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Derived equivalences of algebras

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Abstract

Derived categories and equivalences between them are the pièce de résistance of modern homological algebra. They are widely used in many branches of mathematics, especially in algebraic geometry and representation theory. In this note, we shall survey some recently developed construction methods of derived equivalences for algebras and rings, with applications to homological conjectures, such as Broué's abelian defect group conjecture and the finitistic dimension conjecture, and to computation of higher algebraic K-groups of algebras and rings.

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1. *Introduction*

Derived categories (or more generally, triangulated categories) and derived equivalences, introduced by Grothendieck around the 1960s and developed substantially by his student Verdier in 1977 (see [**121**]), have nowadays connections with many branches of mathematics and physics, from algebraic geometry, representation theory of groups and algebras to mirror symmetry in string theory [**23**, **65**, **78**, **99**]. In representation theory, Rickard's Morita theory for derived categories of rings (see [**108**, **110**]) and Keller's Morita theory for differential graded algebras (see [**71**, **73**]) provide powerful tools to understand derived module categories and equivalences of both rings and differential graded rings. However, the following fundamental question in the study of derived categories and derived equivalences still remains:

Main question: How can we construct derived equivalences for algebras?

Of course, by Rickard's Morita theory for derived categories of rings [**108**], this question is reduced to the question of both finding tilting complexes and determining their endomorphism rings. But the latter seems quite difficult to be realized in practice. This can be seen from one open problem, namely the famous Broué's abelian defect group conjecture in the modular representation theory of finite groups, which says that the module categories of a block algebra A of a finite group algebra and its Brauer correspondent B should have equivalent derived categories if their common defect group is abelian (see [**111**]). This conjecture is considered as one of the central, but also hardest, problems in the modular representation theory of finite groups. Though this conjecture is verified for symmetric groups by Chuang and Rouquier in

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[34] and for many other cases (for instance, see [79, 85, 93, 98, 127]), it seems to be far away from being solved completely. From this conjecture one may get a little flavor of the difficulty about finding suitable tilting complexes and determining their endomorphism algebras. For further information on the developments of Broué's abelian defect group conjecture, we refer the reader to [116].

In the last few decades, there are many attempts made toward constructions of a class of or all tilting complexes and derived equivalences for special algebras. For example, for self-injective algebras with radical-square zero, every tilting complex is a shift of a free module. For representation-finite, standard self-injective algebras, a derived classification was presented in [5]. For Brauer tree algebras, a lot of tilting complexes were constructed in [113] (see also the references therein). For preprojective algebras of Dynkin type, a complete determination of tilting complexes was given in [1]. Also, from some idempotent ideals, tilting complexes were constructed and their endomorphism rings were described (see, for example, [56, 57, 80, 129]). There are also individual efforts toward constructing new derived equivalences from given ones by applying operations on algebras. For instance, Rickard used tensor products and trivial extensions to produce new derived equivalences in [108, 109], Barot and Lenzing employed one-point extensions to transfer a derived equivalence to a new one in [11]. Regrettably, I cannot pursue all references here.

In this note we shall survey some general methods for constructions of derived equivalences of algebras. We mainly concentrate on the following aspects:

(1) Given a kind of short 'exact' sequences, we construct derived equivalences of the endomorphism rings of objects involved in the sequences. This includes constructing derived equivalences from almost split sequences (see [8] for definition) and certain Nakayama-stable idempotent elements. The construction reveals actually an intrinsic connection among almost split sequences, BB-tilting modules and derived equivalences. In this aspect, we also present Dugas' and Grant's methods for constructing derived equivalences of symmetric algebras.

(2) Given a derived (or stable) equivalence of algebras, we construct (or lift it to) a new derived equivalence of resulting algebras, according to information of the given equivalences. This includes lifting a stable equivalence of Morita type to a derived equivalence, passing to quotient algebras and extending a derived equivalence between corner algebras to a derived equivalence between given algebras.

(3) Construction of tilting complexes over pullback algebras through the ones over their constituent algebras. This includes a kind of gluing idempotent elements of derived equivalent algebras.

Finally, we survey advances in applications of derived equivalences to Broué's abelian defect group conjecture and the finitistic dimension conjecture, and to calculations of higher algebraic K-groups of matrix subrings.

The note is organized as follows: In Section 2, we fix some notation and recall necessary definitions needed in the paper. In Section 3, we recall briefly the history of developments about derived categories and equivalences applied in the representation theory of algebras. In particular, we quote Rickard's theorem on derived equivalences for rings and display some derived invariants. In this section, we also mention two conjectures: Broué's abelian defect group conjecture and the Bondal–Orlov conjecture. In Section 4, we present a variety of methods for constructing titling complexes and derived equivalences of algebras and rings. In particular, we construct derived equivalences between the endomorphism rings of objects involved in short exact sequences, including almost split sequences and certain triangles. We restate Hoshino–Kato's construction of tilting complexes using ν-stable idempotents. Also, we present constructions of derived equivalences for Auslander–Yoneda algebras by using almost ν-stable derived equivalences and by passing to quotient algebras. In Section 5, we survey methods of lifting stable equivalences of Morita type to derived equivalences for Frobenius-finite algebras. Also, we extend derived equivalences between smaller algebras of the form eAe with e an idempotent in an algebra A to derived equivalences between the whole algebras themselves. Finally, we mention constructions of derived equivalences for tensor products given by Ladkani in [**80**] and for Milnor squares by Hu-Xi in [**62**]. In Section 6, we mention three applications of derived equivalences of algebras and rings. This includes an approach to Broué's abelian defect group conjecture and to the finitistic dimension conjecture on finite-dimensional algebras, and to higher algebraic K-groups of subrings of matrix rings. In this section, derived equivalences between subrings of matrix rings are also given.

2. *Triangulated categories and equivalences*

We briefly recall some definitions and fix notation needed in this paper.

2.1. *Notation*

Let $\mathcal{C}$ be an additive category.

By a subcategory of $\mathcal{C}$ we mean a full subcategory $\mathcal{B}$ of $\mathcal{C}$ closed under isomorphisms, that is, if $X \in \mathcal{B}$ and $Y \in \mathcal{C}$ with $Y \simeq X$, then $Y \in \mathcal{B}$.

Given two morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}$, we denote the composition of f with g by $fg : X \to Z$. The induced morphisms $\mathrm{Hom}_{\mathcal{C}}(Z, f) : \mathrm{Hom}_{\mathcal{C}}(Z, X) \to \mathrm{Hom}_{\mathcal{C}}(Z, Y)$ and $\mathrm{Hom}_{\mathcal{C}}(f, Z) : \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$ are denoted by f^* and f_*, respectively. As in [**48**], the composition of a functor $F : \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ with another functor $G : \mathcal{D} \to \mathcal{E}$ between categories $\mathcal{D}$ and $\mathcal{E}$ is denoted by GF which is a functor from $\mathcal{C}$ to $\mathcal{E}$.

If M is an object of $\mathcal{C}$, we denote by $\mathrm{add}(M)$ the full subcategory of $\mathcal{C}$ consisting of objects that are direct summands of direct sums of finitely many copies of M, and by $\mathrm{End}_{\mathcal{C}}(M)$ the endomorphism ring of M.

A complex $X^\bullet = (X^i, d_X^i)_{i\in\mathbb{Z}}$ over $\mathcal{C}$ is a sequence of objects X^i in $\mathcal{C}$ with morphisms $d_X^i : X^i \to X^{i+1}$ such that $d_X^i d_X^{i+1} = 0$ for all $i \in \mathbb{Z}$. These d_X^i are called the *differentials* of $X^\bullet$. A complex $X^\bullet = (X^i, d_X^i)$ over $\mathcal{C}$ is called a *radical complex* if all differentials d_X^i are radical morphisms. Recall that a morphism $f : X \to Y$ in $\mathcal{C}$ is called a *radical morphism* if, for any object Z and morphisms $h : Z \to X$ and $g : Y \to Z$ in $\mathcal{C}$, the composition hfg is not an isomorphism.

As usual, a morphism $f^\bullet : X^\bullet \to Y^\bullet$ between two complexes $X^\bullet$ and $Y^\bullet$ is a family $f^\bullet = (f^i : X^i \to Y^i)_{i\in\mathbb{Z}}$ of morphisms in $\mathcal{C}$ such that $f^i d_Y^i = d_X^i f^{i+1}$ for all $i \in \mathbb{Z}$. Note that morphisms between complexes are also called cochain maps in the literature. Two morphisms can be composed degreewise. Let $\mathscr{C}(\mathcal{C})$ be the category of all complexes over $\mathcal{C}$ with cochain maps, and $\mathscr{K}(\mathcal{C})$ the homotopy category of $\mathscr{C}(\mathcal{C})$. If $\mathcal{C}$ is an abelian category, we denote by $\mathscr{D}(\mathcal{C})$ the derived category of $\mathcal{C}$ which is, by definition, the localization of $\mathscr{K}(\mathcal{C})$ at all quasi-isomorphisms. It is well known that both $\mathscr{K}(\mathcal{C})$ and $\mathscr{D}(\mathcal{C})$ are triangulated categories (see the next section for definition).

We denote by $\mathscr{C}^b(\mathcal{C})$, $\mathscr{C}^+(\mathcal{C})$ and $\mathscr{C}^-(\mathcal{C})$ the categories of bounded, lower-bounded and upper bounded complexes over $\mathcal{C}$, respectively. Similarly, we have the categories $\mathscr{K}^{\mathrm{b}}(\mathcal{C})$, $\mathscr{K}^+(\mathcal{C})$ and $\mathscr{K}^-(\mathcal{C})$ as well as $\mathscr{D}^{\mathrm{b}}(\mathcal{C})$, $\mathscr{D}^+(\mathcal{C})$ and $\mathscr{D}^-(\mathcal{C})$.

Now, we fix some terminology on approximations in the sense of Auslander–Smalø.

Let $\mathcal{D}$ be a subcategory of $\mathcal{C}$, and X be an object in $\mathcal{C}$. Recall that a morphism $f : D \to X$ in $\mathcal{C}$ is called a *right $\mathcal{D}$-approximation* of X if $D \in \mathcal{D}$ and the induced map $f^* = \mathrm{Hom}_{\mathcal{C}}(-, f)$: $\mathrm{Hom}_{\mathcal{C}}(D', D) \to \mathrm{Hom}_{\mathcal{C}}(D', X)$ is surjective for every object $D' \in \mathcal{D}$. A morphism $f : X \to Y$ in $\mathcal{C}$ is said to be *right minimal* if any morphism $g : X \to X$ with $gf = f$ is an automorphism. A minimal right $\mathcal{D}$-approximation of X is a right $\mathcal{D}$-approximation of X, which is right minimal. Dually, there are the notions of *left $\mathcal{D}$-approximations* and *minimal left $\mathcal{D}$-approximations*. The subcategory $\mathcal{D}$ is said to be *functorially finite* in $\mathcal{C}$ if every object in $\mathcal{C}$ has a right and left $\mathcal{D}$-approximation. Note that the right and minimal right approximations were termed precover and cover, respectively, in ring theory.

2.2. *Triangulated categories and derived functors*

The original impulse to develop the 'derived' formalism came from the need to find a suitable formulation of Grothendieck's coherent duality theory. Let us recall the definition of triangulated categories.

Again, let $\mathcal{C}$ be an additive category. Suppose that $[1]$ is an automorphism of $\mathcal{C}$. The automorphism $[1]$ is usually called a *shift functor* of $\mathcal{C}$. A sextuple (X, Y, Z, u, v, w) in $\mathcal{C}$ is given by objects $X, Y, Z \in \mathcal{C}$ and morphisms $u : X \to Y, v : Y \to Z$ and $w : Z \to X[1]$. A morphism of sextuples from (X, Y, Z, u, v, w) to (X', Y', Z', u', v', w') is a triple $(f : X \to X', g : Y \to Y', h : Z \to Z')$ of morphisms in $\mathcal{C}$ such that $fu' = ug, gv' = vh, w(f[1]) = hw'$.

If in this situation f, g and h are isomorphisms in $\mathcal{C}$, then the morphism of the sextuple is called an *isomorphism*.

Definition 2.1 [**121**]. An additive category $\mathcal{C}$ with a shift functor $[1]$ and a set Σ of sextuples is called a triangulated category if Σ satisfies the following conditions. The elements of Σ are then called triangles.

(TR1)
- (a) Every morphism $u : X \to Y$ in $\mathcal{C}$ can be embedded into a triangle (X, Y, Z, u, v, w).
- (b) The sextuple $(X, X, 0, 1_X, 0, 0)$ is a triangle, where 1_X denotes the identity morphism from X to itself.
- (c) Every sextuple isomorphic to a triangle is a triangle.

(TR2) If (X, Y, Z, u, v, w) is a triangle, then $(Y, Z, X[1], v, w, -u[1])$ is a triangle.

(TR3) Given two triangles (X, Y, Z, u, v, w) and (X', Y', Z', u', v', W') and morphisms $f : X \to X', g : Y \to Y'$ such that $fu' = ug$, there exists a morphism (f, g, h) from the first triangle to the second.

(TR4) (The octahedral axiom) Given triangles $(X, Y, Z', u, i, i'), (Y, Z, X', v, j, j')$ and $(X, Z, Y', u.v, k, k')$, there exist morphisms $f : Z' \to Y', g : Y' \to X'$ such that the following diagram commutes and the third row is a triangle.

$$\begin{tikzcd}
Y'[-1] \arrow[r, "{k'[-1]}"] \arrow[d, "{g[-1]}"] & X \arrow[r, equal] \arrow[d, "u"] & X \arrow[d, "uv"] & & \\
X'[-1] \arrow[r, "{j'[-1]}"] & Y \arrow[r, "v"] \arrow[d, "i"'] & Z \arrow[r, "j"] \arrow[d, "k"'] & X' \arrow[r, "j'"] \arrow[d, equal] & Y[1] \arrow[d, "{i[1]}"] \\
& Z' \arrow[r, "f"] \arrow[d, "i'"] & Y' \arrow[r, "g"] \arrow[d, "k'"] & X' \arrow[r, "{j'(i[1])}"] & Z'[1] \\
& X[1] \arrow[r, equal] & X[1] & &
\end{tikzcd}$$

Throughout the note, for a triangulated category, its shift functor is denoted by $[1]$ universally.

To compare one triangulated category with another triangulated category, we also have the notion of 'exact' functors.

