Lambda e}(\Lambda e, eX)$. As in [31, Corollary 3.2(2)], we can show that $\mathrm{Tor}_j^{e\Lambda e}(\Lambda e, eX)$ is a Λ/J-module for each $j \geqslant 1$. Thus, we see from [31, Lemma 2.4] (the bound there should be raised by 1) that

$$\begin{aligned} \mathrm{pd}({}_\Lambda \Lambda e \otimes_{e\Lambda e} eX) &\leqslant b + 1 + \max\bigl\{\mathrm{pd}({}_\Lambda \Lambda e \otimes_{e\Lambda e} Q_j), \mathrm{pd}\bigl({}_\Lambda \mathrm{Tor}_j^{e\Lambda e}(\Lambda e, eX)\bigr)\bigr\} \\ &\leqslant b + 1 + a + 1 = a + b + 2 \end{aligned}$$

since $\Lambda e \otimes_{e\Lambda e} Q_j$ is projective. This implies $\mathrm{pd}({}_\Lambda JX) \leqslant \max\{\mathrm{pd}({}_\Lambda J \otimes_\Lambda X), \mathrm{pd}({}_\Lambda K_X)\} + 1 \leqslant a + b + 3$.

Hence we have

$$\mathrm{pd}({}_\Lambda X) \leqslant \max\bigl\{\mathrm{pd}({}_\Lambda JX), \mathrm{pd}({}_\Lambda X/JX)\bigr\} \leqslant a + b + 3,$$

and $\mathrm{gldim}(\Lambda) \leqslant a + b + 3$. This finishes the proof of Lemma 4.5. □

5. Affine Hecke algebras

In this section, we verify that the extended affine Hecke algebras of type A_{n-1} (associated with general or special linear groups) are affine cellular; then we use this fact to prove the main result of this section, Theorem 5.8, on cohomology of affine Hecke algebras. These algebras are of fundamental importance in various parts of algebra and of number theory, but their rather complicated structure is far from being well-understood. We first will exhibit their cellular structure, using in particular Lusztig's cell theory [21–23] and N.H. Xi's proof [32] of a conjecture of Lusztig; the latter will identify the algebras B_j. In this way we get new parameter sets for the simple representations. Previous classifications, in the case of the quantum parameter q not being a root of unity, have been achieved by Kazhdan and Lusztig [15], who proved the

Deligne–Langlands conjecture in this case, using equivariant homology, and by Ginzburg using equivariant K-theory, see also the papers [26,27]. Recently, N.H. Xi in [32] extended this classification to the case of q not being a root of the Poincaré polynomial. The general case later on has been settled in a completely different way by Ariki and Mathas [1], who worked with cyclotomic Hecke algebras, that is, with a certain infinite family of finite dimensional quotient algebras.

The second main result of this section is an application of our homological theory. Working over a field and assuming q not to be a root of the Poincaré polynomial (as in N.H. Xi's extension of the Deligne–Langlands classification), the extended affine Hecke algebra will turn out to satisfy the affine analogue of being quasi-hereditary; hence it has finite global dimension, and, more strongly, its derived category has a stratification.

5.1. *Affine Hecke algebras and Kazhdan–Lusztig basis*

We first recall the general definitions of Hecke algebra of a Coxeter system and of asymptotic algebras. Then we specialise for the rest of this section to particular extended affine Hecke algebras in type A.

Let R be the Laurent polynomial ring $\mathbb{Z}[q, q^{-1}]$ over $\mathbb{Z}$ in one variable q.

Let (W, S) be a Coxeter system with S the set of simple reflections. The *Hecke algebra* $\mathscr{H}$ of (W, S) over R is an associative R-algebra, with a free basis $\{T_w \mid w \in W\}$ and relations

$$\big(T_s - q^2\big)(T_s + 1) = 0 \quad \text{if } s \in S,$$
$$T_w T_u = T_{wu} \quad \text{if } \ell(wu) = \ell(w) + \ell(u).$$

The elements T_w turn out to be invertible. Let $^-$ be the $\mathbb{Z}$-linear ring homomorphism of R defined by $q \longmapsto q^{-1}$. Then there is a $\mathbb{Z}$-linear automorphism $^-$ of order two on $\mathscr{H}$ given by

$$\sum_w a_w T_w \longmapsto \sum_w \bar{a}_w T_{w^{-1}}^{-1}, \quad a_w \in R.$$

For each element $w \in W$, there is a unique element C_w in $\mathscr{H}$ such that $\overline{C}_w = C_w$ and $C_w = q^{-\ell(w)} \sum_{y \leqslant w} P_{y,w}(q^2) T_y$, where $P_{y,w}$ is a polynomial in q of degree at most $\frac{1}{2}(\ell(w) - \ell(y) - 1)$ if $\ell(w) > \ell(y)$ and $P_{w,w} = 1$.

The basis $\{C_w \mid w \in W\}$ is called the *Kazhdan–Lusztig basis* of the Hecke algebra $\mathscr{H}$ and the polynomials $P_{y,w}$ are the *Kazhdan–Lusztig polynomials*.

For $w, u \in W$, we write

$$C_w C_u = \sum_{z \in W} h_{w,u,z} C_z,$$

where $h_{w,u,z}$ are elements in R.

For (W, S) crystallographic (see [21, 3.1]), the *asymptotic Hecke algebra* $\mathscr{B}$ of (W, S) has been defined (by Lusztig) as follows: $\mathscr{B}$ has a free $\mathbb{Z}$-basis $\{t_w \mid w \in W\}$. Multiplication is given by

$$t_w t_u = \sum_z \gamma_{w,u,z} t_z,$$

where $\gamma_{w,u,z}$ are certain non-negative integers (see [22] for further information).

