Ringel modules and homological subcategories

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Abstract

Given a good n-tilting module T over a ring A, let B be the endomorphism ring of T, it is an open question whether the kernel of the left-derived functor $T \otimes_B^{\mathbb{L}} -$ between the derived module categories of B and A could be realized as the derived module category of a ring C via a ring epimorphism $B \to C$ for $n \geq 2$. In this paper, we first provide a uniform way to deal with the above question both for tilting and cotilting modules by considering a new class of modules called Ringel modules, and then give criterions for the kernel of $T \otimes_B^{\mathbb{L}} -$ to be equivalent to the derived module category of a ring C with a ring epimorphism $B \to C$. Using these characterizations, we display both a positive example of n-tilting modules from noncommutative algebra, and a counterexample of n-tilting modules from commutative algebra to show that, in general, the open question may have a negative answer. As another application of our methods, we consider the dual question for cotilting modules, and get corresponding criterions and counterexamples. The case of cotilting modules, however, is much more complicated than the case of tilting modules.

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1 Introduction

As is well known, tilting theory has had significant applications in many branches of mathematics (see [1]), and the key objectives in this theory are tilting modules, or more generally, tilting complexes or objects. Given a good tilting module T over a ring A, let B be the endomorphism ring of T, if T is classical, then a beautiful theorem of Happel says that the derived module category $\mathscr{D}(B)$ of B is triangle equivalent to the derived module category $\mathscr{D}(A)$ of A (see [18]). Thus one can use derived invariants to understand homological, geometric and numerical properties of A through B, or conversely, of B through A. This theorem also tells that one cannot get new derived categories from classical tilting modules. For infinitely generated tilting modules, Bazzoni, Mantese and Tonolo recently show a remarkable result: $\mathscr{D}(A)$ can be regarded as a full subcategory or a quotient category of $\mathscr{D}(B)$ (see [6]). Moreover, it is proved in [11] that if the projective dimension of T is at most 1, then there is a homological ring epimorphism $\lambda : B \to C$ of rings such that the kernel of the total left-derived functor $T \otimes_B^{\mathbb{L}} -$, as a full triangulated subcategory of $\mathscr{D}(B)$, can be realized as the derived module category $\mathscr{D}(C)$ of C. Thus, for (infinitely generated) good tilting

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modules of projective dimension at most 1, Happel's theorem now has a new appearance and can be featured as a recollement of derived module categories:

$$\begin{tikzcd} \mathscr{D}(C) \arrow[r, "D(\lambda_*)"] & \mathscr{D}(B) \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "{}_AT\otimes^{\mathbb{L}}_B-"] & \mathscr{D}(A) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

However, for tilting modules of higher projective dimension, the existence of the above recollement is unknown (see the first open question in [11]). On the one hand, the argument used in [11] actually does not work any more for the general case because the proof there involves a two-term complex which depends on the projective dimension. Thus some new ideas are necessary for attacking the general situation. On the other hand, neither positive examples nor counterexamples to this general case are known to experts. So, it is quite mysterious whether the above recollement still exists for a good tilting module of projective dimension at least 2.

In the present paper, we shall consider this question in detail. In fact, our discussion is implemented in the framework of Ringel modules (see Definition 4.1). This provides us a way to deal with the above question uniformly for higher tilting and cotilting modules. We first provide characterizations of when the kernel of the functor $T\otimes^{\mathbb{L}}_B-$ can be realized as the derived module category of a ring C with a homological ring epimorphism $B\to C$, and then use these criterions to give positive and negative examples to the above question for tilting modules of projective dimension bigger than 1. Finally, as another application of our criterions, we shall consider the above question for cotilting modules.

Before stating our main results precisely, we first introduce notation and recall some definitions.

Let A be a ring with identity, and let n be a natural number. A left A-module T is called an *n-tilting* A-module (see [15]) if the following three conditions are satisfied:

$(T1)$ There is an exact sequence

$$0\longrightarrow P_n\longrightarrow\cdots\longrightarrow P_1\xrightarrow{\sigma}P_0\xrightarrow{\pi}T\longrightarrow 0$$

of A-modules such that all P_i are projective, that is, the projective dimension of T is at most n;

$(T2)$ $\mathrm{Ext}^j_A(T,T^{(I)})=0$ for all $j\geq 1$ and nonempty sets I, where $T^{(I)}$ denotes the direct sum of I copies of T;

$(T3)$ There is an exact sequence

$$0\longrightarrow {}_AA\xrightarrow{\omega}T_0\longrightarrow T_1\longrightarrow\cdots\longrightarrow T_n\longrightarrow 0$$

of A-modules such that T_i is isomorphic to a direct summand of a direct sum of copies of T for all $0\leq i\leq n$.

An n-tilting module T is said to be *good* if $(T3)$ can be replaced by

$(T3)'$ there is an exact sequence

$$0\longrightarrow {}_AA\xrightarrow{\omega}T_0\longrightarrow T_1\longrightarrow\cdots\longrightarrow T_n\longrightarrow 0$$

of A-modules such that T_i is isomorphic to a direct summand of a finite direct sum of copies of T for all $0\leq i\leq n$. A good n-tilting module T is said to be *classical* if the modules P_i in $(T1)$ are finitely generated (see [10, 19]).

For any given tilting A-module T with $(T1)$-$(T3)$, the module $T':=\bigoplus_{i=0}^n T_i$ is a good n-tilting module which is equivalent to the given one, that is, T and T' generate the same tilting class in the category of A-modules (see [6]).

Let T be an n-tilting A-module and B the endomorphism ring of ${}_AT$. In general, the total right-derived functor $\mathbb{R}\mathrm{Hom}_A(T,-)$ does not define a triangle equivalence between the (unbounded) derived category $\mathscr{D}(A)$ of A and the derived category $\mathscr{D}(B)$ of B. However, if ${}_AT$ is good, then $\mathbb{R}\mathrm{Hom}_A(T,-)$ is fully faithful and induces a triangle equivalence between the derived category $\mathscr{D}(A)$ and the Verdier quotient of $\mathscr{D}(B)$ modulo the kernel $\mathrm{Ker}(T\otimes^{\mathbb{L}}_B-)$ of the total left-derived functor $T\otimes^{\mathbb{L}}_B-$ (see [6, Theorem 2.2]). Furthermore, the functor $\mathbb{R}\mathrm{Hom}_A(T,-):\mathscr{D}(A)\to\mathscr{D}(B)$ is an equivalence if and only if T is a classical tilting module if and only if $\mathrm{Ker}(T\otimes^{\mathbb{L}}_B-)$ vanishes (see [6]). From this point of view, the category $\mathrm{Ker}(T\otimes^{\mathbb{L}}_B-)$ measures the difference between the derived categories $\mathscr{D}(A)$ and $\mathscr{D}(B)$.

Motivated by the main result in [11], we introduce the following notion. A full triangulated subcategory $\mathcal{X}$ of $\mathscr{D}(B)$ is said to be *homological* if there is a homological ring epimorphism $B\to C$ of rings such that the restriction functor $\mathscr{D}(C)\to\mathscr{D}(B)$ induces a triangle equivalence from $\mathscr{D}(C)$ to $\mathcal{X}$. Thus, if the projective dimension of a good tilting module ${}_AT$ is at most 1, then the subcategory $\mathrm{Ker}(T\otimes^{\mathbb{L}}_B-)$ of $\mathscr{D}(B)$ is homological. Now, in terms of homological subcategories, our question can be restated as follows:

Question. *Is the full triangulated subcategory* $\mathrm{Ker}(T\otimes^{\mathbb{L}}_B-)$ *of* $\mathscr{D}(B)$ *always homological for any good n-tilting A-module T with $n\geq 2$? Here, B is the endomorphism ring of the module T.*

Let us first give several characterizations for $\mathrm{Ker}(T\otimes^{\mathbb{L}}_B-)$ to be homological.

Theorem 1.1. *Suppose that A is a ring and n is a natural number. Let T be a good n-tilting A-module, and let B be the endomorphism ring of ${}_AT$. Then the following are equivalent:*

(1) *The full triangulated subcategory* $\mathrm{Ker}(T\otimes^{\mathbb{L}}_B -)$ *of* $\mathscr{D}(B)$ *is homological.*

(2) *The category consisting of the B-modules Y with* $\mathrm{Tor}^B_m(T,Y)=0$ *for all $m\geq 0$ is an abelian subcategory of the category of all B-modules.*

(3) *The m-th cohomology of the complex* $\mathrm{Hom}_A(P^\bullet,A)\otimes_A T_B$ *vanishes for all $m\geq 2$, where the complex $P^\bullet$ is a deleted projective resolution of ${}_AT$.*

(4) *The kernel K of the homomorphism* $\mathrm{Coker}(\varphi_0)\longrightarrow \mathrm{Coker}(\varphi_1)$ *induced from* $\sigma: P_1\to P_0$ *in* $(T1)$ *satisfies* $\mathrm{Ext}^m_{B^{\mathrm{op}}}(T,K)=0$ *for all $m\geq 0$, where* $\varphi_i:\mathrm{Hom}_A(P_i,A)\otimes_A T\longrightarrow \mathrm{Hom}_A(P_i,T)$ *is the composition map under the identification of ${}_AT_B$ with* $\mathrm{Hom}_A(A,T)$ *for $i=0,1$.*

In particular, if $n=2$, then (1) holds if and only if $\mathrm{Ext}^2_A(T,A)\otimes_A T=0$.

We remark that if the category $\mathrm{Ker}(T\otimes^{\mathbb{L}}_B -)$ is homological in $\mathscr{D}(B)$, then the generalized localization $\lambda: B\to B_T$ of B at the module T_B exists (see Definition 3.4) and is homological, and therefore there is a recollement of derived module categories:

$$\begin{tikzcd} \mathscr{D}(B_T) \arrow[r, "D(\lambda_*)"] & \mathscr{D}(B) \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "{}_AT\otimes^{\mathbb{L}}_B-"] & \mathscr{D}(A) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

where $D(\lambda_*)$ stands for the restriction functor induced by λ. Thus, Theorem 1.1 can be regarded as a kind of generalization of [11, Theorem 1.1 (1)], and also gives an explanation why [11, Theorem 1.1 (1)] holds.

As a consequence of Theorem 1.1, we have the following corollary in which (1) extends [11, Theorem 1.1 (1)], while our new contribution to (2) is the necessity part of the statement.

Corollary 1.2. *Suppose that A is a ring and n is a natural number. Let T be a good n-tilting A-module, and let B be the endomorphism ring of ${}_AT$.*

(1) *If ${}_AT$ decomposes into $M\oplus N$ such that the projective dimension of ${}_AM$ is at most 1 and that the first syzygy of ${}_AN$ is finitely generated, then the category* $\mathrm{Ker}(T\otimes^{\mathbb{L}}_B -)$ *is homological.*

(2) *Suppose that A is commutative. If* $\mathrm{Hom}_A(T_{i+1},T_i)=0$ *for all T_i in $(T3)'$ with $1\leq i\leq n-1$, then the category* $\mathrm{Ker}(T\otimes^{\mathbb{L}}_B -)$ *is homological if and only if the projective dimension of ${}_AT$ is at most 1, that is, ${}_AT$ is a 1-tilting module.*

A remarkable consequence of Corollary 1.2 is that we can get an answer to the above-mentioned question. In fact, in Section 7.1, we display an example of an n-tilting module T for each $n\geq 2$ and shows that $\mathrm{Ker}(T\otimes^{\mathbb{L}}_B -)$ is not homological.

Dually, there is the notion of (good) cotilting modules of finite injective dimension over arbitrary rings. This notion involves injective cogenerators of module categories. As is known, there is no nice duality between infinitely generated tilting and cotilting modules. This means that methods for dealing with tilting modules may not work dually with cotilting modules. Nevertheless, we shall use methods in this paper to deal with cotilting modules with respect to some "nice" injective cogenerators. Our methods cover particularly cotilting modules over Artin algebras. Here, our main concern again is when the induced subcategories of derived categories of the endomorphism rings of good cotilting modules are homological, or equivalently, the existence of a recollement similar to [11, Theorem 1.1 (1)].

Our consideration is focused on (infinitely generated) cotilting modules over Artin algebras A. Let D be the usual duality of an Artin algebra. The dual module $D(A_A)$ is an injective cogenerator for the category of A-modules, and called the *ordinary injective cogenerator*. Our main result for cotilting modules is as follows.

Theorem 1.3. *Suppose that A is an Artin algebra. Let U be a good 1-cotilting A-module with respect to the ordinary injective cogenerator for the category of A-modules. Set* $R:=\mathrm{End}_A(U)$ *and* $M:=\mathrm{Hom}_A(U,D(A))$. *Then the universal localization $\lambda: R\to R_M$ of R at the module ${}_RM$ is homological, and there exists a recollement of derived module categories:*

$$\begin{tikzcd} \mathscr{D}(R_M) \arrow[r, "D(\lambda_*)"] & \mathscr{D}(R) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}(A) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

where $D(\lambda_*)$ stands for the restriction functor induced by λ.

As is known, over an Artin algebra, each 1-cotilting module is equivalent to the dual of a 1-tilting right module (see [1, Chapter 11, Section 4.15]). However, we cannot get Theorem 1.3 from the result [11, Theorem 1.1 (1)] because the relationship between the endomorphism ring of an infinitely generated 1-cotlting module and the one of the corresponding 1-tilting right module is unknown.

For a more general formulation of Theorem 1.3 on higher cotilting modules, one may see Corollary 6.3 and the diagram (‡) above Corollary 6.3. For higher cotiltig modules, we also give conditions and counterexamples for subcategories from cotilting modules not to be homological, though additional attention is needed.

The contents of this paper are sketched as follows. In Section 2, we fix notation, recall some definitions and prove some homological formulas. In Section 3, we introduce bireflective and homological subcategories in derived categories of rings, and discuss when bireflective subcategories are homological. In Section 4, we introduce a new class of modules, called Ringel modules, and establish a crucial result, Proposition 4.4, which is used not only to decide if a bireflective subcategory is homological, but also to investigate higher tilting and cotilting modules in the later considerations. In Section 5, we apply the results in previous sections to good tilting modules and show Theorem 1.1 as well as Corollary 1.2. At the end of this section, we point out an example which shows that there do exist higher tilting modules satisfying the conditions of Corollary 1.2 (1). In Section 6, we first apply our results in Section 4 to cotilting modules in a general setting, and then prove Theorem 1.3 for Artin algebras. It is worth noting that, for cotilting A-modules U, recollements of $\mathscr{D}(\mathrm{End}_A(U))$ may depend on the choices of injective cogenerators to which the cotilting modules are referred. In this section, we also give conditions for the subcategories from cotilting modules not to be homological. This is a preparation for constructing counterexamples in the next section. In Section 7, we apply our results in Section 5 to good tilting modules T over commutative rings, and give a counterexample to show that, in general, $\mathrm{Ker}(T\otimes^{\mathbb{L}}_B -)$ may not be realized as the derived module category of a ring C with a homological ring epimorphism $B\to C$. For higher cotilting modules, the same situation occurs. More precisely, we shall use results in Section 6 to display a counterexample which demonstrates that, in general, the corresponding subcategories from cotilting modules cannot be realizable as derived module categories of rings. This section ends with a few open questions closely related to the results in this paper.

2 Preliminaries

In this section, we briefly recall some definitions, basic facts and notation used in this paper. For unexplained notation employed in this paper, we refer the reader to [11] and the references therein.

2.1 Notation

Let $\mathcal{C}$ be an additive category.

Throughout the paper, a full subcategory $\mathcal{B}$ of $\mathcal{C}$ is always assumed to be closed under isomorphisms, that is, if $X\in\mathcal{B}$ and $Y\in\mathcal{C}$ with $Y\simeq X$, then $Y\in\mathcal{B}$.

Let X be an object in $\mathcal{C}$. Denote by $\mathrm{add}(X)$ the full subcategory of $\mathcal{C}$ consisting of all direct summands of finite coproducts of copies of M. If $\mathcal{C}$ admits small coproducts (that is, coproducts indexed over sets exist in $\mathcal{C}$), then we denote by $\mathrm{Add}(X)$ the full subcategory of $\mathcal{C}$ consisting of all direct summands of small coproducts of copies of X. Dually, if $\mathcal{C}$ admits small products, then we denote by $\mathrm{Prod}(X)$ the full subcategory of $\mathcal{C}$ consisting of all direct summands of small products of copies of X.

Given two morphisms $f: X\to Y$ and $g: Y\to Z$ in $\mathcal{C}$, we denote the composite of f and g by fg which is a morphism from X to Z. The induced morphisms $\mathrm{Hom}_{\mathcal{C}}(Z,f):\mathrm{Hom}_{\mathcal{C}}(Z,X)\to\mathrm{Hom}_{\mathcal{C}}(Z,Y)$ and $\mathrm{Hom}_{\mathcal{C}}(f,Z):\mathrm{Hom}_{\mathcal{C}}(Y,Z)\to\mathrm{Hom}_{\mathcal{C}}(X,Z)$ are denoted by f^* and f_*, respectively.

We denote the composition of a functor $F:\mathcal{C}\to\mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ with a functor $G:\mathcal{D}\to\mathcal{E}$ between categories $\mathcal{D}$ and $\mathcal{E}$ by GF which is a functor from $\mathcal{C}$ to $\mathcal{E}$. Let $\mathrm{Ker}(F)$ and $\mathrm{Im}(F)$ be the kernel and image of the functor F, respectively. In particular, $\mathrm{Ker}(F)$ is closed under isomorphisms in $\mathcal{C}$. In this note, we require that $\mathrm{Im}(F)$ is closed under isomorphisms in $\mathcal{D}$.

Suppose that $\mathcal{Y}$ is a full subcategory of $\mathcal{C}$. Let $\mathrm{Ker}(\mathrm{Hom}_{\mathcal{C}}(-,\mathcal{Y}))$ be the left orthogonal subcategory with respect to $\mathcal{Y}$, that is, the full subcategory of $\mathcal{C}$ consisting of the objects X such that $\mathrm{Hom}_{\mathcal{C}}(X,Y)=0$ for all objects Y in $\mathcal{Y}$. Similarly, we can define the right orthogonal subcategory $\mathrm{Ker}(\mathrm{Hom}_{\mathcal{C}}(\mathcal{Y},-))$ of $\mathcal{C}$ with respect to $\mathcal{Y}$.

Let $\mathscr{C}(\mathcal{C})$ be the category of all complexes over $\mathcal{C}$ with chain maps, and $\mathscr{K}(\mathcal{C})$ the homotopy category of $\mathscr{C}(\mathcal{C})$. As usual, we denote by $\mathscr{C}^b(\mathcal{C})$ the category of bounded complexes over C, and by $\mathscr{K}^b(\mathcal{C})$ the homotopy category of $\mathscr{C}^b(\mathcal{C})$. When $\mathcal{C}$ is abelian, the derived category of $\mathcal{C}$ is denoted by $\mathscr{D}(\mathcal{C})$, which is the localization of $\mathscr{K}(\mathcal{C})$ at all quasi-isomorphisms. It is well known that both $\mathscr{K}(\mathcal{C})$ and $\mathscr{D}(\mathcal{C})$ are triangulated categories. For a triangulated category, its shift functor is denoted by $[1]$ universally.

If $\mathcal{T}$ is a triangulated category with small coproducts, then, for an object U in $\mathcal{T}$, we denote by $\mathrm{Tria}(U)$ the smallest full triangulated subcategory of $\mathcal{T}$ containing U and being closed under small coproducts.

Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are triangulated categories with small coproducts. If $F:\mathcal{T}\to\mathcal{T}'$ is a triangle functor which commutes with small coproducts, then $F(\mathrm{Tria}(U))\subseteq \mathrm{Tria}(F(U))$ for every object U in $\mathcal{T}$.

2.2 Homological formulas

In this paper, all rings considered are assumed to be associative and with identity, and all ring homomorphisms preserve identity. Unless stated otherwise, all modules are referred to left modules.

Let R be a ring. We denote by R-Mod the category of all unitary left R-modules, by Ω_R^n the n-th syzygy operator of R-Mod for $n\in\mathbb{N}$, and regard Ω_R^0 as the identity operator of R-Mod.

If M is an R-module and I is a nonempty set, then we denote by $M^{(I)}$ and M^I the direct sum and product of I copies of M, respectively. If $f:M\to N$ is a homomorphism of R-modules, then the image of $x\in M$ under f is denoted by $(x)f$ instead of $f(x)$. The endomorphism ring of the R-module M is denoted by $\mathrm{End}_R(M)$. Thus M becomes a natural R-$\mathrm{End}_R(M)$-bimodule. Similarly, if N_R is a right R-module, then, by our convention, N is a left $(\mathrm{End}(N_R))^{\mathrm{op}}$- right R-bimodule.

As usual, we simply write $\mathscr{C}(R)$, $\mathscr{K}(R)$ and $\mathscr{D}(R)$ for $\mathscr{C}(R\text{-Mod})$, $\mathscr{K}(R\text{-Mod})$ and $\mathscr{D}(R\text{-Mod})$, respectively, and identify R-Mod with the subcategory of $\mathscr{D}(R)$ consisting of all stalk complexes concentrated in degree zero. Let $\mathscr{C}(R\text{-proj})$ be the full subcategory of $\mathscr{C}(R)$ consisting of those complexes such that all of their terms are finitely generated projective R-modules.

For each $n\in\mathbb{Z}$, we denote by $H^n(-):\mathscr{D}(R)\to R\text{-Mod}$ the n-th cohomology functor. A complex $X^\bullet$ is said to be *acyclic (or exact)* if $H^n(X^\bullet)=0$ for all $n\in\mathbb{Z}$.

In the following, we shall recall some definitions and basic facts about derived functors defined on derived module categories. For more details and proofs, we refer to [9, 25, 1, 13].

Recall that $\mathscr{K}(R)_P$ (respectively, $\mathscr{K}(R)_I$) denotes the smallest full triangulated subcategory of $\mathscr{K}(R)$ which

(i) contains all the bounded-above (respectively, bounded-below) complexes of projective (respectively, injective) R-modules, and

(ii) is closed under arbitrary direct sums (respectively, direct products).

Let $\mathscr{K}(R)_C$ be the full subcategory of $\mathscr{K}(R)$ consisting of all acyclic complexes. Then $(\mathscr{K}(R)_P,\mathscr{K}(R)_C)$ forms a hereditary torsion pair in $\mathscr{K}(R)$ in the following sense:

(a) Both $\mathscr{K}(R)_P$ and $\mathscr{K}(R)_C$ are full triangulated subcategories of $\mathscr{K}(R)$.

(b) $\mathrm{Hom}_{\mathscr{K}(R)}(M^\bullet,N^\bullet)=0$ for $M^\bullet\in\mathscr{K}(R)_P$ and $N^\bullet\in\mathscr{K}(R)_C$.

(c) For each $X^\bullet\in\mathscr{K}(R)$, there exists a distinguished triangle in $\mathscr{K}(R)$:

$$ {}_pX^\bullet \xrightarrow{\alpha_{X^\bullet}} X^\bullet \longrightarrow {}_cX^\bullet \longrightarrow ({}_pX^\bullet)[1] $$

such that ${}_pX^\bullet\in\mathscr{K}(R)_P$ and ${}_cX^\bullet\in\mathscr{K}(R)_C$.

In particular, for each complex $X^\bullet$ in $\mathscr{K}(R)$, the chain map ${}_pX^\bullet\xrightarrow{\alpha_{X^\bullet}}X^\bullet$ is a quasi-isomorphism in $\mathscr{K}(R)$. The complex ${}_pX^\bullet$ is called the *projective resolution* of $X^\bullet$ in $\mathscr{D}(R)$. For example, if X is an R-module, then we can choose ${}_pX$ to be a deleted projective resolution of ${}_RX$.

Note also that the property (b) implies that each quasi-isomorphism between complexes in $\mathscr{K}(R)_P$ is an isomorphism in $\mathscr{K}(R)$, that is a chain homotopy equivalence in $\mathscr{K}(R)$.

Dually, the pair $(\mathscr{K}(R)_C,\mathscr{K}(R)_I)$ is a hereditary torsion pair in $\mathscr{K}(R)$. This means that, for each $X^\bullet$ in $\mathscr{D}(R)$, there exists a complex ${}_iX^\bullet\in\mathscr{K}(R)_I$ together with a quasi-isomorphism $\beta_{X^\bullet}:X^\bullet\to{}_iX^\bullet$. The complex ${}_iX^\bullet$ is called the *injective resolution* of $X^\bullet$ in $\mathscr{D}(R)$.

More important, the composition functors

$$ \mathscr{K}(R)_P\hookrightarrow\mathscr{K}(R)\longrightarrow\mathscr{D}(R) \quad\text{and}\quad \mathscr{K}(R)_I\hookrightarrow\mathscr{K}(R)\longrightarrow\mathscr{D}(R) $$

are equivalences of triangulated categories, and the canonical localization functor $q:\mathscr{K}(R)\to\mathscr{D}(R)$ induces an isomorphism $\mathrm{Hom}_{\mathscr{K}(R)}(X^\bullet,Y^\bullet)\xrightarrow{\simeq}\mathrm{Hom}_{\mathscr{D}(R)}(X^\bullet,Y^\bullet)$ of abelian groups whenever either $X^\bullet\in\mathscr{K}(R)_P$ or $Y^\bullet\in\mathscr{K}(R)_I$.

For a triangle functor $F:\mathscr{K}(R)\to\mathscr{K}(S)$, we define its *total left-derived functor* $\mathbb{L}F:\mathscr{D}(R)\to\mathscr{D}(S)$ by $X^\bullet\mapsto F({}_pX^\bullet)$, and its *total right-derived functor* $\mathbb{R}F:\mathscr{D}(R)\to\mathscr{D}(S)$ by $X^\bullet\mapsto F({}_iX^\bullet)$. Specially, if F preserves acyclicity, that is, $F(X^\bullet)$ is acyclic whenever $X^\bullet$ is acyclic, then F induces a triangle functor $D(F):\mathscr{D}(R)\to\mathscr{D}(S)$ defined by

$X^\bullet \mapsto F(X^\bullet)$. In this case, up to natural isomorphism, we have $\mathbb{L}F = \mathbb{R}F = D(F)$, and simply call $D(F)$ the *derived functor* of F.

Let $M^\bullet$ be a complex of R-S-bimodules. Then, the tensor functor and the Hom-functor

$$M^\bullet \otimes_S^\bullet - : \mathscr{K}(S) \to \mathscr{K}(R) \quad \text{and} \quad \mathrm{Hom}_R^\bullet(M^\bullet, -) : \mathscr{K}(R) \to \mathscr{K}(S)$$

form a pair of adjoint triangle functors. For the concise definitions of the tensor and Hom complex of two complexes, we refer, for example, to [13, Section 2.1]. For simplicity, if $Y \in S$-Mod and $X \in R$-Mod, we denote $M^\bullet \otimes_S^\bullet Y$ and $\mathrm{Hom}_R^\bullet(M^\bullet, X)$ by $M^\bullet \otimes_S Y$ and $\mathrm{Hom}_R(M^\bullet, X)$, respectively.

Denote by $M^\bullet \otimes_S^{\mathbb{L}} -$ the total left-derived functor of $M^\bullet \otimes_S^\bullet -$, and by $\mathbb{R}\mathrm{Hom}_R(M^\bullet, -)$ the total right-derived functor of $\mathrm{Hom}_R^\bullet(M^\bullet, -)$. Note that $\big(M^\bullet \otimes_S^{\mathbb{L}} -, \mathbb{R}\mathrm{Hom}_R(M^\bullet, -)\big)$ is still an adjoint pair of triangle functors.

The following result is freely used, but not explicitly stated in the literature. Here, we will arrange it as a lemma for later reference. For the idea of its proof, we refer to [25, Generalized Existence Theorem 10.5.9].

Lemma 2.1. *Let R and S be rings, and let $H : \mathscr{K}(R) \longrightarrow \mathscr{K}(S)$ be a triangle functor.*

(1) *Define $\mathcal{L}_H$ to be the full subcategory of $\mathscr{K}(R)$ consisting of all complexes $X^\bullet$ such that the chain map $H(\alpha_{X^\bullet}) : H({}_pX^\bullet) \longrightarrow H(X^\bullet)$ is a quasi-isomorphism in $\mathscr{K}(S)$. Then*

(i) *$\mathcal{L}_H$ is a triangulated subcategory of $\mathscr{K}(R)$ containing $\mathscr{K}(R)_P$.*

(ii) *$\mathcal{L}_H \cap \mathscr{K}(R)_C = \{X^\bullet \in \mathscr{K}(R)_C \mid H(X^\bullet) \in \mathscr{K}(S)_C\}$.*

(iii) *There exists a commutative diagram of triangle functors:*

$$\begin{tikzcd}
\mathscr{K}(R)_P \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & \mathscr{D}(R) \arrow[d, "\mathbb{L}H"] \\
\mathcal{L}_H/\mathcal{L}_H \cap \mathscr{K}(R)_C \arrow[r, "D(H)"] & \mathscr{D}(S)
\end{tikzcd}$$

where $\mathcal{L}_H/\mathcal{L}_H \cap \mathscr{K}(R)_C$ denotes the Verdier quotient of $\mathcal{L}_H$ by $\mathcal{L}_H \cap \mathscr{K}(R)_C$, and where $D(H)$ is defined by $X^\bullet \mapsto H(X^\bullet)$ for $X^\bullet \in \mathcal{L}_H$.

(2) *Define $\mathcal{R}_H$ to be the full subcategory of $\mathscr{K}(R)$ consisting of all complexes $X^\bullet$ such that the chain map $H(\beta_{X^\bullet}) : H(X^\bullet) \to H({}_iX^\bullet)$ is a quasi-isomorphism in $\mathscr{K}(S)$. Then*

(i) *$\mathcal{R}_H$ is a triangulated subcategory of $\mathscr{K}(R)$ containing $\mathscr{K}(R)_I$.*

(ii) *$\mathcal{R}_H \cap \mathscr{K}(R)_C = \{X^\bullet \in \mathscr{K}(R)_C \mid H(X^\bullet) \in \mathscr{K}(S)_C\}$.*

(iii) *There exists a commutative diagram of triangle functors:*

$$\begin{tikzcd}
\mathscr{K}(R)_I \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & \mathscr{D}(R) \arrow[d, "\mathbb{R}H"] \\
\mathcal{R}_H/\mathcal{R}_H \cap \mathscr{K}(R)_C \arrow[r, "D(H)"] & \mathscr{D}(S)
\end{tikzcd}$$

where $\mathcal{R}_H/\mathcal{R}_H \cap \mathscr{K}(R)_C$ denotes the Verdier quotient of $\mathcal{R}_H$ by $\mathcal{R}_H \cap \mathscr{K}(R)_C$, and where $D(H)$ is defined by $X^\bullet \mapsto H(X^\bullet)$ for $X^\bullet \in \mathcal{R}_H$.