Definition 2.2. (1) Let $\mathcal{T} = (\mathcal{T}, \Sigma)$ and $\mathcal{T}' = (\mathcal{T}', \Sigma')$ be two triangulated categories. An additive functor $F : \mathcal{T} \to \mathcal{T}'$ between $\mathcal{T}$ and $\mathcal{T}'$ is called a triangle functor if there is an invertible natural transformation $\alpha : F[1] \to [1]F$ such that $(FX, FY, FX, Fu, Fv, (Fw)\alpha_X,) \in \Sigma'$ whenever $(X, Y, Z, u, v, w) \in \Sigma$.

(2) Two triangulated categories $\mathcal{T}$ and $\mathcal{T}'$ are said to be triangle-equivalent if there is a triangle functor $F : \mathcal{T} \to \mathcal{T}'$ which is an equivalence of categories. In this case, we say that F is a triangle equivalence, or $\mathcal{T}$ and $\mathcal{T}'$ are triangle-equivalent or equivalent as triangulated categories.

If F is a triangle equivalence, then its quasi-inverse is also a triangle equivalence, and F sends triangles to triangles. Triangle functors between triangulated categories may have some surprising properties. For example, any full triangle functor between triangulated categories is faithful so long as it does not take any non-zero object to zero (see [**108**]). For further information on triangulated categories and derived categories, we refer the reader to [**55**] or to the original book of Verdier [**121**]. Also, Hartshorne [**53**], Beilinson, Bernstein and Deligne [**15**] and Iversen [**66**] give introductions to these concepts.

A general method to get triangulated categories is to form quotients of Frobenius categories by their projective objects (an exact category is Frobenius if it has enough injectives and enough projectives and the two classes coincide). Here, the shift functor is just the Heller operator and the triangles are constructed by using the Heller operator on the quotient category. For more details of this construction, we refer the reader to [**48**] where all techniques are presented. Such a class of triangulated categories was called *algebraic* triangulated categories in [**76**]. As an example of Frobenius categories, we mention that the category of finite-dimensional A-modules over a self-injective algebra A is a Frobenius category. Recall that a k-algebra A over a field k is said to be *self-injective* if the regular left module is injective. Another general way to obtain triangulated categories is from algebraic topology. These are triangulated categories which are equivalent to full triangulated subcategories of the homotopy categories of stable model categories. Such triangulated categories were termed *topological* and may not be algebraic [**118**]. There are also triangulated categories which are neither algebraic nor topological (see [**94**]).

The notion of algebraic triangulated categories is somewhat relevant to the axiom (TR3). Note that the morphism h in (TR3) is not unique relative to a given pair (f, g) with $fu' = ug$. Even for the homotopy category $\mathscr{K}(\mathcal{A})$ of an additive category $\mathcal{A}$, Neeman pointed out in [**95**] that, given a pair (f, g) between distinguished triangles (that is, triangles induced from mapping cones of $u : X \to Y$ in $\mathscr{K}(\mathcal{A})$), there is a natural set of choices for the third map h in $\mathscr{K}(\mathcal{A})$, which is closed under addition and composition. The construction of these naturally good completions uses the fact that $\mathscr{K}(\mathcal{A})$ is a quotient of the category of complexes and cochain maps. Since an algebraic triangulated category is the quotient $\underline{\mathcal{B}}$ of a Frobenius category $\mathcal{B}$, the triangle induced from a morphism $\underline{u} : X \to Y$ in $\underline{\mathcal{B}}$ can be described explicitly by the pushout of a morphism $u : X \to Y$ and an injective 'envelope' $i_X : X \to I(X)$ of X in $\mathcal{B}$. Following the idea of Neeman in [**95**], Dugas constructed a unique good completion h for each pair (f, g) in an algebraic triangulated category $\mathcal{T}$. For more details, we refer to [**40**, Section 3; **95**]. The uniqueness of h relative to a pair (f, g) gives a way to identify the endomorphism ring of the two-term complex $X \xrightarrow{u} Y$ of objects in $\mathcal{T}$ with the endomorphism ring of the object Z in the triangle (X, Y, Z, u, v, w).

Examples of algebraic triangulated categories are homotopy categories and derived categories of schemes and abelian categories. Keller showed in [**76**] that differential graded categories are a source of algebraic triangulated categories. Namely homotopy and derived categories of modules over differential graded algebras and differential graded categories are algebraic triangulated categories. We refer the reader to [**71**] for further details on differential graded categories and derived categories over them. By a theorem of Porta [**103**, Theorem 1.2], every algebraic triangulated category which is well generated in the sense of [96, Definition 1.15, p. 15] is equivalent to a localization of the derived category of a small differential graded category. An analogous result of Porta's theorem for topological triangulated categories has recently been proved by Heider in [54].

2.3. *Derived categories of rings*

In this note, we are mainly interested in a special class of algebraic triangulated categories, namely derived categories of abelian categories or more restrictively the derived categories of the module categories over algebras and rings. Since they are the main objects in this note, we recall a few details of derived module categories.

All rings considered in the note are associative and with identity, and all ring homomorphisms preserve identity. Unless stated otherwise, modules are referred to left modules.

Let R be a ring. We denote by R-Mod the category of all unitary R-modules. By our convention of the composition of two morphisms, if $f : M \to N$ is a homomorphism of R-modules, then the image of $x \in M$ under f is denoted by $(x)f$ instead of $f(x)$. Thus $\mathrm{Hom}_A(M, N)$ is naturally an $\mathrm{End}_A(M)$-$\mathrm{End}_A(N)$-bimodule.

Let R-mod be the full subcategory of R-Mod consisting of all finitely generated R-modules. We denote by R-$\underline{\mathrm{mod}}$ the stable module category of R-mod. By definition, R-$\underline{\mathrm{mod}}$ has the same objects as R-mod, but the hom-sets $\underline{\mathrm{Hom}}_R(X, Y)$ are given by the quotients of $\mathrm{Hom}_R(X, Y)$ modulo those morphisms that factorize through a projective R-module.

Two finite-dimensional algebras A and B over a fixed field (or generally, two Artin algebras A and B) are said to be *stably equivalent* if their stable module categories A-$\underline{\mathrm{mod}}$ and B-$\underline{\mathrm{mod}}$ are equivalent. By an Artin algebra we mean an algebra A over a commutative Artin ring k such that A is a finitely generated k-module.

We shall simply write $\mathscr{C}(R)$ for the categories of all complexes of R-modules. Let $\mathscr{K}(R)$ and $\mathscr{D}(R)$ be the homotopy and derived categories of R-Mod, respectively. The module category R-Mod is fully embedded into $\mathscr{D}(R)$ by considering modules as stalk complexes concentrated in degree zero.

Note that $\mathscr{C}(R)$ is an abelian category and that $\mathscr{K}(R)$ and $\mathscr{D}(R)$ are triangulated categories. Similarly, one has the triangulated categories $\mathscr{D}^{\mathrm{b}}(R)$, $\mathscr{D}^{+}(R)$ and $\mathscr{D}^{-}(R)$. The triangles in these categories are given as follows. If $(X^\bullet, d_X^\bullet)$ and $(Y^\bullet, d_Y^\bullet)$ are two complexes over R-Mod, then the mapping cone of a morphism $h^\bullet : X^\bullet \to Y^\bullet$ of complexes, denoted by $\mathrm{Con}(h^\bullet)$, gives rise to a sextuple

$$X^\bullet \xrightarrow{h^\bullet} Y^\bullet \longrightarrow \mathrm{Con}(h^\bullet) \longrightarrow X^\bullet[1]$$

in $\mathscr{K}(R)$, called a *distinguished triangle*. The triangles in $\mathscr{K}(R)$ and $\mathscr{D}(R)$ are exactly those sextuples which are isomorphic to distinguished triangles.

For each $n \in \mathbb{Z}$, the nth cohomology functor from $\mathscr{D}(R)$ to R-Mod is denoted by $H^n(-)$. Certainly, this functor is naturally isomorphic to the Hom-functor $\mathrm{Hom}_{\mathscr{D}(R)}(R, -[n])$.

Now, we recall some basic facts about derived functors of derived module categories of rings. For details and proofs, we refer to [71, 123].

Let $\mathscr{K}(R)_P$ (respectively, $\mathscr{K}(R)_I$) be the smallest full triangulated subcategory of $\mathscr{K}(R)$ which

(i) contains all bounded above (respectively, bounded below) complexes of projective (respectively, injective) R-modules, and
(ii) is closed under arbitrary direct sums (respectively, direct products).

It is known that $\mathscr{K}(R)_P$ is contained in $\mathscr{K}(R\text{-Proj})$, where R-Proj is the full subcategory of R-Mod consisting of all projective R-modules. Moreover, the composition functors

$$\mathscr{K}(R)_P \hookrightarrow \mathscr{K}(R) \longrightarrow \mathscr{D}(R) \quad \text{and} \quad \mathscr{K}(R)_I \hookrightarrow \mathscr{K}(R) \longrightarrow \mathscr{D}(R)$$

are equivalences of triangulated categories. Thus, for each complex $X^\bullet$ in $\mathscr{D}(R)$, there exists a complex ${}_pX^\bullet \in \mathscr{K}(R)_P$ together with a quasi-isomorphism ${}_pX^\bullet \to X^\bullet$, and a complex ${}_iX^\bullet \in \mathscr{K}(R)_I$ together with a quasi-isomorphism $X^\bullet \to {}_iX^\bullet$. The complex ${}_pX^\bullet$ is called the *projective resolution* of $X^\bullet$ in $\mathscr{K}(R)$. For example, if X is an R-module, then there is an exact sequence of R-modules

$$\cdots \longrightarrow P^{-n} \longrightarrow P^{-n+1} \longrightarrow \cdots \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow X \longrightarrow 0$$

with all P_j projective R-modules and we can take ${}_pX$ to be the complex:

$$\cdots \longrightarrow P^{-n} \longrightarrow P^{-n+1} \longrightarrow \cdots \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow 0,$$

which is called a *deleted projective resolution* of the module ${}_RX$.

If either $X^\bullet \in \mathscr{K}(R)_P$ or $Y^\bullet \in \mathscr{K}(R)_I$, then the canonical localization functor from $\mathscr{K}(R)$ to $\mathscr{D}(R)$ induces an isomorphism: $\mathrm{Hom}_{\mathscr{K}(R)}(X^\bullet, Y^\bullet) \simeq \mathrm{Hom}_{\mathscr{D}(R)}(X^\bullet, Y^\bullet)$.

For any triangle functor $H : \mathscr{K}(R) \to \mathscr{K}(S)$, there is a total left-derived functor $\mathbb{L}H : \mathscr{D}(R) \to \mathscr{D}(S)$ defined by $X^\bullet \mapsto H({}_pX^\bullet)$, and a total right-derived functor $\mathbb{R}H : \mathscr{D}(R) \to \mathscr{D}(S)$ defined by $X^\bullet \mapsto H({}_iX^\bullet)$. Observe that, if H preserves acyclicity, that is, $H(X^\bullet)$ is acyclic whenever $X^\bullet$ is acyclic, then H induces a triangle functor $D(H) : \mathscr{D}(R) \to \mathscr{D}(S)$ defined by $X^\bullet \mapsto H(X^\bullet)$. In this case, $\mathbb{L}H = \mathbb{R}H = D(H)$ up to natural isomorphism. As usual, $D(H)$ is called the *derived functor* of H.

Let $M^\bullet$ be a complex of R-S-bimodules. Then the tensor functor and Hom-functor

$$M^\bullet \otimes_S^\bullet - : \mathscr{K}(S) \longrightarrow \mathscr{K}(R) \quad \text{and} \quad \mathrm{Hom}_R^\bullet(M^\bullet, -) : \mathscr{K}(R) \longrightarrow \mathscr{K}(S)$$

form an adjoint pair of triangle functors. Denote by $M^\bullet \otimes_S^{\mathbf{L}} -$ the left-derived functor of $M^\bullet \otimes_S^\bullet -$, and by $\mathbb{R}\mathrm{Hom}_R(M^\bullet, -)$ the right-derived functor of $\mathrm{Hom}_R^\bullet(M^\bullet, -)$. Then $(M^\bullet \otimes_S^{\mathbf{L}} -, \mathbb{R}\mathrm{Hom}_R(M^\bullet, -))$ is an adjoint pair of triangle functors.

If $M^\bullet$ is just a bimodule, we simply write $M \otimes_S^{\mathbf{L}} -$ for $M^\bullet \otimes_S^{\mathbf{L}} -$ and $\mathbb{R}\mathrm{Hom}_R(M, -)$ for $\mathbb{R}\mathrm{Hom}_R(M^\bullet, -)$.

Note that the tensor functor and Hom-functor defined on homotopy categories of R-modules can be extended to functors on derived module categories.

3. *Derived categories and derived equivalences of algebras*

In this section we briefly recall the history of developments of derived equivalences in the representation theory of algebras. In particular, we recall Rickard's Morita theory for derived categories and collect some invariants of derived equivalences. Also, we mention two important conjectures which are in terms of derived equivalences, in order to have a feeling on the ubiquity and importance of derived equivalences.

3.1. *Tilting modules*

The notion of tilting modules of projective dimension at most 1 played an important role in the development of tilting theory. It was first introduced by Brenner and Butler in [**21**]. One of the main aims in [**21**] was to generalize systematically some results of Auslander, Platzeck and Reiten in [**7**] where they studied the reflection functors without diagrams, while the reflection functors was initiated by Bernstein, Gelfand and Ponomarev in [**17**]. This might be considered as the first step toward the general notion of tilting modules. After the work of Brenner and Butler on tilting modules, Happel and Ringel simplified the axioms of tilting modules defined in [**21**] and gave a much simple formulation of the definition of tilting modules (see [**51**]), and Bongartz gave another treatment of some results in [**51**]. Further generalization of the notion of tilting modules of finite projective dimension was given in [**91**].

Definition 3.1 [**21**, **51**, **91**]. Let A be a finite-dimensional algebra over a field. An A-module $T \in A$-mod is called a tilting module if the following conditions are satisfied.

(1) $\mathrm{pd}_A(T) \leqslant n < \infty$,
(2) $\mathrm{Ext}^i_A(T,T) = 0$ for all $i \geqslant 1$, and
(3) there is an exact sequence $0 \to {}_AA \to T_0 \to T_1 \to \cdots \to T_n \to 0$ with $T_i \in \mathrm{add}(T)$.

If $n = 1$, Condition (3) can be replaced by

(3') the number of non-isomorphic indecomposable direct summands of T equals the number of non-isomorphic simple A-modules (see [**51**; **114**, Chapter 4]).

Tilting modules have many applications in and connections with other algebraic areas. For instance, tilting modules appears in the study of highest weight categories of semisimple Lie algebras and algebraic groups, where tilting modules are described as modules with filtrations by standard modules and by co-standard modules, respectively. For more details, we refer the reader to [**2**, **37**, **38**].