Now we prepare for the definition of extended Hecke algebras. Let (W', S) be a Coxeter system with S the set of simple reflections. Suppose there is an abelian group Ω acting on (W', S). Then the semidirect product is the *extended Coxeter group* $W = \Omega \ltimes W'$. The length function and the Bruhat–Chevalley order on W' are extended to W: $\ell(\omega w) = \ell(w)$, and $\omega w \leqslant \omega' w'$ if and only if $\omega = \omega'$ and $w \leqslant w'$, where ω, ω' are in Ω and w, w' are in W'.

It is known that we may write $W' = W_0 \ltimes P$ for an abelian subgroup P of W such that W_0 is the finite Weyl group, that is, the quotient $N_G(T)/T$ of the normaliser $N_G(T)$ of a torus T of the corresponding group G. In case of (W', S) being of type $\widetilde{A}_{n-1}$, the W_0 is isomorphic to the symmetric group on n letters. The polynomial $\sum_{w \in W_0} q^{\ell(w)} \in R$ is called the *Poincaré polynomial* of the Hecke algebra $\mathscr{H}$ of (W, S).

The *extended affine Hecke algebra* $\mathscr{H}_R(n, q)$ of type A_{n-1} is the Hecke algebra of (W, S) with (W', S) being the Coxeter system of type $\widetilde{A}_{n-1}$ ($n \geqslant 3$) and Ω the cyclic group $\mathbb{Z}_n$ of order n in case we work with special linear groups, or the infinite cyclic group $\mathbb{Z}$ in case of general linear groups; in either case the cyclic group is acting by rotations of the Coxeter graph. The corresponding asymptotic Hecke algebra of type $\widetilde{A}_{n-1}$ will be denoted by $\mathscr{B}(n)$.

If k is a commutative $\mathbb{Z}$-algebra, we shall write $\mathscr{H}_k(n, q)$ for the algebra $k \otimes_{\mathbb{Z}} \mathscr{H}_R(n, q)$. Thus $\mathscr{H}_k(n, q)$ is a $k[q, q^{-1}]$-algebra with the basis $\{T_w \mid w \in W\}$ and the above relations. Similarly, we write $\mathscr{B}_k(n)$ for the algebra $k \otimes_{\mathbb{Z}} \mathscr{B}(n)$.

If I is a subset of S, we denote by W^I the *parabolic subgroup* of W generated by elements in I. By $\mathscr{H}_R(W^I)$ we denote the subalgebra of $\mathscr{H}_R(n, q)$ generated by all C_w with $w \in W^I$.

5.2. Cells and asymptotic algebras

Many basic results about Hecke algebras are in terms of Lusztig's cell theory to be recalled now for our particular situation (see [21]). As it is natural in type A, we will derive the affine cellular structure from the cell theory.

Let w and u be in W. We write $w \leqslant_L u$ if there is a chain $w = w_0, w_1, \ldots, w_r = u$, possibly $r = 0$, of elements in W such that for each $j < r$, C_{w_j} occurs with non-zero coefficient when expanding the product $C_s C_{w_{j+1}}$ into a linear combination of basis elements, for some $s \in S$. Then $\leqslant_L$ is a pre-order on W. The pre-order $\leqslant_R$ is defined by setting $w \leqslant_R u$ if and only if $w^{-1} \leqslant_L u^{-1}$. The pre-order $\leqslant_{LR}$ is defined by the union of $\leqslant_L$ and $\leqslant_R$, that is, given $x, x' \in W$, we say that $x \leqslant_{LR} x'$ if there is a sequence $x = x_0, x_1, \ldots, x_n = x'$ of elements in W such that for each j, $1 \leqslant j \leqslant n$, we have $x_{j-1} \leqslant_L x_j$ or $x_{j-1} \leqslant_R x_j$.

The equivalence relation $\sim_L$ is defined by $w \sim u$ if and only if $w \leqslant_L u$ and $u \leqslant_L w$. The equivalence classes of $\sim_L$ are called the *left cells* of W. Similarly, one defines the equivalence relation $\sim_R$ and the *right cells* of W, and the equivalence relation $\sim_{LR}$ and the *two-sided cells* of W. Note that each pre-order defined above induces a partial order on the set of the corresponding equivalence classes in W.

From now on we will consider the case (W, S) with (W', S) being the Coxeter system of type $\widetilde{A}_{n-1}$ ($n \geqslant 3$) and Ω the cyclic group $\mathbb{Z}_n$ of order n or the infinite cyclic group, acting by rotations of the Coxeter graph. It has been shown in [30] that there is a bijection between the set of two-sided cells of W and the set of partitions of n. In fact, this bijection is described by a map σ from W to the set of all partitions of n. There is a partial order, the *dominance order*, denoted by $\trianglelefteq$, on the set of all partitions of n. This dominance order is compatible with the order $\leqslant_{LR}$ by a result in [30]; for $w, u \in W$, we have $w \leqslant_{LR} u$ if and only if $\sigma(u) \trianglelefteq \sigma(w)$.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ be a partition of n and $\mathbf{c}$ the corresponding two-sided cell in W. We denote by w_λ the longest element of the Young subgroup of W associated to λ. Then $w_\lambda \in \mathbf{c}$.