Note that if H commutes with arbitrary direct sums, then $\mathcal{L}_H$ is closed under arbitrary direct sums in $\mathscr{K}(R)$. Dually, if H commutes with arbitrary direct products, then $\mathcal{R}_H$ is closed under arbitrary direct products in $\mathscr{K}(R)$.

From Lemma 2.1, we see that, up to natural isomorphism, the action of the functor $\mathbb{L}H$ (respectively, $\mathbb{R}H$) on a complex $X^\bullet$ in $\mathcal{L}_H$ (respectively, $\mathcal{R}_H$) is the same as that of the functor H on $X^\bullet$. Based on this point of view, we obtain the following result which will be applied in our later proofs.

Corollary 2.2. *Let R and S be two rings. Suppose that (F, G) is a pair of adjoint triangle functors with $F : \mathscr{K}(S) \to \mathscr{K}(R)$ and $G : \mathscr{K}(R) \to \mathscr{K}(S)$. Let $\theta : FG \to Id_{\mathscr{K}(R)}$ and $\varepsilon : (\mathbb{L}F)(\mathbb{R}G) \to Id_{\mathscr{D}(R)}$ be the counit adjunctions. If $X^\bullet \in \mathcal{R}_G$ and $G(X^\bullet) \in \mathcal{L}_F$, then there exists a commutative diagram in $\mathscr{D}(R)$:*

$$\begin{tikzcd}
(\mathbb{L}F)(\mathbb{R}G)(X^\bullet) \arrow[r, "\varepsilon_{X^\bullet}"] \arrow[d, "\simeq"'] & X^\bullet \arrow[d, equal] \\
FG(X^\bullet) \arrow[r, "\theta_{X^\bullet}"] & X^\bullet
\end{tikzcd}$$

Proof. It follows from $X^\bullet \in \mathcal{R}_G$ that the quasi-isomorphism $\beta_{X^\bullet}: X^\bullet \to {}_iX^\bullet$ in $\mathscr{K}(R)$ induces a quasi-isomorphism $G(\beta_{X^\bullet}): G(X^\bullet) \to G({}_iX^\bullet)$ in $\mathscr{K}(S)$. Since $(\mathscr{K}(S)_P, \mathscr{K}(S)_C)$ is a hereditary torsion pair in $\mathscr{K}(S)$, there exists a homomorphism ${}_pG(\beta_{X^\bullet}): {}_pG(X^\bullet) \to {}_pG({}_iX^\bullet)$ in $\mathscr{K}(S)$ such that the following diagram is commutative:

$$\begin{tikzcd}
{}_pG(X^\bullet) \arrow[r, "\alpha_{G(X^\bullet)}"] \arrow[d, "{}_pG(\beta_{X^\bullet})"'] & G(X^\bullet) \arrow[d, "G(\beta_{X^\bullet})"] \\
{}_pG({}_iX^\bullet) \arrow[r, "\alpha_{G({}_iX^\bullet)}"] & G({}_iX^\bullet)
\end{tikzcd}$$

Note that ${}_pG(\beta_{X^\bullet})$ is a quasi-isomorphism in $\mathscr{K}(S)$ since all the other chain maps in the above diagram are quasi-isomorphisms. By the property (b) related to the pair $(\mathscr{K}(S)_P, \mathscr{K}(S)_C)$, we know that ${}_pG(\beta_{X^\bullet})$ is an isomorphism in $\mathscr{K}(S)$, and therefore the chain map $F({}_pG(\beta_{X^\bullet})): F({}_pG(X^\bullet)) \longrightarrow F({}_pG({}_iX^\bullet))$ is an isomorphism in $\mathscr{K}(R)$.

Now, we can easily construct the following commutative diagram in $\mathscr{K}(R)$:

$$\begin{tikzcd}
 & F({}_pG(X^\bullet)) \arrow[r, "F(\alpha_{G(X^\bullet)})"] \arrow[d, "F({}_pG(\beta_{X^\bullet}))"', "\simeq"] & FG(X^\bullet) \arrow[r, "\theta_{X^\bullet}"] \arrow[d, "FG(\beta_{X^\bullet})"] & X^\bullet \arrow[d, "\beta_{X^\bullet}"] \\
(\mathbb{L}F)(\mathbb{R}G)(X^\bullet) \arrow[r, equal] & F({}_pG({}_iX^\bullet)) \arrow[r, "F(\alpha_{G({}_iX^\bullet)})"] & FG({}_iX^\bullet) \arrow[r, "\theta_{{}_iX^\bullet}"] & {}_iX^\bullet
\end{tikzcd}$$

Since $G(X^\bullet) \in \mathcal{L}_F$ by assumption, the chain map $F(\alpha_{G(X^\bullet)})$ is a quasi-isomorphism in $\mathscr{K}(R)$, and is an isomorphism in $\mathscr{D}(R)$. Clearly, the quasi-isomorphism $\beta_{X^\bullet}$ is an isomorphism in $\mathscr{D}(R)$.

Furthermore, the counit $\varepsilon_{X^\bullet}: (\mathbb{L}F)(\mathbb{R}G)(X^\bullet) \longrightarrow X^\bullet$ is actually given by the composite of the following homomorphisms in $\mathscr{D}(R)$:

$$(\mathbb{L}F)(\mathbb{R}G)(X^\bullet) = F({}_pG({}_iX^\bullet)) \xrightarrow{F(\alpha_{G({}_iX^\bullet)})} FG({}_iX^\bullet) \xrightarrow{\theta_{{}_iX^\bullet}} {}_iX^\bullet \xrightarrow{(\beta_{X^\bullet})^{-1}} X^\bullet.$$

Define

$$\tau = \big(F({}_pG(\beta_{X^\bullet}))\big)^{-1} F(\alpha_{G(X^\bullet)}):\ (\mathbb{L}F)(\mathbb{R}G)(X^\bullet) \longrightarrow FG(X^\bullet)$$

which is an isomorphism in $\mathscr{D}(R)$. It follows that there exists a commutative diagram in $\mathscr{D}(R)$:

$$\begin{tikzcd}
(\mathbb{L}F)(\mathbb{R}G)(X^\bullet) \arrow[r, "\varepsilon_{X^\bullet}"] \arrow[d, "\tau"] & X^\bullet \arrow[d, equal] \\
FG(X^\bullet) \arrow[r, "\theta_{X^\bullet}"] & X^\bullet
\end{tikzcd}$$

This finishes the proof. □

As a preparation for our later proofs, we mention the following three homological formulas which are related to derived functors or total derived functors. The first one is taken from [16, Theorem 3.2.1, Theorem 3.2.13, Remark 3.2.27].

Lemma 2.3. *Let R and S be rings. Suppose that M is an S-R-bimodule and I is an injective S-module.*

(1) *If N is an R-module, then*

$$\mathrm{Hom}_S(\mathrm{Tor}_i^R(M,N), I) \simeq \mathrm{Ext}_R^i(N, \mathrm{Hom}_S(M,I)) \text{ for all } i \geq 0.$$

(2) *If L is an R^{op}-module which has a finitely generated projective resolution in R^{op}-*Mod*, then*

$$\mathrm{Hom}_S(\mathrm{Ext}_R^i(L,M), I) \simeq \mathrm{Tor}_i^R(L, \mathrm{Hom}_S(M,I)) \text{ for all } i \geq 0.$$

The next formula is proved in [13, Section 2.1].

Lemma 2.4. *Let R and S be rings. Suppose that $X^\bullet$ is a bounded complex of R-S-bimodules. If $X^\bullet \in \mathscr{C}^b(R\text{-proj})$, then there is a natural isomorphism of functors:*

$$\mathrm{Hom}_R(X^\bullet, R) \otimes^\bullet_R - \xrightarrow{\simeq} \mathrm{Hom}^\bullet_R(X^\bullet, -) : \mathscr{C}(R) \to \mathscr{C}(S).$$

In particular,

$$\mathrm{Hom}_R(X^\bullet, R) \otimes^{\mathbb{L}}_R - \xrightarrow{\simeq} \mathbb{R}\mathrm{Hom}_R(X^\bullet, -) : \mathscr{D}(R) \to \mathscr{D}(S).$$

The last formula is useful for us to calculate the cohomology groups of tensor products of complexes.

Lemma 2.5. *Let n be an integer, and let S be a ring and M an S^{op}-module. Suppose that $Y^\bullet := (Y^i)_{i\in\mathbb{Z}}$ is a complex in $\mathscr{C}(S)$ such that $Y^i = 0$ for all $i \geq n+1$, and $\mathrm{Tor}^S_j(M, Y^i) = 0$ for all $i \in \mathbb{Z}$ and $j \geq 1$. Let $m \in \mathbb{Z}$ with $m < n$. If $\mathrm{Tor}^S_t\big(M, H^{m+t}(Y^\bullet)\big) = 0 = \mathrm{Tor}^S_{t-1}\big(M, H^{m+t}(Y^\bullet)\big)$ for $0 \leq t \leq n-m-1$, then $H^m(M \otimes_S Y^\bullet) \simeq \mathrm{Tor}^S_{n-m}\big(M, H^n(Y^\bullet)\big)$.*

Proof. Suppose that $Y^\bullet$ is the following form:

$$\cdots \longrightarrow Y^{m-1} \xrightarrow{d^{m-1}} Y^m \xrightarrow{d^m} Y^{m+1} \longrightarrow \cdots \longrightarrow Y^{n-1} \xrightarrow{d^{n-1}} Y^n \longrightarrow 0 \longrightarrow \cdots$$

For $i \in \mathbb{Z}$, define $C_i := \mathrm{Coker}(d^{i-1}) = Y^i/\mathrm{Im}(d^{i-1})$ and $I_i := \mathrm{Im}(d^i)$. Then we have two short exact sequences of S-modules for each $i \in \mathbb{Z}$:

$$(a) \quad 0 \longrightarrow H^i(Y^\bullet) \longrightarrow C_i \xrightarrow{\pi_i} I_i \longrightarrow 0 \quad \text{and} \quad (b) \quad 0 \longrightarrow I_i \overset{\lambda_i}{\hookrightarrow} Y^{i+1} \longrightarrow C_{i+1} \longrightarrow 0.$$

Clearly, $H^i(Y^\bullet) = \mathrm{Ker}(\pi_i\lambda_i)$, and $d^i : Y^i \to Y^{i+1}$ is just the composite of the canonical surjection $Y^i \to C_i$ with $\pi_i\lambda_i : C_i \to Y^{i+1}$.

(1) We claim that if $M \otimes_S H^i(Y^\bullet) = 0$, then $H^i(M \otimes_S Y^\bullet) \simeq \mathrm{Tor}^S_1(M, C_{i+1})$.

In fact, since $M \otimes_S - : S\text{-Mod} \to \mathbb{Z}\text{-Mod}$ is right exact, the sequence

$$M \otimes_S Y^{i-1} \xrightarrow{1\otimes d^{i-1}} M \otimes_S Y^i \longrightarrow M \otimes_S C_i \longrightarrow 0$$

is exact, that is, $\mathrm{Coker}(1 \otimes d^{i-1}) \simeq M \otimes_S C_i$. This implies that $H^i(M \otimes_S Y^\bullet) \simeq \mathrm{Ker}(1 \otimes \pi_i\lambda_i)$ where

$$1 \otimes \pi_i\lambda_i = (1 \otimes \pi_i)(1 \otimes \lambda_i) : M \otimes_S C_i \to M \otimes_S Y^{i+1},$$

which is the composite of $1 \otimes \pi_i : M \otimes_S C_i \longrightarrow M \otimes_S I_i$ with $1 \otimes \lambda_i : M \otimes_S I_i \longrightarrow M \otimes_S Y^{i+1}$.

Assume that $M \otimes_S H^i(Y^\bullet) = 0$. Then $1 \otimes \pi_i$ is an isomorphism and $\mathrm{Ker}(1 \otimes \pi_i\lambda_i) \simeq \mathrm{Ker}(1 \otimes \lambda_i)$. Now, we apply $M \otimes_S -$ to the sequence (b), and get the following exact sequence:

$$\mathrm{Tor}^S_1(M, Y^{i+1}) \longrightarrow \mathrm{Tor}^S_1(M, C_{i+1}) \longrightarrow M \otimes_S I_i \xrightarrow{1\otimes\lambda_i} M \otimes_S Y^{i+1}$$

Since $\mathrm{Tor}^S_1(M, Y^{i+1}) = 0$ by assumption, we obtain $\mathrm{Tor}^S_1(M, C_{i+1}) \simeq \mathrm{Ker}(1 \otimes \lambda_i)$. It follows that

$$H^i(M \otimes_S Y^\bullet) \simeq \mathrm{Ker}(1 \otimes \pi_i\lambda_i) \simeq \mathrm{Ker}(1 \otimes \lambda_i) \simeq \mathrm{Tor}^S_1(M, C_{i+1}).$$

This finishes the claim (1).

(2) We show that, for any $j \geq 1$, if $\mathrm{Tor}^S_j(M, H^i(Y^\bullet)) = 0 = \mathrm{Tor}^S_{j-1}(M, H^i(Y^\bullet))$, then

$$\mathrm{Tor}^S_j(M, C_i) \xrightarrow{\simeq} \mathrm{Tor}^S_{j+1}(M, C_{i+1}).$$

This follows from applying $M \otimes_S -$ to the exact sequences (a) and (b), respectively, together with our assumptions on $Y^\bullet$.

(3) Let $m \in \mathbb{Z}$ with $m \leq n-1$. Suppose that

$$\mathrm{Tor}^S_t\big(M, H^{m+t}(Y^\bullet)\big) = 0 = \mathrm{Tor}^S_{t-1}\big(M, H^{m+t}(Y^\bullet)\big) \quad \text{for} \quad 0 \leq t \leq n-m-1.$$

Then, by taking $t = 0$, we have $M \otimes_S H^m(Y^\bullet) = 0$. Thanks to (1), we have $H^m(M \otimes_S Y^\bullet) \simeq \mathrm{Tor}^S_1(M, C_{m+1})$. Since $Y^i = 0$ for $i \geq n+1$, it follows that $H^n(Y^\bullet) = C_n$. This implies that if $n-m = 1$, then $H^m(M \otimes_S Y^\bullet) \simeq \mathrm{Tor}^S_{n-m}\big(M, H^n(Y^\bullet)\big)$.

Now, suppose $n-m \geq 2$. For $1 \leq t \leq n-m-1$, we see from (2) that $\mathrm{Tor}^S_t(M, C_{m+t}) \xrightarrow{\simeq} \mathrm{Tor}^S_{t+1}(M, C_{m+t+1})$. Thus

$$\mathrm{Tor}^S_1(M, C_{m+1}) \simeq \mathrm{Tor}^S_2(M, C_{m+2}) \simeq \cdots \simeq \mathrm{Tor}^S_{n-m-1}(M, C_{n-1}) \simeq \mathrm{Tor}^S_{n-m}(M, C_n).$$

Consequently, $H^m(M \otimes_S Y^\bullet) \simeq \mathrm{Tor}^S_1(M, C_{m+1}) \simeq \mathrm{Tor}^S_{n-m}(M, C_n) = \mathrm{Tor}^S_{n-m}(M, H^n(Y^\bullet))$. This finishes the proof of Lemma 2.5. $\square$

2.3 Relative Mittag-Leffler modules

Now, we recall the definition of relative Mittag-Leffler modules (see [17], [2]).

Definition 2.6. A right R-module M is said to be *R-Mittag-Leffler* if the canonical map

$$\rho_I:\ M\otimes_R R^I \longrightarrow M^I,\ m\otimes(r_i)_{i\in I}\mapsto (mr_i)_{i\in I}\ \text{ for }\ m\in M,\ r_i\in R,$$

is injective for any nonempty set I.

A right R-module M is said to be *strongly R-Mittag-Leffler* if the m-th syzygy of M is R-Mittag-Leffler for every $m\geq 0$.

By [17, Theorem 1], a right R-module M is R-Mittag-Leffler if and only if, for any finitely generated submodule X of M_R, the inclusion $X\to M$ factorizes through a finitely presented right R-module. This implies that if M is finitely presented, then it is R-Mittag-Leffler. Actually, for such a module M, the above map ρ_I is always bijective (see [16, Theorem 3.2.22]). Further, if the ring R is right noetherian, then each right R-module is R-Mittag-Leffler since each finitely generated right R-module is finitely presented.

In the next lemma, we shall collect some basic properties of Mittag-Leffler modules for later use.

Lemma 2.7. *Let R be a ring and M a right R-module. Then the following statements are true.*

(1) *If M is R-Mittag-Leffler, then so is each module in* $\mathrm{Add}(M_R)$. *In particular, each projective right R-module is R-Mittag-Leffler.*

(2) *The first syzygy of M in R^{op}-Mod is R-Mittag-Leffler if and only if* $\mathrm{Tor}_1^R(M,R^I)=0$ *for every nonempty set I.*

(3) *M is strongly R-Mittag-Leffler if and only if M is R-Mittag-Leffler and* $\mathrm{Tor}_i^R(M,R^I)=0$ *for each $i\geq 1$ and every nonempty set I.*

(4) *If M is finitely generated, then M is strongly R-Mittag-Leffler if and only if M has a finitely generated projective resolution.*

Proof. (1) follows from the fact that tensor functors commute with direct sums.

(2) Note that the first syzygy $\Omega_R(M)$ of M depends on the choice of projective presentations of M_R. However, the "R-Mittag-Leffler" property of $\Omega_R(M)$ is independent of the choice of projective presentations of M_R. This is due to (1) and Schanuel's Lemma in homological algebra.

So, we choose an exact sequence

$$0\longrightarrow K_1\stackrel{f}{\longrightarrow}P_1\longrightarrow M\longrightarrow 0$$

of right R-modules with P_1 projective, and shall show that K_1 is R-Mittag-Leffler if and only if $\mathrm{Tor}_1^R(M,R^I)=0$ for any nonempty set I.

Obviously, we can construct the following exact commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Tor}_1^R(M,R^I) \arrow[r] & K_1\otimes_R R^I \arrow[r, "f\otimes 1"] \arrow[d, "\rho_2"] & P_1\otimes_R R^I \arrow[r] \arrow[d, "\rho_1"] & M\otimes_R R^I \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & {K_1}^I \arrow[r, "f^I"] & {P_1}^I \arrow[r] & M^I \arrow[r] & 0
\end{tikzcd}$$

where ρ_i, $1\leq i\leq 2$, are the canonical maps (see Definition 2.6). Since the projective module P_1 is R-Mittag-Leffler by (1), the map ρ_1 is injective. This means that ρ_2 is injective if and only if so is $f\otimes 1$. Clearly, the former is equivalent to that K_1 is R-Mittag-Leffler, while the latter is equivalent to that $\mathrm{Tor}_1^R(M,R^I)=0$. This finishes the proof of (2).

(3) For each $i\geq 0$, let $\Omega_R^i(M)$ stand for the i-th syzygy of M in R^{op}-Mod. Then, for each nonempty set I, we always have

$$\mathrm{Tor}_{i+1}^R(M,R^I)\simeq \mathrm{Tor}_1^R(\Omega_R^i(M),\,R^I).$$

Now (3) follows immediately from (2).

(4) The sufficient condition is clear. Now suppose that M is strongly R-Mittag-Leffler. We need only to show that the first syzygy of M is finitely generated, that is, M is finitely presented. However, this follows from the fact that the inclusion map $M\hookrightarrow M$ factorizes through a finitely presented right R-module. $\square$

A special class of strongly Mittag-Leffler modules is the class of tilting modules. The following result can be concluded from [2, Corollary 9.8], which will play an important role in our proof of the main result.

Lemma 2.8. *If M is a tilting right R-module, then M is strongly R-Mittag-Leffler.*

As a corollary of Lemmas 2.8 and 2.7 (4), we obtain the following result which is a generalization of [11, Corollary 4.7].

Corollary 2.9. *Let M be a tilting right R-module. If M is finitely generated, then M is classical.*

Proof. Suppose that M_R is finitely generated. Then we can get an exact sequence $(T3)'$ from $(T3)$ by using the argument in [11, Corollary 4.7] repeatedly. This shows that M_R is actually a good tilting module. Since M is strongly R-Mittag-Leffler, it follows from Lemma 2.7 (4) that M admits a finitely generated projective resolution. Clearly, such a resolution can be chosen to be of finite length since M has finite projective dimension. This implies that M_R is classical. $\square$

3 Homological subcategories of derived module categories

In this section, we shall give the definitions of bireflective and homological subcategories of derived module categories. In particular, we shall establish some applicable criterions for bireflective subcategories to be homological.

Let R and S be arbitrary rings.

Let $\lambda : R \to S$ be a homomorphism of rings. We denote by $\lambda_* : S\text{-Mod} \to R\text{-Mod}$ the restriction functor induced by λ, and by $D(\lambda_*) : \mathscr{D}(S) \to \mathscr{D}(R)$ the derived functor of the exact functor λ_*. Recall that λ is a *ring epimorphism* if $\lambda_* : S\text{-Mod} \to R\text{-Mod}$ is fully faithful. This is equivalent to saying that the multiplication map $S \otimes_R S \to S$ is an isomorphism in S-Mod.

Two ring epimorphisms $\lambda : R \to S$ and $\lambda' : R \to S'$ are said to be *equivalent* if there is an isomorphism $\psi : S \to S'$ of rings such that $\lambda' = \lambda\psi$. Note that there is a bijection between the equivalence classes of ring epimorphisms staring from R and bireflective full subcategories of R-Mod, and that there is a bijection between bireflective full subcategories of R-Mod and the abelian full subcategories of R-Mod which are closed under arbitrary direct sums and direct products (see, for example, [11, Lemma 2.1]).

Recall that a ring epimorphism $\lambda : R \to S$ is called *homological* if $\mathrm{Tor}_i^R(S,S) = 0$ for all $i > 0$. This is equivalent to that the functor $D(\lambda_*) : \mathscr{D}(S) \to \mathscr{D}(R)$ is fully faithful, or that $S \otimes_R^{\mathbb{L}} S \simeq S$ in $\mathscr{D}(S)$. It is known that $D(\lambda_*)$ has a left adjoint $S \otimes_R^{\mathbb{L}} -$ and a right adjoint $\mathbb{R}\mathrm{Hom}_R(S, -)$.

Let $\mathcal{Y}$ be a full triangulated subcategory of $\mathscr{D}(R)$. We say that $\mathcal{Y}$ is *bireflective* if the inclusion $\mathcal{Y} \to \mathscr{D}(R)$ admits both a left adjoint and a right adjoint.

Combining [8, Chapter I, Proposition 2.3] with [11, Section 2.3], we know that a full triangulated subcategory $\mathcal{Y}$ of $\mathscr{D}(R)$ is bireflective if and only if there exists a recollement of triangulated categories of the form

$$\begin{tikzcd} \mathcal{Y} \arrow[r, "i_*"] & \mathscr{D}(R) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathcal{X} \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

where i_* is the inclusion functor. Here, by a recollement of triangulated categories (see [7]) we mean that there are six triangle functors between triangulated categories in the following diagram:

$$\begin{tikzcd} \mathcal{Y} \arrow[r, "i_*=i_!"] & \mathscr{D}(R) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^!=j^*"] & \mathcal{X} \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

such that

(1) $(i^*, i_*), (i_!, i^!), (j_!, j^!)$ and (j^*, j_*) are adjoint pairs,

(2) i_*, j_* and $j_!$ are fully faithful functors,

(3) $i^! j_* = 0$ (and thus also $j^! i_! = 0$ and $i^* j_! = 0$), and

(4) for each object $X \in \mathscr{D}(R)$, there are two canonical distinguished triangles in $\mathscr{D}(R)$:

$$i_! i^!(X) \longrightarrow X \longrightarrow j_* j^*(X) \longrightarrow i_! i^!(X)[1], \qquad j_! j^!(X) \longrightarrow X \longrightarrow i_* i^*(X) \longrightarrow j_! j^!(X)[1],$$

where $i_! i^!(X) \to X$ and $j_! j^!(X) \to X$ are counit adjunction morphisms, and where $X \to j_* j^*(X)$ and $X \to i_* i^*(X)$ are unit adjunction morphisms.

Note that $\mathcal{X}$ is always equivalent to the full subcategory $\mathrm{Ker}\big(\mathrm{Hom}_{\mathscr{D}(R)}(-,\mathcal{Y})\big)$ of $\mathscr{D}(R)$ as triangulated categories (for example, see [11, Lemma 2.6]). But here we do not require that the triangulated category $\mathcal{X}$ must be a subcategory of $\mathscr{D}(R)$ in general. For more examples of recollements related to homological ring epimorphisms, we refer the reader to [12].

Clearly, if $\mathcal{Y}$ is homological (see Definition in Section 1), then it is bireflective. Let us now consider the converse of this statement.

From now on, we assume that $\mathcal{Y}$ is a **bireflective subcategory** of $\mathscr{D}(R)$, and define $\mathscr{E} := \mathcal{Y} \cap R\text{-Mod}$.

It is easy to see that $\mathcal{Y}$ is closed under isomorphisms, arbitrary direct sums and direct products in $\mathscr{D}(R)$. This implies that $\mathscr{E}$ also has the above properties in R-Mod. Moreover, $\mathscr{E}$ always admits the "2 out of 3" property: For an arbitrary short exact sequence in R-Mod, if any two of its three terms belong to $\mathscr{E}$, then the third one belongs to $\mathscr{E}$. By [11, Lemma 2.1], $\mathscr{E}$ is an abelian subcategory of R-Mod if and only if $\mathscr{E}$ is closed under kernels (respectively, cokernels) in R-Mod. This is also equivalent to saying that there exists a unique ring epimorphism $\lambda : R \to S$ (up to equivalence) such that $\mathscr{E}$ is equal to $\mathrm{Im}(\lambda_*)$.

If $\mathcal{Y}$ is homological via a homological ring epimorphism $\lambda : R \to S$, then $\mathcal{Y} = \mathrm{Im}\big(D(\lambda_*)\big)$ and $\mathscr{E} = \mathrm{Im}(\lambda_*)$. In this case, $\mathscr{E}$ must be a full, abelian subcategory of R-Mod.

In general, for a bireflective subcategory $\mathcal{Y}$ in $\mathscr{D}(R)$, the category $\mathscr{E}$ may not be abelian. This means that bireflective subcategories in $\mathscr{D}(R)$ may not be homological. Alternatively, we can reach this point by looking at differential graded rings: By the proof of [8, Chapter IV, Proposition 1.1], the complex $i^*(R)$ is a compact generator of $\mathcal{Y}$. In particular, we have $\mathcal{Y} = \mathrm{Tria}(i^*(R))$. It follows from [1, Chapter 5, Theorem 8.5] that there exists a dg (differential graded) ring such that its dg derived category is equivalent to $\mathcal{Y}$ as triangulated categories. In general, this dg ring may have non-trivial cohomologies in other degrees besides the degree 0. In other words, the category $\mathcal{Y}$ may not be realized by the derived module category of an ordinary ring.

Let $i_* : \mathcal{Y} \to \mathscr{D}(R)$ be the inclusion functor with $i^* : \mathscr{D}(R) \to \mathcal{Y}$ as its left adjoint. Define $\Lambda := \mathrm{End}_{\mathscr{D}(R)}(i^*(R))$. Then, associated with $\mathcal{Y}$, there is a ring homomorphism defined by

$$\delta :\ R \longrightarrow \Lambda, \quad r \mapsto i^*(\cdot r) \ \text{for} \ r \in R,$$

where $\cdot r : R \to R$ is the right multiplication by r map. This ring homomorphism induces a functor

$$\delta_* : \Lambda\text{-Mod} \longrightarrow R\text{-Mod},$$

called the restriction functor.

The following result is motivated by [22, Section 6 and Section 7].

Lemma 3.1. *The following statements hold true.*

(1) *For each $Y^\bullet \in \mathcal{Y}$, we have $H^n(Y^\bullet) \in \mathrm{Im}(\delta_*)$ for all $n \in \mathbb{Z}$. In particular, $H^n(i^*(R))$ is an R-Λ-bimodule for all $n \in \mathbb{Z}$.*

(2) *Let $\eta_R : R \to i_*i^*(R)$ be the unit adjunction morphism with respect to the adjoint pair (i^*, i_*). Then $\Lambda \simeq H^0(i^*(R))$ as R-Λ-bimodules, and there exists a commutative diagram of R-modules:*

$$\begin{tikzcd} R \arrow[r, "\delta"] \arrow[dr, "H^0(\eta_R)"'] & \Lambda \arrow[d, "\simeq"] \\ & H^0(i^*(R)) \end{tikzcd}$$

(3) *If $H^0(i^*(R)) \in \mathcal{Y}$, then $H^n(i^*(R)) = 0$ for all $n \geq 1$, the homomorphism δ is a ring epimorphism and*

$$\mathcal{Y} = \{Y^\bullet \in \mathscr{D}(R) \mid H^m(Y^\bullet) \in \mathrm{Im}(\delta_*) \ \textit{for all} \ m \in \mathbb{Z}\}.$$

Proof. The proof of Lemma 3.1 is derived from [22, Section 6 and Section 7], where $\mathcal{Y}$ is related to a set of two-term complexes in $\mathscr{C}(R\text{-proj})$.

By our convention, the full subcategory $\mathrm{Im}(\delta_*)$ of R-Mod is required to be closed under isomorphisms in R-Mod. Let $\eta_R : R \to i_*i^*(R) = i^*(R)$ be the unit adjunction morphism.

(1) Let $Y^\bullet \in \mathcal{Y}$. Then we obtain the following isomorphisms for each $n \in \mathbb{Z}$:

$$\mathrm{Hom}_{\mathscr{D}(R)}(i^*(R), Y^\bullet[n]) \xrightarrow{\simeq} \mathrm{Hom}_{\mathscr{D}(R)}(R, i_*(Y^\bullet)[n]) = \mathrm{Hom}_{\mathscr{D}(R)}(R, Y^\bullet[n]) \simeq H^n(Y^\bullet),$$

where the first isomorphism is given by $\mathrm{Hom}_{\mathscr{D}(R)}(\eta_R, Y^\bullet[n])$, which is actually an isomorphism of R-modules. Since $\mathrm{Hom}_{\mathscr{D}(R)}(i^*(R), Y^\bullet[n])$ is a left Λ-module, we clearly have $H^n(Y^\bullet) \in \mathrm{Im}(\delta_*)$. If $Y^\bullet = i^*(R)$, then one can check that the composite of the following isomorphisms

$$(*) \quad \mathrm{Hom}_{\mathscr{D}(R)}(i^*(R), i^*(R)[n]) \simeq \mathrm{Hom}_{\mathscr{D}(R)}(R, i_*i^*(R)[n])) = \mathrm{Hom}_{\mathscr{D}(R)}(R, i^*(R)[n]) \simeq H^n(i^*(R))$$

is an isomorphism of R-Λ-bimodules. This implies that $H^n(i^*(R))$ is an R-Λ-bimodule.