3.2. *Happel's investigation of derived module categories of algebras*

Tilting modules may be regarded as a starting point of the general Morita theory of derived categories of algebras. Around 1986, Happel published his works [**47**, **48**] and initiated investigations of the representation theory of finite-dimensional algebras from the view point of derived categories. Since then there have been discovered lots of beautiful results in this direction. We will survey some of them in the following sections.

Recall that two rings R and S are called *derived equivalent* if their derived categories $\mathscr{D}^{\mathrm{b}}(R)$ and $\mathscr{D}^{\mathrm{b}}(S)$ are equivalent as triangulated categories. For finite-dimensional algebras (or more generally, Artin algebras) A and B, they are derived equivalent if and only if $\mathscr{D}^{\mathrm{b}}(A\text{-mod})$ and $\mathscr{D}^{\mathrm{b}}(B\text{-mod})$ are equivalent as triangulated categories (see [**108**, Corollary 8.3]).

For tilting modules over finite-dimensional algebras, Happel proved the following result.

Theorem 3.2 [**47**]. *Let A be a finite-dimensional algebra over a field and T be a tilting A-module, and let $B := \mathrm{End}_A(T)$. Then*

(1) *A and B are derived equivalent.*
(2) *$|\mathrm{gl.dim}(A) - \mathrm{gl.dim}(B)| \leqslant \mathrm{pd}_A(T)$.*

Also, the derived categories $\mathscr{D}^{\mathrm{b}}(H\text{-mod})$ of finite-dimensional hereditary algebras H over algebraically closed fields were completely described by the Auslander–Reiten quivers of H in terms of indecomposable H-modules in H-mod (see [**47**]). Roughly speaking, $\mathscr{D}^{\mathrm{b}}(H\text{-mod})$ can be obtained by suitable gluing of $\mathbb{Z}$ copies of the category of indecomposable H-modules in H-mod.

3.3. *Happel's Theorem extended by Cline–Parshall–Scott*

Happel's result, Theorem 3.2, was proved for finite-dimensional algebras and finite-dimensional modules. In [**35**], Cline, Parshall and Scott extended Happel's result to arbitrary rings and tilting modules. For an arbitrary ring R, the notion of tilting modules is defined in a very similar way.

Definition 3.3 [**35**]. Let R be a ring. An R-module T is called a tilting module if the following conditions are satisfied:

(1) There is a projective resolution $0 \to P_n \to \cdots \to P_1 \to P_0 \to T \to 0$ of T, where all P_j are finitely generated projective R-modules,
(2) $\mathrm{Ext}^i_R(T,T) = 0$ for all $i \geqslant 1$, and
(3) there is an exact sequence $0 \to {}_RR \to T_0 \to T_1 \to \cdots \to T_n \to 0$ with $T_i \in \mathrm{add}(T)$.

For tilting modules over rings, it was shown in [**35**] that Happel's result, Theorem 3.2, remains true. We point out that, if one extends the definition of finitely generated tilting modules to that of infinitely generated tilting modules, then Condition (2) was strengthened and Theorem 3.2 is no longer true (see [**13**]). In this case, the derived category of the endomorphism ring of an infinitely generated tilting R-module is much bigger than the derived category of R, and actually forms a recollement (see [**13**, **24**, **25**]).

3.4. *Rickard's Morita theory of derived categories and equivalences*

In a very general setting, Rickard furthered the notion of tilting modules and dealt with derived equivalences between arbitrary rings. He introduced the key notion of tilting complexes and established a beautiful Morita theory for derived categories of rings. This provides a more general framework for understanding algebraic and homological properties of algebras from the view point of derived module categories.

To describe derived equivalent rings, we first recall Rickard's definition of tilting complexes (see [**108**]), which generalize the notion of tilting modules.

Definition 3.4. Let R be a ring. A complex $T^\bullet \in \mathscr{K}^{\mathrm{b}}(R\text{-proj})$ is called a tilting complex over R if

(1) $\mathrm{Hom}_{\mathscr{D}(R)}(T^\bullet, T^\bullet[n]) = 0$ for all $n \neq 0$, and
(2) $\mathrm{add}(T^\bullet)$ generates $\mathscr{K}^{\mathrm{b}}(R\text{-proj})$ as a triangulated category, that is, the smallest triangulated subcategory of $\mathscr{K}^{\mathrm{b}}(R\text{-proj})$ containing $\mathrm{add}(T^\bullet)$ coincides with $\mathscr{K}^{\mathrm{b}}(R\text{-proj})$.

Thus a deleted projective resolution $P^\bullet$ of a tilting R-module T is a tilting complex, and we have $T \simeq P^\bullet$ in $\mathscr{D}(R)$, and $\mathrm{End}_R(T) \simeq \mathrm{End}_{\mathscr{D}(R)}(P^\bullet)$. In this sense, the notion of tilting complexes is a natural generalization of tilting modules.

The following theorem, due to Rickard, is a very useful description of derived equivalences for rings.

Theorem 3.5 [**108**, Theorem 6.4]. *For two rings A and B, the following are equivalent:*

(a) *$\mathscr{D}^{\mathrm{b}}(A)$ and $\mathscr{D}^{\mathrm{b}}(B)$ are equivalent as triangulated categories.*
(b) *$\mathscr{D}^{-}(A)$ and $\mathscr{D}^{-}(B)$ are equivalent as triangulated categories.*
(c) *$\mathscr{K}^{\mathrm{b}}(A\text{-proj})$ and $\mathscr{K}^{\mathrm{b}}(B\text{-proj})$ are equivalent as triangulated categories.*
(d) *$\mathscr{K}^{\mathrm{b}}(A\text{-Proj})$ and $\mathscr{K}^{\mathrm{b}}(B\text{-Proj})$ are equivalent as triangulated categories.*
(e) *There is a tilting complex $T^\bullet \in \mathscr{K}^{\mathrm{b}}(A\text{-proj})$ such that $B \simeq \mathrm{End}_{\mathscr{K}^{\mathrm{b}}(A\text{-proj})}(T^\bullet)$.*

Thus, given a derived equivalence F between A and B, there is a unique (up to isomorphism) tilting complex $T^\bullet$ over A such that $FT^\bullet = B$ by Theorem 3.5(c). This complex $T^\bullet$ is called a *tilting complex associated to F*. Transparently, the two conditions in Definition 3.4 are satisfied by a deleted projective resolution of a tilting module (see Definition 3.3(2) and (3)). Also, the stack complex B satisfies the two conditions in Definition 3.4. Since F is a triangle equivalence, $T^\bullet$ also satisfies the two conditions. This may explain why the conditions in the definition of tilting complexes are naturally required.

Theorem 3.5 is, in fact, an analog of the Morita theorem on module categories of rings. Recall from [**92**] that two rings R and S are Morita equivalent (that is, R-Mod $\simeq S$-Mod) if and only if there is a finitely generated projective R-module ${}_RP$ such that

(1) P generates R-Mod (or R-proj), that is, every R-module (or finitely generated projective R-module) is a homomorphic image of a direct sum of copies of P.
(2) $S \simeq \mathrm{End}_R(P)$.

Thus Theorem 3.5(e) can be regarded as a derived version of Morita theorem. Also, it follows from Theorem 3.5(b) that $\mathscr{D}(R)$ and $\mathscr{D}(S)$ are equivalent as triangulated categories if (b) holds true since every R-module M has a projective resolution $P^\bullet$ which is isomorphic to M in $\mathscr{D}^-(R)$.

Slightly general formulation of the above Morita Theorem is as follows: if $\mathcal{A}$ is an abelian category admitting all set-indexed coproducts and P is a compact (that is, $\mathrm{Hom}_{\mathcal{A}}(P,-)$ commutes with all set-indexed coproducts) projective generator of $\mathcal{A}$, then the functor $\mathrm{Hom}_{\mathcal{A}}(P,-) : \mathcal{A} \to \mathrm{End}_{\mathcal{A}}(P)$-Mod is an equivalence. An analog of this result for triangulated categories $\mathcal{T}$ was given by Keller in [**71**], in which $\mathcal{T}$ is assumed to be algebraic, P is replaced by a compact generator T of $\mathcal{T}$ and $\mathrm{End}_{\mathcal{A}}(P)$ is replaced by the differential graded algebra $\mathbb{R}\mathrm{Hom}(T,T)$, where the homology of $\mathbb{R}\mathrm{Hom}(T,T)$ is $\bigoplus_{p\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{T}}(T,T[p])$. In particular, if the algebraic triangulated category $\mathcal{T}$ is the closure of T under forming extensions, shifts (in both directions) and direct summands. Then $\mathcal{T}$ is equivalent to the category of perfect complexes over $\mathbb{R}\mathrm{Hom}(T,T)$ as triangulated categories. Recall that an object T is a generator in $\mathcal{T}$ if, for any object M with $\mathrm{Hom}_{\mathcal{T}}(M,T[n]) = 0$ for all $n \in \mathbb{Z}$, we have $M = 0$.

Morita equivalences of rings can be described by tensor products of bimodules, as shown by the following theorem, due to Morita [**92**].

THEOREM 3.6. *Let R and S be two rings. The following are equivalent.*

(1) *R and S are Morita equivalent.*
(2) *There are two R-S-bimodule ${}_RP_S$ and ${}_SQ_R$ and isomorphisms of bimodules*

$$\theta : P \otimes_S Q \longrightarrow R \text{ and } \phi : Q \otimes_S P \longrightarrow S$$

such that for all $x, y \in P$ and $f, g \in Q$,

$$((x \otimes f)\theta)y = x((f \otimes y)\phi) \text{ and } f((x \otimes g)\theta) = ((f \otimes x)\phi)g.$$

Compared with Theorem 3.6, derived equivalences can also be characterized in terms of tensor products of complexes of bimodules.

THEOREM 3.7 [**110**]. *Let A and B be two k-algebras over a commutative ring k such that they are flat over k. Then the following are equivalent:*

(1) *A and B are derived equivalent.*
(2) *There are two complexes $P^\bullet \in \mathscr{D}^{\mathrm{b}}(A \otimes_k B^{\mathrm{op}})$ and $Q^\bullet \in \mathscr{D}^{\mathrm{b}}(B \otimes_k A^{\mathrm{op}})$ such that*

$$P^\bullet \otimes^{\mathbf{L}}_B Q^\bullet \simeq {}_AA_A \text{ in } \mathscr{D}^{\mathrm{b}}(A \otimes_k A^{\mathrm{op}}) \text{ and } Q^\bullet \otimes^{\mathbf{L}}_A P^\bullet \simeq {}_BB_B \text{ in } \mathscr{D}^{\mathrm{b}}(B \otimes_k B^{\mathrm{op}}).$$

Note that the functor $Q^\bullet \otimes^{\mathbf{L}}_A - : \mathscr{D}^{\mathrm{b}}(A) \to \mathscr{D}^{\mathrm{b}}(B)$ is an equivalence. It is not known if this equivalence coincides with the given derived equivalence in (1) of Theorem 3.7, but they have the isomorphic images on objects (see [**110**, Corollary 3.5]).

The above description of derived equivalences suggests the following definition: Let A and B be two k-algebras over a commutative ring k such that they are flat over k. A complex

$P^\bullet \in \mathscr{D}^{\mathrm{b}}(A \otimes_k B^{^{\mathrm{op}}})$ is called a *two-sided tilting complex* over $A \otimes_k B^{^{\mathrm{op}}}$ if there is a complex $Q^\bullet \in \mathscr{D}^{\mathrm{b}}(B \otimes_k A^{^{\mathrm{op}}})$ such that

$$P^\bullet \otimes_B^{\mathbf{L}} Q^\bullet \simeq {}_AA_A \text{ in } \mathscr{D}^{\mathrm{b}}(A \otimes_k A^{^{\mathrm{op}}}) \text{ and } Q^\bullet \otimes_A^{\mathbf{L}} P^\bullet \simeq {}_BB_B \text{ in } \mathscr{D}^{\mathrm{b}}(B \otimes_k B^{^{\mathrm{op}}}).$$

A derived equivalence F between k-algebras A and B is said to be *standard* if there is a two-sided tilting complex $Q^\bullet$ over $B \otimes_k A^{^{\mathrm{op}}}$ such that F and $Q^\bullet \otimes_A^{\mathbf{L}} -$ are naturally equivalent. For a discussion on standard derived equivalences, one may see [**110**]. In general, we have the following fact: let A and B be algebras over a commutative ring k such that A is a projective k-module. If A and B are derived equivalent, then Keller gave an explicit construction of a two-sided tilting complex $P^\bullet$ such that $P^\bullet \otimes_B^{\mathbf{L}} - : \mathscr{D}(B) \to \mathscr{D}(A)$ is a triangle equivalence (see [**72**, **74**] for details).

3.5. *Invariants of derived equivalences of algebras*

For two derived equivalent algebras, one algebra may be representation-finite, and the other may be representation-infinite. They may also have different global dimensions. Though derived equivalent algebras may have significant differences in homological aspects and in algebraic structures, they can still have many common features in many other aspects. Here we shall list some of invariants of derived equivalences, which will be useful to adjudge whether two algebras are not derived equivalent. A property $\mathscr{P}$ is said to be *invariant* under derived equivalences provided that if a ring (or an algebra) A has the property $\mathscr{P}$ then so do all rings (or algebras) B which are derived equivalent to A.

The next theorem collects a few invariants of derived equivalences.

THEOREM 3.8. *The following are invariants of derived equivalences between rings.*

(1) *The Hochschild* (*co-*) *homology and cyclic homology. In particular, the centers of rings* (*see* [**73**, **110**]).
(2) *The number of non-isomorphic simple modules if we are restricted to Artin algebras.*
(3) *Finiteness of global* (*or finitistic*) *dimensions* (*see* [**48**, **69**, **102**]).
(4) *The Cartan determinants, and the characteristic polynomials of Coxeter matrices if the Cartan matrices of Artin algebras are invertible* (*see* [**50**, *Lemma* 4.1]; *for a detailed proof, see* [**128**, *Proposition* 6.8.9]).
(5) *Algebraic K-groups and G-theory* (*see* [**41**]).
(6) *Symmetry of algebras over an arbitrary field* (*respectively, self-injectivity of algebras over an algebraically closed field*) (*see* [**3**, **108**]).
(7) *Finite-dimensional gentle algebras over a field* (*see* [**117**]).
(8) *The identity component of the algebraic group of outer automorphisms of finite-dimensional algebras* (*see* [**64**]).