Let $L_1, L_2, \ldots, L_m$ be a list of all left cells in $\mathbf{c}$ with $w_\lambda \in L_1$. Then $R_j := L_j^{-1}$, $1 \leqslant j \leqslant m$, are a list of all right cells in $\mathbf{c}$. Lusztig has shown that each left cell and each right cell in $\mathbf{c}$ contains exactly one involution element d in W with certain properties. These are called the *distinguished involutions* in $\mathbf{c}$. We denote by $D_{\mathbf{c}}$ the set of all distinguished involutions in $\mathbf{c}$, and by D the set of all distinguished involutions. Let $A_{jl} = R_j \cap L_l$ for $1 \leqslant j, l \leqslant m$. Then $A_{jl}^{-1} = A_{lj}$, and there is a bijection $\phi_{jl} : A_{jl} \longrightarrow A_{11}$ defined in [32]. For $x \in A_{jl}$, $\phi_{jl}(x)^{-1} = \phi_{lj}(x^{-1})$. Note also that $\mathbf{c}$ is the disjoint union of all A_{jl}. Each A_{jj} contains a unique distinguished involution element d_j.

In [21, 2.1], a function $a : W \longrightarrow \mathbb{N}$ has been defined, which now is called Lusztig's a-function. We will use the following result that has been proved in [22].

Lemma 5.1. *Let W be an extended Coxeter group.*

(1) *If $\mathbf{c}$ is a two-sided cell in W, then $a(w) = a(u)$ for all $w, u \in \mathbf{c}$. In particular, $a(z) = a(z^{-1})$ for all $z \in W$.*
(2) *For any $x \in W$, we have*

$$\mathscr{H} C_x \subseteq \sum_{\substack{y \in W \\ y \leqslant_L x}} RC_y, \qquad C_x \mathscr{H} \subseteq \sum_{\substack{y \in W \\ y \leqslant_R x}} RC_y, \qquad \mathscr{H} C_x \mathscr{H} \subseteq \sum_{\substack{y \in W \\ y \leqslant_{LR} x}} RC_y.$$

(3) *If $w \leqslant_{LR} u$ in W, then $a(w) \geqslant a(u)$. If $w \leqslant_L u$ and $a(w) = a(u)$, then $w \sim_L u$. Similarly, if $w \leqslant_R u$ and $a(w) = a(u)$, then $w \sim_R u$. In particular, if $w \leqslant_{LR} u$ in W and $a(u) = a(w)$, then $w \sim_{LR} u$.*
(4) *Let w, u and z be in a two-sided cell $\mathbf{c}$. If $h_{w,u,z} \neq 0$, then $z \sim_R w$ and $z \sim_L u$.*

Let $\mathbf{c}$ be a two-sided cell in W. Then the free $\mathbb{Z}$-submodule $\mathscr{B}_{\mathbf{c}}$ of $\mathscr{B}$ generated by all t_w with $w \in \mathbf{c}$ is a two-sided ideal in $\mathscr{B}$, and it has an identity $\sum_{d \in D_{\mathbf{c}}} t_d$, where as above $D_{\mathbf{c}} = D \cap \mathbf{c}$. Thus $\mathscr{B}$ is a direct sum of $\mathscr{B}_{\mathbf{c}}$, where the sum ranges over all two-sided cells of W (see [22]). $\mathscr{B}_{\mathbf{c}}$ is called the *asymptotic Hecke algebra of the cell* $\mathbf{c}$. The canonical projection from $\mathscr{B}$ to $\mathscr{B}_{\mathbf{c}}$ will be denoted by $\pi_{\mathbf{c}}$.

In the following lemma we collect some properties of the structure constants of Hecke algebras and asymptotic Hecke algebras.

Lemma 5.2. *Let $\mathbf{c}$ be a two-sided cell in W.*

(1) *For x_1, x_2, x_3, z' in W such that $z' \in \mathbf{c}$ and $x_2 \in \mathbf{c}$, we have*

$$\sum_{z \in \mathbf{c}} h_{x_1,x_2,z} \gamma_{z,x_3,z'} = \sum_{z \in \mathbf{c}} h_{x_1,z,z'} \gamma_{x_2,x_3,z}.$$

(2) *Suppose $y, z \in \mathbf{c}$. Then $\sum_{d \in D_{\mathbf{c}}} \gamma_{d,z,z} = 1$, and $\gamma_{d,y,z} = 0$ if $y \neq z$.*

Proof. Statement (1) follows from [22, 2.4(d)] and [11, Proposition 3.4.4]. Statement (2) is a consequence of the fact that $\sum_{d \in D_{\mathbf{c}}} t_d$ is the identity of $\mathscr{B}_{\mathbf{c}}$. □

5.3. The extended affine Hecke algebras of type A are affine cellular

In this subsection we are going to demonstrate that our general framework covers the extended affine Hecke algebras of type A.

We keep the notation of the previous subsection and set up some more notation. For a subset J of W, we denote by $\mathscr{B}_J(n)$ the $\mathbb{Z}$-submodule of $\mathscr{B}$ generated by all basis elements t_w with $w \in J$. Let $B := \mathscr{B}_{A_{11}}$, where as before $A_{11} = R_1 \cap L_1$. Lusztig has shown that B is a commutative algebra with a $\mathbb{Z}$-linear involution $t_w \longmapsto t_{w^{-1}}$ for $w \in A_{11}$. The following result has been conjectured by Lusztig (in a more general form) and it has been proved (in type A) by N.H. Xi [32]:

Theorem 5.3. *(See N.H. Xi [32].) Let* $\mathbf{c}$ *be a two-sided cell in* W *and* λ *the corresponding partition of* n. *Let* $\mu = (\mu_1, \ldots, \mu_{r'})$ *be the dual partition of* λ, *and define* $n_\lambda := n!/(\mu_1! \cdots \mu_{r'}!)$.