(2) In $(*)$, we take $n = 0$. This gives the first part of (2). For the second part of (2), we note that there exists the following commutative diagram of R-modules:

$$\begin{tikzcd} \mathrm{Hom}_R(R,R) \arrow[r, "i^*"] \arrow[dr, "\mathrm{Hom}_{\mathscr{D}(R)}(R,\eta_R)"'] & \mathrm{Hom}_{\mathscr{D}(R)}(i^*(R), i^*(R)) \arrow[d, "\simeq"] \\ & \mathrm{Hom}_{\mathscr{D}(R)}(R, i_*i^*(R)) \end{tikzcd}$$

which implies the diagram in (2) if we identify $\mathrm{Hom}_R(R,R)$, $\mathrm{Hom}_{\mathscr{D}(R)}(R, i_*i^*(R))$ and $\mathrm{Hom}_{\mathscr{D}(R)}(R, \eta_R)$ with R, $H^0(i^*(R))$ and $H^0(\eta_R)$, respectively.

(3) Define

$$\mathcal{Y}' := \{Y^\bullet \in \mathscr{D}(R) \mid H^m(Y^\bullet) \in \mathrm{Im}(\delta_*) \text{ for all } m \in \mathbb{Z}\}.$$

It follows from (1) that $\mathcal{Y} \subseteq \mathcal{Y}'$.

Suppose $H^0(i^*(R)) \in \mathcal{Y}$. We shall prove that $\mathcal{Y}' \subseteq \mathcal{Y}$, and so $\mathcal{Y} = \mathcal{Y}'$.

In fact, from (2) we see that $\Lambda \simeq H^0(i^*(R))$ as R-modules, and so ${}_R\Lambda \in \mathcal{Y}$. Note that the derived functor $D(\delta_*) : \mathscr{D}(\Lambda) \to \mathscr{D}(R)$ admits a right adjoint, and therefore it commutes with arbitrary direct sums. Since $\mathcal{Y}$ is a full triangulated subcategory of $\mathscr{D}(R)$ closed under arbitrary direct sums in $\mathscr{D}(R)$, it follows from $\mathscr{D}(\Lambda) = \mathrm{Tria}({}_\Lambda\Lambda)$ and ${}_R\Lambda \in \mathcal{Y}$ that $\mathrm{Im}\big(D(\delta_*)\big) \subseteq \mathcal{Y}$. In particular, $\mathrm{Im}(\delta_*) \subseteq \mathcal{Y}$.

To prove $\mathcal{Y}' \subseteq \mathcal{Y}$, we shall use the following statements (a)-(d) mentioned in [3, Lemma 4.6]. For the definitions of homotopy limits and homotopy colimits in triangulated categories, we refer to [9, Section 2].

(a) By canonical truncations, one can show that each bounded complex over R can be generated by its cohomologies, that is, if $M^\bullet \in \mathscr{C}^b(R)$, then $M^\bullet$ belongs to the smallest full triangulated subcategory of $\mathscr{D}(R)$ containing $H^n(M^\bullet)$ with all $n \in \mathbb{Z}$.

(b) Any bounded-above complex over R can be expressed as the homotopy limit of its bounded "quotient" complexes, which are obtained from canonical truncations.

(c) Any bounded-below complex over R can be expressed as the homotopy colimit of its bounded "sub" complexes, which are obtained from canonical truncations.

(d) Any complex is generated by a bounded-above complex and a bounded-below complex obtained by canonical truncations.

Recall that $\mathcal{Y}$ is a full triangulated subcategory of $\mathscr{D}(R)$ closed under arbitrary direct sums and direct products in $\mathscr{D}(R)$. Therefore it is closed under taking homotopy limits and homotopy colimits in $\mathscr{D}(R)$. Now, by the fact $\mathrm{Im}(\delta_*) \subseteq \mathcal{Y}$ and the above statements (a)-(d), we can show that $\mathcal{Y}' \subseteq \mathcal{Y}$. Thus $\mathcal{Y} = \mathcal{Y}'$.

Next, we shall show that $H^n(i^*(R)) = 0$ for all $n \geq 1$. The idea of the proof given here is essentially taken from [22, Lemma 6.4].

On the one hand, from the adjoint pair (i^*, i_*), we can obtain a triangle in $\mathscr{D}(R)$:

$$X^\bullet \longrightarrow R \xrightarrow{\eta_R} i^*(R) \longrightarrow X^\bullet[1].$$

It is cleat that the unit η_R induces an isomorphism $\mathrm{Hom}_{\mathscr{D}(R)}(i^*(R), Y^\bullet[n]) \simeq \mathrm{Hom}_{\mathscr{D}(R)}(R, Y^\bullet[n])$ for each $Y^\bullet \in \mathcal{Y}$ and $n \in \mathbb{Z}$. This implies that $\mathrm{Hom}_{\mathscr{D}(R)}(X^\bullet, Y^\bullet[n]) = 0$ for $Y^\bullet \in \mathcal{Y}$ and $n \in \mathbb{Z}$.

On the other hand, by the canonical truncation at degree 0, we obtain a distinguished triangle of the following form in $\mathscr{D}(R)$:

$$i^*(R)^{\leq 0} \xrightarrow{\alpha} i^*(R) \xrightarrow{\beta} i^*(R)^{\geq 1} \longrightarrow i^*(R)^{\leq 0}[1]$$

such that $H^s\big(i^*(R)^{\leq 0}\big) \simeq \begin{cases} 0 & \text{if } s \geq 1, \\ H^s(i^*(R)) & \text{if } s \leq 0, \end{cases}$ and $H^t\big(i^*(R)^{\geq 1}\big) \simeq \begin{cases} 0 & \text{if } t \leq 0, \\ H^t(i^*(R)) & \text{if } t \geq 1. \end{cases}$

It follows that $\eta_R\beta = 0$ and that there exists a homomorphism $\gamma : R \to i^*(R)^{\leq 0}$ such that $\gamma\alpha = \eta_R$. Since $i^*(R) \in \mathscr{Y} = \mathscr{Y}'$, we know that $i^*(R)^{\leq 0} \in \mathscr{Y}$ and $\mathrm{Hom}_{\mathscr{D}(R)}(X^\bullet, i^*(R)^{\leq 0}) = 0$. Consequently, there exists a homomorphism $\theta : i^*(R) \to i^*(R)^{\leq 0}$ such that $\gamma = \eta_R\,\theta$. So, we have the following diagram in $\mathscr{D}(R)$:

$$\begin{tikzcd}
 & & i^*(R)^{\leq 0} \arrow[d, "\alpha"'] & \\
X^\bullet \arrow[r] & R \arrow[ur, dashed, "\gamma"] \arrow[r, "\eta_R"] \arrow[dr, dashed, "0"] & i^*(R) \arrow[r] \arrow[u, bend right, "\theta"'] \arrow[d, "\beta"'] & X^\bullet[1] \\
 & & i^*(R)^{\geq 1} \arrow[d] & \\
 & & i^*(R)^{\leq 0}[1] &
\end{tikzcd}$$

Further, one can check that $\eta_R\theta\alpha = \gamma\alpha = \eta_R$. Since $\eta_R : R \to i_*i^*(R) = i^*(R)$ is a unit morphism, we infer that $\theta\alpha = Id_{i^*(R)}$, and so

$$H^n(\theta\alpha) = H^n(\theta)H^n(\alpha) = Id_{H^n(i^*(R))} \text{ for any } n \in \mathbb{Z}.$$

This means that $H^n(\theta) : H^n(i^*(R)) \to H^n\big(i^*(R)^{\leq 0}\big)$ is injective. Observe that $H^n\big(i^*(R)^{\leq 0}\big) = 0$ for $n \geq 1$. Hence $H^n(i^*(R)) = 0$ for $n \geq 1$.

Finally, we shall prove that $\delta : R \to \Lambda$ is a ring epimorphism.

Clearly, the δ is a ring epimorphism if and only if for every Λ-module M, the induced map $\mathrm{Hom}_R(\delta, M) : \mathrm{Hom}_R(\Lambda, M) \longrightarrow \mathrm{Hom}_R(R, M)$ is an isomorphism. Observe that $\mathrm{Hom}_R(\delta, M)$ is always surjective. To see that this map is also injective, we shall use the commutative diagram in (2) and show that the induced map

$$\mathrm{Hom}_R\big(H^0(\eta_R), M\big) : \mathrm{Hom}_R\big(H^0(i^*(R)), M\big) \longrightarrow \mathrm{Hom}_R(R, M)$$

is injective. That is, we have to prove that if $f_i : H^0(i^*(R)) \to M$, with $i = 1, 2$, are two homomorphisms in R-Mod such that $H^0(\eta_R)f_1 = H^0(\eta_R)f_2$, then $f_1 = f_2$.

Now, we describe the map $H^0(\eta_R)$. Recall that $H^n(i^*(R)) = 0$ for all $n \geq 1$. Without loss of generality, we may assume that $i^*(R)$ is of the following form $\big($up to isomorphism in $\mathscr{D}(R)\big)$:

$$\cdots \longrightarrow V^{-n} \xrightarrow{d^{-n}} V^{-n+1} \longrightarrow \cdots \longrightarrow V^{-1} \xrightarrow{d^{-1}} V^0 \longrightarrow 0 \longrightarrow \cdots$$

From the canonical truncation, we can obtain the following distinguished triangle in $\mathscr{D}(R)$:

$$V^{\bullet\,\leq -1} \longrightarrow i^*(R) \xrightarrow{\pi} H^0(i^*(R)) \longrightarrow V^{\bullet\,\leq -1}[1]$$

where $V^{\bullet\,\leq -1}$ is of the form:

$$\cdots \longrightarrow V^{-n} \longrightarrow V^{-n+1} \longrightarrow \cdots \longrightarrow V^{-2} \longrightarrow \mathrm{Ker}(d^{-1}) \longrightarrow 0 \longrightarrow \cdots$$

and π is the chain map induced by the canonical surjection $V^0 \to H^0(i^*(R)) = \mathrm{Coker}(d^{-1})$. Applying $H^0(-) = \mathrm{Hom}_{\mathscr{D}(R)}(R, -)$ to the above triangle, we see that $H^0(\eta_R) = \eta_R\pi$ in $\mathscr{D}(R)$ and that $H^0(\pi)$ is an isomorphism of R-modules.

Suppose that $H^0(\eta_R)f_1 = H^0(\eta_R)f_2 : R \to M$ with $f_i : H^0(i^*(R)) \to M$ for $i = 1, 2$. Then $\eta_R\pi f_1 = \eta_R\pi f_2$. From the proof of (2), we have $\mathrm{Im}(\delta_*) \subseteq \mathscr{Y}$. Thus ${}_RM \in \mathscr{Y}$ since M is an Λ-module. Note that the unit $\eta_R : R \to i_*i^*(R) = i^*(R)$ induces an isomorphism $\mathrm{Hom}_{\mathscr{D}(R)}(i^*(R), M) \simeq \mathrm{Hom}_{\mathscr{D}(R)}(R, M)$. Thus $\pi f_1 = \pi f_2$ and $H^0(\pi)f_1 = H^0(\pi)f_2$. It follows from the isomorphism of $H^0(\pi)$ that $f_1 = f_2$. This means that $\mathrm{Hom}_R\big(H^0(\eta_R), M\big)$ is injective, and thus δ is a ring epimorphism. This finishes the proof of (3). $\square$

In the following, we shall systematically discuss when bireflective subcategories of derived categories are homological. Note that some partial answers have been given in the literature, for example, see [22, Theorem 0.7 and Proposition 5.6], [3, Proposition 1.7] and [11, Proposition 3.6]. Let us first mention the following criterions.

Lemma 3.2. *Let $\mathscr{Y}$ be a bireflective subcategory of $\mathscr{D}(R)$, and let $i^*: \mathscr{D}(R) \to \mathscr{Y}$ be a left adjoint of the inclusion $\mathscr{Y} \hookrightarrow \mathscr{D}(R)$. Then the following are equivalent:*

(1) *$\mathscr{Y}$ is homological.*

(2) *$H^m(i^*(R)) = 0$ for any $m \neq 0$.*

(3) *$H^0(i^*(R)) \in \mathscr{Y}$ and $H^m(i^*(R)) = 0$ for any $m < 0$.*

(4) *$H^0(i^*(R)) \in \mathscr{Y}$, and the associated ring homomorphism $\delta: R \to \mathrm{End}_{\mathscr{D}(R)}(i^*(R))$ is a homological ring epimorphism.*

(5) *There exists a ring epimorphism $\lambda: R \to S$ such that ${}_RS \in \mathscr{Y}$ and $i^*(R)$ is isomorphic in $\mathscr{D}(R)$ to a complex $Z^\bullet := (Z^n)_{n\in\mathbb{Z}}$ with $Z^i \in S\text{-Mod}$ for $i = 0,1$.*

(6) *$\mathscr{E} := \mathscr{Y} \cap R\text{-Mod}$ is an abelian subcategory of R-Mod such that $i^*(R)$ is isomorphic in $\mathscr{D}(R)$ to a complex $Z^\bullet := (Z^n)_{n\in\mathbb{Z}}$ with $Z^i \in \mathscr{E}$ for $i = 0,1$.*

In particular, if one of the above conditions is fulfilled, then $\mathscr{Y}$ can be realized as the derived category of $\mathrm{End}_{\mathscr{D}(R)}(i^*(R))$ via δ.

Proof. It follows from the proof of [3, Proposition 1.7] that (1) and (2) are equivalent, and that (2) implies both (3) and (4). By Lemma 3.1 (3), we know that (3) implies (2).

Now, we show that (4) implies (1). In fact, since $H^0(i^*(R)) \in \mathscr{Y}$, it follows from Lemma 3.1 (3) that

$$\mathscr{Y} = \{Y^\bullet \in \mathscr{D}(R) \mid H^m(Y^\bullet) \in \mathrm{Im}(\delta_*) \text{ for all } m \in \mathbb{Z}\},$$

where $\delta: R \to \Lambda := \mathrm{End}_{\mathscr{D}(R)}(i^*(R))$ is the associated ring homomorphism. By assumption, δ is a homological ring epimorphism, and therefore the derived functor $D(\delta_*): \mathscr{D}(\Lambda) \to \mathscr{D}(R)$ is fully faithful. Furthermore, we know from [3, Lemma 4.6] that

$$\mathrm{Im}\big(D(\delta_*)\big) = \{Y^\bullet \in \mathscr{D}(R) \mid H^m(Y^\bullet) \in \mathrm{Im}(\delta_*) \text{ for all } m \in \mathbb{Z}\}.$$

Thus $\mathscr{Y} = \mathrm{Im}\big(D(\delta_*)\big) \subseteq \mathscr{D}(R)$, that is, $\mathscr{Y}$ is homological by definition. Hence (4) implies (1).

Consequently, we have proved that (1)-(4) in Lemma 3.2 are equivalent.

Note that (5) and (6) are equivalent because $\mathscr{E}$ is an abelian subcategory of R-Mod if and only if there is a ring epimorphism $\lambda: R \to S$ such that $\mathscr{E} = \mathrm{Im}(\lambda_*)$ (see [11, Lemma 2.1]).

In the following, we shall prove that (1) implies (5) and that (5) implies (2).

Suppose that $\mathscr{Y}$ is homological, that is, there exists a homological ring epimorphism $\lambda: R \to S$ such that the functor $D(\lambda_*): \mathscr{D}(S) \to \mathscr{D}(R)$ induces a triangle equivalence from $\mathscr{D}(S)$ to $\mathscr{Y}$. Thus $\mathscr{Y} = \mathrm{Im}(D(\lambda_*))$. Since $i^*(R) \in \mathscr{Y}$, we have $i^*(R) \in \mathrm{Im}(D(\lambda_*))$. It follows that there exists a complex $Z^\bullet := (Z^n)_{n\in\mathbb{Z}} \in \mathscr{C}(S)$ such that $i^*(R) \simeq Z^\bullet$ in $\mathscr{D}(R)$. This shows (5).

It remains to show that (5) implies (2). The idea of the following proof arises from the proof of [11, Proposition 3.6].

Let $\lambda: R \to S$ be a ring epimorphism satisfying the assumptions in (5). We may identify $\mathrm{Im}(\lambda_*)$ with S-Mod since $\lambda_*: S\text{-Mod} \to R\text{-Mod}$ is fully faithful. Let $Z^\bullet$ be a complex in $\mathscr{C}(R)$ such that $Z^\bullet \simeq i^*(R)$ in $\mathscr{D}(R)$. We may assume that $Z^\bullet := (Z^n, d^n)_{n\in\mathbb{Z}}$ such that $Z^n \in S\text{-Mod}$ for $n = 0,1$, and define $\varphi = \mathrm{Hom}_{\mathscr{D}(R)}(\lambda, Z^\bullet): \mathrm{Hom}_{\mathscr{D}(R)}(S, Z^\bullet) \longrightarrow \mathrm{Hom}_{\mathscr{D}(R)}(R, Z^\bullet)$. We claim that the map φ is surjective.

In fact, there is a commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{K}(R)}(S,Z^\bullet) \arrow[r, "q_1"] \arrow[d, "\varphi'"] & \mathrm{Hom}_{\mathscr{D}(R)}(S,Z^\bullet) \arrow[d, "\varphi"] \\
\mathrm{Hom}_{\mathscr{K}(R)}(R,Z^\bullet) \arrow[r, "q_2"] & \mathrm{Hom}_{\mathscr{D}(R)}(R,Z^\bullet),
\end{tikzcd}$$

where $\varphi' = \mathrm{Hom}_{\mathscr{K}(R)}(\lambda, Z^\bullet)$, and where q_1 and q_2 are induced by the localization functor $q: \mathscr{K}(R) \to \mathscr{D}(R)$. Clearly, the q_2 is a bijection. To prove that φ is surjective, it is sufficient to show that φ' is surjective.

Let $\overline{f^\bullet} := \overline{(f^i)} \in \mathrm{Hom}_{\mathscr{K}(R)}(R, Z^\bullet)$ with $(f^i)_{i\in\mathbb{Z}}$ a chain map from R to $Z^\bullet$. Then $f^i = 0$ for any $i \neq 0$ and $f^0 d^0 = 0$. Since Z^0 is an S-module, we can define $g: S \to Z^0$ by $s \mapsto s(1)f^0$ for $s \in S$. One can check that g is a homomorphism of R-modules with $f^0 = \lambda g$, as is shown in the following visual diagram:

$$\begin{tikzcd}
 & & R \arrow[r, "\lambda"] \arrow[d, "f^0"'] & S \arrow[dl, dashed, "g"'] & & \\
\cdots \arrow[r] & Z^{-1} \arrow[r, "d^{-1}"] & Z^0 \arrow[r, "d^0"] & Z^1 \arrow[r, "d^1"] & Z^2 \arrow[r] & \cdots
\end{tikzcd}$$

Since $\lambda : R \to S$ is a ring epimorphism and since Z^1 is an S-module, the induced map $\mathrm{Hom}_R(\lambda, Z^1) : \mathrm{Hom}_R(S, Z^1) \to \mathrm{Hom}_R(R, Z^1)$ is a bijection. Thus, from this bijection together with $\lambda g d^0 = f^0 d^0 = 0$, it follows that $gd^0 = 0$. Now, we can define a morphism $\overline{g^\bullet} := \overline{(g^i)} \in \mathrm{Hom}_{\mathscr{K}(R)}(S, Z^\bullet)$, where $(g^i)_{i\in\mathbb{Z}}$ is the chain map with $g^0 = g$ and $g^i = 0$ for any $i \neq 0$. Thus $\overline{f^\bullet} = \lambda \overline{g^\bullet}$. This shows that φ' is surjective. Consequently, the map φ is surjective, and the induced map

$$\mathrm{Hom}_{\mathscr{D}(R)}(\lambda, i^*(R)) : \mathrm{Hom}_{\mathscr{D}(R)}(S, i^*(R)) \to \mathrm{Hom}_{\mathscr{D}(R)}(R, i^*(R))$$

is surjective since $Z^\bullet \simeq i^*(R)$ in $\mathscr{D}(R)$.

Finally, we shall prove that $i^*(R) \simeq S$ in $\mathscr{D}(R)$. In particular, this will give rise to $H^m(i^*(R)) \simeq H^m(S) = 0$ for any $m \neq 0$, and therefore show (2). So, it suffices to prove that $i^*(R) \simeq S$ in $\mathscr{D}(R)$.

Indeed, let $i_* : \mathscr{Y} \to \mathscr{D}(R)$ be the inclusion, and let $\eta_R : R \to i_* i^*(R)$ be the unit with respect to the adjoint pair (i^*, i_*). Clearly, $i^*(R) = i_* i^*(R)$ in $\mathscr{D}(R)$. Since we have proved that $\mathrm{Hom}_{\mathscr{D}(R)}(\lambda, i^*(R))$ is surjective, there exists a homomorphism $v : S \to i_* i^*(R)$ in $\mathscr{D}(R)$ such that $\eta_R = \lambda v$. Furthermore, since ${}_R S$ belongs to $\mathscr{Y}$ by assumption, we see that $\mathrm{Hom}_{\mathscr{D}(R)}(\eta_R, S) : \mathrm{Hom}_{\mathscr{D}(R)}(i^*(R), S) \to \mathrm{Hom}_{\mathscr{D}(R)}(R, S)$ is an isomorphism. Thus there exists a homomorphism $u : i_* i^*(R) \to S$ in $\mathscr{D}(R)$ such that $\lambda = \eta_R u$. This yields the following commutative diagram in $\mathscr{D}(R)$:

$$\begin{tikzcd}
R \arrow[r, equal] \arrow[d, "\eta_R"'] & R \arrow[r, equal] \arrow[d, "\lambda"'] & R \arrow[d, "\eta_R"] \\
i_* i^*(R) \arrow[r, dashed, "u"] & S \arrow[r, dashed, "v"] & i_* i^*(R),
\end{tikzcd}$$

which shows that $\eta_R = \eta_R uv$ and $\lambda = \lambda vu$. Since $\mathrm{Hom}_{\mathscr{D}(R)}(\eta_R, i^*(R))$: $\mathrm{Hom}_{\mathscr{D}(R)}(i^*(R), i^*(R)) \to \mathrm{Hom}_{\mathscr{D}(R)}(R, i_* i^*(R))$ is an isomorphism, we clearly have $uv = 1_{i_* i^*(R)}$. Note that $\mathrm{Hom}_R(\lambda, S) : \mathrm{Hom}_R(S, S) \to \mathrm{Hom}_R(R, S)$ is bijective since $\lambda : R \to S$ is a ring epimorphism. It follows from $\lambda = \lambda vu$ that $vu = 1_S$. Thus the map u is an isomorphism in $\mathscr{D}(R)$, and $i^*(R) = i_* i^*(R) \simeq S$ in $\mathscr{D}(R)$. This shows that (5) implies (2).

Hence all the statements in Lemma 3.2 are equivalent. This finishes the proof. $\square$.

Now, we mention a special bireflective subcategory of $\mathscr{D}(R)$, which is constructed from complexes of finitely generated projective R-modules. For the proof, we refer to [8, Chapter III, Theorem 2.3; Chapter IV, Proposition 1.1]. See also [11, Lemma 2.8].

Lemma 3.3. *Let Σ be a set of complexes in $\mathscr{C}^b(R\text{-proj})$. Define $\mathscr{Y} := \mathrm{Ker}\big(\mathrm{Hom}_{\mathscr{D}(R)}(\mathrm{Tria}(\Sigma), -)\big)$. Then $\mathscr{Y}$ is bireflective and equal to the full subcategory of $\mathscr{D}(R)$ consisting of complexes $Y^\bullet$ in $\mathscr{D}(R)$ such that $\mathrm{Hom}_{\mathscr{D}(R)}(P^\bullet, Y^\bullet[n]) = 0$ for every $P^\bullet \in \Sigma$ and $n \in \mathbb{Z}$.*

To develop properties of the bireflective subcategories of $\mathscr{D}(R)$ in Lemma 3.3, we shall define the so-called generalized localizations, which is motivated by a discussion with Silvana Bazzoni in 2012. In fact, this notion was first discussed in [21] under the name "homological localizations" for a set of complexes in $\mathscr{C}^b(R\text{-proj})$, and is related to both the telescope conjecture and algebraic K-theory. The reason for not choosing the adjective word "homological" in this note is that we have reserved this word for ring epimorphisms.

Definition 3.4. Let R be a ring, and let Σ be a set of complexes in $\mathscr{C}(R)$. A homomorphism $\lambda_\Sigma : R \to R_\Sigma$ of rings is called a *generalized localization* of R at Σ provided that

(1) λ_Σ is Σ-exact, that is, if $P^\bullet$ belongs to Σ, then $R_\Sigma \otimes_R P^\bullet$ is exact as a complex over R_Σ, and

(2) λ_Σ is universally Σ-exact, that is, if S is a ring together with a Σ-exact homomorphism $\varphi : R \to S$, then there exists a unique ring homomorphism $\psi : R_\Sigma \to S$ such that $\varphi = \lambda_\Sigma \psi$.

If Σ consists only of two-term complexes in $\mathscr{C}^b(R\text{-proj})$, then the generalized localization of R at Σ is the *universal localization* of R at Σ in the sense of Cohn (see [14]). It was proved in [14] that universal localizations always exist. However, generalized localizations may not exist in general. For a counterexample, we refer the reader to [21, Example 15.2].

We remark that, in Definition 3.4 (1), if Σ consists of complexes in $\mathscr{C}^b(R\text{-proj})$, then, for each $P^\bullet := (P^i)_{i\in\mathbb{Z}} \in \Sigma$, the complex $R_\Sigma \otimes_R P^\bullet$ is actually split exact as a complex over R_Σ since $R_\Sigma \otimes_R P^i$ is a projective R_Σ-module for each i. Further, by Definition 3.4 (2), if $\lambda_i : R \to R_i$ is a generalized localization of R at Σ for $i = 1, 2$, then λ_1 and λ_2 are equivalent, that is, there exists a ring isomorphism $\rho : R_1 \to R_2$ such that $\lambda_2 = \lambda_1 \rho$.

Suppose that $\mathscr{U}$ is a set of R-modules each of which possesses a finitely generated projective resolution of finite length. For each $U \in \mathscr{U}$, we choose such a projective resolution ${}_pU$ of finite length, and set $\Sigma := \{{}_pU \mid U \in \mathscr{U}\} \subseteq \mathscr{C}^b(R\text{-proj})$, and let $R_{\mathscr{U}}$ be the generalized localization of R at Σ. If ${}_pU'$ is another choice of finitely generated projective resolution of finite length for U, then the generalized localization of R at $\Sigma' := \{_pU' \mid U \in \mathcal{U}\}$ is isomorphic to $R_{\mathcal{U}}$, that is, $R_{\mathcal{U}}$ does not depend on the choice of projective resolutions of U. Thus, we may say that $R_{\mathcal{U}}$ is the *generalized localization of R at $\mathcal{U}$*.

Generalized localizations have the following simple properties (compare with [11, Theorem 3.1 and Lemma 3.2]).

Lemma 3.5. *Let R be a ring and let Σ be a set of complexes in $\mathscr{C}^b(R\text{-proj})$. Assume that the generalized localization $\lambda_\Sigma : R \to R_\Sigma$ of R at Σ exists. Then the following hold.*

(1) *For any homomorphism $\varphi : R_\Sigma \to S$ of rings, the ring homomorphism $\lambda_\Sigma\varphi : R \to S$ is Σ-exact.*

(2) *The ring homomorphism λ_Σ is a ring epimorphism.*

(3) *Define $\Sigma^* := \{\mathrm{Hom}_R(P^\bullet, R) \mid P^\bullet \in \Sigma\}$. Then λ_Σ is also the generalized localization of R at the set Σ^*. In particular, $R_{\Sigma^*} \simeq R_\Sigma$ as rings.*

Proof. (1) For each $P^\bullet \in \Sigma$, we have the following isomorphisms of complexes of S-modules:

$$S \otimes_R P^\bullet \simeq (S \otimes_{R_\Sigma} R_\Sigma) \otimes_R P^\bullet \simeq S \otimes_{R_\Sigma} (R_\Sigma \otimes_R P^\bullet).$$

Since $R_\Sigma \otimes_R P^\bullet$ is split exact in $\mathscr{C}(R_\Sigma)$, we see that $S \otimes_R P^\bullet$ is also split exact in $\mathscr{C}(S)$. This means that the ring homomorphism $\lambda_\Sigma\varphi$ is Σ-exact.

(2) Assume that $\varphi_i : R_\Sigma \to S$ is a ring homomorphism for $i = 1, 2$, such that $\lambda_\Sigma\varphi_1 = \lambda_\Sigma\varphi_2$. It follows from (1) that $\lambda_\Sigma\varphi_i$ is Σ-exact. By the property (2) in Definition 3.4, we obtain $\varphi_1 = \varphi_2$. This implies that λ_Σ is a ring epimorphism.

(3) Note that $P^\bullet$ is in $\mathscr{C}^b(R\text{-proj})$. It follows from Lemma 2.4 that, for any homomorphism $R \to S$ of rings, there are the following isomorphisms of complexes:

$$\mathrm{Hom}_R(P^\bullet, R) \otimes_R S \simeq \mathrm{Hom}_R(P^\bullet, S) \simeq \mathrm{Hom}_R(P^\bullet, \mathrm{Hom}_S({}_SS_R, S)) \simeq \mathrm{Hom}_S(S \otimes_R P^\bullet, S).$$

This implies that the complex $\mathrm{Hom}_R(P^\bullet, R) \otimes_R S$ is (split) exact in $\mathscr{C}(S^{\mathrm{op}})$ if and only if so is the complex $S \otimes_R P^\bullet$ in $\mathscr{C}(S)$. Now, (3) follows immediately from the definition of generalized localizations. $\square$

In the following, we shall establish a relation between bireflective subcategories of $\mathscr{D}(R)$ and generalized localizations. In particular, the statements (3) and (4) in Lemma 3.6 below will be useful for discussions in the next section and the proof of Theorem 1.1.