Thus, in order to understand some property of a given algebra (or mathematical object), one may pass to its derived equivalent algebras (or mathematical objects) which might be easy to handle. For example, to understand properties of coherent sheaves over weighted projective lines $\mathbb{X}$, Geigle and Lenzing employed Ringel's tubular algebras (see [**114**] for definition) because the derived category of coherent sheaves over $\mathbb{X}$ is triangle-equivalent to the derived module category of a tubular algebra (see [**43**, **82**]). Another example is the well-known work of Beilinson who reduced the study of derived category of coherent sheaves over $\mathbb{P}^n$ to the one of a triangular matrix algebra (see [**14**]). Further examples of applications of derived equivalences can be found in Section 6.

Derived equivalences were initiated from algebraic geometry and also widely used in algebraic geometry. Many geometric invariants were discovered to be preserved under derived equivalences. Let us just mention one of them. For the proof of the following result, one may see [**65**, Proposition 4.1]

Proposition 3.9. *Let X and Y be two smooth projective varieties over a field, and let $\mathscr{D}^{\mathrm{b}}(X)$ denote the bounded derived category of coherent sheaves over X. If $\mathscr{D}^{\mathrm{b}}(X) \simeq \mathscr{D}^{\mathrm{b}}(Y)$ as triangulated categories, then*

(1) *X and Y have the same dimension.*
(2) *The canonical rings of X and Y are isomorphic and in particular, the Kodaira dimension of X and Y are equal (see [**65**, Proposition 6.1; **99**] for a proof).*

For the definitions of the dimension and canonical ring of a variety, we refer to [**65**, Section 6.5].

3.6. *Broué's abelian defect group conjecture and Bondal–Orlov conjecture*

Derived equivalences appear in different branches of mathematics. To get a feeling of the ubiquity of derived equivalences, we mention two conjectures from the modular representation theory of finite groups and algebraic geometry, respectively. Both of them may reflect the importance of derived equivalences and difficulty of how to find (suitable) tilting complexes such that their endomorphism algebras have the desired properties, even though we have the Morita theory of derived categories in hand (see Theorem 3.5). Note that each of the conjectures requires just to prove an existence of a derived equivalence.

Broué's abelian defect group conjecture is considered as one of the central problems in the modular representation theory of finite groups, and remains unsolved.

Let k be an algebraically closed field of characteristic $p > 0$, and let G be a finite group with p dividing the order of G. Suppose that B is a block of the group algebra kG with the defect group P. By a result due to Brauer, there is a unique block b of the group algebra $kN_G(P)$ of $N_G(P)$, the normalizer of P in G, such that b has the same defect group P and the restriction from $\mathscr{D}^{\mathrm{b}}(B)$ to $\mathscr{D}^{\mathrm{b}}(b)$ is faithful. This is the Brauer correspondence, which provides a bijection between blocks B of kG with the defect group P and blocks b of $kN_G(P)$ with the defect group P. Motivated by the study of isometry of finite groups, Broué introduced a conjecture [**23**].

Broue's Abelian Defect Group Conjecture: If P is abelian, then B and b are derived equivalent.

Roughly speaking, Broué's conjecture predicates that two symmetric algebras are derived equivalent under a 'commutativity' condition. This conjecture was verified for symmetric groups in [**34**] and for alternating groups in [**85**], where the proof is based on the consideration in [**34**]. For more examples, we refer to the recent paper [**127**] and to the old preprint [**98**] as well as the papers [**79**, **93**] and the references therein. For further information on the developments of the conjecture, we refer to [**116**] and the home page of Rickard: https://people.maths.bris.ac.uk/~majcr/adgc/adgc.html. In general, the conjecture seems to be far away from being completely solved.

Now, we mention the conjecture of Bondal–Orlov on derived equivalences in (non-commutative) crepant resolutions.

Let Y and Y^+ be three-dimensional smooth varieties related by a flop. One may consider the derived categories $\mathscr{D}^{\mathrm{b}}(coh(Y))$ and $\mathscr{D}^{\mathrm{b}}(coh(Y^+))$ of coherent sheaves over Y and Y^+, respectively. In a paper (see [**20**]), Bondal and Orlov proposed the following conjecture.

Bondal–Orlov conjecture: The bounded derived categories $\mathscr{D}^{\mathrm{b}}(coh(Y))$ and $\mathscr{D}^{\mathrm{b}}(coh(Y^+))$ are equivalent.

This conjecture was proven by Bridgeland (see [**22**]), and later extended to non-commutative crepant resolutions by Van den Bergh in [**16**].

Suppose that k is an algebraically closed field and R is an integral Gorenstein k-algebra. Van den Bergh defined a *non-commutative crepant resolution* of R to be an algebra $A = \mathrm{End}_R(M)$ where M is a reflexive R-module, A has finite global dimension, and A is a maximal Cohen–Macaulay R-module.

THEOREM 3.10 (van den Bergh [**16**]). *If R is three-dimensional and has terminal singularities, then all crepant resolutions of R (commutative as well as non-commutative) are derived equivalent.*

4. *Constructions of tilting complexes and derived equivalences for algebras*

By Theorem 3.5, one important ingredient of constructing derived equivalences is to find tilting complexes. In this section, we survey a variety of methods for constructing derived equivalences and tilting complexes for algebras and rings.

4.1. *Hoshino–Kato's construction of tilting complexes*

We first mention a construction of tilting complexes obtained from idempotent elements initiated by Okuyama [**98**], and further developed by Hoshino–Kato [**56**, **57**]. This gives rise, in fact, to a two-term tilting complex. Note that two-term tilting complexes are often used to study Brauer tree algebras (for example, see [**129**]).

Assume that A is a finite-dimensional algebra over a field and $e \in A$ is an idempotent element. Let $f : P \to A$ be a right $\mathrm{add}(Ae)$-approximation of A. We consider f as a complex $P^\bullet$ of projective A-modules with A in degree 0 and P in degree -1, that is, $P^\bullet$ is the mapping cone of f. Note that P can be chosen as a direct sum of finitely many copies of Ae and that the cokernel of f is A/AeA. We define $T^\bullet := Ae[1] \oplus P^\bullet$. Then the following result holds.

PROPOSITION 4.1 [**56**]. *Let A be a finite-dimensional algebra over a field k and e an idempotent element in A. Then $T^\bullet = Ae[1] \oplus P^\bullet$ is a tilting complex if and only if* $\mathrm{Hom}_A(A/AeA, Ae) = 0$.

A special case of Proposition 4.1 is that an idempotent element $e \in A$ satisfies $\nu(Ae) \simeq Ae$, where ν is the Nakayama functor $D\mathrm{Hom}_A(-, A)$ with D the usual k-duality. In this situation, it follows from the natural isomorphism $\mathrm{Hom}_A(P, M) \simeq D\mathrm{Hom}_A(M, \nu P)$ for P a projective A-module that

$$\mathrm{Hom}_A(A/AeA, Ae) \simeq D\mathrm{Hom}_A(A/AeA, \nu(Ae)) \simeq \mathrm{Hom}_A(Ae, A/AeA) = 0.$$

This case appears for symmetric algebras A (for example, group algebras of finite groups over a field) because we always have $\nu(Ae) \simeq Ae$ for any idempotent element e in a symmetric algebra A.

This construction is closely related to mutations in a triangulated category studied in [**67**], see also [**19**, **52**]. But we warn the reader that mutations in [**67**] do not have to produce derived equivalences of algebras in general. For conditions of when a mutation gives a derived equivalence, we refer the reader to, for instance, [**58**, Corollary 4.11] and the preprint [**81**].

For constructions of two-term tilting complexes over symmetric algebras, one may also see [**112**, **129**].

4.2. *Derived equivalences from short exact sequences*

In this section, we present a general method to construct derived equivalences of rings from each short exact sequence in an additive category. The rings involved in this construction are defined by the objects in the sequence. In particular, we get a nice relation between almost split sequences (see [8] for definition) and derived equivalences.

Now let us recall the definition of relatively split sequences from [60]. Throughout this section, we assume that $\mathcal{C}$ is an additive category and $\mathcal{D}$ is a full subcategory of $\mathcal{C}$.

Definition 4.2. A sequence

$$X \xrightarrow{f} M \xrightarrow{g} Y$$

of morphisms between objects in $\mathcal{C}$ is said to be $\mathcal{D}$-split if the following three conditions are satisfied:

(1) $M \in \mathcal{D}$;
(2) f is a left $\mathcal{D}$-approximation of X, and g is a right $\mathcal{D}$-approximation of Y; and
(3) f is a kernel of g, and g is a cokernel of f.

Typical examples of $\mathcal{D}$-split sequences are almost split sequences introduced by Auslander and Reiten, these sequences are the most important ones in the representation theory of Artin algebras (see, for example, [8] for more details). Let A be an Artin algebra and $0 \to X \to M \to Y \to 0$ be an almost split sequence in A-mod. If we take $\mathcal{C} = A$-mod and $\mathcal{D} = \operatorname{add}(M)$, then the sequence $X \to M \to Y$ is a $\mathcal{D}$-split sequence in $\mathcal{C}$. This follows immediately from properties of almost split sequences. Another example of $\mathcal{D}$-split sequences reads as follows: If P is a projective-injective A-module, then any exact sequence $0 \to X \to P' \to M \to 0$ of A-modules with $P' \in \operatorname{add}(P)$ is an $\operatorname{add}(P)$-split sequence in A-mod.

The significance of $\mathcal{D}$-split sequences is given by the following result.

Theorem 4.3 [60]. *Let M be an object in $\mathcal{C}$. Suppose that $X \to M' \to Y$ is an add(M)-split sequence in $\mathcal{C}$. Then the endomorphism rings $\operatorname{End}_{\mathcal{C}}(M \oplus X)$ and $\operatorname{End}_{\mathcal{C}}(M \oplus Y)$ are derived equivalent.*

In fact, we can say more about the derived equivalence in Theorem 4.3, namely the derived equivalence between the endomorphism rings is given by a tilting module of projective dimension at most 1. Thus, by Brenner–Butler's Theorem in [21], we have two torsion pairs, defined by the tilting module, for the module categories of the endomorphism rings.

Let A be an Artin algebra. Recall from [60, Section 4] that an A-module T is called an *n-BB-tilting module* (after the names of Brenner and Butler) if it is of the form $T = P \oplus \tau^{-}(S)$, where τ is the Auslander–Reiten translation, S is a simple, non-injective A-module such that $\operatorname{Ext}^i_A(D(A), S) = 0$ for $0 \leqslant i \leqslant n-1$ and $\operatorname{Ext}^j_A(S, S) = 0$ for $1 \leqslant j \leqslant n$ and P is the direct sum of all indecomposable projective A-modules which are not isomorphic to the projective cover of S. In case that S is projective, a 1-BB-tilting module is called an *APR-tilting module* (after the name of Auslander, Platzeck and Reiten).

Corollary 4.4. *Let A be an Artin algebra. If $0 \to X \to M \to Y \to 0$ is an almost split sequence in A-mod and $N \in A$-mod such that $X, Y \notin \operatorname{add}(N)$, then $\operatorname{End}_A(X \oplus M \oplus N)$ is derived equivalent to $\operatorname{End}_A(M \oplus Y \oplus N)$ via a 1-BB-tilting module.*

In Corollary 4.4, if $N = 0$, then $\operatorname{End}_A(X \oplus M)$ is derived equivalent to $\operatorname{End}_A(M \oplus Y)$ by a 1-BB-tilting module. Thus Corollary 4.4 reveals a substantial relation among almost split sequences, BB-tilting modules and derived equivalences. Note that n-BB-tilting modules appear in constructions of derived equivalences by several consecutive almost split sequences (see [60, Proposition 4.3]).

The proof of Theorem 4.3 is based on a general fact.

LEMMA 4.5 [**60**]. *Let M be an object in $\mathcal{C}$. Suppose*

$$X \xrightarrow{f} M_n \longrightarrow \cdots \longrightarrow M_2 \xrightarrow{t} M_1 \xrightarrow{g} Y$$

is a (not necessarily exact) sequence of morphisms in $\mathcal{C}$ with $M_i \in \mathrm{add}(M)$, satisfying the following conditions.

(1) *The morphism $f: X \to M_n$ is a left $\mathrm{add}(M)$-approximation of X, and the morphism $g: M_1 \to Y$ is a right $\mathrm{add}(M)$-approximation of Y;*
(2) *Put $V := M \oplus X$ and $W := M \oplus Y$. There are two induced exact sequences*

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{C}}(V, X) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{C}}(V, M_n) \to \cdots \to \mathrm{Hom}_{\mathcal{C}}(V, M_1) \xrightarrow{g_*} \mathrm{Hom}_{\mathcal{C}}(V, Y),$$

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{C}}(Y, W) \xrightarrow{g^*} \mathrm{Hom}_{\mathcal{C}}(M_1, W) \to \cdots \to \mathrm{Hom}_{\mathcal{C}}(M_n, W) \xrightarrow{f^*} \mathrm{Hom}_{\mathcal{C}}(X, W).$$

Then the endomorphism rings $\mathrm{End}_{\mathcal{C}}(M \oplus X)$ and $\mathrm{End}_{\mathcal{C}}(M \oplus Y)$ are derived-equivalent via a tilting module of projective dimension at most n.

Observe that triangles in a triangulated category are a natural generalization of exact sequences in an abelian category and that Auslander–Reiten triangle (see [**48**]) in a derived category is a natural generalization of almost split sequences in a module category. So one may ask whether the above result, Corollary 4.4, holds true for triangles. Unfortunately, if almost split sequences are replaced by Auslander–Reiten triangles in Corollary 4.4, then the result is no longer true in general. To get a more general statement, two generalizations of Theorem 4.3 were done in different directions. One is to use subalgebras of the endomorphism algebras, and the other is to pass to quotient algebras of the endomorphism algebras.

The first case was carried out by Yiping Chen in [**32**] for exact sequences in abelian categories, and then extended to triangles in triangulated categories by Shengyong Pan in [**100**]. The second case will be discussed in Section 4.3. Recently, two further generalizations are given in [**28**] for relatively exact sequences in additive categories and in [**101**] for a class of Beilinson–Green algebras, respectively. Now, we first pursue the construction in [**28**].

The following is a slight generalization of $\mathcal{D}$-split sequences.

DEFINITION 4.6 [**28**]. Let $\mathcal{D}$ be a full subcategory of $\mathcal{C}$. A sequence

$$X \xrightarrow{f} M_0 \xrightarrow{g} Y$$

of objects and morphisms in $\mathcal{C}$ is called a $\mathcal{D}$-exact sequence provided that

(1) $M_0 \in \mathcal{D}$.
(2) The following two sequences of abelian groups are exact:

$$(\dagger) \qquad 0 \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X \oplus M,\, X) \xrightarrow{f^*} \mathrm{Hom}_{\mathcal{C}}(X \oplus M,\, M_0) \xrightarrow{g^*} \mathrm{Hom}_{\mathcal{C}}(X \oplus M,\, Y)$$

$$(\ddagger) \qquad 0 \longrightarrow \mathrm{Hom}_{\mathcal{C}}(Y,\, M \oplus Y) \xrightarrow{g_*} \mathrm{Hom}_{\mathcal{C}}(M_0,\, M \oplus Y) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{C}}(X,\, M \oplus Y)$$

for every object M in $\mathcal{D}$.