(1) $\mathscr{B}_{\mathbf{c}}(n)$ *is isomorphic to an* $n_\lambda \times n_\lambda$ *matrix algebra over a commutative ring* $B := \mathscr{B}_{R_1 \cap L_1}(n)$ *with identity. The isomorphism is given by* $t_w \longmapsto E_{jl}(t_{\phi_{jl}(w)})$ *if* $w \in A_{jl}$, *where* $E_{jl}(b)$ *denotes a square matrix whose* (j,l)*-entry is* b *and all other entries are zero.*
(2) *The ring* B *in* (1) *is an affine commutative* $\mathbb{Z}$*-algebra. It is isomorphic to a tensor product of rings of the form* $\mathbb{Z}[X_1, X_2, \ldots, X_{s+1}]/(X_s X_{s+1} - 1)$.

Statement (1) is taken from [32, Theorem 2.3.2, p. 16], where B is the algebra $\mathscr{B}_{R_1 \cap L_1}(n)$.

Statement (2) is essentially [32, Theorem 8.2.1] that B is isomorphic to the representation ring of an algebraic group, which is a product of general linear groups GL_t. The representation ring of a general linear group GL_t is known to be isomorphic to $\mathbb{Z}[X_1, \ldots, X_{t-1}, X_t, X_t^{-1}]$, see for instance [6, Exercise 23.36(d), p. 379]. The case of special linear groups is done in [32, Section 8.4].

Thus each element in $\mathscr{B}_{\mathbf{c}}(n)$ is a matrix over B. So we identify t_w for $w \in A_{jl}$ with $E_{jl}(t_{\phi_{jl}(w)})$. We may use Theorem 5.3 to label the basis element C_w with $w \in \mathbf{c}$, that is, we write $\widetilde{E_{jl}}(t_{\phi_{jl}(w)})$ for C_w with $w \in A_{jl}$. Let $I_{\mathbf{c}}$ be the identity matrix corresponding to the element $\sum_{d \in D_{\mathbf{c}}} t_d$, and $\widetilde{I_{\mathbf{c}}}$ be the matrix corresponding to the element $\sum_{d \in D_{\mathbf{c}}} C_d$. Note that $I_{\mathbf{c}} = \widetilde{I_{\mathbf{c}}}$ as matrices over B, but they are considered as elements in different sets, in order to avoid any confusion (see notations in Section 3.2).

The next lemma follows from Lemma 5.1 which indicates how to label the indices of a matrix corresponding to a basis element in $\mathscr{B}(n)$.

Lemma 5.4. *Suppose* $\mathbf{c}$ *is a two-sided cell in* W *with* $w \in \mathbf{c}$. *If* $u, z \in W$ *such that* $h_{w,u,z} \neq 0$ *then* $z \in A_{jq}$, *where* j *is given by* $w \in A_{jl}$ *and* q *is given by* $u \in A_{pq}$.

In the following, we shall rewrite the homomorphism φ defined by Lusztig in [22] in our setup, as in Section 3.2.

Recall that D denotes the distinguished involutions in W', and that $\mathscr{B}_R(n) = R \otimes_{\mathbb{Z}} \mathscr{B}(n)$ is a direct sum of $\mathscr{B}_{\mathbf{c}}(n)$, where $\mathbf{c}$ runs over all two-sided cells of W, and $\mathscr{B}_{\mathbf{c}}(n)$ is the R-module spanned by all t_w with $w \in \mathbf{c}$. There is a well-defined injective homomorphism, due to Lusztig, $\varphi : \mathscr{H}_R(n,q) \longrightarrow \mathscr{B}_R(n)$:

$$\varphi(C_w) = \sum_{\substack{d \in D \\ z \in W \\ a(z) = a(d)}} h_{w,d,z} t_z$$

of R-algebras with identity.

Let $\mathbf{c}$ be a two-sided cell of W. We denote by $\mathscr{H}_R(n,q)^{\leqslant \mathbf{c}}$ the free R-submodule of $\mathscr{H}_R(n,q)$ generated by all C_w with $w \leqslant_{LR} w'$ for some $w' \in \mathbf{c}$, and by $\mathscr{H}_R(n,q)^{<\mathbf{c}}$ the free R-submodule of $\mathscr{H}_R(n,q)$ generated by all C_w with $w <_{LR} w'$ for some $w' \in \mathbf{c}$. Then both $\mathscr{H}_R(n,q)^{\leqslant \mathbf{c}}$ and $\mathscr{H}_R(n,q)^{<\mathbf{c}}$ are ideals in $\mathscr{H}_R(n,q)$. We denote by $\mathscr{H}_R(n,q)^{\mathbf{c}}$ the quotient $\mathscr{H}_R(n,q)^{\leqslant \mathbf{c}}/\mathscr{H}_R(n,q)^{<\mathbf{c}}$. Thus $\mathscr{H}_R(n,q)^{\mathbf{c}}$ has an R-basis $\{[C_w] \mid w \in \mathbf{c}\}$, and the multiplication in $\mathscr{H}_R(n,q)^{\mathbf{c}}$ is given by

$$[C_w][C_u] = \sum_{z\in\mathbf{c}} h_{w,u,z}[C_z]$$

for all $w, u \in \mathbf{c}$.