Lemma 3.6. *Let Σ be a set of complexes in $\mathscr{C}^b(R\text{-proj})$, and let $j_! : \mathrm{Tria}(\Sigma) \to \mathscr{D}(R)$ be the inclusion. Define $\mathcal{Y} := \mathrm{Ker}\big(\mathrm{Hom}_{\mathscr{D}(R)}(\mathrm{Tria}(\Sigma), -)\big)$. Then the following are true.*

(1) *There exists a recollement of triangulated categories:*

$$\begin{tikzcd} \mathcal{Y} \arrow[r, "i_*"] & \mathscr{D}(R) \arrow[l, bend right, "i^*"'] \arrow[l, bend left] \arrow[r] & \mathrm{Tria}(\Sigma) \arrow[l, bend right, "j_!"'] \arrow[l, bend left] \end{tikzcd}$$

where (i^*, i_*) is a pair of adjoint functors with i_* the inclusion.

(2) *The associated ring homomorphism $\delta : R \to \Lambda := \mathrm{End}_{\mathscr{D}(R)}(i^*(R))$ induced by i^* admits the following property: For any Σ-exact ring homomorphism $\varphi : R \to S$, there exists a ring homomorphism $\psi : \Lambda \to S$ such that $\varphi = \delta\psi$.*

(3) *If $H^0(i^*(R)) \in \mathcal{Y}$, then δ is a generalized localization of R at Σ. In particular, if the subcategory $\mathcal{Y}$ of $\mathscr{D}(R)$ is homological, then δ is a generalized localization of R at Σ.*

(4) *Define $\Sigma^* := \{\mathrm{Hom}_R(P^\bullet, R) \in \mathscr{C}^b(R^{\mathrm{op}}\text{-proj}) \mid P^\bullet \in \Sigma\}$ and $\mathcal{Y}' := \mathrm{Ker}\big(\mathrm{Hom}_{\mathscr{D}(R^{\mathrm{op}})}(\mathrm{Tria}(\Sigma^*), -)\big)$. Then $\mathcal{Y}$ is homological in $\mathscr{D}(R)$ if and only if so is $\mathcal{Y}'$ in $\mathscr{D}(R^{\mathrm{op}})$.*

Proof. (1) can be concluded from [11, Lemma 2.6 and Lemma 2.8].

(2) The proof here is motivated by [22, Lemma 7.3]. Let $\varphi : R \to S$ be a Σ-exact ring homomorphism. Since $S \otimes_R P^\bullet$ is exact in $\mathscr{C}(S)$ for $P^\bullet \in \Sigma$, we have $S \otimes_R^{\mathbb{L}} P^\bullet = S \otimes_R P^\bullet \simeq 0$ in $\mathscr{D}(S)$. Further, the functor $S \otimes_R^{\mathbb{L}} - : \mathscr{D}(R) \to \mathscr{D}(S)$ commutes with arbitrary direct sums, so $S \otimes_R^{\mathbb{L}} X^\bullet \simeq 0$ for each $X^\bullet \in \mathrm{Tria}(\Sigma)$.

Let $\mathscr{D}(R)/\mathrm{Tria}(\Sigma)$ denote the Verdier quotient of $\mathscr{D}(R)$ by the full triangulated subcategory $\mathrm{Tria}(\Sigma)$. It follows from the recollement in (1) that i^* induces a triangle equivalence:

$$\mathscr{D}(R)/\mathrm{Tria}(\Sigma) \xrightarrow{\simeq} \mathcal{Y}.$$

Since $S\otimes^{\mathbb{L}}_R-$ sends $\mathrm{Tria}(\Sigma)$ to zero, there exists a triangle functor $F:\mathscr{Y}\to\mathscr{D}(S)$ together with a natural isomorphism of triangle functors:

$$\Phi:\ S\otimes^{\mathbb{L}}_R - \xrightarrow{\simeq} F\,i^*:\ \mathscr{D}(R)\longrightarrow\mathscr{D}(S).$$

This clearly induces the following canonical ring homomorphisms:

$$\Lambda:=\mathrm{End}_{\mathscr{D}(R)}(i^*(R))\xrightarrow{F}\mathrm{End}_{\mathscr{D}(S)}\big(F(i^*(R))\big)\simeq\mathrm{End}_{\mathscr{D}(S)}(S\otimes^{\mathbb{L}}_R R)\simeq\mathrm{End}_{\mathscr{D}(S)}(S)\simeq S$$

where the first isomorphism is induced by the natural isomorphism $\Phi_R: S\otimes^{\mathbb{L}}_R R\longrightarrow F(i^*(R))$ in $\mathscr{D}(S)$. Now, we define $\psi:\Lambda\to S$ to be the composite of the above ring homomorphisms. Then it is easy to check that $\varphi=\delta\psi$. Consequently, the δ has the property mentioned in (2).

(3) Assume that $H^0(i^*(R))\in\mathscr{Y}$. By Lemma 3.1 (3), the map δ is a ring epimorphism. Combining this with (2), we know that δ satisfies the condition (2) in Definition 3.4. To see that δ is the generalized localization of R at Σ, we have to show that δ satisfies the condition (1) in Definition 3.4, that is, δ is Σ-exact.

In fact, by Lemma 3.1 (2), we have $\Lambda\simeq H^0(i^*(R))$ as R-modules. This gives rise to ${}_R\Lambda\in\mathscr{Y}$. Note that $\mathrm{Hom}_{\mathscr{D}(R)}(X^\bullet,Y^\bullet)=0$ for $X^\bullet\in\mathrm{Tria}(\Sigma)$ and $Y^\bullet\in\mathscr{Y}$. In particular, we have $\mathrm{Hom}_{\mathscr{D}(R)}(P^\bullet,\Lambda[n])=0$ for any $P^\bullet\in\Sigma$ and $n\in\mathbb{Z}$. It follows that $H^n(\mathrm{Hom}_R(P^\bullet,\Lambda))\simeq\mathrm{Hom}_{\mathscr{K}(R)}(P^\bullet,\Lambda[n])\simeq\mathrm{Hom}_{\mathscr{D}(R)}(P^\bullet,\Lambda[n])=0$, and therefore the complex $\mathrm{Hom}_R(P^\bullet,\Lambda)$ is exact. Since $P^\bullet\in\mathscr{C}^b(R\text{-proj})$, we have $\mathrm{Hom}_R(P^\bullet,\Lambda)\in\mathscr{C}^b(\Lambda^{\mathrm{op}}\text{-proj})$. This implies that $\mathrm{Hom}_R(P^\bullet,\Lambda)$ is split exact, and therefore the complex $\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(\mathrm{Hom}_R(P^\bullet,\Lambda),\Lambda)$ over Λ is split exact. Now, we claim that the latter complex is isomorphic to the complex $\Lambda\otimes_R P^\bullet$ in $\mathscr{C}(\Lambda)$. Actually, this follows from the following general fact in homological algebra:

For any finitely generated projective R-module P, there exists a natural isomorphism of Λ-modules:

$$\Lambda\otimes_R P\longrightarrow\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(\mathrm{Hom}_R(P,\Lambda),\Lambda),\quad x\otimes p\mapsto[f\mapsto x(p)f]$$

for $x\in\Lambda$, $p\in P$ and $f\in\mathrm{Hom}_R(P,\Lambda)$. Consequently, the complex $\Lambda\otimes_R P^\bullet$ is exact in $\mathscr{C}(\Lambda)$, and thus δ is Σ-exact. Hence δ is a generalized localization of R at Σ.

Clearly, the second part of Lemma 3.6 (3) follows from the equivalences of (1) and (4) in Lemma 3.2.

(4) We shall only prove the necessity of (4) since the sufficiency of (4) can be proved similarly.

Suppose that $\mathscr{Y}$ is homological in $\mathscr{D}(R)$. It follows from Lemma 3.2 (4) and Lemma 3.6 (3) that the ring homomorphism $\delta:R\to\Lambda$ is not only a homological ring epimorphism, but also a generalized localization of R at Σ. Moreover, by Lemma 3.5 (3), the map δ is also a generalized localization of R at Σ^*.

Note that $\mathscr{Y}'$ is a bireflective subcategory of $\mathscr{D}(R^{\mathrm{op}})$ by Lemma 3.3. Now, let $\mathbf{L}$ be a left adjoint of the inclusion $\mathscr{Y}'\to\mathscr{D}(R^{\mathrm{op}})$. To show that $\mathscr{Y}'$ is homological in $\mathscr{D}(R^{\mathrm{op}})$, we employ the equivalences of (1) and (4) in Lemma 3.2, and prove that

(a) $H^0(\mathbf{L}(R))\in\mathscr{Y}'$ and

(b) the ring homomorphism $\delta':R\longrightarrow\Lambda':=\mathrm{End}_{\mathscr{D}(R^{\mathrm{op}})}(\mathbf{L}(R))$ induced by $\mathbf{L}$ is homological.

Clearly, under the assumption (a), we see from (3) that δ' is a generalized localization of R at Σ^*. Since δ is also a generalized localization of R at Σ^*, there exists a ring isomorphism $\rho:\Lambda'\longrightarrow\Lambda$ such that $\delta=\delta'\rho$. Note that δ is homological. It follows that δ' is homological.

It remains to show (a). In fact, since $H^0(\mathbf{L}(R))\simeq\Lambda'$ as right R-modules by Lemma 3.1 (2), it is sufficient to prove that the right R-module Λ' belongs to $\mathscr{Y}'$. However, by (1) and Lemma 3.3, we have

$$\mathscr{Y}'=\{Y^\bullet\in\mathscr{D}(R^{\mathrm{op}})\mid\mathrm{Hom}_{\mathscr{D}(R^{\mathrm{op}})}\big(\mathrm{Hom}_R(P^\bullet,R),\,Y^\bullet[n]\big)=0\text{ for }P^\bullet\in\Sigma\text{ and }n\in\mathbb{Z}\},$$

and by the isomorphism ρ and $\delta=\delta'\rho$, we get $\Lambda'\simeq\Lambda$ as right R-modules. Consequently, to show $\Lambda'_R\in\mathscr{Y}'$, it is enough to show that Λ_R belongs to $\mathscr{Y}'$, that is, we have to prove that $\mathrm{Hom}_{\mathscr{D}(R^{\mathrm{op}})}\big(\mathrm{Hom}_R(P^\bullet,R),\,\Lambda[n]\big)=0$ for any $P^\bullet\in\Sigma$ and $n\in\mathbb{Z}$.

Let $P^\bullet\in\Sigma$, and set $P^{\bullet *}:=\mathrm{Hom}_R(P^\bullet,R)$. Since $P^\bullet$ is a complex in $\mathscr{C}^b(R\text{-proj})$, we see from Lemma 2.4 that $\mathrm{Hom}_{R^{\mathrm{op}}}(P^{\bullet *},\Lambda)\simeq\Lambda\otimes_R P^\bullet$ as complexes in $\mathscr{C}(\Lambda)$, and therefore there exist the following isomorphisms:

$$\mathrm{Hom}_{\mathscr{D}(R^{\mathrm{op}})}\big(P^{\bullet *},\,\Lambda[n]\big)\simeq\mathrm{Hom}_{\mathscr{K}(R^{\mathrm{op}})}\big(P^{\bullet *},\,\Lambda[n]\big)\simeq H^n(\mathrm{Hom}_{R^{\mathrm{op}}}(P^{\bullet *},\Lambda))\simeq H^n(\Lambda\otimes_R P^\bullet).$$

Since $\delta:R\to\Lambda$ is a generalized localization of R at Σ, the complex $\Lambda\otimes_R P^\bullet$ is exact in $\mathscr{C}(\Lambda)$, that is, $H^n(\Lambda\otimes_R P^\bullet)=0$ for any $n\in\mathbb{Z}$. Thus $\mathrm{Hom}_{\mathscr{D}(R^{\mathrm{op}})}\big(P^{\bullet *},\,\Lambda[n]\big)=0$ for $n\in\mathbb{Z}$. Thus $\Lambda_R\in\mathscr{Y}'$, and the proof of the necessity of (4) is completed. $\square$

As an application of Lemma 3.6 (3), we have the following result which says that generalized localizations can be constructed from homological ring epimorphisms.

Corollary 3.7. *Let $\lambda : R \to S$ be a homological ring epimorphism. Suppose that ${}_RS$ has a finitely generated projective resolution of finite length. Let $P^\bullet$ be a complex in $\mathscr{C}^b(R\text{-proj})$, which is isomorphic in $\mathscr{D}(R)$ to the mapping cone of λ. Then λ is a generalized localization of R at $P^\bullet$.*

Proof. Since λ is homological and $P^\bullet$ is isomorphic to the mapping cone of λ in $\mathscr{D}(R)$, it follows from [23, Section 4] that there is a recollement of triangulated categories:

$$\begin{tikzcd}
\mathscr{D}(S) \arrow[r, "D(\lambda_*)"] & \mathscr{D}(R) \arrow[l, bend right, "S\otimes^{\mathbb{L}}_R -"'] \arrow[l, bend left] \arrow[r] & \mathrm{Tria}(P^\bullet) \arrow[l, bend right, "j_!"'] \arrow[l, bend left]
\end{tikzcd}$$

where $j_!$ is the inclusion. This shows that $\mathscr{Y} := \mathrm{Ker}\big(\mathrm{Hom}_{\mathscr{D}(R)}(\mathrm{Tria}(P^\bullet), -)\big)$ is equivalent to $\mathscr{D}(S)$. Thus $\mathscr{Y}$ is homological. Note that $S \otimes^{\mathbb{L}}_R R \simeq S$ and $\mathrm{End}_R({}_RS) \simeq S$. By Lemma 3.6 (3), we know that λ is a generalized localization of R at $P^\bullet$. $\square$

4 Ringel modules

This section is devoted to preparations for proofs of our main results in this paper. First, we introduce a special class of modules, called Ringel modules, which can be constructed from both good tilting and cotilting modules, and then discuss certain bireflective subcategories (of derived module categories) arising from Ringel modules. Finally, we shall describe when these subcategories are homological. In particular, we shall establish a key proposition, Proposition 4.4, which will be applied in later sections.

Throughout this section, let R be an arbitrary ring, M an R-module and S the endomorphism ring of ${}_RM$. Then M becomes naturally an R-S-bimodule. Further, let n be an arbitrary but fixed natural number.

Definition 4.1. The R-module M is called an *n-Ringel module* provided that the following three conditions are fulfilled:

$(R1)$ there exists an exact sequence

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

of R-modules such that $P_i \in \mathrm{add}({}_RR)$ for all $0 \le i \le n$,

$(R2)$ $\mathrm{Ext}^j_R(M,M) = 0$ for all $j \ge 1$, and

$(R3)$ there exists an exact sequence

$$0 \longrightarrow {}_RR \longrightarrow M_0 \stackrel{\nu}{\longrightarrow} M_1 \longrightarrow \cdots \longrightarrow M_n \longrightarrow 0$$

of R-modules such that $M_i \in \mathrm{Prod}({}_RM)$ for all $0 \le i \le n$.

An n-Ringel R-module M is said to be *perfect* if the ring S is right noetherian; and *good* if

$(R4)$ the right S-module M is strongly S-Mittag-Leffler (see Definition 2.6).

Classical tilting modules are good Ringel modules. Conversely, for a Ringel module M, if each M_i in $(R3)$ is isomorphic to a direct summand of finite direct products of copies of M, then M becomes a classical tilting module (see Introduction).

If a Ringel R-module M has the property $\mathrm{Prod}({}_RM) = \mathrm{Add}({}_RM)$ (for example, M_S is of finite length), then ${}_RM$ is a tilting module. In this case, ${}_RM$ is even classical (see Corollary 2.9).

Moreover, if the ring S is right noetherian (see the statements following Definition 2.6), then any right S-module is S-Mittag-Leffler. Thus each perfect Ringel R-module must be good.

It is worth noting that good tilting (or cotilting) modules may not be Ringel modules because it may not be finitely generated. For example, the infinitely generated $\mathbb{Z}$-module $\mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z}$ is a good tilting module, but not a Ringel module. Clearly, the good 1-cotilting $\mathbb{Z}$-module $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$ is not a Ringel module.

Assume that ${}_RM$ satisfies $(R1)$. Then M is isomorphic in $\mathscr{D}(R)$ to the following complex of finitely generated projective R-modules:

$$\cdots \longrightarrow 0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0 \longrightarrow \cdots$$

It follows from Lemma 3.3 that $\mathscr{Y} := \{Y^\bullet \in \mathscr{D}(R) \mid \mathrm{Hom}_{\mathscr{D}(R)}(M, Y^\bullet[m]) = 0 \text{ for all } m \in \mathbb{Z}\}$ is a bireflective subcategory of $\mathscr{D}(R)$.

Now, assume that M satisfies both $(R1)$ and $(R2)$. Then the functors

$$\mathbf{G} := {}_RM\otimes^{\mathbb{L}}_S - :\ \mathscr{D}(S)\longrightarrow \mathscr{D}(R) \quad \text{and} \quad \mathbf{H} := \mathbb{R}\mathrm{Hom}_R(M,-):\ \mathscr{D}(R)\longrightarrow \mathscr{D}(S)$$

induce a triangle equivalence: $\mathscr{D}(S)\xrightarrow{\simeq}\mathrm{Tria}({}_RM)$ (see [1, Chapter 5, Corollary 8.4, Theorem 8.5]). Moreover, $\mathscr{Y}=\mathrm{Ker}(\mathbf{H})$ since $H^m\big(\mathbb{R}\mathrm{Hom}_R(M,Y^\bullet)\big)\simeq \mathrm{Hom}_{\mathscr{D}(R)}(M,Y^\bullet[m])$ for each $Y^\bullet\in\mathscr{D}(R)$ and $m\in\mathbb{Z}$.

Thus, by Lemma 3.6 (1) and (3) as well as Lemma 3.2, we have the following useful result for constructing recollements of derived module categories.

Lemma 4.2. *Suppose that the R-module M satisfies $(R1)$ and $(R2)$. Then there exists a recollement of triangulated categories:*

$$(*)\qquad \begin{tikzcd} \mathscr{Y} \arrow[r, "i_*"] & \mathscr{D}(R) \arrow[l, bend right, "i^*"'] \arrow[l, bend left] \arrow[r, "\mathbf{H}"] & \mathscr{D}(S) \arrow[l, bend right, "\mathbf{G}"'] \arrow[l, bend left] \end{tikzcd}$$

where (i^*,i_*) is a pair of adjoint functors with i_* the inclusion.

If, in addition, the category $\mathscr{Y}$ is homological in $\mathscr{D}(R)$, then the generalized localization $\lambda: R\to R_M$ of R at M exists and is homological, which induces a recollement of derived module categories:

$$\begin{tikzcd} \mathscr{D}(R_M) \arrow[r, "D(\lambda_*)"] & \mathscr{D}(R) \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "\mathbf{H}"] & \mathscr{D}(S) \arrow[l, bend right, "\mathbf{G}"'] \arrow[l, bend left] \end{tikzcd}$$

In the following, we shall consider when the category $\mathscr{Y}$ is homological. In general, this category is not homological since the category

$$\mathscr{E} := \mathscr{Y}\cap R\text{-Mod} = \{Y\in R\text{-Mod}\mid \mathrm{Ext}^m_R(M,Y)=0 \text{ for all } m\geq 0\}$$

may not be an abelian subcategory of R-Mod. So, we need to impose some additional conditions on the module M.

By Lemma 3.2, whether $\mathscr{Y}$ is homological is completely determined by the cohomology groups of $i_*i^*(R)$. So, to calculate these cohomology groups efficiently, we shall concentrate on good Ringel modules.

From now on, we assume that ${}_RM$ is a **good** n-Ringel module, and define $M^\bullet$ to be the complex

$$\cdots\longrightarrow 0\longrightarrow M_0\xrightarrow{\nu} M_1\longrightarrow\cdots\longrightarrow M_n\longrightarrow 0\longrightarrow\cdots$$

arising from $(R3)$ in Definition 4.1, where M_i is in degree i for $0\leq i\leq n$.

First of all, we establish the following result.

Lemma 4.3. *The following statements are true.*

(1) *For each $X\in\mathrm{Prod}({}_RM)$, the evaluation map $\theta_X: M\otimes_S\mathrm{Hom}_R(M,X)\longrightarrow X$ is injective and* $\mathrm{Coker}(\theta_X)\in\mathscr{E}$.

(2)
$$H^j(i_*i^*(R))\simeq\begin{cases} 0 & \text{if } j<0,\\ H^{j+1}\big({}_RM\otimes_S\mathrm{Hom}_R(M,M^\bullet)\big) & \text{if } j>0.\end{cases}$$

(3) *For $n=0$, the complex $i_*i^*(R)$ is isomorphic in $\mathscr{D}(R)$ to the stalk complex* $\mathrm{Coker}(\theta_{M_0})$. *For $n\geq 1$, the complex $i_*i^*(R)$ is isomorphic in $\mathscr{D}(R)$ to a complex of the form*

$$0\longrightarrow E^0\longrightarrow E^1\longrightarrow\cdots\longrightarrow E^{n-1}\longrightarrow 0$$

with $E^m\in\mathscr{E}$ for $0\leq m\leq n-1$.

Proof. Recall that M is an R-S-bimodule with $S=\mathrm{End}_R(M)$. So we have a pair of adjoint functors:

$${}_RM\otimes_S - : S\text{-Mod}\longrightarrow R\text{-Mod} \quad\text{and}\quad \mathrm{Hom}_R(M,-): R\text{-Mod}\longrightarrow S\text{-Mod}.$$

This can be naturally extended to a pair of adjoint triangle functors between homotopy categories:

$${}_RM\otimes_S - : \mathscr{K}(S)\longrightarrow\mathscr{K}(R) \quad\text{and}\quad \mathrm{Hom}_R(M,-):\mathscr{K}(R)\longrightarrow\mathscr{K}(S).$$

By passing to derived categories, we obtain the derived functors $\mathbf{G}$ and $\mathbf{H}$, respectively. Further, let

$$\theta : M \otimes_S \mathrm{Hom}_R(M,-) \longrightarrow Id_{R\text{-Mod}} \quad \text{and} \quad \varepsilon : \mathbf{GH} \longrightarrow Id_{\mathscr{D}(R)}$$

be the counit adjunctions with respect to $\big(M \otimes_S -, \mathrm{Hom}_R(M,-)\big)$ and $(\mathbf{G},\mathbf{H})$, respectively.

Note that, for each $X^\bullet \in \mathscr{D}(R)$, it follows from the recollement $(*)$ in Lemma 4.2 that there exists a canonical distinguished triangle in $\mathscr{D}(R)$:

$$\mathbf{GH}(X^\bullet) \xrightarrow{\varepsilon_{X^\bullet}} X^\bullet \longrightarrow i_* i^*(X^\bullet) \longrightarrow \mathbf{GH}(X^\bullet)[1].$$

(1) Let $X \in \mathrm{Prod}({}_RM)$. To verify that θ_X is injective, it is sufficient to show that

$$\theta_{M^I} : M \otimes_S \mathrm{Hom}_R(M, M^I) \longrightarrow M^I$$

is injective for any nonempty set I. Since $\mathrm{Hom}_R(M,M^I) \simeq \mathrm{Hom}_R(M,M)^I$, the injection of θ_{M^I} is equivalent to saying that the canonical map $\rho_I : M \otimes_S S^I \longrightarrow M^I$, defined in Definition 2.6, is injective. This holds exactly if M is S-Mittag-Leffler. However, the axiom $(R4)$ ensures that M is S-Mittag-Leffler. Thus $\theta_X : M \otimes_S \mathrm{Hom}_R(M,X) \longrightarrow X$ is injective.

To prove $\mathrm{Coker}(\theta_X) \in \mathscr{E} := \mathscr{Y} \cap R\text{-Mod}$, we demonstrate that there is the following commutative diagram in $\mathscr{D}(R)$:

$$(a) \qquad \begin{array}{ccccccc}
\mathbf{GH}(X) & \xrightarrow{\varepsilon_X} & X & \longrightarrow & i_* i^*(X) & \longrightarrow & \mathbf{GH}(X)[1] \\
\simeq \downarrow & & \| & & \downarrow \simeq & & \downarrow \simeq \\
M \otimes_S \mathrm{Hom}_R(M,X) & \xrightarrow{\theta_X} & X & \longrightarrow & \mathrm{Coker}(\theta_X) & \longrightarrow & M \otimes_S \mathrm{Hom}_R(M,X)[1]
\end{array}$$

With the help of this diagram and the recollement $(*)$ in Lemma 4.2, we have $i_* i^*(X) \in \mathscr{Y}$, and therefore

$$i_* i^*(X) \simeq \mathrm{Coker}(\theta_X) \in \mathscr{Y} \cap R\text{-Mod} = \mathscr{E}.$$

This will finish the proof of (1). So we shall prove the existence of the above diagram (a).

In fact, we shall first show that there exists a commutative diagram (b) in $\mathscr{D}(R)$:

$$(b) \qquad \begin{array}{ccc}
\mathbf{GH}(X) & \xrightarrow{\varepsilon_X} & X \\
\simeq \downarrow & & \| \\
M \otimes_S \mathrm{Hom}_R(M,X) & \xrightarrow{\theta_X} & X
\end{array}$$

This can be seen as follows: In Corollary 2.2, we take $F := {}_RM \otimes_S -$ and $G := \mathrm{Hom}_R(M,-)$. Then $\mathbf{G} = \mathbb{L}F$ and $\mathbf{H} = \mathbb{R}G$. To prove the existence of (b), it suffices to prove $X \in \mathscr{R}_G$ and $G(X) \in \mathscr{L}_F$. For the definitions of $\mathscr{R}_G$ and $\mathscr{L}_F$, we refer to Lemma 2.1.

Observe that $X \in \mathscr{R}_G$ if and only if $\mathrm{Ext}^j_R(M,X) = 0$ for any $j > 0$. Since $X \in \mathrm{Prod}({}_RM)$, it suffices to show that $\mathrm{Ext}^j_R(M,M^I) = 0$ for any $j > 0$ and any set I. This follows from $\mathrm{Ext}^j_R(M,M^I) \simeq \mathrm{Ext}^j_R(M,M)^I = 0$ by the axiom $(R2)$. Thus $X \in \mathscr{R}_G$.

Note that $G(X) \in \mathscr{L}_F$ if and only if $\mathrm{Tor}^S_j(M, G(X)) = 0$ for any $j > 0$. Since $X \in \mathrm{Prod}({}_RM)$ and G commutes with arbitrary direct products in R-Mod, we have $G(X) \in \mathrm{Prod}({}_SS)$. This means that, to prove $G(X) \in \mathscr{L}_F$, it is sufficient to check $\mathrm{Tor}^S_j(M,S^I) = 0$ for any $j > 0$ and any set I. However, since M is a good Ringel module, the right S-module M is strongly S-Mittag-Leffler by the axiom $(R4)$, and therefore $\mathrm{Tor}^S_j(M,S^I) = 0$ by Lemma 2.7 (3). This shows $G(X) \in \mathscr{L}_F$.

Hence, by Corollary 2.2, the diagram (b) does exist. Now, by the recollement $(*)$ in Lemma 4.2, we can extend ε_X to a canonical triangle in $\mathscr{D}(R)$: $\mathbf{GH}(X) \xrightarrow{\varepsilon_X} X \longrightarrow i_* i^*(X) \longrightarrow \mathbf{GH}(X)[1]$. Since each short exact sequence in R-Mod induces a canonical triangle in $\mathscr{D}(R)$:

$$M \otimes_S \mathrm{Hom}_R(M,X) \xrightarrow{\theta_X} X \longrightarrow \mathrm{Coker}(\theta_X) \longrightarrow M \otimes_S \mathrm{Hom}_R(M,X)[1],$$

the diagram (a) follows from the commutative diagram (b).

(2) Since M is a Ringel R-module, it follows from $(R3)$ that there is a quasi-isomorphism $R \to M^\bullet$ in $\mathscr{K}(R)$. Consequently, we can form the following commutative diagram in $\mathscr{D}(R)$:

$$(c)\qquad \begin{tikzcd} \mathbf{GH}(R) \arrow[r, "\varepsilon_R"] \arrow[d, "\simeq"'] & R \arrow[d, "\simeq"] \\ \mathbf{GH}(M^\bullet) \arrow[r, "\varepsilon_{M^\bullet}"] & M^\bullet \end{tikzcd}$$

Next, using Corollary 2.2 again, we shall show that there exists a commutative diagram in $\mathscr{D}(R)$:

$$(d)\qquad \begin{tikzcd} \mathbf{GH}(M^\bullet) \arrow[r, "\varepsilon_{M^\bullet}"] \arrow[d, "\simeq"'] & M^\bullet \arrow[d, equal] \\ M \otimes_S \mathrm{Hom}_R(M, M^\bullet) \arrow[r, "\theta_{M^\bullet}"] & M^\bullet \end{tikzcd}$$

By Corollary 2.2, we need only to show that $M^\bullet \in \mathcal{R}_G$ and $G(M^\bullet) \in \mathcal{L}_F$.

On the one hand, by the axiom $(R3)$ of Definition 4.1, $M^\bullet$ is a bounded complex such that each term of it belongs to $\mathrm{Prod}(M)$. On the other hand, by Lemma 2.1, the categories $\mathcal{R}_G$ and $\mathcal{L}_F$ are triangulated subcategories of $\mathscr{K}(R)$ and $\mathscr{K}(S)$, respectively. Thus, to prove that $M^\bullet \in \mathcal{R}_G$ and $G(M^\bullet) \in \mathcal{L}_F$, it is enough to prove that $X \in \mathcal{R}_G$ and $G(X) \in \mathcal{L}_F$ for any $X \in \mathrm{Prod}({}_RM)$. Clearly, the latter has been shown in (1). Thus (d) follows directly from Corollary 2.2.