Note that $\mathcal{D}$-split sequences in $\mathcal{C}$ are $\mathcal{D}$-exact sequences in $\mathcal{C}$ because Condition (2) in Definition 4.6 implies $fg = 0$ and, moreover, if f is a kernel of g and g is a cokernel of f, then Condition (2) holds automatically. But the converse is not true: Since every short exact sequence $0 \to X \to M \to Y \to 0$ in an abelian category is an $\mathrm{add}(M)$-exact sequence, we get not only the ubiquity of relatively exact sequences, but also examples of $\mathcal{D}$-exact sequences which are not $\mathcal{D}$-split sequences. For instance, we take $A = k[T_1, T_2]/(T_1^2, T_2^2, T_1T_2)$ with k a field, $\mathcal{C} = A$-mod and $X = A/\mathrm{rad}(A)$, then the canonical sequence $0 \to X \to A \to A/(T_1) \to 0$ is exact, but Condition (2) in Definition 4.2 is not satisfied. Thus this sequence is not $\mathrm{add}({}_AA)$-split.

In the following, we shall focus on the most interesting case where $\mathcal{D} = \mathrm{add}(M)$ for M an object in $\mathcal{C}$. Compared with Theorem 4.3, an $\mathrm{add}(M)$-exact sequence (not necessarily an $\mathrm{add}(M)$-split sequence) does not have to provide a derived equivalence between the endomorphism rings of the objects in the sequence. However, we shall prove that there does exist a derived equivalence between certain subrings of the corresponding endomorphism rings.

PROPOSITION 4.7 [**28**]. *Let M be an object in $\mathcal{C}$, and let $X \xrightarrow{f} M_0 \xrightarrow{g} Y$ be an $\mathrm{add}(M)$-exact sequence in $\mathcal{C}$. Set*

$$R := \left\{ \begin{pmatrix} h_1 & h_2 \\ fh_3 & h_4 \end{pmatrix} \in \begin{pmatrix} \mathrm{End}_{\mathcal{C}}(M) & \mathrm{Hom}_{\mathcal{C}}(M,X) \\ \mathrm{Hom}_{\mathcal{C}}(X,M) & \mathrm{End}_{\mathcal{C}}(X) \end{pmatrix} \middle| \begin{matrix} h_3 \in \mathrm{Hom}_{\mathcal{C}}(M_0,M) \text{ and there exists} \\ h_5 \in \mathrm{End}_{\mathcal{C}}(M_0) \text{ such that } h_4 f = fh_5 \end{matrix} \right\}$$

and

$$S := \left\{ \begin{pmatrix} h_1 & h_2\, g \\ h_3 & h_4 \end{pmatrix} \in \begin{pmatrix} \mathrm{End}_{\mathcal{C}}(M) & \mathrm{Hom}_{\mathcal{C}}(M,Y) \\ \mathrm{Hom}_{\mathcal{C}}(Y,M) & \mathrm{End}_{\mathcal{C}}(Y) \end{pmatrix} \middle| \begin{matrix} h_2 \in \mathrm{Hom}_{\mathcal{C}}(M,M_0) \text{ and there exists} \\ h_5 \in \mathrm{End}_{\mathcal{C}}(M_0) \text{ such that } gh_4 = h_5 g \end{matrix} \right\}.$$

Then R and S are subrings of $\mathrm{End}_{\mathcal{C}}(M \oplus X)$ and $\mathrm{End}_{\mathcal{C}}(M \oplus Y)$, respectively, and are derived equivalent.

Recall that the *dominant dimension* of an Artin algebra A is defined to be the supremum of the numbers n in a minimal injective resolution

$$0 \longrightarrow_A A \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_{n-1} \longrightarrow \cdots$$

of A such that $I_0, \dots, I_{n-1}$ is projective.

Proposition 4.7 can be applied to construct derived equivalent algebras such that one algebra is of dominant dimension at least 2 and the other has dominant dimension 1. Thus one can construct examples to show that derived equivalences do not preserve generalized symmetric algebras in general. This answered a question by Fang and Koenig (see [**28**, Section 5] for details). Recall that an algebra is called a *generalized symmetric algebra* if it is of the form $\mathrm{End}_A(A \oplus M)$ with A a symmetric algebra and M an A-module (see [**42**]). We reminded the reader that derived equivalences preserve symmetric algebras over a field (see Theorem 3.8(6)).

4.3. *Derived equivalences from triangles and cohomological approximations*

Almost split sequences in module categories were naturally generalized to Auslander–Reiten triangles in derived module categories (see [**48**]). As mentioned, we cannot get derived equivalence from an Auslander–Reiten triangle just by forming the corresponding endomorphism algebras. On the other hand, when dealing with Yoneda algebras or Koszul duality of algebras (or more generally, the endomorphism algebras of objects in an orbit category), we not only confront an endomorphism algebra, but also need higher cohomological groups (or homomorphisms of higher dimensional shifts). For example, the Koszul dual $\mathrm{Ext}^*(k,k)$ of $k[X]/(X^2)$ is the endomorphism algebra $\mathrm{End}(k)$ plus higher cohomologies $\mathrm{Ext}^i(k,k)$, $i \geqslant 1$, and therefore it is the polynomial algebra $k[X]$. Thus a natural question is how to modify the construction in the previous section to get derived equivalences, in a general context, for a kind of Yoneda algebras, including trivial extensions, Veronese algebras and quotients of Yoneda algebras.

In this section, we present a method for constructing derived equivalences from triangles by forming certain quotient algebras of generalized Yoneda algebras, that is, Φ-Auslander–Yoneda algebras. Let us first recall some relevant notions and notation in [**61**].

Let Φ be a subset of $\mathbb{N} := \{0, 1, 2, \dots\}$. Following [**61**], Φ is called an *admissible subset* of $\mathbb{N}$ if $0 \in \Phi$ and, for any $a.b, c \in \Phi$ with $a + b + c \in \Phi$, we have $a + b \in \Phi$ if and only if $b + c \in \Phi$.

Examples of admissible subsets in $\mathbb{N}$ are $\{0, 1, \dots, n\}$, $n\mathbb{N} := \{nx \mid x \in \mathbb{N}\}$ for any n, and the subset $\{0, a, b\}$ of $\mathbb{N}$ for any $a, b \in \mathbb{N}$. But $\{0, 1, 2, 4\}$ is not an admissible subset of $\mathbb{N}$.

Clearly, the definition of admissible sets in $\mathbb{N}$ can be extended to the one in $\mathbb{Z}$ (or more generally, in a monoid). For simplicity of our presentation in this note, we will work almost exclusively with admissible subsets of $\mathbb{N}$. Now, let us define an associative algebra with identity for each admissible subset Φ of $\mathbb{N}$ and each object X in a triangulated category.

Let $\mathcal{T}$ be a triangulated k-category with k a commutative ring, and let X be an object in $\mathcal{T}$. For any admissible subset Φ of $\mathbb{N}$, we define

$$R(\mathcal{T},\Phi,X) := \bigoplus_{i\in\Phi} \mathrm{Hom}_{\mathcal{T}}(X, X[i]).$$

The multiplication on $R(\mathcal{T},\Phi,X)$ is given by

$$(f_i)_{i\in\Phi}\cdot(g_j)_{j\in\Phi} = (h_l)_{l\in\Phi}, \quad \text{where} \quad h_l = \sum_{\substack{u,v\in\Phi\\ u+v=l}} f_u(g_v[u]).$$

The associativity of the multiplication on $R(\mathcal{T},\Phi,X)$ is guaranteed by the admissibility of Φ. The condition $0\in\Phi$ ensures that this algebra has identity. Following [**61**], the algebra $R(\mathcal{T},\Phi,X)$ is called the Φ-*Auslander–Yoneda algebra* of X.

Auslander–Yoneda algebras capture many known classes of algebras. Let $\mathcal{T}=\mathscr{D}(A)$ with A a ring, and let X be an A-module, if $\Phi=\mathbb{N}$, then $R(\mathcal{T},\mathbb{N},X)$ is just the Yoneda algebra $\mathrm{Ext}^*_A(X)$ of X with the usual concatenation of exact sequences as its multiplication. If $\Phi=\{0,1,\dots,n\}$, then $R(\mathcal{T},\Phi,X)$ is the quotient algebra of the Yoneda algebra $\mathrm{Ext}^*_A(X)$ of X by the ideal $\bigoplus_{i>n}\mathrm{Ext}^i_A(X,X)$. If $\Phi=\{0,a\}$, then $R(\mathcal{T},\Phi,X)$ is precisely the trivial extension of $\mathrm{End}_A(X)$ by the bimodule $\mathrm{Ext}^a_A(X,X)$. If $\Phi=n\mathbb{N}$, then $R(\mathcal{T},\Phi,X)$ is just the nth Veronese algebra of $\mathrm{Ext}^*_A(X)$. For properties of Veronese algebras of graded algebras in commutative algebra, we refer to [**9**].

We warn the reader that, in general, $R(\mathcal{T},\Phi,X)$ is neither a subalgebra nor a quotient algebra of the Yoneda algebra $R(\mathcal{T},\mathbb{N},X)$ of X. For example, if we take $\Phi=\{0,1,3\}$, $A=k[t]/(t^2)$ and $X=k$, then $R(\mathcal{T},\Phi,X)$ is neither a subalgebra nor a quotient of the Yoneda algebra $\mathrm{Ext}^*_{\mathcal{T}}(X)$ of X.

Given an admissible subset Φ of $\mathbb{N}$ (or $\mathbb{Z}$), one may define a Φ-orbit category $\mathcal{T}/\Phi$ of $\mathcal{T}$ in a natural way: The objects of $\mathcal{T}/\Phi$ are the same as the objects of $\mathcal{T}$, the Hom-set of two objects X and Y is defined as

$$\mathrm{Hom}_{\mathcal{T}/\Phi}(X,Y) := \bigoplus_{i\in\Phi}\mathrm{Hom}_{\mathcal{T}}(X,Y[i]),$$

and the composition of morphisms is defined in an obvious way: For $f\in\mathrm{Hom}_{\mathcal{T}}(X,Y[i]), g\in \mathrm{Hom}_{\mathcal{T}}(X,Y[j])$, with $i,j\in\Phi$, we define

$$f\cdot g = \begin{cases} f\cdot g[i] & \text{if } i+j\in\Phi,\\ 0 & \text{otherwise.}\end{cases}$$

Now, one has to check the associativity of this composition, but this is guaranteed by the admissibility of Φ (see [**61**] for details). So the Auslander–Yoneda algebra $R(\mathcal{T},\Phi,X)$ is nothing else than the endomorphism algebra of X in the Φ-orbit category $\mathcal{T}/\Phi$ of $\mathcal{T}$. Here, an open question arises naturally and is left to the interested reader:

Question: For which admissible sets Φ of $\mathbb{Z}$ are the orbit categories $\mathcal{T}/\Phi$ of the triangulated category $\mathcal{T}$ again triangulated?

This seems to be a hard question. For further information on orbit triangulated categories, we refer to the paper [**75**].

Now, we fix an admissible subset Φ of $\mathbb{N}$, a triangulated k-category $\mathcal{T}$ with k a field, an object M in $\mathcal{T}$ and a triangle in $\mathcal{T}$:

$$X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \longrightarrow X[1].$$

Then we define a diagonal morphism $\tilde{\alpha}$ and a skew-diagonal morphism $\tilde{\beta}$.

$$\tilde{\alpha} = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} : X \oplus M \to M_1 \oplus M, \quad \tilde{\beta} = \begin{pmatrix} 0 & \beta \\ 1 & 0 \end{pmatrix} : M_1 \oplus M \to M \oplus Y.$$

Recall from [**61**] that a morphism $f : X \to D$ in $\mathcal{T}$ is called a *left* $(\mathrm{add}(M), \Phi)$*-approximation* of X if $D \in \mathrm{add}(M)$ and for any $i \in \Phi$ and any morphism $h : X \to D'[i]$ with $D' \in \mathrm{add}(M)$, there is a morphism $h' : D \to D'[i]$ such that $h = fh'$. Similarly, we have the definition of *right* $(\mathrm{add}(M), \Phi)$*-approximations*. So an $(\mathrm{add}(M), \Phi)$-approximation of X must be an $\mathrm{add}(M)$-approximation of X in the sense of Auslander–Smalø (see Section 2.1). Thus $(\mathrm{add}(M), \Phi)$-approximations might be considered as a kind of cohomological approximation if we think of $\mathrm{Hom}_{\mathscr{D}(A)}(X, Y[n])$ being just the cohomological group $\mathrm{Ext}^n_A(X, Y)$ for A-modules X and Y.

Compared with Theorem 4.3, the following result provides a derived equivalence between quotient algebras of Auslander–Yoneda algebras, which is constructed from triangles in a triangulated category.

THEOREM 4.8 [**58**]. *For the above-given triangle in $\mathcal{T}$, assume that the two conditions are satisfied.*

(i) $\mathrm{Hom}_{\mathcal{T}}(M, X[i]) = \mathrm{Hom}_{\mathcal{T}}(Y[-i], M) = 0$ *for all* $0 \neq i \in \Phi$, *and*

(ii) *the morphism α is a left* $(\mathrm{add}(M), \Phi)$*-approximation of X and β is a right* $(\mathrm{add}(M), -\Phi)$*-approximation of Y.*

Let I be the ideal of $R(\mathcal{T}, \Phi, M \oplus X)$ consisting of all elements $(x_i)_{i\in\Phi}$ such that $x_i = 0$ for $0 \neq i \in \Phi$, x_0 factorizes through $\mathrm{add}(M)$ *and $x_0\tilde{\alpha} = 0$, and let J be the ideal of $R(\mathcal{T}, \Phi, M \oplus Y)$ consisting of all elements $(y_i)_{i\in\Phi}$ such that $y_i = 0$ for $0 \neq i \in \Phi$, y_0 factorizes through* $\mathrm{add}(M)$ *and $\tilde{\beta}y_0 = 0$. Then $R(\mathcal{T}, \Phi, M \oplus X)/I$ and $R(\mathcal{T}, \Phi, M \oplus Y)/J$ are derived equivalent.*

Theorem 4.8 supplies a large class of derived equivalences by flexible choices of Φ. Moreover, there are many cases where both I and J vanish. For example, when dealing with exact sequences in the module categories of rings, we have $I = 0$ and $J = 0$.