Note that $\mathscr{B}_{\mathbf{c}}(n)$ is a $\mathscr{B}_{\mathbf{c}}(n)$-$\mathscr{H}_R(n,q)^{\mathbf{c}}$-bimodule via

$$t_w \cdot [C_u] := \sum_{z\in\mathbf{c}} h_{w,u,z} t_z$$

for $w \in \mathbf{c}$, $u \in W$ (see [23, 1.4(b)], or [33, 2.1(d)]). This simply expresses the left regular representation of $\mathscr{H}_R(n,q)^{\mathbf{c}}$. Note that our right module structure on $\mathscr{B}_{\mathbf{c}}(n)$ is the same as the one in [33, 2.1(d)]. In fact, if $h_{w,u,z} \neq 0$, then $z \leqslant_R w$. Thus, if $a(z) = a(w)$, then $z \in \mathbf{c}$ by Lemma 5.1.

The algebra homomorphism φ induces an algebra homomorphism from $\mathscr{H}_R(n,q)^{\mathbf{c}}$ to $\mathscr{B}_{\mathbf{c}}(n)$:

$$\psi_{\mathbf{c}} : \mathscr{H}_R(n,q)^{\mathbf{c}} \longrightarrow \mathscr{B}_{\mathbf{c}}(n), \qquad [C_w] \longmapsto \sum_{d\in D_{\mathbf{c}},\, z\in\mathbf{c}} h_{w,d,z} t_z = t_w \cdot \Bigl[\sum_{d\in D_{\mathbf{c}}} C_d\Bigr].$$

In terms of matrix language, we see that for $w \in A_{jl}$,

$$\psi_{\mathbf{c}} : \widetilde{E_{jl}}(t_{\phi_{jl}(w)}) \longmapsto E_{jl}(t_{\phi_{jl}(w)}) \cdot \widetilde{I}_{\mathbf{c}} = E_{jl}(t_{\phi_{jl}(w)}) \biggl(\biggl(\sum_{d\in\mathbf{c}} t_d\biggr) \cdot \widetilde{I}_{\mathbf{c}}\biggr),$$

where $\widetilde{I}_{\mathbf{c}}$ is the identity matrix for $\mathscr{B}_{\mathbf{c}}(n)$.

In the following we shall prove that the multiplication rule in $\mathscr{H}_R(n,q)^{\mathbf{c}}$ is that of a swich algebra $\tilde{\Lambda}$ as in Section 3.2. The corresponding algebra Λ is the asymptotic algebra of the cell $\mathbf{c}$, as defined by Lusztig. So, at this point we see that Lusztig's asymptotic algebra is an asymptotic algebra in our general sense, and thus no confusion should arise by using the same term for both objects.

The following lemma is inspired by a result in [11].

Lemma 5.5. *In $\mathscr{B}_{\mathbf{c}}(n)$ we have*

$$t_w \cdot [C_u] = t_w \biggl(\biggl(\sum_{d\in\mathbf{c}} t_d\biggr) \cdot \widetilde{I}_{\mathbf{c}}\biggr) t_u$$

for all $w, u \in \mathbf{c}$.

Proof. Since $\sum_{d\in\mathbf{c}} t_d$ is the identity of $\mathscr{B}_{\mathbf{c}}(n)$, there are, by definition, equalities

$$t_w\left(\left(\sum_{d\in\mathbf{c}} t_d\right)\cdot\widetilde{I_{\mathbf{c}}}\right)t_u=\left(t_w\cdot\left(\sum_{d\in D_{\mathbf{c}}}[C_d]\right)\right)t_u=\sum_{d\in\mathbf{c},\,z,z'\in\mathbf{c}} h_{w,d,z'}\gamma_{z',u,z}t_z.$$

Now, we use Lemma 5.2(1) to replace $\sum_{z'} h_{w,d,z'}\gamma_{z',u,z}$ by $\sum_{z'} h_{w,z',z}\gamma_{d,u,z'}$, and get that

$$t_w\left(\left(\sum_{d\in\mathbf{c}} t_d\right)\cdot\widetilde{I_{\mathbf{c}}}\right)t_u=\sum_{d,z,z'} h_{w,z',z}\gamma_{d,u,z'}t_z=\sum_{z,z'} h_{w,z',z}\left(\sum_{d}\gamma_{d,u,z'}\right)t_z.$$

Then, it follows from Lemma 5.2(2) that

$$t_w\left(\left(\sum_{d\in\mathbf{c}} t_d\right)\cdot\widetilde{I_{\mathbf{c}}}\right)t_u=\sum_{z\in\mathbf{c}} h_{w,u,z}t_z=t_w\cdot[C_u].$$

This finishes the proof. □

When Lemma 5.5 gets translated into matrix language, the left-hand side of the equation in Lemma 5.5 expresses the multiplication $\widetilde{E_{jl}(t_{\phi_{jl}(w)})}\cdot\widetilde{E_{pq}(t_{\phi_{pq}(u)})}$ in $\mathscr{H}_R(n,q)^{\mathbf{c}}$ for $w\in A_{jl}$ and $u\in A_{pq}$, and the right-hand side is just the product $E_{jl}(t_{\phi_{jl}(w)})\Psi_{\mathbf{c}}E_{pq}(t_{\phi_{pq}(u)})$ in the usual matrix algebra $\mathscr{B}_{\mathbf{c}}(n)$, where $\Psi_{\mathbf{c}}$ is the matrix representing $(\sum_{d\in\mathbf{c}} t_d)\cdot\sum_{d\in\mathbf{c}}[C_d]$. Note that the map defined by $[C_w]\longmapsto[C_{w^{-1}}]$ is an R-involution of $\mathscr{H}_R(n,q)^{\mathbf{c}}$. This is induced from the R-involution $*$ of $\mathscr{H}_R(n,q)$, which sends $C_w\longmapsto C_{w^{-1}}$.