Note that $\theta_X : M \otimes_S \mathrm{Hom}_R(M, X) \longrightarrow X$ is injective by (1). Since $M_i \in \mathrm{Prod}({}_RM)$ by the axiom $(R3)$, each map θ_{M_i} is injective for $0 \leq i \leq n$. This clearly induces a complex $\mathrm{Coker}(\theta_{M^\bullet})$ of the form:

$$0 \longrightarrow \mathrm{Coker}(\theta_{M_0}) \xrightarrow{\partial_0} \mathrm{Coker}(\theta_{M_1}) \xrightarrow{\partial_1} \cdots \longrightarrow \mathrm{Coker}(\theta_{M_{n-1}}) \xrightarrow{\partial_{n-1}} \mathrm{Coker}(\theta_{M_n}) \longrightarrow 0 \ \text{ in } \ \mathscr{C}(R)$$

such that there is an exact sequence of complexes over R:

$$0 \longrightarrow M \otimes_S \mathrm{Hom}_R(M, M^\bullet) \xrightarrow{\theta_{M^\bullet}} M^\bullet \longrightarrow \mathrm{Coker}\big(\theta_{M^\bullet}\big) \longrightarrow 0.$$

Since each exact sequence of complexes over R can be naturally extended to a canonical triangle in $\mathscr{D}(R)$, we obtain a triangle in $\mathscr{D}(R)$:

$$(e)\quad M \otimes_S \mathrm{Hom}_R(M, M^\bullet) \xrightarrow{\theta_{M^\bullet}} M^\bullet \longrightarrow \mathrm{Coker}\big(\theta_{M^\bullet}\big) \longrightarrow M \otimes_S \mathrm{Hom}_R(M, M^\bullet)[1].$$

Certainly, we also have a canonical triangle in $\mathscr{D}(R)$ from the recollement $(*)$ in Lemma 4.2:

$$(f)\quad \mathbf{GH}(R) \longrightarrow R \longrightarrow i_*i^*(R) \longrightarrow \mathbf{GH}(R)[1].$$

So, combining (c), (d), (e) with (f), one can easily construct the following commutative diagram in $\mathscr{D}(R)$:

$$\begin{tikzcd} \mathbf{GH}(R) \arrow[r, "\varepsilon_R"] \arrow[d, "\simeq"'] & R \arrow[r] \arrow[d, "\simeq"] & i_*i^*(R) \arrow[r] \arrow[d, "\simeq"] & \mathbf{GH}(R)[1] \arrow[d, "\simeq"] \\ M \otimes_S \mathrm{Hom}_R(M, M^\bullet) \arrow[r, "\theta_{M^\bullet}"] & M^\bullet \arrow[r] & \mathrm{Coker}(\theta_{M^\bullet}) \arrow[r] & M \otimes_S \mathrm{Hom}_R(M, M^\bullet)[1] \end{tikzcd}$$

In particular, we have $i_*i^*(R) \simeq \mathrm{Coker}(\theta_{M^\bullet})$ in $\mathscr{D}(R)$, and therefore

$$H^j(i_*i^*(R)) \simeq H^j\big(\mathrm{Coker}(\theta_{M^\bullet})\big) \text{ for any } j \in \mathbb{Z}.$$

This implies that $H^j(i_*i^*(R)) = 0$ for $j < 0$ or $j > n$.

Now, combining (e) with $R \simeq M^\bullet$ in $\mathscr{D}(R)$, we obtain a triangle in $\mathscr{D}(R)$:

$$M \otimes_S \mathrm{Hom}_R(M, M^\bullet) \longrightarrow R \longrightarrow \mathrm{Coker}\big(\theta_{M^\bullet}\big) \longrightarrow M \otimes_S \mathrm{Hom}_R(M, M^\bullet)[1].$$

Applying the cohomology functor H^j to this triangle, one can check that

$$H^j(i_*i^*(R)) \simeq H^j\big(\mathrm{Coker}(\theta_{M^\bullet})\big) \simeq H^{j+1}(M \otimes_S \mathrm{Hom}_R(M, M^\bullet)) \text{ for any } j > 0.$$

Thus (2) follows.

(3) For $n = 0$, the conclusion follows from $i_*i^*(R) \simeq \mathrm{Coker}(\theta_{M^\bullet})$ trivially. So, we may assume $n \geq 1$. By the final part of the proof of (2), we know that

$$i_*i^*(R) \simeq \mathrm{Coker}(\theta_{M^\bullet}) \text{ in } \mathscr{D}(R) \quad \text{and} \quad H^n\big(\mathrm{Coker}(\theta_{M^\bullet})\big) \simeq H^{n+1}(M \otimes_S \mathrm{Hom}_R(M, M^\bullet)).$$

Since the $(n+1)$-term of the complex $M \otimes_S \mathrm{Hom}_R(M, M^\bullet)$ is zero, we see that $H^n\big(\mathrm{Coker}(\theta_{M^\bullet})\big) = 0$. This implies that the $(n-1)$-th differential ∂_{n-1} of the complex $\mathrm{Coker}(\theta_{M^\bullet})$ is surjective. It follows that $\mathrm{Coker}(\theta_{M^\bullet})$ is isomorphic in $\mathscr{D}(R)$ to the following complex:

$$(\dagger) \quad 0 \longrightarrow \mathrm{Coker}(\theta_{M_0}) \xrightarrow{\partial_0} \mathrm{Coker}(\theta_{M_1}) \xrightarrow{\partial_1} \cdots \longrightarrow \mathrm{Coker}(\theta_{M_{n-2}}) \xrightarrow{\partial_{n-2}} \mathrm{Ker}(\partial_{n-1}) \longrightarrow 0.$$

Since $M_m \in \mathrm{Prod}({}_RM)$ for $0 \leq m \leq n$ by the axiom $(R3)$, we see from (1) that $\mathrm{Coker}(\theta_{M_m}) \in \mathscr{E}$. Note that $\mathscr{E}$ is always closed under kernels of surjective homomorphisms in R-Mod. Thus $\mathrm{Ker}(\partial_{n-1}) \in \mathscr{E}$. This means that $(\dagger)$ is a bounded complex with all of its terms in $\mathscr{E}$.

Consequently, the complex $i_*i^*(R)$ is isomorphic in $\mathscr{D}(R)$ to the complex $(\dagger)$ with the required form in Lemma 4.3 (3). This finishes the proof. □

Remark. By the proof of Lemma 4.3 (2), we see that the complex ${}_RM \otimes_S \mathrm{Hom}_R(M, M^\bullet)$ is isomorphic in $\mathscr{D}(R)$ to both ${}_RM \otimes^{\mathbb{L}}_S \mathrm{Hom}_R(M, M^\bullet)$ and $\mathbf{GH}(R)$. This implies that, up to isomorphism, the cohomology groups $H^j\big({}_RM \otimes_S \mathrm{Hom}_R(M, M^\bullet)\big)$, for $j \in \mathbb{Z}$, are independent of the choice of the complex $M^\bullet$ which arises in the axiom $(R3)$ of Definition 4.1.

With the help of Lemma 3.2 and Lemma 4.3, we can prove the following key proposition.

Proposition 4.4. *The following statements are equivalent:*

(1) *The full triangulated subcategory $\mathscr{Y}$ of $\mathscr{D}(R)$ is homological.*

(2) *The category $\mathscr{E}$ is an abelian subcategory of R-*Mod*.*

(3) $H^j\big({}_RM \otimes_S \mathrm{Hom}_R(M, M^\bullet)\big) = 0$ *for any* $j \geq 2$.

(4) *The kernel of the homomorphism* $\partial_0 : \mathrm{Coker}(\theta_{M_0}) \longrightarrow \mathrm{Coker}(\theta_{M_1})$ *induced from* ν *belongs to* $\mathscr{E}$.

Proof. The equivalences of (1) and (2) follow from those of (1) and (6) in Lemma 3.2 together with Lemma 4.3 (3), while the equivalences of (1) and (3) follow from those of (1) and (2) in Lemma 3.2 together with Lemma 4.3 (2). Now we prove that (1) and (4) are equivalent. By Lemma 4.3 (2) and the equivalence of (1) and (3) in Lemma 3.2, we see that (1) is equivalent to $H^0(i_*i^*(R)) \in \mathscr{Y}$. By the proof of Lemma 4.3 (2), we infer that $H^0(i_*i^*(R)) \simeq H^0(\mathrm{Coker}(\theta_{M^\bullet})) \simeq \mathrm{Ker}(\partial_0)$. Thus, (1) is equivalent to $\mathrm{Ker}(\partial_0) \in \mathscr{Y} \cap \text{-Mod}R = \mathscr{E}$. □

As a consequence of Proposition 4.4, we have the following handy characterizations.

Corollary 4.5. *Assume that the projective dimension of ${}_RM$ is equal to n. Then the following are true.*

(1) *If $n \leq 1$, then $\mathscr{Y}$ is always homological.*

(2) *If $n = 2$, then $\mathscr{Y}$ is homological if and only if* $M \otimes_S \mathrm{Ext}^2_R(M, R) = 0$.

(3) *Suppose that $n \geq 3$ and* $\mathrm{Tor}^S_i(M, \mathrm{Ext}^j_R(M, R)) = 0$ *for* $2 \leq j \leq n-1$ *and* $0 \leq i \leq j-2$. *Then $\mathscr{Y}$ is homological if and only if*

$$\mathrm{Tor}^S_k(M, \mathrm{Ext}^n_R(M, R)) = 0 \quad \textit{for} \quad 0 \leq k \leq n-2.$$

Proof. The key point in the proof is to check when the j-th cohomology group $H^j\big({}_RM \otimes_S \mathrm{Hom}_R(M, M^\bullet)\big)$ vanishes for $j \geq 2$. Note that $H^j\big(M \otimes_S \mathrm{Hom}_R(M, M^\bullet)\big) = 0$ for all $j > n$.

For $n \leq 1$, the conclusion in Corollary 4.5 is clear. So, we suppose $n \geq 2$. By the axiom $(R2)$, we have $\mathrm{Ext}^j_R(M, M) = 0$ for all $j \geq 1$. It follows that $\mathrm{Ext}^j_R(M, M^I) \simeq \mathrm{Ext}^j_R(M, M)^I = 0$ for any nonempty set I, and therefore $\mathrm{Ext}^j_R(M, X) = 0$ for any $X \in \mathrm{Prod}(M)$.

By the axiom $(R3)$, there exists an exact sequence in R-Mod:

$$0 \longrightarrow R \longrightarrow M_0 \longrightarrow M_1 \longrightarrow \cdots \longrightarrow M_n \longrightarrow 0$$

such that $M_i \in \mathrm{Prod}(M)$ for $0 \leq i \leq n$. Since $\mathrm{Ext}^j_R(M,X) = 0$ for any $X \in \mathrm{Prod}(M)$ and $j \geq 1$, we know that the following complex $\mathrm{Hom}_R(M,M^\bullet)$:

$$0 \longrightarrow \mathrm{Hom}_R(M,M_0) \longrightarrow \mathrm{Hom}_R(M,M_1) \longrightarrow \mathrm{Hom}_R(M,M_2) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_R(M,M_n) \longrightarrow 0$$

satisfies that $H^j\big(\mathrm{Hom}_R(M,M^\bullet)\big) \simeq \mathrm{Ext}^j_R(M,R)$ for each $j \geq 1$.

(2) Let $n = 2$. Consider the complex $M \otimes_S \mathrm{Hom}_R(M, M^\bullet)$:

$$0 \longrightarrow M \otimes_S \mathrm{Hom}_R(M,M_0) \longrightarrow M \otimes_S \mathrm{Hom}_R(M,M_1) \longrightarrow M \otimes_S \mathrm{Hom}_R(M,M_2) \longrightarrow 0.$$

Since the functor ${}_RM \otimes_S - : S\text{-Mod} \to R\text{-Mod}$ is right exact, we have

$$H^2\big(M \otimes_S \mathrm{Hom}_R(M, M^\bullet)\big) \simeq M \otimes_S H^2\big(\mathrm{Hom}_R(M, M^\bullet)\big) \simeq M \otimes_S \mathrm{Ext}^2_R(M,R).$$

Now, the statement (2) follows from the equivalences of (1) and (3) in Proposition 4.4.

(3) Under the assumption of (3), we claim that

$$H^m\big(M \otimes_S \mathrm{Hom}_R(M, M^\bullet)\big) \simeq \mathrm{Tor}^S_{n-m}(M, \mathrm{Ext}^n_R(M,R)) \quad \text{for} \quad 2 \leq m \leq n.$$

Consequently, the statement (3) will follow from the equivalences of (1) and (3) in Proposition 4.4.

In the following, we shall apply Lemma 2.5 to prove this claim. Define $Y^\bullet := \mathrm{Hom}_R(M, M^\bullet)$. This is a complex over S with $Y^i = \mathrm{Hom}_R(M,M_i)$ for $0 \leq i \leq n$ and $Y^i = 0$ for $i \geq n+1$. Moreover, since the right S-module M is strongly S-Mittag-Leffler by the axiom $(R4)$, it follows from the proof of Lemma 4.3 (1) that

$$\mathrm{Tor}^S_k\big(M, \mathrm{Hom}_R(M,X)\big) = 0 \ \text{ for all } k \geq 1 \text{ and } X \in \mathrm{Prod}(M).$$

This implies that $\mathrm{Tor}^S_k(M,Y^i) = 0$ for all $i \in \mathbb{Z}$ and $k \geq 1$.

Recall that $H^j(Y^\bullet) \simeq \mathrm{Ext}^j_R(M,R)$ for all $j \geq 1$. By assumption, we obtain

$$\mathrm{Tor}^S_i(M, H^j(Y^\bullet)) = 0 \quad \text{for} \quad 2 \leq j \leq n-1 \quad \text{and} \quad 0 \leq i \leq j-2.$$

Clearly, this implies that, for each $2 \leq m \leq n-1$, we have

$$\mathrm{Tor}^S_t\big(M, H^{m+t}(Y^\bullet)\big) = 0 = \mathrm{Tor}^S_{t-1}\big(M, H^{m+t}(Y^\bullet)\big) \text{ for } 0 \leq t \leq n-m-1.$$

It follows from Lemma 2.5 that $H^m(M \otimes_S Y^\bullet) \simeq \mathrm{Tor}^S_{n-m}\big(M, H^n(Y^\bullet)\big) \simeq \mathrm{Tor}^S_{n-m}(M, \mathrm{Ext}^n_R(M,R))$.

To finish the proof of the claim, it remains to prove $H^n(M \otimes_S Y^\bullet) \simeq M \otimes_S \mathrm{Ext}^n_R(M,R)$. However, since the functor $M \otimes_S -$ is right exact and since $Y^i = 0$ for $i \geq n+1$, we see that $H^n(M \otimes_S Y^\bullet) \simeq M \otimes_S H^n(Y^\bullet) \simeq M \otimes_S \mathrm{Ext}^n_R(M,R)$. This finishes the proof of the above-mentioned claim. Thus (3) holds. $\square$

As another consequence of Proposition 4.4, we mention the following result which is not used in this note, but of its own interest.

Corollary 4.6. (1) *If $M_0 \in \mathrm{Add}({}_RM)$, then ${}_RM$ is a classical tilting module.*

(2) *If $M_1 \in \mathrm{Add}({}_RM)$, then $\mathcal{Y}$ is homological in $\mathscr{D}(R)$.*

Proof. (1) Suppose $M_0 \in \mathrm{Add}({}_RM)$. We claim that $\mathrm{Coker}(\theta_{M_0}) = 0$. In fact, since ${}_RM$ is finitely generated by the axiom $(R1)$, the functor $\mathrm{Hom}_R(M,-) : R\text{-Mod} \to S\text{-Mod}$ commutes with arbitrary direct sums. It follows that the evaluation map $\theta_X : M \otimes_S \mathrm{Hom}_R(M,X) \longrightarrow X$ is an isomorphism for each $X \in \mathrm{Add}({}_RM)$. Since $M_0 \in \mathrm{Add}({}_RM)$, the map $\theta_{M_0} : M \otimes_S \mathrm{Hom}_R(M,M_0) \longrightarrow M_0$ is an isomorphism, and therefore $\mathrm{Coker}(\theta_{M_0}) = 0$. Combining this with the proof of Proposition 4.4, we have $H^0(i_*i^*(R)) \simeq \mathrm{Ker}(\partial_0) = 0$. Note that $\mathrm{End}_{\mathscr{D}(R)}(i^*(R)) \simeq H^0(i^*(R)) = H^0(i_*i^*(R))$ as R-modules by Lemma 3.1 (2). This implies that $\mathrm{End}_{\mathscr{D}(R)}(i^*(R)) = 0$ and so $\mathcal{Y} = 0$ by Lemma 3.1 (1). Now, it follows from Lemma 4.2 that $\mathbb{R}\mathrm{Hom}_R(M,-) : \mathscr{D}(R) \longrightarrow \mathscr{D}(S)$ is a triangle equivalence. Consequently, ${}_RM$ is a classical tilting module by [1, Chapter 5, Theorem 4.1].

(2) It follows from the proof of (1) that $\mathrm{Coker}(\theta_{M_1}) = 0$. Thus (2) follows from Proposition 4.4 and Lemma 4.3 (1). $\square$

5 Application to tilting modules: Proofs of Theorem 1.1 and Corollary 1.2

In this section, we first develop some properties of (good) tilting modules, and then give a method to construct good Ringel modules. With these preparations in hand, we finally apply Proposition 4.4 to prove Theorem 1.1 and Corollary 1.2.

Throughout this section, A will be a ring and n a natural number. In addition, we assume that T is a **good** n-tilting A-module with $(T1),(T2)$ and $(T3)'$. Let $B := \mathrm{End}_A(T)$.

First of all, we shall mention a few basic properties of good tilting modules in the following lemma. For proofs, we refer to [1, Chapter 11, Lemma 2.7], [6, Proposition 1.4, Lemma 1.5] and [5, Proposition 3.5].

Lemma 5.1. *The following hold true for the tilting module ${}_AT$.*

(1) *The torsion class* $T^{\perp} := \{X \in A\text{-Mod} \mid \mathrm{Ext}^i_A(T,X) = 0 \ for\ all\ i \geq 1\}$ *in* A-Mod *is closed under arbitrary direct sums in* A-Mod*.*

(2) *The right B-module T has a finitely generated projective resolution of length at most n:*

$$0 \longrightarrow \mathrm{Hom}_A(T_n,T) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_A(T_1,T) \longrightarrow \mathrm{Hom}_A(T_0,T) \longrightarrow T_B \longrightarrow 0$$

with $T_i \in \mathrm{add}({}_AT)$ *for all* $0 \leq i \leq n$*.*

(3) *The map* $A^{\mathrm{op}} \to \mathrm{End}_{B^{\mathrm{op}}}(T)$*, defined by* $a \mapsto [t \mapsto at]$ *for* $a \in A$ *and* $t \in T$*, is an isomorphism of rings. Moreover,* $\mathrm{Ext}^i_{B^{\mathrm{op}}}(T,T) = 0$ *for all* $i \geq 1$*.*

(4) *If* $T_n = 0$ *in the axiom* $(T3)'$*, then* ${}_AT$ *is an* $(n-1)$*-tilting module.*

Let us introduce some notation which will be used throughout this section.
Define

$$G := {}_AT \otimes^{\mathbb{L}}_B - : \mathscr{D}(B) \to \mathscr{D}(A), \quad H := \mathbb{R}\mathrm{Hom}_A(T,-) : \mathscr{D}(A) \to \mathscr{D}(B),$$

$$Q^\bullet := \ \cdots \longrightarrow 0 \longrightarrow \mathrm{Hom}_A(T,T_0) \longrightarrow \mathrm{Hom}_A(T,T_1) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_A(T,T_n) \longrightarrow 0 \longrightarrow \cdots$$

where $\mathrm{Hom}_A(T,T_i)$ is of degree i for $0 \leq i \leq n$, and $Q^{\bullet *} := \mathrm{Hom}_B(Q^\bullet, B) \in \mathscr{C}(B^{\mathrm{op}}\text{-proj})$. Clearly, $Q^{\bullet *}$ is isomorphic in $\mathscr{C}^b(B^{\mathrm{op}}\text{-proj})$ to the complex

$$\cdots \longrightarrow 0 \longrightarrow \mathrm{Hom}_A(T_n,T) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_A(T_1,T) \longrightarrow \mathrm{Hom}_A(T_0,T) \longrightarrow 0 \longrightarrow \cdots$$

The following result is due to Bazzoni [6, Theorem 2.2], which says that, in general, $\mathscr{D}(A)$ is not equivalent to $\mathscr{D}(B)$, but a full subcategory of $\mathscr{D}(B)$.

Lemma 5.2. *The functor* $H : \mathscr{D}(A) \to \mathscr{D}(B)$ *is fully faithful, and* $\mathrm{Im}(H) = \mathrm{Ker}(\mathrm{Hom}_{\mathscr{D}(B)}(\mathrm{Ker}(G),-))$.

The next result supplies a way to understand good tilting modules T by some special objects or by subcategories of derived module categories. In particular, the category $\mathrm{Ker}(G)$ is a bireflective subcategory of $\mathscr{D}(B)$.

Lemma 5.3. *For the tilting A-module T, we have the following:*

(1) $H(A) \simeq Q^\bullet$ *in* $\mathscr{D}(B)$ *and* $\mathrm{Hom}_{\mathscr{D}(B)}(Q^\bullet, Q^\bullet[m]) = 0$ *for any* $m \neq 0$*.*

(2) $\mathrm{Ker}(G) = \{Y^\bullet \in \mathscr{D}(B) \mid \mathrm{Hom}_{\mathscr{D}(B)}(Q^\bullet, Y^\bullet[i]) = 0 \ for\ all\ i \in \mathbb{Z}\}$.

(3) *Let* $j_! : \mathrm{Tria}(Q^\bullet) \longrightarrow \mathscr{D}(B)$ *and* $i_* : \mathrm{Ker}(G) \longrightarrow \mathscr{D}(B)$ *be the inclusions. Then there exists a recollement of triangulated categories together with a triangle equivalence:*

$$(\star) \qquad \begin{tikzcd} \mathrm{Ker}(G) \arrow[r, "i_*"] & \mathscr{D}(B) \arrow[l, bend right, "i^*"'] \arrow[l, bend left] \arrow[r, "j^!"] & \mathrm{Tria}(Q^\bullet) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \arrow[r, "G\,j_*", "\simeq"'] & \mathscr{D}(A) \end{tikzcd}$$

such that $G\,j_*\,j^!$ *is naturally isomorphic to* G*.*

Proof. We remark that Lemma 5.3 is implied in [6]. For convenience of the reader, we give a proof here.

(1) By the axiom $(T3)'$, the stalk complex A is quasi-isomorphic in $\mathscr{C}(A)$ to the complex $T^\bullet$ of the form:

$$\cdots \longrightarrow 0 \longrightarrow T_0 \longrightarrow T_1 \longrightarrow \cdots \longrightarrow T_n \longrightarrow 0 \longrightarrow \cdots$$

where $T_i \in \mathrm{add}(T)$ is in degree i for $0 \le i \le n$. Further, by the axiom $(T2)$, we have $T_i \in T^{\perp} := \{X \in A\text{-Mod} \mid \mathrm{Ext}^i_A(T,X) = 0 \text{ for all } i \geq 1\}$. It follows from Lemma 2.1 (1) that $H(A) \simeq H(T^\bullet) \simeq \mathrm{Hom}_A(T,T^\bullet) = Q^\bullet$ in $\mathscr{D}(B)$. Since the functor H is fully faithful by Lemma 5.2, we obtain

$$\mathrm{Hom}_{\mathscr{D}(B)}(Q^\bullet, Q^\bullet[m]) \simeq \mathrm{Hom}_{\mathscr{D}(B)}(H(A),H(A)[m]) \simeq \mathrm{Hom}_{\mathscr{D}(A)}(A,A[m]) \simeq \mathrm{Ext}^m_A(A,A) = 0$$

for any $m \neq 0$. This shows (1).

(2) Since $Q^\bullet \in \mathscr{C}^b(B\text{-proj})$ and since $Q^{\bullet *}$ is quasi-isomorphic to T_B by Lemma 5.1 (2), we have the following natural isomorphisms of triangle functors:

$$\mathbb{R}\mathrm{Hom}_B(Q^\bullet, -) \xrightarrow{\simeq} Q^{\bullet *} \otimes^{\mathbb{L}}_B - \xrightarrow{\simeq} {}_{\mathbb{Z}}T \otimes^{\mathbb{L}}_B - : \mathscr{D}(B) \longrightarrow \mathscr{D}(\mathbb{Z}),$$

where the first isomorphism follows from Lemma 2.4. Note that $H^m(\mathbb{R}\mathrm{Hom}_{\mathscr{D}(B)}(Q^\bullet, Y^\bullet)) \simeq \mathrm{Hom}_{\mathscr{D}(B)}(Q^\bullet, Y^\bullet[m])$ for $m \in \mathbb{Z}$ and $Y^\bullet \in \mathscr{D}(B)$. This shows (2).

(3) Since $Q^\bullet \in \mathscr{C}^b(B\text{-proj})$, we know from (2) and Lemma 3.6 (1) that there exists a recollement of triangulated categories:

$$(\star\star) \qquad \begin{tikzcd} \mathrm{Ker}(G) \arrow[r, "i_*"] & \mathscr{D}(B) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^!"] & \mathrm{Tria}(Q^\bullet) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

On the one hand, by the correspondence of recollements and TTF (torsion, torsion-free) triples (see, for example, [11, Section 2.3]), we infer from $(\star\star)$ that $\mathrm{Im}(j_*) = \mathrm{Ker}(\mathrm{Hom}_{\mathscr{D}(B)}(\mathrm{Ker}(G),-))$ and that the functor $j_* : \mathrm{Tria}(Q^\bullet) \longrightarrow \mathrm{Im}(j_*)$ is a triangle equivalence with the restriction of $j^!$ to $\mathrm{Im}(j_*)$ as its quasi-inverse. On the other hand, it follows from Lemma 5.2 that $\mathrm{Im}(H) = \mathrm{Ker}(\mathrm{Hom}_{\mathscr{D}(B)}(\mathrm{Ker}(G),-))$ and the functor $H : \mathscr{D}(A) \longrightarrow \mathrm{Im}(H)$ is a triangle equivalence with the restriction of G to $\mathrm{Im}(H)$ as its quasi-inverse. Consequently, we see that $\mathrm{Im}(j_*) = \mathrm{Im}(H)$ and the composition $G\,j_* : \mathrm{Tria}(Q^\bullet) \longrightarrow \mathscr{D}(A)$ of j_* with G is also a triangle equivalence.

It remains to check
$$G \xrightarrow{\simeq} G\,j_* j^! : \mathscr{D}(B) \longrightarrow \mathscr{D}(A).$$
In fact, for any $X^\bullet \in \mathscr{D}(B)$, by the recollement $(\star\star)$, there exists a canonical triangle in $\mathscr{D}(B)$:

$$i_* i^!(X^\bullet) \longrightarrow X^\bullet \longrightarrow j_* j^!(X^\bullet) \longrightarrow i_* i^!(X^\bullet)[1].$$

Since $\mathrm{Im}(i_* i^!) = \mathrm{Im}(i_*) = \mathrm{Ker}(G)$, we know that $G(X^\bullet) \xrightarrow{\simeq} G\,j_* j^!(X^\bullet)$ in $\mathscr{D}(B)$. This proves (3). $\square$

Next, we shall investigate when the subcategory $\mathrm{Ker}(G)$ of $\mathscr{D}(B)$ is homological. The following result conveys that this discussion can be proceeded along the right B-module T.

Lemma 5.4. *The category* $\mathrm{Ker}(G)$ *is a homological subcategory of* $\mathscr{D}(B)$ *if and only if* $\mathrm{Ker}\big(\mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(T,-)\big)$ *is a homological subcategory of* $\mathscr{D}(B^{\mathrm{op}})$.

Proof. In Lemma 3.6, we take $R := B$ and $\Sigma := \{Q^\bullet\}$. Then $\Sigma^* = \{Q^{\bullet *}\}$ where $Q^{\bullet *} := \mathrm{Hom}_B(Q^\bullet, B)$. Since $Q^{\bullet *}$ is quasi-isomorphic to T_B by Lemma 5.1 (2), we infer that $Q^{\bullet *} \xrightarrow{\simeq} T_B$ in $\mathscr{D}(B^{\mathrm{op}})$ and that there exists a natural isomorphism of triangle functors:

$$\mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(T,-) \xrightarrow{\simeq} \mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(Q^{\bullet *},-) : \mathscr{D}(B^{\mathrm{op}}) \longrightarrow \mathscr{D}(\mathbb{Z}).$$

This implies that

$$\mathrm{Ker}\big(\mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(T,-)\big) = \mathrm{Ker}\big(\mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(Q^{\bullet *},-)\big) = \{Y^\bullet \mid \mathrm{Hom}_{\mathscr{D}(B^{\mathrm{op}})}(Q^{\bullet *}, Y^\bullet[m]) = 0 \text{ for } m \in \mathbb{Z}\}.$$

Thus Lemma 5.4 follows from Lemmas 3.3 and 3.6 (4). $\square$

Next, we point out that each good tilting module naturally corresponds to a good Ringel module. This guarantees that we can apply Proposition 4.4 to show Theorem 1.1.

Lemma 5.5. *The right B-module T_B is a good n-Ringel module.*

Proof. By Lemma 5.1 (2), the axiom $(R1)$ holds for T_B, and the projective dimension of T_B is at most n. Moreover, by Lemma 5.1 (3), the axiom $(R2)$ also holds for T_B. Now, we check the axiom $(R3)$ for T_B.

In fact, according to the axiom $(T1)$, the module ${}_AT$ admits a projective resolution of A-modules:

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow T \longrightarrow 0$$

with $P_i \in \mathrm{Add}({}_AA)$ for $0 \leq i \leq n$. Since $\mathrm{Ext}_A^j(T,T) = 0$ for each $j \geq 1$ by the axiom $(T2)$, it follows that the sequence

$$0 \longrightarrow B \longrightarrow \mathrm{Hom}_A(P_0,T) \longrightarrow \mathrm{Hom}_A(P_1,T) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_A(P_n,T) \longrightarrow 0$$

of right B-modules is exact. Note that $\mathrm{Hom}_A(P_i,T) \in \mathrm{Prod}(T_B)$ due to $P_i \in \mathrm{Add}({}_AA)$. This means that the axiom $(R3)$ holds for T_B. Thus the right B-module T_B is an n-Ringel module.

It remains to prove that T_B is good, that is, T_B satisfies the axiom $(R4)$.

Actually, by Lemma 5.1 (3), the map $A^{\mathrm{op}} \longrightarrow \mathrm{End}_{B^{\mathrm{op}}}(T)$, defined by $a \mapsto [t \mapsto at]$ for $a \in A$ and $t \in T$, is an isomorphism of rings. Further, it follows from Lemma 2.8 that the right A^{op}-module T is strongly A^{op}-Mittag-Leffler. Hence, the right $\mathrm{End}_{B^{\mathrm{op}}}(T)$-module T is strongly $\mathrm{End}_{B^{\mathrm{op}}}(T)$-Mittag-Leffler. Thus, by definition, the n-Ringel B^{op}-module T is good. $\square$

Remark. If ${}_AT$ is infinitely generated, then the right B-module T is not a tilting module. In fact, it follows from Lemma 5.1 (2) that T_B is finitely generated. Suppose contrarily that T_B is a tilting right B-module. Then, by Corollary 2.9, the right B-module T_B is classical, and therefore ${}_AT$ is classical by Lemma 5.1 (2)-(3). This is a contradiction.

Now, with the previous preparations, we are in the position to prove Theorem 1.1.

Proof of Theorem 1.1. We shall use Proposition 4.4 to show the equivalences in Theorem 1.1.