COROLLARY 4.9 [**58**]. *Let A be an Artin algebra and $M \in A$-mod. Suppose*

$$0 \to X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \to 0$$

is an exact sequence in A-mod such that α is a left $(\mathrm{add}(M), \Phi)$*-approximation of X and β is a right* $(\mathrm{add}(M), -\Phi)$*-approximation of Y in $\mathscr{D}^{\mathrm{b}}(A\text{-mod})$, and that* $\mathrm{Ext}^i_A(M, X) = 0 = \mathrm{Ext}^i_A(Y, M)$ *for all $0 \neq i \in \Phi$. Then the Φ-Auslander–Yoneda algebras of $X \oplus M$ and $M \oplus Y$ are derived equivalent.*

If $\Phi = \{0\}$ and the sequence in Corollary 4.9 is an almost split sequence, then we recover the derived equivalence in Corollary 4.4. Another special case of Corollary 4.9 is the situation of self-injective algebras. So we re-obtain the result [**61**, Corollary 3.4].

COROLLARY 4.10 [**61**]. *If A is a self-injective Artin algebra and X is an A-module, then the Φ-Auslander–Yoneda algebras of $A \oplus X$ and $A \oplus \Omega^i(X)$ are derived equivalent for any admissible subset Φ of $\mathbb{N}$ and any integer i, where Ω is the syzygy operator.*

Starting with an algebraic triangulated category $\mathcal{T}$, a triangle endofunctor $F : \mathcal{T} \to \mathcal{T}$ and a finite admissible subset Φ of $\mathbb{N}$, Pan and Peng defined the Φ*-Beilinson–Green algebra* $G(\Phi, X)$ associated to an object X of $\mathcal{T}$: It is a lower triangular matrix ring with (i, j)-entry $\mathrm{Hom}_{\mathcal{T}}(X, F^{i-j}X)$ when $i - j \in \Phi$ and 0 otherwise. In a way similar to Corollary 4.9, it was shown in [**101**] that one can get a derived equivalence of Φ-Beilinson–Green algebras from a triangle in $\mathcal{T}$ with certain approximation properties. The assumption that the triangulated category considered is algebraic has played a role in the discussion in [**101**]. The interested reader can consult the details in [**40**, **101**].

There are further generalizations of Theorem 4.8 in two directions: One is to extend it to n-angulated categories which are more general than triangulated categories. This was carried out in [**31**]. The other is to introduce one or two auto-functors of $\mathcal{T}$ into the definition of Φ-Auslander–Yoneda algebras, so that results on derived equivalences can be applied in a more general context (see [**61**, Appendix A]). For example, when the Auslander–Reiten translation on derived module categories of hereditary algebras is involved, then preprojective algebras, introduced by Gelfand and Ponomarev in [**44**], are covered (see, for example, [**10**]). For further details of these generalizations, the interested reader is referred to the papers [**31**, **61**].

Very recently, Yiping Chen and Wei Hu use ghost ideals and further the construction of derived equivalences from short 'exact' sequences to a very wide variety of triangles. For details, we refer to the paper [**33**].

4.4. *Derived equivalences for symmetric algebras*

In this section, we present two constructions of derived equivalences for symmetric algebras given by Dugas in [**40**] and Grant in [**45**], respectively.

Suppose that $\mathcal{T}$ is an algebraic, Krull–Remak–Schmidt triangulated k-category. Let $\mathcal{D}$ be a full subcategory of $\mathcal{T}$. We denote by $\langle\mathcal{D}\rangle$ the smallest additive subcategory of $\mathcal{T}$ containing $\cup_{i\in\mathbb{Z}}\mathcal{D}[i]$. In [**40**], a slightly different approximation was introduced and the following result was proved.

THEOREM 4.11 [**40**]. *If $\mathcal{T}$ contains a triangle*

$$X \xrightarrow{f} M' \xrightarrow{g} Y \longrightarrow X[1],$$

with $M'\in\langle M\rangle$ for some $M\in\mathcal{T}$, satisfying the two conditions

(a) *f is a left $\langle M\rangle$-approximation of X; and*
(b) *g is a right $\langle M\rangle$-approximation of Y,*

then

(1) *$R(\mathcal{T},\mathbb{Z},X\oplus M)$ and $R(\mathcal{T},\mathbb{Z},M\oplus Y)$ are derived equivalent.*
(2) *For any $M''\in\langle M\rangle$ with $M'\in\operatorname{add}(M'')$, the rings $\Lambda:=\operatorname{End}_{\mathcal{T}}(M''\oplus X)$ and $\Gamma:=\operatorname{End}_{\mathcal{T}}(M''\oplus Y)$ are derived equivalent.*

One ingredient of the proof of Theorem 4.11 is based on an analysis of uniqueness of the completion of h in (TR3) of Definition 2.1. Here, the assumption that the triangulated category considered is algebraic plays a role in the discussion.

Compared with Theorem 4.8, Theorem 4.11 provides derived equivalences between two endomorphism algebras instead of quotient algebras, and can also be applied to get derived equivalences between symmetric algebras.

COROLLARY 4.12 [**40**, Theorem 5.2]. *Suppose that A is a finite-dimensional, symmetric k-algebra over a field k and that X and M are any complexes in $\mathscr{K}^{\mathrm{b}}(A\text{-proj})$. Then there exists a left $\langle M\rangle$-approximation $f:X\to M'$ of X in $\mathscr{K}(A)$. If Y is the mapping cone of f, then*

(1) *$R(\mathscr{K}(A),\mathbb{Z},X\oplus M)$ and $R(\mathscr{K}(A),\mathbb{Z},Y\oplus M)$ are derived equivalent, symmetric algebras.*

(2) $\mathrm{End}_{\mathscr{K}(A)}(X \oplus M'')$ *and* $\mathrm{End}_{\mathscr{K}(A)}(Y \oplus M'')$ *are derived equivalent, symmetric algebras for any* $M'' \in \langle M \rangle$ *with* $M' \in \mathrm{add}(M'')$.

A surprising point in this result is the symmetry of the endomorphism algebras. However, this follows from the following simple but very interesting observation.

PROPOSITION 4.13 [**40**, Proposition 5.1]. *Let A be a finite-dimensional, symmetric k-algebra over a field k. Then, for any $X^\bullet \in \mathscr{K}^{\mathrm{b}}(A\text{-proj})$, the rings $\mathrm{End}_{\mathscr{K}(A)}(X^\bullet)$ and $R(\mathscr{K}(A), \mathbb{Z}, X^\bullet)$ are finite-dimensional, symmetric k-algebras.*

For applications of this construction in Calabi–Yau categories, one may find details in [**40**].

Now, we mention another construction of derived equivalences given by Grant in [**45**]. Let A be a symmetric algebra and P a projective A-module. Suppose that $E := \mathrm{End}_A(P)$ is a twisted-periodic algebra of period n, that is, there exists an integer $n \geqslant 1$, an automorphism σ of the algebra E and an exact sequence

$$P_{n-1} \xrightarrow{d_{n-1}} P_{n-2} \longrightarrow \cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0$$

of projective E-E-bimodules such that $\mathrm{Coker}(d_1) = E$ and $\mathrm{Ker}(d_{n-1})$ is the σ-twisted bimodule E_σ. Here the underlying k-space of E_σ is E, the left E-module structure on E_σ is the regular one and the right E-module structure is induced from the automorphism σ. We define $P^* := \mathrm{Hom}_A(P, A)$ and denote by $Y^\bullet$ the complex

$$P_{n-1} \xrightarrow{d_{n-1}} P_{n-2} \longrightarrow \cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0,$$

where P_i is in degree $-i$. Let $f : Y^\bullet \to E$ be the morphism of chain complexes of E-E-bimodules given by the isomorphism $\mathrm{Coker}(d_1) \simeq E$, and let g be the following composition of morphisms of complexes of A-A-bimodules:

$$P \otimes_E Y^\bullet \otimes P^* \xrightarrow{P\otimes f \otimes P^*} P \otimes_E E \otimes_E P^* \longrightarrow P \otimes_E P^* \xrightarrow{ev} A,$$

where ev is the evaluation map. Now, one defines $X^\bullet$ to be the mapping cone of g and puts $\Psi_P := X^\bullet \otimes_A^{\mathbf{L}} - : \mathscr{D}^{\mathrm{b}}(A) \to \mathscr{D}^{\mathrm{b}}(A)$.

THEOREM 4.14 [**45**, Theorem 3.9]. *If A is a symmetric algebra over a field, P is a projective A-module and $E := \mathrm{End}_A(P)$ is a twisted-periodic algebra of period n, then $\Psi_P : \mathscr{D}^{\mathrm{b}}(A\text{-mod}) \to \mathscr{D}^{\mathrm{b}}(A\text{-mod})$ is an equivalence of triangulated categories.*

This result tells us that for a symmetric algebra, there are many possibilities to get auto-equivalences of its derived module category. By choosing a special class of projective modules P, some groups can be realized by the corresponding auto-equivalences if one passes to the Grothendieck group of $\mathscr{D}^{\mathrm{b}}(A\text{-mod})$. This provides a kind of categorifications of these groups. It would be interesting to know how all of the functors Ψ_P for all $P \in A$-proj are related to each other. For relations of this construction to the auto-equivalences given by Seidel–Thomas and Huybrechts–Thomas, one is referred to the original paper [**45**].

4.5. *Derived equivalences for Auslander–Yoneda algebras and quotient algebras*

In this section, we introduce two methods to construct new derived equivalences from given ones. One is to form endomorphism algebras, or more generally, Φ-Auslander–Yoneda algebras, and the other is to form quotient algebras of derived equivalent algebras.

Throughout this section, we consider $\mathcal{T} = \mathscr{D}^{\mathrm{b}}(A\text{-mod})$ with A an Artin algebra. In this case, the Φ-Auslander–Yoneda algebra of $X \in A$-mod has the underlying k-space $\bigoplus_{i\in\Phi} \mathrm{Ext}_A^i(X, X)$.

To distinguish which module category we are working with, the Φ-Auslander–Yoneda algebra of X will be written simply as $R(A, \Phi, X)$.

Suppose that $F : \mathscr{D}^{\mathrm{b}}(A\text{-mod}) \to \mathscr{D}^{\mathrm{b}}(B\text{-mod})$ is a derived equivalence between two Artin algebras A and B, with the quasi-inverse functor G. Further, suppose

$$T^{\bullet}: \quad \cdots \longrightarrow 0 \longrightarrow T^{-n} \longrightarrow \cdots \longrightarrow T^{-1} \longrightarrow T^{0} \longrightarrow 0 \longrightarrow \cdots$$

is a radical, tilting complex over A associated to the functor F, and suppose

$$\bar{T}^{\bullet}: \quad \cdots \longrightarrow 0 \longrightarrow \bar{T}^{0} \longrightarrow \bar{T}^{1} \longrightarrow \cdots \longrightarrow \bar{T}^{n} \longrightarrow 0 \longrightarrow \cdots$$

is a radical, tilting complex over B associated to the functor G. We say that the functor F is *almost ν-stable* if $\mathrm{add}(\bigoplus_{i=-1}^{-n} T^i) = \mathrm{add}(\bigoplus_{i=-1}^{-n} \nu_A T^i)$, and $\mathrm{add}(\bigoplus_{i=1}^{n} \bar{T}^i) = \mathrm{add}(\bigoplus_{i=1}^{n} \nu_B \bar{T}^i)$, where ν_A is the Nakayama functor for A.

Given an almost ν-stable derived equivalence $F : \mathscr{D}^{\mathrm{b}}(A) \to \mathscr{D}^{\mathrm{b}}(B)$ between Artin algebras A and B, it was shown in [**59**] that there exists an equivalence functor $\bar{F}$, associated to F, between the stable module categories A-$\underline{\mathrm{mod}}$ and B-$\underline{\mathrm{mod}}$.

THEOREM 4.15 [**61**, Theorem 1.1]. *Suppose that F is an almost ν-stable derived equivalence between Artin algebras A and B. Let $\bar{F} : A\text{-}\underline{\mathrm{mod}} \to B\text{-}\underline{\mathrm{mod}}$ is the stable equivalence associated to F. Let X be an A-module, and define $M := A \oplus X$ and $N := B \oplus \bar{F}(X)$. Further, let Φ be an admissible subset of $\mathbb{N}$. Then*

(1) *$R(A, \Phi, M)$ and $R(B, \Phi, N)$ are derived equivalent.*
(2) *If Φ is finite, then there exists an almost ν-stable derived equivalence between the Φ-Auslander–Yoneda algebras $R(A, \Phi, M)$ and $R(B, \Phi, N)$. In particular, there is an almost ν-stable derived equivalence and a stable equivalence between $\mathrm{End}_A(M)$ and $\mathrm{End}_B(N)$.*

Recall that an *Auslander algebra* is an Artin algebra A such that its global dimension of A is at most 2 and its dominant dimension is at least 2 (for the definition of dominant dimensions, see Section 4.2). Auslander algebras have played an important role in studying representation-finite algebras. In fact, Auslander proved that an Artin algebra A is an Auslander algebras if and only if it is Morita equivalent to an algebra of the form $\mathrm{End}_B(X)$, where B is a representation-finite Artin algebra and X is the direct sum of all non-isomorphic indecomposable B-modules. In this case, X is called an *additive generator* for B-mod. Since Auslander algebras and Yoneda algebras are special classes of Φ-Auslander–Yoneda algebras, Theorem 4.15 supplies a lot of examples of derived equivalences between Auslander algebras, and between Yoneda algebras. For instance, we have the following corollary for self-injective algebras.

COROLLARY 4.16 [**61**]. *Suppose that A and B are self-injective Artin algebras of finite representation type with ${}_AX$ and ${}_BY$ additive generators for A-mod and B-mod, respectively. If A and B are derived equivalent, then*

(i) *the Auslander algebras of A and B are both derived and stably equivalent.*
(ii) *The Yoneda algebra $\mathrm{Ext}_A^*(X)$ of X and the Yoneda algebra $\mathrm{Ext}_B^*(Y)$ of Y are derived equivalent.*

Note that for self-injective algebras, every derived equivalence between them is almost ν-stable (up to shift), and the syzygy functor on stable categories is closely related to the auto-equivalence functor $K \otimes_A -$, where K is a kernel of the multiplication map $A \otimes_k A \to A$. This explains why Theorem 4.15 can be applied to self-injective algebras.

Another natural idea for getting derived equivalences from given ones is to pass to quotient algebras.