Thus we get the following result.

Proposition 5.6. *Let* $\mathbf{c}$ *be a two-sided cell in* W. *Let* Λ *be the matrix algebra* $\mathscr{B}_{\mathbf{c}}(n)$. *Then there is a matrix* Ψ *in* Λ *such that* $\mathscr{H}_R(n,q)^{\mathbf{c}}$ *can be identified with* $\tilde{\Lambda}$. *The multiplication in* $\tilde{\Lambda}$ *is given by* $\tilde{a}\cdot\tilde{b}=a\Psi_{\mathbf{c}}b$ *for all* $a,b\in\Lambda$. *Moreover, the homomorphism defined by Lusztig from* $\mathscr{H}_R(n,q)^{\mathbf{c}}$ *to* $\mathscr{B}_{\mathbf{c}}(n)$ *can be identified with the map from* $\tilde{\Lambda}$ *to* Λ *by multiplying* $\Psi_{\mathbf{c}}$ *from the right.*

Since the dominance order of partitions of n gives a partial order on the set of two-sided cells in W, which is compatible with the partial order $\leqslant_{LR}$, we may choose a linear order on the cells in the following way: Let the set of two-sided cells of W be partitioned as $\mathbf{a}_1\cup\mathbf{a}_2\cup\cdots\cup\mathbf{a}_s$ such that $\mathbf{a}_1$ takes the maximal value, and $\mathbf{a}_2$ the second largest value, and so on. Now we choose a linear order for each $\mathbf{a}_j$, and define the elements of $\mathbf{a}_j$ to be less than those in $\mathbf{a}_{j+1}$. In this way we get a linear order of cells: $\mathbf{c}_1,\mathbf{c}_2,\dots,\mathbf{c}_f$ with the property that $\mathbf{c}_j\leqslant_{LR}\mathbf{c}_l$ implies that $j\leqslant l$. We define $\mathscr{J}'_j$ to be the R-module generated by all C_w with $w\in\mathbf{c}_j$, and $\mathscr{J}_j=\bigoplus_{l=1}^{j}\mathscr{J}'_l$. Then $\mathscr{J}'_j$ is invariant under the involution $*$, $\mathscr{J}_j$ is an ideal in $\mathscr{H}_R(n,q)$ with $\mathscr{J}_j/\mathscr{J}_{j-1}\simeq\mathscr{H}_R(n,q)^{\mathbf{c}_j}$, and the chain

$$\mathscr{J}_1\subseteq\mathscr{J}_2\subseteq\cdots\subseteq\mathscr{J}_f=\mathscr{H}_R(n,q)$$

is a cell chain for $\mathscr{H}_R(n,q)$ by Propositions 5.6 and 2.2. Thus we have proved

Theorem 5.7. *Let* $R=\mathbb{Z}[q,q^{-1}]$. *Then the extended affine Hecke algebra* $\mathscr{H}_R(n,q)$ *of type* A_{n-1} *is an affine cellular* $\mathbb{Z}$*-algebra with respect to the* R*-involution* $*:C_w\longmapsto C_{w^{-1}}$ *for* $w\in W$.

Remark 1. If E is a simple $\mathscr{B}_{\mathbf{c}}(n)$-module such that $\psi_{\mathbf{c}}(\mathscr{H}_R(n,q)^{\mathbf{c}})E \neq 0$, then, by Proposition 3.8, $E_{\psi_{\mathbf{c}}}$ has a unique simple quotient S as an $\mathscr{H}_R(n,q)^{\mathbf{c}}$-module, say $S = E_{\psi_{\mathbf{c}}}/X$, where X is a maximal submodule of $E_{\psi_{\mathbf{c}}}$ with $\Psi_{\mathbf{c}}(\mathscr{H}_R(n,q)^{\mathbf{c}})X = 0$. The module S also is a simple $\mathscr{H}_R(n,q)$-module. Since E can also be regarded as an $\mathscr{H}_R(n,q)$-module via $\varphi\pi_{\mathbf{c}}$ and since $\psi_{\mathbf{c}}$ is induced from the restriction of $\varphi\pi_{\mathbf{c}}$ to $\mathscr{H}_R(n,q)^{\leqslant \mathbf{c}}$, we get that $\varphi(\mathscr{H}_R(n,q)^{\leqslant \mathbf{c}})X = 0$. This means that all composition factors (as an $\mathscr{H}_R(n,q)$-module) of X are modules over $\mathscr{H}_R(n,q)/\mathscr{H}_R(n,q)^{\leqslant \mathbf{c}}$. Thus the $\mathscr{H}_R(n,q)$-module E has one composition factor in the layer $\mathbf{c}$, and (possibly) other composition factors in higher layers $\mathbf{c}'$ with $\mathbf{c}' > \mathbf{c}$.

Remark 2. As mentioned before, an alternative approach to studying simple modules of affine Hecke algebras has been based on the use of cyclotomic Hecke algebras, whose cellular structure has been found and used by Dipper, James, Mathas, Ariki and others. These algebras are finite dimensional quotients of the extended affine Hecke algebras. Moreover, each simple module of the affine Hecke algebra is a simple module of some cyclotomic Hecke algebra, and vice versa. The known cell structures of cyclotomic Hecke algebras do, however, use parameter sets whose cardinality strongly depends on the particular cyclotomic algebra, that is, on the choice of parameters for forming the quotient. So, there cannot exist a finite cell chain on the affine Hecke algebra that induces the cell structures (of varying cell length) on all these quotient algebras.