Recall that we denote by $P^\bullet$ the complex which is the deleted projective resolution of ${}_AT$:

$$\cdots \longrightarrow 0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{\sigma} P_0 \longrightarrow 0 \longrightarrow \cdots$$

appearing in the axiom $(T1)$. Here, P_i is in degree $-i$ for $0 \leq i \leq n$.

By Lemma 5.5, we know that T is a good n-Ringel B^{op}-module and that the exact sequence in the axiom $(R3)$ can be chosen as

$$0 \longrightarrow B_B \longrightarrow \mathrm{Hom}_A(P_0,T) \longrightarrow \mathrm{Hom}_A(P_1,T) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_A(P_n,T) \longrightarrow 0.$$

In particular, the complex $M^\bullet$ in Proposition 4.4 can be chosen to be the following complex:

$$\mathrm{Hom}_A(P^\bullet,T): \ \cdots \longrightarrow 0 \longrightarrow \mathrm{Hom}_A(P_0,T) \longrightarrow \mathrm{Hom}_A(P_1,T) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_A(P_n,T) \longrightarrow 0 \longrightarrow \cdots$$

Now, in Proposition 4.4, we take $R := B^{\mathrm{op}}$, $S := A^{\mathrm{op}}$ and $M := {}_RT_S$. Further, let

$$\mathbf{H} = \mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(T,-) : \mathscr{D}(B^{\mathrm{op}}) \longrightarrow \mathscr{D}(A^{\mathrm{op}}).$$

It follows from Lemma 5.4 that $\mathrm{Ker}(G)$ is homological in $\mathscr{D}(R)$ if and only if so is $\mathrm{Ker}(\mathbf{H})$ in $\mathscr{D}(B^{\mathrm{op}})$. In other words, the statement (1) in Theorem 1.1 is equivalent to the following statement:

$(1')$ The category $\mathrm{Ker}(\mathbf{H})$ is a homological subcategory of $\mathscr{D}(B^{\mathrm{op}})$.

In the following, we shall show that $(1')$ is equivalent to $(2),(3)$ and (4), respectively.

We first show that $(1')$ and (2) are equivalent. In fact, it follows form Proposition 4.4 that $(1')$ is equivalent to

$(2')$ The category $\mathscr{E} := \{Y \in B^{\mathrm{op}}\text{-Mod} \mid \mathrm{Ext}^m_{B^{\mathrm{op}}}(T,Y) = 0 \text{ for all } m \geq 0\}$ is an abelian subcategory of B^{op}-Mod. So, we will show that $(2')$ is equivalent to (2). For this aim, we set $\mathscr{A} := \{X \in B\text{-Mod} \mid \mathrm{Tor}^B_m(T,X) = 0 \text{ for all } m \geq 0\}$, and establish a connection between $\mathscr{A}$ and $\mathscr{E}$. Let $(-)^\vee$ be the dual functor $\mathrm{Hom}_{\mathbb{Z}}(-,\mathbb{Q}/\mathbb{Z}) : \mathbb{Z}\text{-Mod} \longrightarrow \mathbb{Z}\text{-Mod}$.

Now, we claim that $(-)^\vee$ induces two exact functors:

$$(-)^\vee : \mathscr{A} \longrightarrow \mathscr{E} \quad \text{and} \quad (-)^\vee : \mathscr{E} \longrightarrow \mathscr{A}$$

such that $X \in \mathscr{A}$ if and only if $X^\vee \in \mathscr{E}$, and that $Y \in \mathscr{E}$ if and only if $Y^\vee \in \mathscr{A}$, where $X \in B$-Mod and $Y \in B^{\mathrm{op}}$-Mod.

In fact, it is known that $\mathbb{Q}/\mathbb{Z}$ is an injective cogenerator for $\mathbb{Z}$-Mod, and that $(-)^\vee$ admits the following properties:

(a) For each $M \in \mathbb{Z}$-Mod, if $M^\vee = 0$, then $M = 0$.

(b) A sequence $0 \to X_1 \to X_2 \to X_3 \to 0$ of $\mathbb{Z}$-modules is exact if and only if $0 \to (X_3)^\vee \to (X_2)^\vee \to (X_1)^\vee \to 0$ is exact.

On the one hand, for each $X \in B$-Mod, it follows from Lemma 2.3 (1) that

$$(\mathrm{Tor}_m^B(T, X))^\vee \simeq \mathrm{Ext}^m_{B^{\mathrm{op}}}(T, X^\vee) \text{ for all } m \geq 0.$$

This implies that $X \in \mathscr{A}$ if and only if $X^\vee \in \mathscr{E}$. This is due to (a).

On the other hand, since T_B has a finitely generated projective resolution in B^{op}-Mod by Lemma 5.1 (2), it follows from Lemma 2.3 (2) that

$$(\mathrm{Ext}^m_{B^{\mathrm{op}}}(T, Y))^\vee \simeq \mathrm{Tor}_m^B(T, Y^\vee) \ \text{ for all } m \geq 0 \text{ and for any } Y \in B^{\mathrm{op}}\text{-Mod}.$$

This means that $Y \in \mathscr{E}$ if and only if $Y^\vee \in \mathscr{A}$, again due to (a). This finishes the proof of the claim.

Recall that $\mathscr{A}$ always admits the "2 out of 3" property: For an arbitrary short exact sequence in B-Mod, if any two of its three terms belong to $\mathscr{A}$, then so does the third. Moreover, $\mathscr{A}$ is an abelian subcategory of B-Mod if and only if $\mathscr{A}$ is closed under kernels (respectively, cokernels) in B-Mod. Clearly, similar statements hold for the subcategory $\mathscr{E}$ of B^{op}-Mod.

By the above-proved claim, one can easily show that $\mathscr{A}$ is closed under kernels in B-Mod if and only if $\mathscr{E}$ is closed under cokernels in B^{op}-Mod. It follows that $\mathscr{A}$ is an abelian subcategory of B-Mod if and only if $\mathscr{E}$ is an abelian subcategory of B^{op}-Mod. Thus $(2')$ is equivalent to (2), and therefore $(1')$ and (2) are equivalent.

Next, we shall verify that $(1')$ and (3) are equivalent. Actually, it follows form Proposition 4.4 that $(1')$ is also equivalent to the following statement:

$(3')$ $H^j\big(\mathrm{Hom}_{B^{\mathrm{op}}}(T, M^\bullet) \otimes_A T\big) = 0$ for all $j \geq 2$, where $\mathrm{Hom}_{B^{\mathrm{op}}}(T, M^\bullet) := \mathrm{Hom}_{B^{\mathrm{op}}}(T, \mathrm{Hom}_A(P^\bullet, T))$ is the complex of the form:

$$0 \longrightarrow \mathrm{Hom}_{B^{\mathrm{op}}}(T, \mathrm{Hom}_A(P_0, T)) \longrightarrow \mathrm{Hom}_{B^{\mathrm{op}}}(T, \mathrm{Hom}_A(P_1, T)) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_{B^{\mathrm{op}}}(T, \mathrm{Hom}_A(P_n, T)) \longrightarrow 0,$$

with $\mathrm{Hom}_{B^{\mathrm{op}}}(T, \mathrm{Hom}_A(P_i, T))$ in degree i for $0 \leq i \leq n$.

So it suffices to verify that $(3')$ and (3) are equivalent. Clearly, for this purpose, it is enough to show that $\mathrm{Hom}_A(P^\bullet, A) \simeq \mathrm{Hom}_{B^{\mathrm{op}}}(T, \mathrm{Hom}_A(P^\bullet, T))$ as complexes over A^{op}.

Note that there exists a natural isomorphism of additive functors:

$$\mathrm{Hom}_{B^{\mathrm{op}}}(T, \mathrm{Hom}_A(-, T)) \xrightarrow{\simeq} \mathrm{Hom}_{B^{\mathrm{op}}}(\mathrm{Hom}_A(A, T), \mathrm{Hom}_A(-, T)) : A\text{-Mod} \to A^{\mathrm{op}}\text{-Mod}.$$

Moreover, the functor $\Phi := \mathrm{Hom}_A(-, T)$ yields a natural transformation:

$$\mathrm{Hom}_A(-, A) \longrightarrow \mathrm{Hom}_{B^{\mathrm{op}}}(\Phi(A), \Phi(-)) : A\text{-Mod} \to A^{\mathrm{op}}\text{-Mod}.$$

Now we shall show that this transformation is even a natural isomorphism. Clearly, it is sufficient to prove that

$$\Phi : \mathrm{Hom}_A(X, A) \xrightarrow{\simeq} \mathrm{Hom}_{B^{\mathrm{op}}}(\Phi(A), \Phi(X))$$

for any projective A-module X. In the following, we will show that this holds even for any A-module X.

In fact, since T is a good tilting A-module, it follows from the axiom $(T3)'$ that there exists an exact sequence $0 \longrightarrow A \longrightarrow T_0 \longrightarrow T_1$ with $T_i \in \mathrm{add}(T)$ for $i = 0, 1$. By Lemma 5.1 (2), we obtain another exact sequence $\Phi(T_1) \longrightarrow \Phi(T_0) \longrightarrow \Phi(A) \longrightarrow 0$ of B^{op}-modules. This gives rise to the following exact commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_A(X, A) \arrow[r] \arrow[d, "\Phi"] & \mathrm{Hom}_A(X, T_0) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Hom}_A(X, T_1) \arrow[d, "\simeq"] \\
0 \arrow[r] & \mathrm{Hom}_{B^{\mathrm{op}}}(\Phi(A), \Phi(X)) \arrow[r] & \mathrm{Hom}_{B^{\mathrm{op}}}(\Phi(T_0), \Phi(X)) \arrow[r] & \mathrm{Hom}_{B^{\mathrm{op}}}(\Phi(T_1), \Phi(X))
\end{tikzcd}$$

where the isomorphisms in the second and third columns are due to $T_0 \in \mathrm{add}(T)$ and $T_1 \in \mathrm{add}(T)$, respectively. Consequently, the $\Phi : \mathrm{Hom}_A(X, A) \longrightarrow \mathrm{Hom}_{B^{\mathrm{op}}}(\Phi(A), \Phi(X))$ in the first column is an isomorphism. This implies that

$$\mathrm{Hom}_A(-, A) \xrightarrow{\simeq} \mathrm{Hom}_{B^{\mathrm{op}}}(\Phi(A), \Phi(-)) \xrightarrow{\simeq} \mathrm{Hom}_{B^{\mathrm{op}}}(T, \mathrm{Hom}_A(-, T)) : A\text{-Mod} \to A^{\mathrm{op}}\text{-Mod}.$$

Thus $\mathrm{Hom}_A(P^\bullet,A)\simeq \mathrm{Hom}_{B^{\mathrm{op}}}(T,\mathrm{Hom}_A(P^\bullet,T))$ as complexes over A^{op}. Thus $(3')$ is equivalent to (3).

It remains to show that $(1')$ is equivalent to (4).

For each right B-module Y, let $\theta_Y:\mathrm{Hom}_{B^{\mathrm{op}}}({}_AT_B,Y)\otimes_A T\longrightarrow Y$ be the evaluation map. Then it follows from the equivalence of (1) and (4) in Proposition 4.4 that $(1')$ is equivalent to the following statement:

$(4')$ The kernel of the homomorphism $\partial_0:\mathrm{Coker}\big(\theta_{\Phi(P_0)}\big)\longrightarrow \mathrm{Coker}\big(\theta_{\Phi(P_1)}\big)$ induced from the homomorphism $\Phi(\sigma):\Phi(P_0)\longrightarrow\Phi(P_1)$ belongs to $\mathscr{E}$.

Now, we claim that $K\simeq \mathrm{Ker}(\partial_0)$ as right B-modules (see the definition of K in Theorem 1.1 (4)). This will show that $(1')$ and (4) are equivalent.

To check the above isomorphism, we first define the following map for each A-module X:

$$\zeta_X:\mathrm{Hom}_A(X,A)\otimes_A T\longrightarrow \mathrm{Hom}_A(X,T),\ f\otimes t\mapsto [x\mapsto (x)ft]$$

for $f\in\mathrm{Hom}_A(X,A)$, $t\in T$ and $x\in X$. This yields a natural transformation $\zeta:\mathrm{Hom}_A(-,A)\otimes_A T\longrightarrow \mathrm{Hom}_A(-,T)$ from A-Mod to B^{op}-Mod. Clearly, by definition, we have $\varphi_i=\zeta_{P_i}$ for $i=0,1$.

Recall that, under the identification of $\Phi(A)$ with T as A-B-bimodules, the functor Φ induces an isomorphism $\mathrm{Hom}_A(X,A)\xrightarrow{\simeq}\mathrm{Hom}_{B^{\mathrm{op}}}(T,\Phi(X))$ of A^{op}-modules. In this sense, one can easily construct the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_A(X,A)\otimes_A T \arrow[r, "\zeta_X"] \arrow[d, "\Phi\otimes 1", "\simeq"'] & \mathrm{Hom}_A(X,T) \arrow[d, equal] \\
\mathrm{Hom}_{B^{\mathrm{op}}}(T,\Phi(X))\otimes_A T \arrow[r, "\theta_{\Phi(X)}"] & \Phi(X)
\end{tikzcd}$$

This implies that $\mathrm{Coker}(\zeta_X)$ is naturally isomorphic to $\mathrm{Coker}\big(\theta_{\Phi(X)}\big)$ as B^{op}-modules. Since $\varphi_i=\zeta_{P_i}$ for $i=0,1$, we show that $K\simeq\mathrm{Ker}(\partial_0)$ as B^{op}-modules.

Hence, we have proved that the statements (1)-(4) in Theorem 1.1 are equivalent.

Now, suppose $n=2$. Then the complex $P^\bullet$ is of the following form:

$$\cdots\longrightarrow 0\longrightarrow P_2\longrightarrow P_1\longrightarrow P_0\longrightarrow 0\longrightarrow\cdots$$

which is a deleted projective resolution of ${}_AT$. Since (1) and (3) in Theorem 1.1 are equivalent, we see that (1) holds if and only if $H^2\big(\mathrm{Hom}_A(P^\bullet,A)\otimes_A T_B\big)=0$. However, since the tensor functor $-\otimes_A T_B: A^{\mathrm{op}}\text{-Mod}\longrightarrow B^{\mathrm{op}}\text{-Mod}$ is always right exact, we have

$$H^2\big(\mathrm{Hom}_A(P^\bullet,A)\otimes_A T_B\big)\simeq H^2\big(\mathrm{Hom}_A(P^\bullet,A)\big)\otimes_A T\simeq \mathrm{Ext}^2_A(T,A)\otimes_A T.$$

This finishes the proof of Theorem 1.1. $\square$

Remarks. (1) If the category $\mathrm{Ker}({}_AT\otimes^{\mathbb{L}}_B -)$ in Theorem 1.1 is homological in $\mathscr{D}(B)$, then it follows from Lemma 4.2 (see also Lemma 5.3 (3)) that the generalized localization $\lambda: B\to B_T$ of B at the module T_B exists and is homological, which gives rise to a recollement of derived module categories:

$$\begin{tikzcd}
\mathscr{D}(B_T) \arrow[r, "D(\lambda_*)"] & \mathscr{D}(B) \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "{}_AT\otimes^{\mathbb{L}}_B -"] & \mathscr{D}(A) \arrow[l, bend right] \arrow[l, bend left]
\end{tikzcd}$$

(2) Combining the remark following Lemma 4.3 with the proof of Theorem 1.1, we infer that the complex $\mathrm{Hom}_A(P^\bullet,A)\otimes_A T_B$ in Theorem 1.1 is isomorphic in $\mathscr{D}(B^{\mathrm{op}})$ to both $\mathrm{Hom}_A(P^\bullet,A)\otimes^{\mathbb{L}}_A T_B$ and $\mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(T,B)\otimes^{\mathbb{L}}_A T$. This implies that, up to isomorphism, the cohomology group $H^m\big(\mathrm{Hom}_A(P^\bullet,A)\otimes_A T_B\big)$ in Theorem 1.1 (3) is independent of the choice of the projective resolutions of ${}_AT$ for all $m\in\mathbb{Z}$.

(3) By the proof of the equivalence of (1) and (4) in Theorem 1.1, we know that $\mathrm{Coker}(\zeta_X)\simeq\mathrm{Coker}\big(\theta_{\Phi(X)}\big)$ as B^{op}-modules for $X\in A$-Mod. If $X\in\mathrm{Add}({}_AA)$, then $\Phi(X)\in\mathrm{Prod}(T_B)$, and therefore it follows from Lemma 4.3 (1) that $\mathrm{Coker}\big(\theta_{\Phi(X)}\big)$ belongs to $\mathscr{E}:=\{Y\in B^{\mathrm{op}}\text{-Mod}\mid \mathrm{Ext}^m_{B^{\mathrm{op}}}(T,Y)=0 \text{ for all } m\geq 0\}$. Particularly, in Theorem 1.1 (4), we always have $\mathrm{Coker}(\varphi_i)\in\mathscr{E}$ for $i=1,2$. Note that $\mathscr{E}$ is closed under kernels of surjective homomorphisms in B^{op}-Mod. Hence, if the homomorphism $\widetilde{\sigma}:\mathrm{Coker}(\varphi_0)\longrightarrow\mathrm{Coker}(\varphi_1)$ induced from $\sigma: P_1\to P_0$ is surjective, then the kernel K of $\widetilde{\sigma}$ does belong to $\mathscr{E}$, and therefore the category $\mathrm{Ker}(T\otimes^{\mathbb{L}}_B -)$ is homological in $\mathscr{D}(B)$ by the equivalence of (1) and (4) in Theorem 1.1.

Clearly, the maps π and ω in the definition of tilting modules induce two canonical quasi-isomorphisms $\widetilde{\pi}: P^\bullet\longrightarrow T$ and $\widetilde{\omega}: A\longrightarrow T^\bullet$ in $\mathscr{C}(A)$, respectively. Consequently, both $\widetilde{\pi}$ and $\widetilde{\omega}$ are isomorphisms in $\mathscr{D}(A)$.

As a preparation for the proof of Corollary 1.2, we shall first establish the following lemma.

Lemma 5.6. *The complex* $\mathrm{Hom}_A(P^\bullet, A)$ *is isomorphic in* $\mathscr{D}(\mathbb{Z})$ *to the following complex:*

$$\mathrm{Hom}_A(T, T^\bullet): \ \cdots \longrightarrow 0 \longrightarrow \mathrm{Hom}_A(T, T_0) \longrightarrow \mathrm{Hom}_A(T, T_1) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_A(T, T_n) \longrightarrow 0 \longrightarrow \cdots$$

In particular, if A is commutative, then $\mathrm{Hom}_A(P^\bullet, A) \otimes_A T_B \simeq \mathrm{Hom}_A(T, T^\bullet) \otimes^{\mathbb{L}}_A T_B$ *in* $\mathscr{D}(B^{\mathrm{op}})$.

Proof. Since $\widetilde{\pi}$ and $\widetilde{\omega}$ are chain maps in $\mathscr{C}(A)$, we can obtain two chain maps in $\mathscr{C}(\mathbb{Z})$:

$$\mathrm{Hom}_A(P^\bullet, A) \xrightarrow{(\widetilde{\omega})^*} \mathrm{Hom}^\bullet_A(P^\bullet, T^\bullet) \xleftarrow{(\widetilde{\pi})_*} \mathrm{Hom}_A(T, T^\bullet).$$

Now, we claim that both chain maps are quasi-isomorphisms.

To check this claim, we apply the cohomology functor $H^i(-)$ to these chain maps for $i \in \mathbb{Z}$, and construct the following commutative diagram:

$$\begin{tikzcd}
H^i(\mathrm{Hom}_A(P^\bullet, A)) \arrow[r, "H^i((\widetilde{\omega})^*)"] \arrow[d, "\simeq"] & H^i(\mathrm{Hom}^\bullet_A(P^\bullet, T^\bullet)) \arrow[d, "\simeq"] & H^i(\mathrm{Hom}_A(T, T^\bullet)) \arrow[l, "H^i((\widetilde{\pi})_*)"'] \arrow[d, "\simeq"] \\
\mathrm{Hom}_{\mathscr{K}(A)}(P^\bullet, A[i]) \arrow[r, "(\widetilde{\omega})^*"] \arrow[d, "q_1"] & \mathrm{Hom}_{\mathscr{K}(A)}(P^\bullet, T^\bullet[i]) \arrow[d, "q_2"] & \mathrm{Hom}_{\mathscr{K}(A)}(T, T^\bullet[i]) \arrow[l, "(\widetilde{\pi})_*"'] \arrow[d, "q_3"] \\
\mathrm{Hom}_{\mathscr{D}(A)}(P^\bullet, A[i]) \arrow[r, "(\widetilde{\omega})^*", "\simeq"'] & \mathrm{Hom}_{\mathscr{D}(A)}(P^\bullet, T^\bullet[i]) & \mathrm{Hom}_{\mathscr{D}(A)}(T, T^\bullet[i]) \arrow[l, "(\widetilde{\pi})_*"', "\simeq"]
\end{tikzcd}$$

where the maps q_j, for $1 \leq j \leq 3$, are induced by the localization functor $q : \mathscr{K}(A) \to \mathscr{D}(A)$, and where the isomorphisms in the third row are due to the isomorphisms $\widetilde{\omega}$ and $\widetilde{\pi}$ in $\mathscr{D}(A)$.

Since $P^\bullet$ is a bounded complex of projective A-modules, both q_1 and q_2 are bijective. This implies that $H^i((\widetilde{\omega})^*)$ is also bijective, and therefore $(\widetilde{\omega})^*$ is a quasi-isomorphism.

Note that $(\widetilde{\pi})_*$ is a quasi-isomorphism if and only if $H^i((\widetilde{\pi})_*)$ is bijective for each $i \in \mathbb{Z}$. This is also equivalent to saying that q_3 is bijective in the above diagram. Actually, to prove the bijection of q_3, it is enough to show that, for $X \in \mathrm{add}(_AT)$ and $i \in \mathbb{Z}$, the canonical map $\mathrm{Hom}_{\mathscr{K}(A)}(T, X[i]) \longrightarrow \mathrm{Hom}_{\mathscr{D}(A)}(T, X[i])$ induced by q is bijective since $T^\bullet$ is a bounded complex with each term in $\mathrm{add}(_AT)$. However, this follows directly from the axiom $(T2)$. Thus $(\widetilde{\pi})_*$ is a quasi-isomorphism.

Consequently, the complexes $\mathrm{Hom}_A(P^\bullet, A)$ and $\mathrm{Hom}_A(T, T^\bullet)$ are isomorphic in $\mathscr{D}(\mathbb{Z})$.

Now, assume that A is commutative. Then each A-module can be naturally regarded as a right A-module and even as an A-A-bimodule. In particular, the complex $T^\bullet$ can be regarded as a complex of A-A-bimodules. In this sense, both $\widetilde{\pi}: P^\bullet \longrightarrow T$ and $\widetilde{\omega}: A \longrightarrow T^\bullet$ are quasi-isomorphisms of complexes of A-A-bimodules. Moreover, one can check that the chain maps $(\widetilde{\omega})^*$ and $(\widetilde{\pi})_*$ are quasi-isomorphisms in $\mathscr{C}(A^{\mathrm{op}})$. This implies that $\mathrm{Hom}_A(P^\bullet, A) \simeq \mathrm{Hom}_A(T, T^\bullet)$ in $\mathscr{D}(A^{\mathrm{op}})$. Note that $\mathrm{Hom}_A(P^\bullet, A) \otimes_A T_B \simeq \mathrm{Hom}_A(P^\bullet, A) \otimes^{\mathbb{L}}_A T_B$ in $\mathscr{D}(B^{\mathrm{op}})$ (see the above remark (2)). As a result, we have $\mathrm{Hom}_A(P^\bullet, A) \otimes_A T_B \simeq \mathrm{Hom}_A(T, T^\bullet) \otimes^{\mathbb{L}}_A T_B$ in $\mathscr{D}(B^{\mathrm{op}})$. $\square$

Proof of Corollary 1.2. (1) By the remark (3) at the end of the proof of Theorem 1.1, we know that if the homomorphism $\widetilde{\sigma}: \mathrm{Coker}(\varphi_0) \longrightarrow \mathrm{Coker}(\varphi_1)$ induced from $\sigma: P_1 \to P_0$ (see Theorem 1.1 (4)) is surjective, then $\mathrm{Ker}(_AT \otimes^{\mathbb{L}}_B -)$ is homological in $\mathscr{D}(B)$.

Now, we verify this sufficient condition for the good tilting module $_AT$ which satisfies the assumption in (1).

In fact, by assumption, we can assume that $_AM$ has a projective resolution: $0 \longrightarrow P'_1 \xrightarrow{\sigma'} P'_0 \longrightarrow {_AM} \longrightarrow 0$ with $P'_0, P'_1 \in \mathrm{Add}(_AA)$, and that $_AN$ has a projective presentation: $P''_1 \longrightarrow P''_0 \xrightarrow{\sigma''} {_AN} \longrightarrow 0$ with $P''_0 \in \mathrm{Add}(_AA)$ and $P''_1 \in \mathrm{add}(_AA)$. Since $_AT = M \oplus N$, we can choose $\sigma = \begin{pmatrix} \sigma' & 0 \\ 0 & \sigma'' \end{pmatrix} : P'_1 \oplus P''_1 \longrightarrow P'_0 \oplus P''_0$. Recall that $\zeta: \mathrm{Hom}_A(-, A) \otimes_A T \longrightarrow \mathrm{Hom}_A(-, T)$ is a natural transformation from A-Mod to B^{op}-Mod (see the proof of Theorem 1.1). Certainly, if $X \in \mathrm{add}(_AA)$, then ζ_X is an isomorphism, and so $\mathrm{Coker}(\zeta_X) = 0$.

Let $\widetilde{\sigma}': \mathrm{Coker}(\zeta_{P'_0}) \longrightarrow \mathrm{Coker}(\zeta_{P'_1})$ and $\widetilde{\sigma}'': \mathrm{Coker}(\zeta_{P''_0}) \longrightarrow \mathrm{Coker}(\zeta_{P''_1})$ be the homomorphisms induced from σ' and σ'', respectively. By definition, we have $\varphi_i = \zeta_{P_i}$ for $i = 0, 1$, and

$$\widetilde{\sigma} = \begin{pmatrix} \widetilde{\sigma}' & 0 \\ 0 & \widetilde{\sigma}'' \end{pmatrix} : \mathrm{Coker}(\zeta_{P'_0}) \oplus \mathrm{Coker}(\zeta_{P''_0}) \longrightarrow \mathrm{Coker}(\zeta_{P'_1}) \oplus \mathrm{Coker}(\zeta_{P''_1}).$$

Now, we show that $\widetilde{\sigma}$ is surjective, or equivalently, both $\widetilde{\sigma}'$ and $\widetilde{\sigma}''$ are surjective. In fact, since $P_1'' \in \mathrm{add}({}_AA)$, we see that $\mathrm{Coker}(\zeta_{P_1''}) = 0$. Thus $\widetilde{\sigma}''$ is surjective. As ${}_AM$ is a direct summand of ${}_AT$ and of projective dimension at most 1, it follows from the axiom $(T2)$ that the map $\mathrm{Hom}_A(\sigma', T) : \mathrm{Hom}_A(P_0', T) \longrightarrow \mathrm{Hom}_A(P_1', T)$ is surjective. This implies that $\widetilde{\sigma}'$ is a surjection. Consequently, $\widetilde{\sigma}$ is surjective. Thus $\mathrm{Ker}({}_AT \otimes_B^{\mathbb{L}} -)$ is homological in $\mathscr{D}(B)$. This finishes the proof of (1).

(2) Suppose that $\mathrm{Ker}({}_AT \otimes_B^{\mathbb{L}} -)$ in Theorem 1.1 is homological. By Theorem 1.1, we have $H^m\big(\mathrm{Hom}_A(P^\bullet, A) \otimes_A T_B\big) = 0$ for all $m \geq 2$. In the sequel, we shall show that if $H^n\big(\mathrm{Hom}_A(P^\bullet, A) \otimes_A T_B\big) = 0$, then $T_n = 0$.

In fact, since A is commutative, it follows from the proof of Lemma 5.6 that $\mathrm{Hom}_A(P^\bullet, A) \simeq \mathrm{Hom}_A(T, T^\bullet)$ in $\mathscr{D}(A^{\mathrm{op}})$. Note that the tensor functor $- \otimes_A T_B : A^{\mathrm{op}}\text{-Mod} \longrightarrow B^{\mathrm{op}}\text{-Mod}$ is right exact. This means that

$$0 = H^n\big(\mathrm{Hom}_A(P^\bullet, A) \otimes_A T_B\big) \simeq H^n(\mathrm{Hom}_A(P^\bullet, A)) \otimes_A T \simeq H^n(\mathrm{Hom}_A(T, T^\bullet)) \otimes_A T.$$

In particular, we have $H^n(\mathrm{Hom}_A(T_n, T^\bullet)) \otimes_A T_n = 0$, due to $T_n \in \mathrm{add}({}_AT)$.

Recall that the complex $\mathrm{Hom}_A(T_n, T^\bullet)$ is of the form

$$\cdots \longrightarrow 0 \longrightarrow \mathrm{Hom}_A(T_n, T_0) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_A(T_n, T_{n-1}) \longrightarrow \mathrm{Hom}_A(T_n, T_n) \longrightarrow 0 \longrightarrow \cdots$$

As $\mathrm{Hom}_A(T_n, T_{n-1}) = 0$ by our assumption in Corollary 1.2 (2), we obtain $H^n(\mathrm{Hom}_A(T_n, T^\bullet)) = \mathrm{Hom}_A(T_n, T_n)$. Thus $\mathrm{End}_A(T_n) \otimes_A T_n = 0$. It follows from the surjective map

$$\mathrm{End}_A(T_n) \otimes_A T_n \longrightarrow T_n, \ f \otimes x \mapsto (x)f \quad \text{for } f \in \mathrm{End}_A(T_n) \ \text{ and } \ x \in T_n$$

that $T_n = 0$. This finishes the proof of the above claim.

By our assumption, we have $\mathrm{Hom}_A(T_{i+1}, T_i) = 0$ for $1 \leq i \leq n-1$. Now, we can proceed by induction on n to show that $T_j = 0$ for $2 \leq j \leq n$. Thus, by Lemma 5.1 (4), T is a 1-tilting module, that is, the projective dimension of ${}_AT$ is at most 1.

The sufficiency of Corollary 1.2 (2) follows from Theorem 1.1, see also [11, Theorem 1.1 (1)]. This finishes the proof of Corollary 1.2. $\square$

Let us end this section by constructing an example of infinitely generated n-tilting modules T such that $\mathrm{Ker}(T \otimes_B^{\mathbb{L}} -)$ are homological.