Suppose that A is an Artin algebra and I is an ideal in A. Let $\overline{A} := A/I$. Then the category $\overline{A}$-mod can be regarded as a full subcategory of A-mod. Moreover, there is a canonical functor from A-mod to $\overline{A}$-mod which sends $X \in A$-mod to $\overline{X} := X/IX$. This functor induces a functor $^- : \mathscr{C}(A) \to \mathscr{C}(\overline{A})$, which sends $X^\bullet$ to the quotient complex $\overline{X}^\bullet := X^\bullet/IX^\bullet$, where $IX^\bullet = (IX^i)_{i\in\mathbb{Z}}$ is a sub-complex of $X^\bullet$. The action of $^-$ on a chain map can be defined canonically. For a complex $X^\bullet \in \mathscr{C}(A)$, there is a canonical exact sequence of complexes of A-modules:

$$0 \longrightarrow IX^\bullet \xrightarrow{i^\bullet} X^\bullet \xrightarrow{\pi^\bullet} \overline{X}^\bullet \longrightarrow 0.$$

Thus, for a complex $Y^\bullet$ of $\overline{A}$-modules, there is the exact sequence:

$$0 \longrightarrow \mathrm{Hom}_{\mathscr{C}(A)}(\overline{X}^\bullet, Y^\bullet) \xrightarrow{\pi^*} \mathrm{Hom}_{\mathscr{C}(A)}(X^\bullet, Y^\bullet) \xrightarrow{i^*} \mathrm{Hom}_{\mathscr{C}(A)}(IX^\bullet, Y^\bullet).$$

Now, it follows from $Y^\bullet \in \mathscr{C}(\overline{A})$ that the map $i^* = 0$, and consequently π^* is an isomorphism. Moreover, π^* actually induces an isomorphism between $\mathrm{Hom}_{\mathscr{K}(A)}(\overline{X}^\bullet, Y^\bullet)$ and $\mathrm{Hom}_{\mathscr{K}(A)}(X^\bullet, Y^\bullet)$.

Let $X^\bullet$ and $X'^\bullet$ be two complexes of A-modules. Then there is a natural map

$$\eta : \mathrm{Hom}_{\mathscr{K}(A)}(X^\bullet, X'^\bullet) \longrightarrow \mathrm{Hom}_{\mathscr{K}(A)}(\overline{X}^\bullet, \overline{X'}^\bullet),$$

which is the composition of $\pi^\bullet_* : \mathrm{Hom}_{\mathscr{K}(A)}(X^\bullet, X'^\bullet) \to \mathrm{Hom}_{\mathscr{K}(A)}(X^\bullet, \overline{X'}^\bullet)$ with the map $(\pi^*)^{-1}$. In particular, if $X^\bullet = X'^\bullet$, then we get a homomorphism of algebras

$$\eta : \mathrm{End}_{\mathscr{K}(A)}(X^\bullet) \longrightarrow \mathrm{End}_{\mathscr{K}(A)}(\overline{X}^\bullet).$$

Now, let $T^\bullet$ be a tilting complex over A, and let $B := \mathrm{End}_{\mathscr{K}(A)}(T^\bullet)$. By the above discussion, there is an algebra homomorphism

$$\eta : \mathrm{End}_{\mathscr{K}(A)}(T^\bullet) \longrightarrow \mathrm{End}_{\mathscr{K}(A)}(\overline{T}^\bullet).$$

Let J_I be the kernel of η, which is an ideal of B. We define $\overline{B} := B/J_I$. Then we have the following derived equivalence for quotient algebras.

THEOREM 4.17 [**61**]. *Let A be an Artin algebra and I an ideal in A, and let $T^\bullet$ be a tilting complex over A with the endomorphism algebra $B = \mathrm{End}_{\mathscr{K}^{\mathrm{b}}(A)}(T^\bullet)$. Then $\overline{T}^\bullet$ is a tilting complex over $\overline{A}$ and induces a derived equivalence between $\overline{A}$ and $\overline{B}$ if and only if $\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A)}(T^\bullet, IT^\bullet[i]) = 0$ for all $i \neq 0$ and $\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A)}(\overline{T}^\bullet, \overline{T}^\bullet[-1]) = 0$.*

Applying Theorem 4.17 to self-injective algebras, we can obtain derived equivalences between quotient algebras.

COROLLARY 4.18 [**61**]. *Let $F : \mathscr{D}^{\mathrm{b}}(A\text{-mod}) \to \mathscr{D}^{\mathrm{b}}(B\text{-mod})$ be a derived equivalence between two self-injective, basic Artin algebras A and B. Suppose that P is a direct summand of ${}_AA$, and Q is a direct summand of ${}_BB$ such that $F(\mathrm{soc}(P))$ is isomorphic to $\mathrm{soc}(Q)$, where $\mathrm{soc}(P)$ denotes the socle of the module P. Then the quotient algebras $A/\mathrm{soc}(P)$ and $B/\mathrm{soc}(Q)$ are derived equivalent.*

Note that the socle of an indecomposable, projective left ideal of a basic algebra A is always an ideal of A.

In [**89**] the author considered the question of constructing tilting complexes over an algebra A from the ones over its subalgebra B of A. In this case, one assumes conditions on given tilting complexes or on the extension $B \subseteq A$ of algebras. For details, we refer to the original paper [**89**].

4.6. *Ladkani's construction of derived equivalences*

We mention a systematic tool to construct new tilting complexes from existing ones using tensor products. This construction is due to Ladkani (see [**80**]) and generalizes a result of Rickard [**110**]. Here, we follow the approach in [**80**].

Suppose that A and B are k-algebras over a commutative ring k such that B is projective as a k-module. Fix a tilting complex $U^\bullet$ of projective B-modules such that the endomorphism algebra of $U^\bullet$ is projective as a k-module. Then, for any tilting complex $T^\bullet$ of A-modules, Rickard showed in [**110**, Theorem 2.1] that $T^\bullet \otimes_k U^\bullet$ is a tilting complex over the tensor product $A \otimes_k B$, with the endomorphism algebra of the form $\mathrm{End}_{\mathscr{D}^{\mathrm{b}}(A)}(T^\bullet) \otimes_k \mathrm{End}_{\mathscr{D}^{\mathrm{b}}(B)}(U^\bullet)$. Here the tensor product $T^\bullet \otimes_k U^\bullet$ of two complexes $T^\bullet$ and $U^\bullet$ means the total complex of the double complex with (i,j)-term $U^i \otimes_k T^j$.

Now, assume that $U^\bullet$ decomposes as $U^\bullet = U_1^\bullet \oplus U_2^\bullet \oplus \cdots \oplus U_n^\bullet$. Then $T^\bullet \otimes_k U^\bullet = (T^\bullet \otimes_k U_1^\bullet) \oplus (T^\bullet \otimes_k U_2^\bullet) \oplus \cdots \oplus (T^\bullet \otimes_k U_n^\bullet)$. Instead of taking just one tilting complex $T^\bullet$, one may take n tilting complexes over A, say $T_1^\bullet, T_2^\bullet, \ldots, T_n^\bullet$, and replace each $T^\bullet \otimes_k U_i^\bullet$ by $T_i^\bullet \otimes_k U_i^\bullet$. Ladkani proved the following result.

THEOREM 4.19 [**80**, Theorem A]. *Let k be a commutative ring and let A and B be two k-algebras, with B projective as a k-module. Let $U_1^\bullet, U_2^\bullet, \ldots, U_n^\bullet$ be complexes in $\mathscr{D}^{\mathrm{b}}(B)$ satisfying the conditions:*

(i) *$U^\bullet = U_1^\bullet \oplus U_2^\bullet \oplus \cdots \oplus U_n^\bullet$ is a tilting complex in $\mathscr{D}^{\mathrm{b}}(B)$;*
(ii) *the terms U^i of the complex $U^\bullet = (U^i)$ are projective as k-modules;*
(iii) *the endomorphism k-algebra $\mathrm{End}_{\mathscr{D}(B)}(U^\bullet)$ is projective as k-module.*

Let $T_1^\bullet, T_2^\bullet, \ldots, T_n^\bullet$ be tilting complexes over A such that, for any $1 \leqslant i, j \leqslant n$, $\mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A)}(T_i^\bullet, T_j^\bullet[r]) = 0$ for any $r \neq 0$, whenever $\mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(B)}(U_i^\bullet, U_j^\bullet) \neq 0$. Then the complex $(T_1^\bullet \otimes_k U_1^\bullet) \oplus (T_2^\bullet \otimes_k U_2^\bullet) \oplus \cdots \oplus (T_n^\bullet \otimes_k U_n^\bullet)$ is a tilting complex over $A \otimes_k B$, and its endomorphism algebra is given by the matrix algebra

$$\begin{pmatrix} M_{11} & M_{12} & \cdots & M_{1n} \\ M_{21} & M_{22} & \cdots & M_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ M_{n1} & M_{n2} & \cdots & M_{nn} \end{pmatrix}$$

where $M_{ij} := \mathrm{Hom}_{\mathscr{D}(A)}(T_i^\bullet, T_j^\bullet) \otimes_k \mathrm{Hom}_{\mathscr{D}(B)}(U_i^\bullet, U_j^\bullet)$ and the multiplication maps $M_{ij} \otimes_k M_{jl} \to M_{il}$ are given by the obvious compositions.

We should notice that, in general, the above matrix algebra is not a tensor product of two matrix algebras, but rather a componentwise tensor product.

Theorem 4.19 has quite a lot of applications, showing that algebras with quivers being lines can be derived equivalent to algebras with quivers 'rectangle'. As another consequence of Theorem 4.19, we mention the following corollary.

THEOREM 4.20 [**80**, Theorem B]. *Let A be a ring, and let $T_1^\bullet, T_2^\bullet, \ldots, T_n^\bullet$ be tilting complexes over A satisfying $\mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A)}(T_i^\bullet, T_j^\bullet[r]) = 0$ for all $1 \leqslant i < j \leqslant n$ and $r \neq 0$. Then the upper-triangular matrix algebra $T_n(A)$ over A is derived equivalent to the matrix algebra*

$$\begin{pmatrix} \mathrm{End}_{\mathscr{D}^{\mathrm{b}}(A)}(T_1^\bullet) & \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A)}(T_1^\bullet, T_2^\bullet) & \cdots & \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A)}(T_1^\bullet, T_n^\bullet) \\ 0 & \mathrm{End}_{\mathscr{D}^{\mathrm{b}}(A)}(T_2^\bullet) & \cdots & \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A)}(T_2^\bullet, T_n^\bullet) \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \mathrm{End}_{\mathscr{D}^{\mathrm{b}}(A)}(T_n^\bullet) \end{pmatrix}.$$

For more sophisticated applications and calculations of Calabi–Yau dimensions of derived equivalent algebras constructed in this way, we refer the reader to the paper [**80**].

4.7. *Derived equivalences for pullback algebras*

Pullbacks of algebras were used widely in mathematics. In particular, they were applied by Milnor to establish a Mayer–Vietoris exact sequence of algebraic K-groups of rings involved in a pullback diagram. A key ingredient in Milnor's investigation is that projective modules over a pullback algebra can be patched from the ones over constituent algebras (see [**88**, Chapter 2]). In this section, we survey methods of constructing tilting complexes and derived equivalences for pullback algebras through the ones over their constituent algebras. For the details of proofs, we refer the reader to [**62**].

Given homomorphisms $\pi_1 : A_1 \to A_0$ and $\pi_2 : A_2 \to A_0$ of algebras, we may form the pullback algebra A of π_1 and π_2:

$$\begin{array}{ccc} A & \xrightarrow{\lambda_1} & A_1 \\ {\scriptstyle\lambda_2}\downarrow & & \downarrow{\scriptstyle\pi_1} \\ A_2 & \xrightarrow{\pi_2} & A_0 \end{array}$$

that is, $A := \{(x, y) \in A_1 \oplus A_2 \mid (x)\pi_1 = (y)\pi_2\}$, and for any homomorphisms $f_i : B \to A_i$ of algebras with $f_1\pi_1 = f_2\pi_2$, there is a unique homomorphism $f : B \to A$ of algebras such that $f_i = f\lambda_i$ for $i = 1, 2$.

Recall that a *Milnor square* of algebras is a pullback diagram of homomorphisms of algebras such that one of π_1 and π_2 is surjective.

Concerning construction of derived equivalences for pullback algebras, the following result is proved in [**62**].

THEOREM 4.21 [**62**]. *Given the above Milnor square of Artin algebras with π_1 surjective, let $T_i^\bullet$ be a basic, radical tilting complex over A_i with $B_i := \mathrm{End}_{\mathscr{K}^{\mathrm{b}}(A_i)}(T_i^\bullet)$ for $0 \leqslant i \leqslant 2$. If there is an isomorphism $A_0 \otimes^\bullet_{A_i} T_i^\bullet \simeq T_0^\bullet$ of complexes for $i = 1, 2$, such that $T_0^\bullet$ is a direct sum of shifts of projective A_0-modules (for example, A_0 is semisimple), then there exist homomorphisms $B_1 \xrightarrow{\eta_1} B_0 \xleftarrow{\eta_2} B_2$ of Artin algebras with η_1 surjective such that the pullback algebra B of η_1 and η_2 is derived equivalent to the pullback algebra A of π_1 and π_2.*

Theorem 4.21 provides us with a machinery to produce many new derived equivalences from given ones. For example, one can glue vertices or unify arrows or identify socle elements to produce new derived equivalences for resulting algebras (see [**62**, Sections 4] for details). In the following we shall survey the case of gluing vertices as an illustration of Theorem 4.21.