5.4. *Cohomology of extended affine Hecke algebras*

Finally, we are going to use our homological theory of affine cellular algebras in order to go beyond cell theory and to derive homological properties of extended affine Hecke algebras. Here we work with a field of characteristic zero instead of $R = \mathbb{Z}[q, q^{-1}]$. Then we can show that for almost all choices of the quantum parameter q, all layers in the cell chain satisfy all the idempotent conditions. Hence the parameter varieties are the full affine spaces, the global dimension is finite and there is a stratification of derived module categories. The precise result is as follows.

Theorem 5.8. *Assume that k is a field of characteristic zero, $q \in k$, and $\sum_{w \in W_0} q^{\ell(w)} \neq 0$. Then all cells in the cell chain of $\mathscr{H}_k(n,q)$ correspond to idempotent ideals, which all have idempotent generators. In particular, the parameter set of simple $\mathscr{H}_k(n,q)$-modules equals the parameter set of simple modules of the asymptotic algebra, and so it is a finite union of affine spaces. Moreover, $\mathscr{H}_k(n,q)$ is of finite global dimension and its derived module category admits a stratification whose sections are the derived module categories of the algebras B_j.*

The assumption made here is exactly that under which N.H. Xi [33] recently has been able to verify the validity of Kazhdan and Lusztig's Deligne–Langlands classification. We are going to make essential use of crucial ideas in his proof.

Note that the extended affine Hecke algebra is noetherian. Therefore, its left and its right global dimension coincide.

A cell chain as in the theorem may be called a 'heredity chain'.

Proof of Theorem 5.8. It is sufficient to verify the validity of the conditions in Theorem 4.4. We fix j and write $\mathbf{c}$ for $\mathbf{c}_j$. By Theorem 5.3, $\mathscr{B}_{\mathbf{c}}(n)$ is the matrix algebra over a commutative algebra $B_{\mathbf{c}}$. This algebra $B_{\mathbf{c}}$ is isomorphic to $k[X_1, \ldots, X_s, X_{s+1}, X_{s+1}^{-1}, \ldots, X_{n_{\mathbf{c}}}, X_{n_{\mathbf{c}}}^{-1}]$, which has finite global dimension. Moreover, this ring satisfies the first condition in Theorem 4.4. Now we shall show that $\mathscr{J}_j/\mathscr{J}_{j-1} \simeq \mathscr{H}_k(n,q)^{\mathbf{c}}$ is idempotent and has a non-zero idempotent element.

First, we claim that for two different two-sided cells $\mathbf{c}$ and $\mathbf{c}'$ in W with $a(\mathbf{c}) = \mathbf{a}(\mathbf{c}')$, we have $C_u C_w \equiv 0 \pmod{\mathscr{H}_k(n,q)^{>a(\mathbf{c})}}$ for all $w \in \mathbf{c}$ and $u \in \mathbf{c}'$. In fact, by Lemma 5.1(2) and (3), we can write $C_u C_w$ in two ways:

$$C_u C_w = \sum_{v \in W,\, v \leqslant_L w} h_{u,w,v} C_v = \sum_{v \leqslant_L w,\, a(v)=a(w)} h_{u,w,v} C_v + \sum_{v \leqslant_L w,\, a(v)>a(w)} h_{u,w,v} C_v,$$

$$C_u C_w = \sum_{v \in W,\, v \leqslant_R u} h_{u,w,v} C_v = \sum_{v \leqslant_R u,\, a(v)=a(u)} h_{u,w,v} C_v + \sum_{v \leqslant_R w,\, a(v)>a(w)} h_{u,w,v} C_v.$$

It follows from Lemma 5.1(3) that

$$\sum_{v \leqslant_L w,\, a(v)=a(w)} h_{u,w,v} C_v \in \mathscr{H}_k(n,q)^{\mathbf{c}} \quad \text{and} \quad \sum_{v \leqslant_R u,\, a(v)=a(u)} h_{u,w,v} C_v \in \mathscr{H}_k(n,q)^{\mathbf{c}'}.$$

Since $\mathbf{c}$ and $\mathbf{c}'$ are different, the first term in each expression must vanish by comparing the two expressions of $C_u C_w$. Thus we get the claim.