Let A be an arbitrary ring with a classical n-tilting A-module T'. Suppose ${}_AT' = M \oplus N$ with M a nonzero A-module of projective dimension at most 1. Let I be an infinite set, and let $T := M^{(I)} \oplus N$. Then T is a good n-tilting module. Since T satisfies Corollary 1.2 (1), we see that $\mathrm{Ker}(T \otimes_B^{\mathbb{L}} -)$ is homological in $\mathscr{D}(B)$.

6 Applications to cotilting modules

Our main purpose in this section is to show Theorem 1.3 and develop some conditions which can be used to decide if subcategories induced from cotilting modules are homological or not. We also provide an example to show that recollements provided by cotilting modules depend upon the choice of injective cogenerators.

6.1 Proof of Theorem 1.3

In this section, we shall apply the results in Section 4 to deal with cotilting modules. First, we shall construct Ringel modules from good cotilting modules, and then use Proposition 4.4 to show the main result, Corollary 6.3, of this section, and finally give the proof of Theorem 1.3.

Suppose that A is a ring and that W is a fixed injective cogenerator for A-Mod. Recall that an A-module W is called a *cogenerator* for A-Mod if, for any A-module Y, there exists an injective homomorphism $Y \to W^I$ in A-Mod with I a set. This is also equivalent to saying that, for any non-zero homomorphism $f : X \to Y$ in A-Mod, there exists a homomorphism $g \in \mathrm{Hom}_A(Y, W)$ such that fg is non-zero.

Let us recall the definition of n-cotilting modules for n a natural number.

Definition 6.1. An A-module U is called an *n-cotilting module* if the following three conditions are satisfied:

$(C1)$ there exists an exact sequence

$$0 \longrightarrow U \longrightarrow I_0 \xrightarrow{\delta} I_1 \longrightarrow \cdots \longrightarrow I_n \longrightarrow 0$$

of A-modules such that I_i is an injective module for every $0 \le i \le n$;

$(C2)$ $\mathrm{Ext}_A^j(U^I, U) = 0$ for each $j \ge 1$ and for every nonempty set I; and

$(C3)$ there exists an exact sequence

$$0 \longrightarrow U_n \longrightarrow \cdots \longrightarrow U_1 \longrightarrow U_0 \longrightarrow W \longrightarrow 0$$

of A-modules, such that $U_i \in \mathrm{Prod}({}_AU)$ for all $0 \le i \le n$.

An n-cotilting A-module U is said to be *good* if it satisfies $(C1)$, $(C2)$ and

$(C3)'$ there is an exact sequence

$$0 \longrightarrow U_n \longrightarrow \cdots \longrightarrow U_1 \longrightarrow U_0 \longrightarrow W \longrightarrow 0$$

of A-modules, such that $U_i \in \mathrm{add}({}_AU)$ for all $0 \le i \le n$.

We say that U is a (good) cotilting A-module if ${}_AU$ is (good) n-cotilting for some $n \in \mathbb{N}$.

We remark that if both W_1 and W_2 are injective cogenerators for A-Mod, then $\mathrm{Prod}(W_1) = \mathrm{Prod}(W_2)$. This implies that the definition of cotilting modules is independent of the choice of injective cogenerators for A-Mod. However, the definition of good cotilting modules relies on the choice of injective cogenerators for A-Mod.

As in the case of tilting modules, for a given n-cotilting A-module U with $(C1)$-$(C3)$, the A-module $U' := \bigoplus_{i=0}^n U_i$ is a good n-cotilting module which is equivalent to the given one in the sense that $\mathrm{Prod}(U) = \mathrm{Prod}(U')$.

From now on, we assume that U is a **good** n-cotilting A-module with $(C_1), (C_2)$ and $(C_3)'$, where the module W in $(C_3)'$ is referred to the fixed injective cogenerator for A-Mod. In this event, we shall call U a *good n-cotilting A-module with respect to W*.

Let $R := \mathrm{End}_A(U)$, $M := \mathrm{Hom}_A(U, W)$ and $\Lambda := \mathrm{End}_A(W)$. Then M is an R-Λ-bimodule.

First of all, we collect some basic properties of good cotilting modules in the following lemma.

Lemma 6.2. *The following hold for the cotilting module U.*

(1) *The R-module M has a finitely generated projective resolution of length at most n:*

$$0 \longrightarrow \mathrm{Hom}_A(U, U_n) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_A(U, U_1) \longrightarrow \mathrm{Hom}_A(U, U_0) \longrightarrow M \longrightarrow 0$$

such that $U_m \in \mathrm{add}({}_AU)$ for all $0 \le m \le n$.

(2) *The Hom-functor $\mathrm{Hom}_A(U, -) : A\text{-Mod} \to R\text{-Mod}$ induces an isomorphism of rings: $\Lambda \simeq \mathrm{End}_R(M)$, and $\mathrm{Ext}_R^i(M, M) = 0$ for all $i \ge 1$.*

(3) *The module M is an n-Ringel R-module.*

Proof. (1) Applying the functor $\mathrm{Hom}_A(U, -)$ to the sequence

$$0 \longrightarrow U_n \longrightarrow \cdots \longrightarrow U_1 \longrightarrow U_0 \longrightarrow W \longrightarrow 0$$

in the axiom $(C_3)'$, we obtain the sequence in (1) with all $\mathrm{Hom}_A(U, U_i) \in \mathrm{add}({}_RR)$. The exactness of this sequence follows directly from the axiom $(C2)$. This also implies that the projective dimension of ${}_RM$ is at most n.

(2) Denote by Ψ the Hom-functor $\mathrm{Hom}_A(U, -) : A\text{-Mod} \to R\text{-Mod}$. Then $\Psi(U) = R$, $\Psi(W) = M$ and, for every $X \in \mathrm{add}({}_AU)$, we have

$$\mathrm{Hom}_A(X, W) \xrightarrow{\simeq} \mathrm{Hom}_R(\Psi(X), \Psi(W)).$$

Clearly, if $n = 0$, then $W = U_0$, $M = \mathrm{Hom}_A(U, U_0)$ as R-modules. In this case, one can easily check (2).

Suppose $n \ge 1$. By (1), the R-module $M = \Psi(W)$ has a finitely generated projective resolution

$$0 \longrightarrow \Psi(U_n) \longrightarrow \cdots \longrightarrow \Psi(U_1) \longrightarrow \Psi(U_0) \longrightarrow \Psi(W) \longrightarrow 0$$

with $U_m \in \mathrm{add}(U)$ for all $0 \le m \le n$. Applying the functor $\mathrm{Hom}_A(-, W)$ to the resolution of W in $(C3)'$, we can construct the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_A(W,W) \arrow[r] \arrow[d, "\Psi"] & \mathrm{Hom}_A(U_0,W) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Hom}_A(U_1,W) \arrow[r] \arrow[d, "\simeq"] & \cdots \arrow[r] & \mathrm{Hom}_A(U_n,W) \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_R(\Psi(W),\Psi(W)) \arrow[r] & \mathrm{Hom}_R(\Psi(U_0),\Psi(W)) \arrow[r] & \mathrm{Hom}_R(\Psi(U_1),\Psi(W)) \arrow[r] & \cdots \arrow[r] & \mathrm{Hom}_R(\Psi(U_n),\Psi(W)) \arrow[r] & 0
\end{tikzcd}$$

where the isomorphisms in the diagram are due to $U_m \in \mathrm{add}({}_AU)$ for $m \leq n$. Since ${}_AW$ is injective, the first row in the diagram is exact. Note that the following sequence

$$0 \longrightarrow \mathrm{Hom}_R(\Psi(W),\Psi(W)) \longrightarrow \mathrm{Hom}_R(\Psi(U_0),\Psi(W)) \longrightarrow \mathrm{Hom}_R(\Psi(U_1),\Psi(W))$$

is always exact since $\Psi(U_1) \longrightarrow \Psi(U_0) \longrightarrow \Psi(W) \longrightarrow 0$ is exact in R-Mod. This implies that the map $\Psi :$ $\mathrm{End}_A(W) \longrightarrow \mathrm{End}_R(\Psi(W))$ is an isomorphism of rings and that the second row in the diagram is also exact. Thus $\mathrm{Ext}^i_R(M,M) = \mathrm{Ext}^i_R(\Psi(W),\Psi(W)) = 0$ for all $i \geq 1$.

(3) We check the axioms $(R1)$-$(R3)$ in Definition 4.1 for M. Clearly, the axioms $(R1)$ and $(R2)$ follow from (1) and (2), respectively. It remains to show the axiom $(R3)$ for M. In fact, by the axiom $(C1)$, there exists an exact sequence of A-modules:

$$0 \longrightarrow U \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_n \longrightarrow 0$$

where I_i is an injective module for $0 \leq i \leq n$. Since W is an injective cogenerator for A-Mod, we have $I_i \in \mathrm{Prod}({}_AW)$. Moreover, from the axiom $(C2)$, we see that $\mathrm{Ext}^j_A(U,U) = 0$ for all $j \geq 1$. This implies that the following sequence

$$0 \longrightarrow R \longrightarrow \mathrm{Hom}_A(U,I_0) \longrightarrow \mathrm{Hom}_A(U,I_1) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_A(U,I_n) \longrightarrow 0$$

is exact. Since the functor $\mathrm{Hom}_A(U,-)$ commutes with arbitrary direct products, it follows from $I_i \in \mathrm{Prod}({}_AW)$ that $\mathrm{Hom}_A(U,I_i) \in \mathrm{Prod}\big({}_R\mathrm{Hom}_A(U,W)\big) = \mathrm{Prod}({}_RM)$. This shows that ${}_RM$ satisfies the axiom $(R3)$. Therefore M is an n-Ringel R-module. $\square$

Observe that, by Lemma 6.2 (2), the ring $\mathrm{End}_R(M)$ can be naturally identified with Λ (up to isomorphism of rings). Now, we define

$$\mathbf{G} := {}_RM \otimes^{\mathbb{L}}_{\Lambda} - : \ \mathscr{D}(\Lambda) \longrightarrow \mathscr{D}(R) \quad \text{and} \quad \mathbf{H} := \mathbb{R}\mathrm{Hom}_R(M,-) : \ \mathscr{D}(R) \longrightarrow \mathscr{D}(\Lambda).$$

Since ${}_RM$ is a Ringel R-module satisfying both $(R1)$ and $(R2)$ in Definition 4.1, it follows from Lemma 4.2 that there exists a recollement of triangulated categories:

$$\begin{tikzcd} \mathrm{Ker}(\mathbf{H}) \arrow[r, "i_*"] & \mathscr{D}(R) \arrow[l, bend right, "i^*"'] \arrow[l, bend left] \arrow[r, "\mathbf{H}"] & \mathscr{D}(\Lambda) \arrow[l, bend right, "\mathbf{G}"'] \arrow[l, bend left] \end{tikzcd}$$

where (i^*, i_*) is a pair of adjoint functors with i_* the inclusion.

If $\mathrm{Ker}(\mathbf{H})$ is homological, then it follows from Lemma 4.2 that the generalized localization $\lambda : R \rightarrow R_M$ of R at M exists and induces a recollement of derived module categories:

$$(\ddagger) \qquad \begin{tikzcd} \mathscr{D}(R_M) \arrow[r, "D(\lambda_*)"] & \mathscr{D}(R) \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "\mathbf{H}"] & \mathscr{D}(\Lambda) \arrow[l, bend right, "\mathbf{G}"'] \arrow[l, bend left] \end{tikzcd}$$

Thus we may construct recollements of derived module categories from good cotilting modules. Here, a problem arises naturally:

Problem: When is $\mathrm{Ker}(\mathbf{H})$ homological in $\mathscr{D}(R)$?

This seems to be a difficult problem because we cannot directly apply Proposition 4.4 to the Ringel module ${}_RM$. The reason is that we do not know whether ${}_RM$ is *good*. Actually, we do not know whether the right Λ-module M is strongly Λ-Mittag-Leffler. Certainly, if Λ is right noetherian, then M is a perfect Ringel R-module (see Definition 4.1), and must be good.

Though we cannot solve this problem entirely, we do have some partial solutions to the problem.

Corollary 6.3. *Suppose that A is a ring together with an injective cogenerator W for A-*Mod*. Let U be a good n-cotilting A-module with respect to W. Suppose that $\Lambda := \mathrm{End}_A(W)$ is a right noetherian ring. Then the following are equivalent:*

(a) $\mathrm{Ker}(\mathbf{H})$ *is homological in* $\mathscr{D}(R)$.

(b) *$H^m\big({}_R\mathrm{Hom}_A(U,W)\otimes_\Lambda \mathrm{Hom}_A(W,I^\bullet)\big)=0$ for all $m\geq 2$, where $I^\bullet$ is a deleted injective coresolution of ${}_AU$:*

$$\cdots\longrightarrow 0\longrightarrow I_0\xrightarrow{\delta} I_1\longrightarrow\cdots\longrightarrow I_n\longrightarrow 0\longrightarrow\cdots$$

with I_i in degree i for all $0\leq i\leq n$.

(c) *The kernel K of the homomorphism* $\mathrm{Coker}(\phi_0)\longrightarrow\mathrm{Coker}(\phi_1)$ *induced from the map $\delta: I_0\to I_1$ satisfies* $\mathrm{Ext}_R^m(M,K)=0$ *for all $m\geq 0$, where $\phi_i:\mathrm{Hom}_A(U,W)\otimes_\Lambda\mathrm{Hom}_A(W,I_i)\longrightarrow\mathrm{Hom}_A(U,I_i)$ is the composition map for $i=0,1$.*

Proof. By the proof of Lemma 6.2 (3), the module $M:=\mathrm{Hom}_A(U,W)$ is an n-Ringel R-module. Moreover, the sequence in the axiom $(R3)$ can be chosen as follows:

$$0\longrightarrow R\longrightarrow\mathrm{Hom}_A(U,I_0)\longrightarrow\mathrm{Hom}_A(U,I_1)\longrightarrow\cdots\longrightarrow\mathrm{Hom}_A(U,I_n)\longrightarrow 0.$$

In this case, the complex $M^\bullet$ can be defined as the following complex:

$$\mathrm{Hom}_A(U,I^\bullet):\ 0\longrightarrow\mathrm{Hom}_A(U,I_0)\longrightarrow\mathrm{Hom}_A(U,I_1)\longrightarrow\cdots\longrightarrow\mathrm{Hom}_A(U,I_n)\longrightarrow 0.$$

Under the assumption that Λ is right noetherian, we know that M is a good Ringel R-module. So it follows from Proposition 4.4 that (a) is equivalent to the following:

(b') $H^j\big({}_RM\otimes_\Lambda\mathrm{Hom}_R(M,M^\bullet)\big)=0$ for any $j\geq 2$, where $M^\bullet:=\mathrm{Hom}_A(U,I^\bullet)$.

To prove that (a) and (b) in Corollary 6.3 are equivalent, it is sufficient to show that (b') and (b) are equivalent. For this purpose, we shall show that $\mathrm{Hom}_R(M,M^\bullet)\simeq\mathrm{Hom}_A(W,I^\bullet)$ as complexes over Λ.

Let $\Psi=\mathrm{Hom}_A(U,-):A\text{-Mod}\to R\text{-Mod}$. Then $\Psi(W)=M$ and $M^\bullet=\Psi(I^\bullet)$. Clearly, the functor Ψ induces a natural transformation

$$\mathrm{Hom}_A(W,-)\longrightarrow\mathrm{Hom}_R(\Psi(W),\Psi(-)):A\text{-Mod}\longrightarrow\Lambda\text{-Mod}.$$

This yields a chain map from $\mathrm{Hom}_A(W,I^\bullet)\longrightarrow\mathrm{Hom}_R(\Psi(W),\Psi(I^\bullet))=\mathrm{Hom}_R(M,M^\bullet)$ in $\mathscr{C}(\Lambda)$, that is,

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_A(W,I_0) \arrow[r]\arrow[d] & \mathrm{Hom}_A(W,I_1) \arrow[r]\arrow[d] & \cdots \arrow[r] & \mathrm{Hom}_A(W,I_n) \arrow[r]\arrow[d] & 0\\
0 \arrow[r] & \mathrm{Hom}_R(\Psi(W),\Psi(I_0)) \arrow[r] & \mathrm{Hom}_R(\Psi(W),\Psi(I_1)) \arrow[r] & \cdots \arrow[r] & \mathrm{Hom}_R(\Psi(W),\Psi(I_n)) \arrow[r] & 0
\end{tikzcd}$$

Note that all I_i are injective A-modules. To verify that this chain map is an isomorphism of complexes, it is enough to show that Ψ induces an isomorphism of Λ-modules:

$$\mathrm{Hom}_A(W,X)\xrightarrow{\simeq}\mathrm{Hom}_R(\Psi(W),\Psi(X))$$

for any injective A-module X. In the following, we shall prove that this holds even for any A-module X.

Suppose $n=0$. By the axiom $(C3)'$, we know that $W=U_0$ as A-modules with $U_0\in\mathrm{add}({}_AU)$. It is clear that $\mathrm{Hom}_A(U_0,X)\xrightarrow{\simeq}\mathrm{Hom}_R(\Psi(U_0),\Psi(X))$ since $U_0\in\mathrm{add}({}_AU)$. Thus $\mathrm{Hom}_A(W,X)\xrightarrow{\simeq}\mathrm{Hom}_R(\Psi(W),\Psi(X))$.

Now, suppose $n\geq 1$. By the axiom $(C3)'$ and Lemma 6.2 (1), there exists an exact sequence $U_1\longrightarrow U_0\longrightarrow W\longrightarrow 0$ of A-modules with $U_0,U_1\in\mathrm{add}({}_AU)$ such that $\Psi(U_1)\longrightarrow\Psi(U_0)\longrightarrow\Psi(W)\longrightarrow 0$ is also exact in R-Mod. From this sequence, we may construct the following exact commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_A(W,X) \arrow[r]\arrow[d, "\Psi"] & \mathrm{Hom}_A(U_0,X) \arrow[r]\arrow[d, "\simeq"] & \mathrm{Hom}_A(U_1,X)\arrow[d, "\simeq"]\\
0 \arrow[r] & \mathrm{Hom}_R(\Psi(W),\Psi(X)) \arrow[r] & \mathrm{Hom}_R(\Psi(U_0),\Psi(X)) \arrow[r] & \mathrm{Hom}_R(\Psi(U_1),\Psi(X))
\end{tikzcd}$$

where the last two vertical maps are isomorphisms since $U_0,U_1\in\mathrm{add}({}_AU)$. This means that $\mathrm{Hom}_A(W,X)\xrightarrow{\simeq}\mathrm{Hom}_R(\Psi(W),\Psi(X))$ for every A-module X.

Consequently, we see that $\mathrm{Hom}_A(W,I^\bullet)\simeq\mathrm{Hom}_R(M,M^\bullet)$ as complexes over Λ. Thus (b') and (b), and therefore, also (a) and (b), are equivalent.

Note that if we identify $\mathrm{Hom}_R(M,M^\bullet)$ with $\mathrm{Hom}_A(W,I^\bullet)$ as complexes over Λ, then the equivalence of (a) and (c) in Corollary 6.3 can be concluded from that of (1) and (4) in Proposition 4.4. Here, we leave the details to the reader. $\square$

As a consequence of Corollary 6.3 (see also Corollary 4.5), we have the following result.

Corollary 6.4. *Let U be a good n-cotilting A-module with respect to the injective cogenerator ${}_AW$. Suppose that $\Lambda := \mathrm{End}_A(W)$ is a right noetherian ring.*

(1) *If ${}_AU = M \oplus N$ such that ${}_AM$ has injective dimension at most* 1 *and that ${}_AN$ has an injective copresentation* $0 \longrightarrow {}_AN \longrightarrow E_0 \longrightarrow E_1$ *with $E_1 \in \mathrm{add}({}_AW)$, then* $\mathrm{Ker}(\mathbf{H})$ *is homological in $\mathscr{D}(R)$.*

(2) *If $n = 2$, then* $\mathrm{Ker}(\mathbf{H})$ *is homological in $\mathscr{D}(R)$ if and only if* $\mathrm{Hom}_A(U,W) \otimes_\Lambda \mathrm{Ext}^2_A(W,U) = 0$.

Proof. The idea of the proof of (1) is very similar to that of Corollary 1.2 (1). Here, we just give a sketch of the proof.

Note that $\mathscr{E} := \{Y \in R\text{-Mod} \mid \mathrm{Ext}^m_R(M,Y) = 0 \text{ for all } m \geq 0\}$ is closed under kernels of surjective homomorphisms in R-Mod, and that $\mathrm{Coker}(\phi_0)$ and $\mathrm{Coker}(\phi_1)$ (see Corollary 6.3 (c)) always belong to $\mathscr{E}$ by Lemma 4.3 (1). Thus, according to the equivalence of (a) and (c) in Corollary 6.3, if we want to show (1), then it suffices to verify that the homomorphism $\widetilde{\delta} : \mathrm{Coker}(\phi_0) \longrightarrow \mathrm{Coker}(\phi_1)$ induced from $\delta : I_0 \to I_1$ is surjective. Actually, this is guaranteed by the assumption that the injective dimension of ${}_AM$ is at most 1 and $E_1 \in \mathrm{add}({}_AW)$. For more details, we refer the reader to the proof of Corollary 1.2 (1).

As to (2), we keep the notation in the proof of Corollary 6.3. Suppose $n = 2$. Then the complex $I^\bullet$ in Corollary 6.3 (b) has the following form

$$\cdots \longrightarrow 0 \longrightarrow I_0 \longrightarrow I_1 \longrightarrow I_2 \longrightarrow 0 \longrightarrow \cdots.$$

By Corollary 6.3, the category $\mathrm{Ker}(\mathbf{H})$ is homological if and only if $H^2\big({}_RM \otimes_\Lambda \mathrm{Hom}_A(W, I^\bullet)\big) = 0$, where $M := \mathrm{Hom}_A(U,W)$. Note that the tensor functor ${}_RM \otimes_\Lambda - : \Lambda\text{-Mod} \longrightarrow R\text{-Mod}$ is right exact. Consequently, we have

$$H^2\big({}_RM \otimes_\Lambda \mathrm{Hom}_A(W, I^\bullet)\big) \simeq M \otimes_\Lambda H^2(\mathrm{Hom}_A(W, I^\bullet)) \simeq M \otimes_\Lambda \mathrm{Ext}^2_A(W,U).$$

This shows (2). $\square$

Finally, we point out a special case for which the ring Λ in Corollary 6.3 is right noetherian.

Let k be a commutative Artin ring. Let $\mathrm{rad}(k)$ be the radical of k (that is, the intersection of all maximal ideals of k), and let J be the injective envelope of $k/\mathrm{rad}(k)$. We say that a k-algebra A is an *Artin k-algebra*, or Artin algebra for short, if A is finitely generated as a k-module.

Suppose that A is an Artin k-algebra. It is well known that the functor $\mathrm{Hom}_k(-,J)$ is a duality between the category A-mod of finitely generated A-modules and that of finitely generated A^{op}-modules. In particular, the dual module $\mathrm{Hom}_k(A_A,J)$ of the right A-module A_A is an injective cogenerator for A-mod, or even for A-Mod. In this case, we shall call $\mathrm{Hom}_k(A_A,J)$ the *ordinary injective cogenerator* for A-Mod.

Note that $\mathrm{End}_A(\mathrm{Hom}_k(A_A,J)) \simeq \mathrm{End}_{A^{\mathrm{op}}}(A)^{\mathrm{op}} \simeq A$ as rings. So, if the module W in Corollary 6.3 is chosen to be the module $\mathrm{Hom}_k(A_A,J)$, then the ring $\Lambda := \mathrm{End}_A(W)$ is isomorphic to A. Since A is an Artin algebra, it is a left and right Artin ring, and certainly a right noetherian ring. Thus Λ is right noetherian and always satisfies the assumption in Corollary 6.3.

Proof of Theorem 1.3. Recall that ${}_AW$ is the ordinary injective cogenerator over the Artin algebra A. According to the above-mentioned facts, the ring $\Lambda := \mathrm{End}_A(W)$ is isomorphic to A, and therefore right noetherian. Since ${}_AU$ is a good 1-cotilting module with respect to W, we know from Corollary 6.4 (1) that the category $\mathrm{Ker}(\mathbf{H})$ is homological. Now, Theorem 1.3 follows from the diagram (‡) above Corollary 6.3. $\square$

Let us end this section by a couple of remarks related to the results in this section.

Remarks. (1) If A is a commutative ring and W is an injective cogenerator for A-Mod, then the dual module $\mathrm{Hom}_A(T,W)$ of a tilting A-module T is always a cotilting A-module. However, there exist cotilting modules over Prüfer domains, which are not equivalent to the dual modules of any tilting modules (see [1, Chapter 11, Section 4.16]). This means that the investigation of infinitely generated cotilting modules cannot be carried out by using dual arguments of infinitely generated tilting modules.

(2) Corollary 6.3 provides actually a recollement of $\mathscr{D}(\mathrm{End}_A(U))$ with $\mathscr{D}(R_M)$ on the left-hand side and $\mathscr{D}(\Lambda)$ on the right-hand side (see (‡) for notation). This recollement depends upon the choice of injective cogenerators for A-Mod. That is, for a fixed cotilting module ${}_AU$, if different injective cogenerators W for A-Mod are chosen in the axiom $(C3)'$, then one may get completely different recollements of $\mathscr{D}(\mathrm{End}_A(U))$.

For example, let $\mathbb{Q}_{(p)}$, $\mathbb{Q}$, $\mathbb{Z}_p$ and $\mathbb{Q}_p$ denote the rings of p-integers, rational numbers, p-adic integers and p-adic numbers, respectively. Recall that $\mathbb{Q}_{(p)}$ is the localization of $\mathbb{Z}$ at the prime ideal $p\mathbb{Z}$. In particular, it is a local Dedekind domain. Moreover, let $E(\mathbb{Z}/p\mathbb{Z})$ be the injective envelope of $\mathbb{Z}/p\mathbb{Z}$, which is an injective cogenerator for the category of $\mathbb{Q}_{(p)}$-modules.

Now, we take $A := \mathbb{Q}_{(p)}$, $T := \mathbb{Q} \oplus E(\mathbb{Z}/p\mathbb{Z})$ and $U := \mathrm{Hom}_A(T, E(\mathbb{Z}/p\mathbb{Z}))$. Due to [11, Section 7.1], we have

(a) the module T is a Bass 1-tilting module over A, and therefore U is an 1-cotilting A-module.

(b) $\mathrm{End}_A(E(\mathbb{Z}/p\mathbb{Z})) \simeq \mathbb{Z}_p$ and $\mathrm{Hom}_A(\mathbb{Q}, E(\mathbb{Z}/p\mathbb{Z})) \simeq \mathbb{Q} \otimes_A \mathrm{End}_A(E(\mathbb{Z}/p\mathbb{Z})) \simeq \mathbb{Q} \otimes_A \mathbb{Z}_p \simeq \mathbb{Q}_p$. Thus $U \simeq \mathbb{Z}_p \oplus \mathbb{Q}_p$ as A-modules.

(c) By [11, Lemma 6.5(3)], there exists an exact sequence of $\mathbb{Z}_p$-modules (and also A-modules):

$$(*') \quad 0 \longrightarrow \mathbb{Z}_p \xrightarrow{\varphi} \mathbb{Q}_p \longrightarrow E(\mathbb{Z}/p\mathbb{Z}) \longrightarrow 0.$$

Note that $\mathbb{Q}_p$ is an injective and flat A-module and that $(*')$ is an injective coresolution of $\mathbb{Z}_p$ as an A-module. This also implies that $W := \mathbb{Q}_p \oplus E(\mathbb{Z}/p\mathbb{Z})$ is an injective cogenerator for A-Mod.

On the one hand, we may consider U as a good 1-cotilting A-module with respect to W. Applying $\mathrm{Hom}_A(U, -)$ to the sequence $(*')$, we get a projective resolution of $\mathrm{Hom}_A(U, E(\mathbb{Z}/p\mathbb{Z}))$ as an $\mathrm{End}_A(U)$-module:

$$0 \longrightarrow \mathrm{Hom}_A(U, \mathbb{Z}_p) \xrightarrow{\varphi^*} \mathrm{Hom}_A(U, \mathbb{Q}_p) \longrightarrow \mathrm{Hom}_A(U, E(\mathbb{Z}/p\mathbb{Z})) \longrightarrow 0.$$

Since both $\mathbb{Q}_p$ and $E(\mathbb{Z}/p\mathbb{Z})$ belong to $\mathrm{add}({}_AW)$, one can use Lemma 6.2 to show that $\mathrm{Hom}_A(U, W)$ is a classical 1-tilting $\mathrm{End}_A(U)$-module such that $\mathrm{End}_{\mathrm{End}_A(U)}(\mathrm{Hom}_A(U, W)) \simeq \mathrm{End}_A(W)$ as rings. It follows that $\mathrm{End}_A(U)$ and $\mathrm{End}_A(W)$ are derived equivalent. In this case, we get a trivial recollement: $\mathscr{D}(\mathrm{End}_A(U)) \xrightarrow{\simeq} \mathscr{D}(\Lambda)$ with $\Lambda := \mathrm{End}_A(W)$. Note that this derived equivalence can also be seen from [20, Theorem 1.1].

On the other hand, we consider U as a good 1-cotilting A-module with respect to $W' := E(\mathbb{Z}/p\mathbb{Z})$. Clearly, the sequence $(*')$ can paly the role in the axiom $(C3)'$. Since $\mathrm{End}_A(E(\mathbb{Z}/p\mathbb{Z})) \simeq \mathbb{Z}_p$, we know from [16, Corollary 2.5.16] that $\mathrm{End}_A(E(\mathbb{Z}/p\mathbb{Z}))$ is a noetherian ring. This implies that U satisfies the assumptions in Corollary 6.4 (1).

By [16, Theorem 3.4.1], one can check that

$$\mathrm{End}_A(\mathbb{Z}_p) \simeq \mathbb{Z}_p,\ \mathrm{Hom}_A(\mathbb{Q}_p, \mathbb{Z}_p) = 0 = \mathrm{Ext}^1_A(\mathbb{Q}_p, \mathbb{Z}_p) = \mathrm{Hom}_A(E(\mathbb{Z}/p\mathbb{Z}), \mathbb{Q}_p),$$

and further that

$$\mathrm{End}_A(U) \simeq \begin{pmatrix} \mathbb{Z}_p & \mathrm{End}_A(\mathbb{Q}_p) \\ 0 & \mathrm{End}_A(\mathbb{Q}_p) \end{pmatrix} \text{ and } \mathrm{End}_A(W) \simeq \begin{pmatrix} \mathrm{End}_A(\mathbb{Q}_p) & \mathrm{End}_A(\mathbb{Q}_p) \\ 0 & \mathbb{Z}_p \end{pmatrix}.$$

Moreover, the universal localization of $\mathrm{End}_A(U)$ at the map φ^*, or at the module $\mathrm{Hom}_A(U, E(\mathbb{Z}/p\mathbb{Z}))$, is isomorphic to $M_2(\mathrm{End}_A(\mathbb{Q}_p))$, the 2×2 matrix ring over $\mathrm{End}_A(\mathbb{Q}_p)$.