Using results of A. Gyoja [13], N.H. Xi proved in [33] that, under our assumptions, for each simple $\mathscr{B}_{\mathbf{c}}(n)$-module E, there is an element $C_w \in W^I$ such that $a(w) = a(\mathbf{c})$ and $\psi_{\mathbf{c}}(C_w)E \neq 0$, where W^I is a finite parabolic subgroup of W. This means that $\mathscr{J}_j/\mathscr{J}_{j-1}$ is an idempotent ideal by Theorem 4.1 and that $\mathscr{H}_k(n,q)^{\mathbf{c}} E_{\psi_{\mathbf{c}}} = E_{\psi_{\mathbf{c}}}$. Let $\mathscr{H}(W^I)$ be the subalgebra of $\mathscr{H}_k(n,q)$ generated by all elements C_u with $u \in W^I$. Now we fix such a simple module E and an element C_w and such a finite parabolic subgroup W^I. In the following we denote the image of an element (or a subset) m of $\mathscr{H}_k(n,q)$ under the canonical map from $\mathscr{H}_k(n,q) \longrightarrow \mathscr{H}_k(n,q)/\mathscr{H}_k(n,q)^{>a(\mathbf{c})}$ by $\bar{m}$. Note that $E_{\psi_{\mathbf{c}}}$ is also an $\overline{\mathscr{H}_k(n,q)}$-module because the space $\mathscr{J}_{j-1}$ spanned by all C_u with $u \in \bigcup_{l=1}^{j-1} \mathbf{c}_l$ contains $\mathscr{H}_k(n,q)^{>a(\mathbf{c})}$. Since $\mathscr{H}(W^I)$ is semisimple by [13] (see also the proof of [33, Theorem 3.2]) and since $\mathscr{H}_k(n,q)^{\geqslant a(\mathbf{c})}$ is an ideal in $\mathscr{H}_k(n,q)$, we deduce that $\overline{\mathscr{H}(W^I)} \cap \overline{\mathscr{H}_k(n,q)^{\geqslant a(\mathbf{c})}}$ is a non-zero semisimple algebra. (Note that if we define $a_j = a(\mathbf{c})$ for $\mathbf{c} \in \mathbf{a}_j$ and if $a(\mathbf{c}) = a_t$, then $\mathscr{H}_k(n,q)^{\geqslant a(\mathbf{c})}$ is spanned by all C_v with $v \in (\mathbf{a}_1 \cup \cdots \cup \mathbf{a}_t)$.) Let e be the identity in $\overline{\mathscr{H}(W^I)} \cap \overline{\mathscr{H}_k(n,q)^{\geqslant a(\mathbf{c})}}$. Suppose $\mathbf{a}_t = \mathbf{c} \cup C'$. We may write $e = e_{\mathbf{c}} + e'$ such that $e_{\mathbf{c}} \in \mathscr{H}_k(n,q)^{\mathbf{c}}$ and $e' \in \mathscr{H}_k(n,q)^{C'}$, and then $\bar{e}_{\mathbf{c}} + \bar{e}' = e = e^2 = \bar{e}_{\mathbf{c}}^2 + (\bar{e}'\bar{e}_{\mathbf{c}} + \bar{e}_{\mathbf{c}}\bar{e}' + (\bar{e}')^2)$. This means that $e_{\mathbf{c}} \equiv e_{\mathbf{c}}^2 \pmod{\mathscr{H}_k(n,q)^{>a(\mathbf{c})}}$ by the above claim and by the fact that $\mathscr{H}_k(n,q)^{>a(\mathbf{c})}$ is an ideal in $\mathscr{H}_k(n,q)$. Note that $\overline{C}_w \in \overline{\mathscr{H}(W^I)} \cap \overline{\mathscr{H}_k(n,q)^{\geqslant a(\mathbf{c})}}$. If $e_{\mathbf{c}} = 0$, then $\psi_{\mathbf{c}}(C_w)E = C_w E_{\psi_{\mathbf{c}}} = \overline{C}_w E_{\psi_{\mathbf{c}}} = \overline{C}_w e E_{\psi_{\mathbf{c}}} = \overline{C_w e'} E_{\psi_{\mathbf{c}}} = \overline{C_w e'} \mathscr{H}_k(n,q)^{\mathbf{c}} E_{\psi_{\mathbf{c}}} = 0$ since $e' \mathscr{H}_k(n,q)^{\mathbf{c}} \subseteq \mathscr{H}_k(n,q)^{>a(\mathbf{c})}$ by the above claim. This contradicts the fact that $\psi_{\mathbf{c}}(C_w)E_{\psi_{\mathbf{c}}} \neq 0$. Thus $e_{\mathbf{c}}$ is a non-zero element in $\mathscr{H}_k(n,q)^{\mathbf{c}}$. Since the space $\mathscr{J}_{j-1}$ contains $\mathscr{H}_k(n,q)^{>a(\mathbf{c})}$, we see that ${e_{\mathbf{c}}}^2 \equiv e_{\mathbf{c}} \pmod{\mathscr{J}_{j-1}}$. This means that $\mathscr{J}_j/\mathscr{J}_{j-1}$ contains a non-zero idempotent. □

Remarks. In parallel and independent work, based on methods of harmonic analysis, Opdam and Solleveld [25] have provided explicit projective resolutions of affine Hecke algebras (over the field of real numbers and with q positive) as bimodules over themselves. This also implies finiteness of global dimension, under the stronger assumptions on q made in [25]. Derived categories and stratifications are not considered in [25].

In the situation of Theorem 5.8, the affine Hecke algebra $\mathscr{H}_k(n,q)$ may be called affine quasi-hereditary, and the statement of Theorem 5.8 implies a whole set of cohomological results analogous to known results about quasi-hereditary algebras and highest weight categories, for example on the vanishing of certain extensions between cell lattices. This affine quasi-hereditary structure on $\mathscr{H}_k(n,q)$ is non-trivial in the following sense: The algebra $\mathscr{H}_k(n,q)$ cannot, in general, be isomorphic to its asymptotic algebra, and its cells cannot in general split off as direct summands, since this would contradict the known description of the centre of $\mathscr{H}_k(n,q)$. Kazhdan–Lusztig theory provides some information about non-vanishing extensions. Our approach may also be used to find non-zero cohomology, which is coming up whenever in our setup the swich algebra $\tilde{\Lambda}$ is acting trivially on a non-zero subspace of a simple Λ-module. It is an open and presumably hard problem to determine between which cells in our cell chain there is non-zero cohomology.

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