Now, we can construct the following non-trivial recollement of derived module categories from the cotilting module U with respect to $W' = E(\mathbb{Z}/p\mathbb{Z})$:

$$\begin{tikzcd} \mathscr{D}(\mathrm{End}_A(\mathbb{Q}_p)) \arrow[r] & \mathscr{D}(\mathrm{End}_A(U)) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}(\mathbb{Z}_p) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

Thus, the recollement $(\ddagger)$ above Corollary 6.3 constructed from a cotilting module U depends on injective cogenerator with respect to which the U is defined.

6.2 Necessary conditions of homological subcategories from cotilting modules

We keep the notation in Section 6.1. For the cotilting module U, we denote by

$$0 \longrightarrow U_n \xrightarrow{\partial_n} U_{n-1} \longrightarrow \cdots \xrightarrow{\partial_2} U_1 \xrightarrow{\partial_1} U_0 \xrightarrow{\partial_0} W \longrightarrow 0$$

the exact sequence in the axiom $(C_3)'$, and by $U^\bullet$ the following complex

$$\cdots \longrightarrow 0 \longrightarrow U_n \xrightarrow{\partial_n} U_{n-1} \longrightarrow \cdots \xrightarrow{\partial_2} U_1 \xrightarrow{\partial_1} U_0 \longrightarrow 0 \longrightarrow \cdots$$

with U_i in degree $-i$ for all $0 \leq i \leq n$. Then ∂_0 induces a canonical quasi-isomorphism $\widetilde{\partial}_0 : U^\bullet \to W$ in $\mathscr{C}(A)$. Recall that the complex $I^\bullet$ in Corollary 6.3 (b) also yields a canonical quasi-isomorphism $\xi : U \to I^\bullet$ in $\mathscr{C}(A)$.

Furthermore, by the proof of the first part of Lemma 5.6, one can show that $\widetilde{\partial}_0$ and ξ do induce the following quasi-isomorphisms

$$(*) \qquad \mathrm{Hom}_A(W, I^\bullet) \xrightarrow{(\widetilde{\partial}_0)_*} \mathrm{Hom}^\bullet_A(U^\bullet, I^\bullet) \xleftarrow{\xi^*} \mathrm{Hom}_A(U^\bullet, U)$$

in $\mathscr{C}(\mathbb{Z})$. Here, we leave checking the details to the reader.

Consequently, the morphism $(\widetilde{\partial}_0)_*(\xi^*)^{-1}: \mathrm{Hom}_A(W,I^\bullet) \longrightarrow \mathrm{Hom}_A(U^\bullet,U)$ in $\mathscr{D}(\mathbb{Z})$ is an isomorphism (compare with Lemma 5.6). Due to the A-Λ-bimodule structure of W, the former complex belongs to $\mathscr{C}(\Lambda)$. However, the latter complex might not be a complex of Λ-modules since $U^\bullet$ is not necessarily a complex of A-Λ-bimodules in general. This means that this isomorphism may not be extended to an isomorphism in $\mathscr{D}(\Lambda)$. Nonetheless, for some special cotilting modules, we do have this isomorphism in $\mathscr{D}(\Lambda)$. For instance, in the case described in the following lemma.

Lemma 6.5. *Suppose that* $\mathrm{Hom}_A(U_i,U_{i+1}) = 0$ *for* $0 \le i < n$.

(1) *There exist a series of ring homomorphisms* $\rho_j : \Lambda \longrightarrow \mathrm{End}_A(U_j)$ *for* $0 \le j \le n$*, such that* $\widetilde{\partial}_0 : U^\bullet \longrightarrow W$ *is a quasi-isomorphism in* $\mathscr{C}(A \otimes_{\mathbb{Z}} \Lambda^{\mathrm{op}})$*. In particular, the complexes* $\mathrm{Hom}_A(W,I^\bullet)$ *and* $\mathrm{Hom}_A(U^\bullet,U)$ *are isomorphic in* $\mathscr{D}(\Lambda)$.

(2) *If* $\mathrm{Ext}^k_A(W,U_k) = \mathrm{Ext}^{k+1}_A(W,U_k) = 0$ *for all* $0 \le k < n$*, then* $\rho_n : \Lambda \longrightarrow \mathrm{End}_A(U_n)$ *is an isomorphism.*

Proof. (1) Set $K_0 := W$, $K_n := U_n$ and $K_m := \mathrm{Ker}(\partial_{m-1})$ for $1 \le m < n$. Then, for each $0 \le i < n$, we have a short exact sequence $0 \longrightarrow K_{i+1} \longrightarrow U_i \xrightarrow{\partial_i} K_i \longrightarrow 0$ of A-modules. In the following, we shall define two ring homomorphisms $\varphi_i : \mathrm{End}_A(K_i) \longrightarrow \mathrm{End}_A(U_i)$ and $\psi_i : \mathrm{End}_A(K_i) \longrightarrow \mathrm{End}_A(K_{i+1})$.

By Lemma 6.2 (1), the sequence

$$0 \longrightarrow \mathrm{Hom}_A(U,K_{i+1}) \longrightarrow \mathrm{Hom}_A(U,U_i) \xrightarrow{\partial_i^*} \mathrm{Hom}_A(U,K_i) \longrightarrow 0$$

is exact. In particular, for $U_i \in \mathrm{add}(U)$, the sequence

$$0 \longrightarrow \mathrm{Hom}_A(U_i,K_{i+1}) \longrightarrow \mathrm{Hom}_A(U_i,U_i) \xrightarrow{\partial_i^*} \mathrm{Hom}_A(U_i,K_i) \longrightarrow 0$$

is exact. Let $f \in \mathrm{End}_A(K_i)$. Then there is a homomorphism $g \in \mathrm{End}_A(U_i)$ such that $\partial_i f = g\partial_i$. We claim that such a g is unique. Actually, if there exists another $g' \in \mathrm{End}_A(U_i)$ such that $\partial_i f = g'\partial_i$. Then $(g-g')\partial_i = 0$, and so the map $g-g'$ factorizes through K_{i+1}. Note that each homomorphism $U_i \to K_{i+1}$ also factorizes through U_{i+1} via ∂_{i+1}. This implies that $g-g' : U_i \to U_i$ factorizes through U_{i+1}. However, since $\mathrm{Hom}_A(U_i,U_{i+1}) = 0$ by assumption, we have $g = g'$. Hence, for a given f, such a g is unique.

Now, we define $\varphi_i : f \mapsto g$ and $\psi_i : f \mapsto h$ where h is the restriction of g to K_{i+1}. This can be illustrated by the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & K_{i+1} \arrow[r, "\lambda_{i+1}"] \arrow[d, dashed, "h"] & U_i \arrow[r, "\partial_i"] \arrow[d, dashed, "g"] & K_i \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & K_{i+1} \arrow[r, "\lambda_{i+1}"] & U_i \arrow[r, "\partial_i"] & K_i \arrow[r] & 0
\end{tikzcd}$$

where λ_{i+1} is the inclusion for $0 \le i \le n-2$ and $\lambda_n := \partial_n$. Clearly, both φ_i and ψ_i are ring homomorphisms.

Recall that $\Lambda := \mathrm{End}_A(W) = \mathrm{End}_A(K_0)$. Furthermore, for $0 \le j \le n$, we define $\rho_j : \Lambda \to \mathrm{End}_A(U_j)$ as follows: If $j = 0$, then $\rho_0 := \varphi_0$; if $j \ge 1$, then ρ_j is defined to be the composite of the following ring homomorphisms:

$$\Lambda \xrightarrow{\psi_0} \mathrm{End}_A(K_1) \xrightarrow{\psi_1} \mathrm{End}_A(K_2) \longrightarrow \cdots \longrightarrow \mathrm{End}_A(K_{j-1}) \xrightarrow{\psi_{j-1}} \mathrm{End}_A(K_j) \xrightarrow{\varphi_j} \mathrm{End}_A(U_j)$$

where φ_n stands for the identity map. By definition, for each $\lambda \in \Lambda$, there exists an exact commutative diagram of A-modules:

$$\begin{tikzcd}
0 \arrow[r] & U_n \arrow[r, "\partial_n"] \arrow[d, "(\lambda)\rho_n"'] & U_{n-1} \arrow[r] \arrow[d, "(\lambda)\rho_{n-1}"] & \cdots \arrow[r, "\partial_2"] & U_1 \arrow[r, "\partial_1"] \arrow[d, "(\lambda)\rho_1"] & U_0 \arrow[r, "\partial_0"] \arrow[d, "(\lambda)\rho_0"] & W \arrow[r] \arrow[d, "\lambda"] & 0 \\
0 \arrow[r] & U_n \arrow[r, "\partial_n"] & U_{n-1} \arrow[r] & \cdots \arrow[r, "\partial_2"] & U_1 \arrow[r, "\partial_1"] & U_0 \arrow[r, "\partial_0"] & W \arrow[r] & 0
\end{tikzcd}$$

Note that U_j is a natural A-$\mathrm{End}_A(U_j)$-bimodule and can be regarded as an A-Λ-bimodule via ρ_j. It follows from the above commutative diagram that ∂_j is a homomorphism of A-Λ-bimodules. This implies that $\widetilde{\partial}_0 : U^\bullet \longrightarrow W$ can be viewed as a quasi-isomorphism in $\mathscr{C}(A \otimes_{\mathbb{Z}} \Lambda^{op})$. In this sense, the quasi-isomorphisms in $(*)$ actually belong to $\mathscr{C}(\Lambda)$. Thus $\mathrm{Hom}_A(W,I^\bullet)$ and $\mathrm{Hom}_A(U^\bullet,U)$ are isomorphic in $\mathscr{D}(\Lambda)$. This finishes (1).

(2) To show that ρ_n is an isomorphism of rings, it suffices to prove that ψ_i is an isomorphism for $0 \le i \le n-1$. Let i be such a fixed number. If $\mathrm{Hom}_A(K_i, U_i) = 0$, then ψ_i is injective. If the induced map $(\lambda_{i+1})_* : \mathrm{Hom}_A(U_i, U_i) \longrightarrow \mathrm{Hom}_A(K_{i+1}, U_i)$ is surjective, then so is ψ_i. Thus, by our assumptions in (2), to show that ψ_i is an isomorphism, it suffices to show that $\mathrm{Hom}_A(K_i, U_i) \simeq \mathrm{Ext}^i_A(W, U_i)$ and that there exists an exact sequence of abelian groups:

$$(**) \quad \mathrm{Hom}_A(U_i, U_i) \xrightarrow{(\lambda_{i+1})_*} \mathrm{Hom}_A(K_{i+1}, U_i) \longrightarrow \mathrm{Ext}^{i+1}_A(W, U_i) \longrightarrow 0.$$

In fact, since $U_s \in \mathrm{add}({}_AU)$ for $0 \le s \le n$, we have $\mathrm{Ext}^r_A(U_s, X) = 0$ for each $r \ge 1$ and $X \in \mathrm{add}({}_AU)$ by the axiom $(C2)$. Now, for $1 \le j \le n$ and $X \in \mathrm{add}({}_AU)$, one can apply $\mathrm{Hom}_A(-, X)$ to the long exact sequence

$$0 \longrightarrow K_j \xrightarrow{\lambda_j} U_{j-1} \longrightarrow \cdots \longrightarrow U_1 \longrightarrow U_0 \longrightarrow W \longrightarrow 0,$$

and get an exact sequence $\mathrm{Hom}_A(U_{j-1}, X) \xrightarrow{(\lambda_j)_*} \mathrm{Hom}_A(K_j, X) \longrightarrow \mathrm{Ext}^j_A(W, X) \longrightarrow 0$ of abelian groups. If we take $j := i$ and $X := U_i$, then $\mathrm{Hom}_A(K_i, U_i) \simeq \mathrm{Ext}^i_A(W, U_i)$ since $\mathrm{Hom}_A(U_{i-1}, U_i) = 0$ by assumption. If we take $j := i+1$ and $X := U_i$, then we get the required sequence $(**)$. This finishes the proof of (2). $\square$

The following result will be used for getting a counterexample which demonstrates that, in general, the category $\mathrm{Ker}(\mathbf{H})$ in Corollary 6.3 may not be homological.

Corollary 6.6. *Keep all the assumptions in Corollary 6.3. Further, suppose that $n \ge 2$ and U has injective dimension exactly equal to n. If $\mathrm{Hom}_A(U_i, U_{i+1}) = \mathrm{Ext}^i_A(W, U_i) = \mathrm{Ext}^{i+1}_A(W, U_i) = 0$ for all $0 \le i < n$, then the category $\mathrm{Ker}(\mathbf{H})$ is not a homological subcategory of $\mathscr{D}(R)$.*

Proof. Suppose contrarily that $\mathrm{Ker}(\mathbf{H})$ is homological in $\mathscr{D}(R)$. Then, by Corollary 6.3, we certainly have $H^n\big({}_R\mathrm{Hom}_A(U, W) \otimes_\Lambda \mathrm{Hom}_A(W, I^\bullet)\big) = 0$. Furthermore, since $\mathrm{Hom}_A(U_i, U_{i+1}) = 0$ for all $0 \le i \le n-1$, we know from Lemma 6.5 (1) that $\mathrm{Hom}_A(W, I^\bullet) \simeq \mathrm{Hom}_A(U^\bullet, U)$ in $\mathscr{D}(\Lambda)$. Thus

$$0 = H^n\big(\mathrm{Hom}_A(U, W) \otimes_\Lambda \mathrm{Hom}_A(W, I^\bullet)\big) \simeq \mathrm{Hom}_A(U, W) \otimes_\Lambda H^n\big(\mathrm{Hom}_A(W, I^\bullet)\big) \simeq \mathrm{Hom}_A(U, W) \otimes_\Lambda H^n\big(\mathrm{Hom}_A(U^\bullet, U)\big).$$

In particular, we have $\mathrm{Hom}_A(U, W) \otimes_\Lambda H^n\big(\mathrm{Hom}_A(U^\bullet, U_n)\big) = 0$, due to $U_n \in \mathrm{add}({}_AU)$. Recall that the complex $\mathrm{Hom}_A(U^\bullet, U_n)$ is of the form

$$0 \longrightarrow \mathrm{Hom}_A(U_0, U) \xrightarrow{(\partial_1)_*} \mathrm{Hom}_A(U_1, U_n) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_A(U_{n-2}, U_n) \xrightarrow{(\partial_{n-1})_*} \mathrm{Hom}_A(U_{n-1}, U_n) \xrightarrow{(\partial_n)_*} \mathrm{Hom}_A(U_n, U_n) \longrightarrow 0$$

with $\mathrm{Hom}_A(U_n, U_n)$ in degree n. Since $\mathrm{Hom}_A(U_{n-1}, U_n) = 0$, we obtain $H^n\big(\mathrm{Hom}_A(U^\bullet, U_n)\big) = \mathrm{End}_A(U_n)$, and so $\mathrm{Hom}_A(U, W) \otimes_\Lambda \mathrm{End}_A(U_n) = 0$. Note that the left Λ-module structure of $\mathrm{End}_A(U_n)$ is defined by the ring homomorphism $\rho_n : \Lambda \longrightarrow \mathrm{End}_A(U_n)$ (see Lemma 6.5 (1)). Since $\mathrm{Ext}^i_A(W, U_i) = \mathrm{Ext}^{i+1}_A(W, U_i) = 0$ for all $0 \le i \le n-1$, we see from Lemma 6.5 (2) that ρ_n is an isomorphism. This implies that

$$\mathrm{Hom}_A(U, W) \otimes_\Lambda \mathrm{End}_A(U_n) \simeq \mathrm{Hom}_A(U, W) \otimes_\Lambda \Lambda \simeq \mathrm{Hom}_A(U, W)$$

and therefore $\mathrm{Hom}_A(U, W) = 0$. Since ${}_AW$ is an injective cogenerator, we must have $U = 0$. This is a contradiction. Thus $\mathrm{Ker}(\mathbf{H})$ is not homological in $\mathscr{D}(R)$. $\square$

7 Counterexamples and open questions

In this section, we shall apply results in the previous sections to give two examples which show that, in general, the category $\mathrm{Ker}({}_AT \otimes^{\mathbb{L}}_B -)$ for an n-tilting module T, or the category $\mathrm{Ker}(\mathbf{H})$ for an n-cotilting module U may not be homological. At the end of this section, we mention a few open questions related to some results in this paper.

Throughout this section, we assume that A is a commutative, noetherian, n-Gorensteion ring for a natural number n. Recall that a ring is called *n-Gorenstein* if the injective dimensions of the regular left and right modules are at most n.

For an A-module M, we denote by $E(M)$ its injective envelope. It is known that if $\mathfrak{p}$ and $\mathfrak{q}$ are two prime ideals of A, then $\mathrm{Hom}_A(E(A/\mathfrak{p}), E(A/\mathfrak{q})) \neq 0$ if and only if $\mathfrak{p} \subseteq \mathfrak{q}$ (see [16, Theorem 3.3.8]). In particular, $E(A/\mathfrak{p}) \simeq E(A/\mathfrak{q})$ if and only if $\mathfrak{p} = \mathfrak{q}$

7.1 Higher n-tilting modules

In the following, we shall apply Corollary 1.2 to provide an example of a good n-tilting A-module T for which the category $\mathrm{Ker}({}_AT\otimes^{\mathbb{L}}_B -)$ in Theorem 1.1 is not homological.

For the n-Gorenstein ring A, it follows from a classical result of Bass that the regular module ${}_AA$ has a minimal injective coresolution of the form:

$$0 \longrightarrow A \longrightarrow \bigoplus_{\mathfrak{p}\in\mathscr{P}_0} E(A/\mathfrak{p}) \longrightarrow \cdots \longrightarrow \bigoplus_{\mathfrak{p}\in\mathscr{P}_n} E(A/\mathfrak{p}) \longrightarrow 0,$$

where $\mathscr{P}_i$ stands for the set of all prime ideals of A with height i (see [4, Theorem 1, Theorem 6.2]). It was pointed out in [24, Introduction] that the A-module

$$T := \bigoplus_{0\leq i\leq n}\ \bigoplus_{\mathfrak{p}\in\mathscr{P}_i} E(A/\mathfrak{p})$$

is an (infinitely generated) n-tilting module.

Clearly, the tilting module ${}_AT$ is good if we define $T_i := \bigoplus_{\mathfrak{p}\in\mathscr{P}_i} E(A/\mathfrak{p})$. Observe that, for $0\leq i<j\leq n$, we have $\mathrm{Hom}_A(E(A/\mathfrak{p}),E(A/\mathfrak{q}))=0$ for $\mathfrak{p}\in P_j$ and $\mathfrak{q}\in P_i$, and therefore $\mathrm{Hom}_A(T_j,T_i)=0$.

Now, we suppose that $n\geq 2$ and the injective dimension of A is exactly equal to n (or equivalently, the Krull dimension of A is exactly n).

Note that $T_i\neq 0$ for all $2\leq i\leq n$ and that T satisfies the assumptions in Corollary 1.2 (2). Since the above injective coresolution of A is minimal, the module ${}_AT$ has projective dimension equal to n (see [5, Proposition 3.5]). By Corollary 1.2 (2), the category $\mathrm{Ker}({}_AT\otimes^{\mathbb{L}}_B -)$ is not homological in $\mathscr{D}(B)$. This means that for this tilting module T, the subcategory $\mathrm{Ker}({}_AT\otimes^{\mathbb{L}}_B -)$ cannot be realized as the derived module category $\mathscr{D}(C)$ of a ring C with a homological ring epimorphism $B\to C$. Thus, for higher n-tilting modules, the answer to the question in Introduction is negative in general.

7.2 Higher n-cotilting modules

Next, we apply Corollary 6.6 to present an example of a good n-cotilting A-module U, for which the category $\mathrm{Ker}(\mathbf{H})$ in Corollary 6.3 is not homological in $\mathscr{D}(R)$.

Assume further that the ring A is *local* with the unique maximal ideal $\mathfrak{m}$. In this case, T_n is an injective cogenerator for A-Mod since $\mathscr{P}_n$ is just the set $\{\mathfrak{m}\}$. This follows from a general statement in commutative algebra: If S is a commutative noetherian ring, then $\bigoplus_{\mathfrak{m}} E(S/\mathfrak{m})$ is an injective cogenerator for S-Mod, where $\mathfrak{m}$ runs over all maximal ideals of S.

Now, we take

$$W := T_n \quad\text{and}\quad U := \mathrm{Hom}_A(T,W) = \bigoplus_{j=0}^{n}\mathrm{Hom}_A(T_j,W).$$

Since ${}_AT$ is an n-tilting A-module, the module ${}_AU$ is an n-cotilting A-module. Furthermore, applying $\mathrm{Hom}_A(-,W)$ to the minimal injective coresolution of ${}_AA$, we get the following exact sequence of A-modules:

$$0\longrightarrow \mathrm{Hom}_A(T_n,W)\longrightarrow \mathrm{Hom}_A(T_{n-1},W)\longrightarrow\cdots\longrightarrow \mathrm{Hom}_A(T_1,W)\longrightarrow \mathrm{Hom}_A(T_0,W)\longrightarrow W\longrightarrow 0.$$

This implies that the cotilting A-module U is good if we define $U_j := \mathrm{Hom}_A(T_j,W)$ for $0\leq j\leq n$ (see the axiom $(C_3)'$ in Definition 6.1).

To see that $\Lambda := \mathrm{End}_A(W)$ is a right noetherian ring, we note that $W=E(A/\mathfrak{m})$ and that Λ is isomorphic to the $\mathfrak{m}$-adic complete of A (see [16, Theorem 3.4.1 (6)]). Since A is noetherian, the ring Λ is also noetherian (see [16, Corollary 2.5.16]).

In the following, we shall prove that ${}_AU$ satisfies all the assumptions in Corollary 6.6. In fact, it suffices to show that, for any $m\geq 0$, we have

(a) $\mathrm{Ext}^m_A(U_r,U_s)=0$ for $0\leq r<s\leq n$.

(b) $\mathrm{Ext}^m_A(W,U_i)=0$ for $0\leq i\leq n-1$, and $\mathrm{Ext}^n_A(W,U_n)\neq 0$.

The reason is the following: According to (b), the injective dimension of U_n is at least n, and therefore exactly n. This means that ${}_AU$ is a cotilting module of injective dimension n. Moreover, from (a) and (b) we can conclude that the assumptions in Corollary 6.6 hold true for U. It then follows from Corollary 6.6 that, for this cotilting module

U, the category $\mathrm{Ker}(\mathbf{H})$ in Corollary 6.3 is not homological in $\mathscr{D}(R)$ with $R := \mathrm{End}_A(U)$. In other words, $\mathrm{Ker}(\mathbf{H})$ cannot be realized as the derived module category $\mathscr{D}(S)$ of a ring S with a homological ring epimorphism $R \to S$.

So, let us verify the above (a) and (b). First, we need the following results about n-Gorenstein rings:

(1) The flat dimension of the A-module T_j is exactly j.

(2) Any flat A-module F admits a minimal injective coresolution of the form

$$0 \longrightarrow {}_AF \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_{n-1} \longrightarrow I_n \longrightarrow 0$$

such that $I_j \in \mathrm{Add}(T_j)$ for all $0 \le j \le n$.

(3) Let $\mathfrak{p}$ and $\mathfrak{q}$ be prime ideals of A. If $\mathfrak{p} \nsubseteq \mathfrak{q}$ or $\mathfrak{q} \nsubseteq \mathfrak{p}$, then $\mathrm{Tor}_m^A(E(A/\mathfrak{p}), E(A/\mathfrak{q})) = 0$ for all $m \ge 0$. Moreover, $\mathrm{Tor}_m^A(E(A/\mathfrak{p}), E(A/\mathfrak{p})) \neq 0$ if and only if m equals the height of $\mathfrak{p}$ in A.

Here, (1) and (2) follow from [26, Proposition 2.1 and Theorem 2.1], while (3) is taken from [16, Lemma 9.4.5 and Theorem 9.4.6].

Since the dual A-module $\mathrm{Hom}_A(F,W)$ of a flat A-module F is injective, we know from (1) that the injective dimension of ${}_AU_j$ is at most j. Since the dual A-module $\mathrm{Hom}_A(I,W)$ of an injective A-module I is always flat (see [16, Corollary 3.2.16 (2)]), we see that the A-module U_j is flat since T_j is injective. It then follows from (2) that U_j admits a minimal injective coresolution of the form

$$0 \longrightarrow U_j \longrightarrow I_{j,0} \longrightarrow I_{j,1} \longrightarrow \cdots \longrightarrow I_{j,j-1} \longrightarrow I_{j,j} \longrightarrow 0$$

with $I_{j,k} \in \mathrm{Add}(T_k)$ for all $0 \le k \le j$.

Now, we show (a). Actually, by Lemma 2.3 (1), we have

$$\mathrm{Ext}_A^m(U_r, U_s) = \mathrm{Ext}_A^m\big(U_r, \mathrm{Hom}_A(T_s, W)\big) \simeq \mathrm{Hom}_A\big(\mathrm{Tor}_m^A(T_s, U_r), W\big) \quad \text{for} \quad m \ge 0.$$

Note that the flatness of U_r implies that $\mathrm{Ext}_A^m(U_r, U_s) = 0$ for $m \ge 1$. It remains to show $\mathrm{Hom}_A(U_r, U_s) = 0$. For this aim, it is sufficient to show $T_s \otimes_A U_r = 0$. Since $T_s := \bigoplus_{\mathfrak{p} \in \mathscr{P}_s} E(A/\mathfrak{p})$ and the functor $- \otimes_A U_r$ commutes with arbitrary direct sums, we have to prove $E(A/\mathfrak{p}) \otimes_A U_r = 0$ for every $\mathfrak{p} \in \mathscr{P}_s$. In fact, since $r < s$ by assumption, we know that $\mathfrak{p} \nsubseteq \mathfrak{q}$ for each $\mathfrak{q} \in \mathscr{P}_k$ with $0 \le k \le r$. It follows from (3) that $\mathrm{Tor}_j^A\big(E(A/\mathfrak{p}), E(A/\mathfrak{q})\big) = 0$ for all $j \ge 0$, and therefore

$$\mathrm{Tor}_j^A(E(A/\mathfrak{p}), T_k) \simeq \bigoplus_{\mathfrak{q} \in \mathscr{P}_k} \mathrm{Tor}_j^A\big(E(A/\mathfrak{p}), E(A/\mathfrak{q})\big) = 0.$$

Since $I_{r,k} \in \mathrm{Add}(T_k)$, we obtain $\mathrm{Tor}_j^A(E(A/\mathfrak{p}), I_{r,k}) = 0$ for all $j \ge 0$. Now, by applying the tensor functor $E(A/\mathfrak{p}) \otimes_A -$ to the minimal injective coresolution of U_r, we can prove $E(A/\mathfrak{p}) \otimes_A U_r = 0$. Thus $T_s \otimes_A U_r = 0$. This finishes the proof of (a).

Finally, we show (b). Let $0 \le i \le n-1$. Recall that $U_i = \mathrm{Hom}_A(T_i, W)$. According to Lemma 2.3 (1), we have

$$\mathrm{Ext}_A^m\big(W, \mathrm{Hom}_A(T_i, W)\big) \simeq \mathrm{Hom}_A\big(\mathrm{Tor}_m^A(T_i, W), W\big) \simeq \mathrm{Hom}_A\Big(\bigoplus_{\mathfrak{p} \in \mathscr{P}_i} \mathrm{Tor}_m^A(E(A/\mathfrak{p}), W), W\Big) \simeq \prod_{\mathfrak{p} \in \mathscr{P}_i} \mathrm{Hom}_A\big(\mathrm{Tor}_m^A(E(A/\mathfrak{p}), W), W\big).$$

Since the ideal $\mathfrak{m}$ is maximal (or of height n), it holds that $\mathfrak{m} \nsubseteq \mathfrak{p}$ for every $\mathfrak{p} \in \mathscr{P}_i$. Hence it follows from (3) that $\mathrm{Tor}_m^A(E(A/\mathfrak{p}), W) = 0$, and therefore $\mathrm{Ext}_A^m(W, U_i) = 0$. Similarly, one can show that

$$\mathrm{Ext}_A^n(W, U_n) = \mathrm{Ext}_A^n(W, \mathrm{Hom}_A(W, W)) \simeq \mathrm{Hom}_A(\mathrm{Tor}_n^A(W, W), W).$$

Since $\mathrm{Tor}_n^A(W, W) = \mathrm{Tor}_n^A\big(E(A/\mathfrak{m}), E(A/\mathfrak{m})\big) \neq 0$ by (3) and since ${}_AW$ is an injective cogenerator, we infer that $\mathrm{Ext}_A^n(W, U_n) \neq 0$. Thus (b) follows.

Consequently, for the n-cotilting A-module U, the subcategory $\mathrm{Ker}(\mathbf{H})$ is not homological in $\mathscr{D}(R)$.

Let us end this paper by the following open questions related to our results in this note.

Question 1. Let A be a ring with identity. Is there a good n-tilting A-module T for $n \ge 2$ such that T is not equivalent to any classical tilting A-module and that $\mathrm{Ker}(T \otimes_B^{\mathbb{L}} -)$ is homological?

Question 2. Is the converse of Corollary 1.2 (1) always true?

For tilting modules over commutative noetherian n-Gorenstein rings, Silvana Bazzoni even guesses a stronger answer: If $\mathrm{Ker}(T \otimes_B^{\mathbb{L}} -)$ is homological in $\mathscr{D}(B)$, then ${}_AT$ should be a 1-tilting module, that is, the module ${}_AN$ in Corollary 1.2 (1) should be zero.

Question 3. Given a good 1-cotilting module U over an **arbitrary ring** A, is there a homological ring epimorphism $\lambda : \mathrm{End}_A(U) \to C$ and a recollement of the following form?

$$\begin{tikzcd} \mathscr{D}(C) \arrow[r, "D(\lambda_*)"] & \mathscr{D}(\mathrm{End}_A(U)) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}(A) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

Note that this reccollement does not involve the derived categories of the endomorphism rings of any injective cogenerators related to U.

Question 4. Given an arbitrary ring A, how to parameterize homological subcategories of $\mathscr{D}(A)$? Equivalently, how to classify homological ring epimorphisms starting from A?

Question 5. Is the Ringel R-module M in Lemma 6.2 always good?

A positive answer to this question would lead to a generalization of Corollary 6.3.

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