Let $A = kQ/\langle\rho\rangle$ be a finite-dimensional algebra over a field k given by quiver $Q = (Q_0, Q_1)$ with relations ρ. Let X be a subset of Q_0, we denote by e_X the idempotent element $\sum_{i\in X} e_i$ in A, where e_i is the primitive idempotent element of A corresponding to the vertex i in Q. Suppose that $\sigma = \{\sigma_1, \dots, \sigma_m\}$ is a partition of X, that is, $\sigma_1, \dots, \sigma_m$ are subsets of X such that $X = \cup_i\sigma_i$ and $\sigma_i \cap \sigma_j = \varnothing$ for $i \neq j$. Let Q^σ be the quiver obtained from Q by just gluing the vertices in σ_t into one vertex, denoted also by σ_t, for $1 \leqslant t \leqslant m$, and keeping all arrows. This means that an arrow in Q_1 with the starting vertex in σ_i and the ending vertex in σ_j becomes an arrows in Q^σ with the starting vertex σ_i and the ending vertex σ_j. Thus the vertex set of Q^σ is the union of the set $Q_0 \setminus X$ and the set $\{\sigma_1, \sigma_2, \dots, \sigma_m\}$, while the arrow set of Q^σ is Q_1 (but the starting and ending vertices may be changed). Then there is a natural homomorphism of algebras:

$$\lambda_\sigma : kQ^\sigma \longrightarrow kQ/\langle\rho\rangle$$

which sends e_i to e_i for $i \notin X$, e_{σ_t} to $\sum_{i\in\sigma_t} e_i$ for $1 \leqslant t \leqslant m$ and preserves all arrows. Clearly, the kernel of λ_σ is contained in $\langle Q_2^\sigma\rangle \subseteq kQ^\sigma$, where Q_2^σ stands for the set of all paths of length 2 of the quiver Q^σ. Let ρ^σ be a set of relations on Q^σ such that $\langle\rho^\sigma\rangle = \mathrm{Ker}(\lambda_\sigma)$. The relations ρ^σ can be obtained in the following way: For each t, let ρ^{σ_t} be the set of relations on Q^σ consisting of all $\alpha\beta$ with α, β being arrows in Q_1 such that the ending vertex of α and the starting vertex of β are different in σ_t. Then $\rho^\sigma = \rho \cup \rho^{\sigma_1} \cup \cdots \cup \rho^{\sigma_m}$. The algebra $A^\sigma := kQ^\sigma/\langle\rho^\sigma\rangle$ is called the *σ-gluing algebra* of A. The above homomorphism λ_σ induces a homomorphism from A^σ to A, denoted again by λ_σ. Observe that $\lambda_\sigma : A^\sigma \to A$ is injective and the image of λ_σ is the subalgebra of A generated by all arrows in Q, the idempotents $\sum_{i\in\sigma_t} e_i$ for $1 \leqslant t \leqslant m$ and $\{e_i \mid i \in Q_0\backslash X\}$. Note that A^σ and A have the same Jacobson radical. This construction has been used in the study of the finitistic dimension conjecture (for example, see [**124**]).

From the definition of gluing algebras, we have the pullback diagram of algebras:

$$\begin{tikzcd} kQ^\sigma/\langle\rho^\sigma\rangle \arrow[r, "\lambda_\sigma"] \arrow[d, "\pi"'] & kQ/\langle\rho\rangle \arrow[d, "\pi"] \\ k^\sigma \arrow[r, "\lambda_\sigma"] & k^X \end{tikzcd}$$

where $\pi : kQ/\langle\rho\rangle \to k^X$ is the canonical homomorphism of algebras, which sends e_i to e_i for $i \in X$, all other idempotents and all arrows to zero. Note that k^X is the semisimple algebra $\bigoplus_{i\in X} k$ and k^σ is just the σ-gluing of k^X.

COROLLARY 4.22. *Let F be a derived equivalence from an algebra $A := kQ/\langle\rho\rangle$ to another algebra $A' := kQ'/\langle\rho'\rangle$. Suppose that X is a subset of Q_0 such that the simple A-modules corresponding to the vertices in X are sent by F to simple A'-modules. Let X' be the set of vertices in Q'_0 corresponding to these simple A'-modules. Let σ be a partition of X and σ' be the corresponding partition of X'. Then the algebras A^σ and $A'^{\sigma'}$ are derived equivalent.*

In Corollary 4.22, we require that F sends some simple modules to simple modules. How can we check this condition? The following proposition provides a solution. For a bounded complex $X^\bullet = (X^i, d_X^i)$ over A-mod and an indecomposable A-module Y, let $[X^i : Y]$ be the number of indecomposable direct summands Z in a decomposition of X^i into indecomposable A-modules, such that Z is isomorphic to Y. Define

$$[X^\bullet : Y] := \sum_{i\in\mathbb{Z}} [X^i : Y].$$

Note that $[X^\bullet : Y]$ is well defined in $\mathscr{C}^b(\Lambda)$ by the Krull–Remak–Schmidt theorem.

PROPOSITION 4.23 [**62**, Lemma 2.4]. *Let $T^\bullet$ be a basic, radical tilting complex over an Artin algebra A, and let $B := \mathrm{End}_{\mathscr{K}^{\mathrm{b}}(A)}(T^\bullet)$. Suppose that $F : \mathscr{D}^{\mathrm{b}}(A\text{-mod}) \to \mathscr{D}^{\mathrm{b}}(B\text{-mod})$ is a derived equivalence induced by $T^\bullet$ and that P is an indecomposable projective A-module with top S. Then $F(S)$ is isomorphic in $\mathscr{D}^{\mathrm{b}}(B\text{-mod})$ to $S'[n]$ for a simple B-module S' and an integer n if and only if $[T^\bullet : P] = 1$.*

A special case of Corollary 4.22 is to attach an algebra simultaneously to derived equivalent algebras. In this way, the resulting algebras are again derived equivalent.

COROLLARY 4.24. *Let F be a derived equivalence between algebras $A := kQ/\langle\rho\rangle$ and $A' := kQ'/\langle\rho'\rangle$ such that F sends the simple A-modules corresponding to vertices in $\bar{Q}_0 \subseteq Q_0$ to the simple A'-modules corresponding to vertices in $\bar{Q}'_0 \subseteq Q'_0$ and $|\bar{Q}_0| = |\bar{Q}'_0|$. Suppose that $C :=$*

$k\Gamma/\langle\rho''\rangle$ is an arbitrary algebra. Let σ be a partition of $\bar{Q}_0 \dot{\cup} \Gamma_0$ and let σ' be the corresponding partition of $\bar{Q}'_0 \dot{\cup} \Gamma_0$. Then the algebras $(A \times C)^{\sigma}$ and $(A' \times C)^{\sigma'}$ are derived equivalent.

Let us display a concrete example to visualize the procedure in Corollary 4.24.

EXAMPLE 4.25. Suppose that A and B are algebras given by quivers with relations, respectively:

$$A: \begin{tikzcd} \overset{1}{\bullet} \arrow[r, shift left, "\gamma"] & \overset{2}{\bullet} \arrow[l, shift left, "\beta"] \arrow[r, shift left, "\delta"] & \overset{3}{\bullet} \arrow[l, shift left, "\alpha"] \end{tikzcd} \qquad B: \begin{tikzcd} \overset{2'}{\bullet} \arrow[r, "\gamma'"] & \overset{3'}{\bullet} \arrow[dl, "\alpha'"] \\ \underset{1'}{\bullet} \arrow[u, "\beta'"] & \end{tikzcd}$$

$$\alpha\delta\alpha, \gamma\delta, \delta\alpha - \beta\gamma; \qquad \alpha'\beta'\gamma'\alpha', \gamma'\alpha'\beta'\gamma'.$$

Then $\dim_k(A) = 12$ and $\dim_k(B) = 13$. By [**62**, Example 5.6], there is a derived equivalence between A and B, which sends the simple A-module corresponding to the vertex 3 to the simple B-module corresponding to the vertex $3'$. Now, we take an algebra C, for example, C is given by the following quiver with a relation

$$\begin{tikzcd} 4\bullet \arrow[r, "\eta"] & \bullet^{3} \arrow[loop right, "\varepsilon"] \end{tikzcd} \qquad \varepsilon^n = 0$$

and glue it at the vertices 3 and $3'$ in A and B, respectively. In this case, the partition σ in Corollary 4.24 is the one such that the vertex 3 in A and the vertex 3 in C form a part and all other parts consist of only one single vertex. Similarly, the partition σ' in Corollary 4.24 is defined. Then, by Corollary 4.24, the gluing algebras Λ and Γ, given by the following quivers with relations, respectively, are derived equivalent:

$$\Lambda: \begin{tikzcd} \overset{1}{\bullet} \arrow[r, shift left, "\gamma"] & \overset{2}{\bullet} \arrow[l, shift left, "\beta"] \arrow[r, shift left, "\delta"] & \overset{3}{\bullet} \arrow[l, shift left, "\alpha"] \arrow[loop right, "\varepsilon"] \\ & & 4\bullet \arrow[u, "\eta"] \end{tikzcd} \qquad \Gamma: \begin{tikzcd} \overset{2'}{\bullet} \arrow[r, "\gamma'"] & \overset{3'}{\bullet} \arrow[dl, "\alpha'"] \arrow[loop right, "\varepsilon'"] \\ \underset{1'}{\bullet} \arrow[u, "\beta'"] & \underset{4'}{\bullet} \arrow[u, "\eta'"] \end{tikzcd}$$

$$\alpha\delta\alpha, \gamma\delta, \delta\alpha - \beta\gamma, \varepsilon^n, \delta\varepsilon, \varepsilon\alpha, \eta\alpha; \qquad \alpha'\beta'\gamma'\alpha', \gamma'\alpha'\beta'\gamma', (\varepsilon')^n, \gamma'\varepsilon', \varepsilon\alpha', \eta'\alpha'.$$

There are other two specific operations for algebras presented by quivers with relations in [**62**], namely unifying arrows and identifying socle elements (see [**62**, Sections 4.2 and 4.3] for details). Both of them are special cases of Theorem 4.21. Here, we just mention an example of identifying socle elements. Observe that the socle element $\alpha\delta$ in A is a complete e_3-cycle, $\alpha'\beta'\gamma'$ in B is a complete e'_3-cycle and ϵ^{n-1} in C is a complete e_3-cycle (see [**62**, Section 4.3] for terminology). Thus we may identify $\alpha\delta$ with ϵ^{n-1} in $A \times C$ and $\alpha'\beta'\gamma'$ with ϵ^{n-1} in $B \times C$, respectively. By [**62**, Theorem 4.8], we get a derived equivalence between the following two algebras $\overline{\Lambda}$ and $\overline{\Gamma}$ which are quotients of Λ and Γ, respectively.

$$\overline{\Lambda}: \begin{tikzcd} \overset{1}{\bullet} \arrow[r, shift left, "\gamma"] & \overset{2}{\bullet} \arrow[l, shift left, "\beta"] \arrow[r, shift left, "\delta"] & \overset{3}{\bullet} \arrow[l, shift left, "\alpha"] \arrow[loop right, "\varepsilon"] \\ & & 4\bullet \arrow[u, "\eta"] \end{tikzcd} \qquad \overline{\Gamma}: \begin{tikzcd} \overset{2'}{\bullet} \arrow[r, "\gamma'"] & \overset{3'}{\bullet} \arrow[dl, "\alpha'"] \arrow[loop right, "\varepsilon'"] \\ \underset{1'}{\bullet} \arrow[u, "\beta'"] & \underset{4'}{\bullet} \arrow[u, "\eta'"] \end{tikzcd}$$

$$\begin{array}{c} \alpha\delta\alpha, \gamma\delta, \delta\alpha - \beta\gamma, \varepsilon^n, \delta\varepsilon, \varepsilon\alpha, \eta\alpha, \\ \alpha\delta - \varepsilon^{n-1}. \end{array} \qquad \begin{array}{c} \alpha'\beta'\gamma'\alpha', \gamma'\alpha'\beta'\gamma', (\varepsilon')^n, \gamma'\varepsilon', \varepsilon\alpha', \eta'\alpha', \\ \alpha'\beta'\gamma' - (\varepsilon')^{n-1}. \end{array}$$

Thus, through these operations, we can construct derived equivalences for resulting algebras. Moreover, these operations can be applied repeatedly. The interested reader is referred to [**62**] for explicit descriptions of these operations.

5. *Derived equivalences and stable equivalences of Morita type*

Rickard showed that if two finite-dimensional, self-injective algebras A and B are derived equivalent then they are also stably equivalent of Morita type. The converse is not true in general. So a natural question is:

How can we get a derived equivalence from a stable equivalence of Morita type between finite-dimensional algebras?

On the one hand, a positive answer to this question will provide a new way to construct derived equivalences. On the other hand, to be able to lift a stable to derived equivalence is of particular interest in an approach to Broué's abelian defect group conjecture and Auslander–Reiten conjecture on stable equivalences. Concerning the former, we refer the reader to Section 6.1. For the latter, we will have a discussion right now in Section 5.1.

For self-injective algebra A, we have $A\text{-}\underline{\mathrm{mod}} \simeq \mathscr{D}^{\mathrm{b}}(A)/\mathscr{K}^{\mathrm{b}}(A\text{-proj})$ (see [**77**, **109**]). But, for a general algebra A, we could not always get $A\text{-}\underline{\mathrm{mod}} \simeq \mathscr{D}^{\mathrm{b}}(A)/\mathscr{K}^{\mathrm{b}}(A\text{-proj})$. This obstacle leads us to introduce the notion of almost ν-stable derived equivalences (see Section 4.5), which can induce stable equivalences of Morita type (see [**59**]).

In this section we first present a method to lift stable equivalences of Morita type to derived equivalences, and then give a general method of extending derived equivalences between algebras of the form eAe to derived equivalences between given algebras A. For convenience, an algebra of the form eAe with e an idempotent in an algebra A will be called a *corner algebra*.

5.1. *Lifting stable to derived equivalences*

In the previous sections, we have seen that new derived equivalences can be constructed from given ones. In this section, we consider how to lift a stable equivalence of Morita type to a derived equivalence.

In [**6**] Asashiba showed a very interesting result which says that, for representation-finite, standard self-injective k-algebras A and B not of type $(D_{3m}, s/3, 1)$ with $m \geqslant 2$ and $3 \nmid s$, each individual stable equivalence from A to B can induce a derived equivalence from A to B. His proof is essentially based on the classification of derived equivalences for representation-finite, standard self-injective algebras in his earlier work [**5**]. The case left by Asashiba is handled recently by Dugas in [**39**]. Thus, for representation-finite, standard self-injective algebras over an algebraically closed field, every stable equivalence can be lifted to a derived equivalence.

In the following, we shall survey some further developments in this direction from [**63**], where the Asashiba–Dugas' result is somehow extended to a large class of algebras, namely Frobenius-finite algebras, including, for example, representation-finite (not necessarily self-injective) algebras and Auslander algebras.

Let A be an Artin algebra. Recall from [**86**] that an A-module X is said to be *ν-stably projective* if $\nu_A^i X$ is projective for all $i \geqslant 0$, where ν_A stands for the Nakayama functor of A. The full subcategory of all ν-stably projective A-modules is denoted by A-stp. An idempotent $e \in A$ is said to be *projectively ν-stable* if Ae is ν-stably projective. Clearly, there is an idempotent element $e \in A$ such that Ae is a basic A-module with $\mathrm{add}(Ae) = A\text{-stp}$. The algebra eAe is then called the *Frobenius part* of A. It is a Frobenius algebra, of course, a self-injective algebra. For a proof, one may see [**63**, Lemma 2.5; **86**]. Note that a Frobenius part of A is uniquely determined by A (up to isomorphism). An algebra A is said to be *Frobenius-finite* (*-tame, or -wild*) if its Frobenius part eAe is representation-finite (-tame, or -wild).

Observe that Frobenius-finite algebras include Auslander algebras and cluster-tilted algebras (see [**63**, Proposition 5.5]). For more examples of Frobenius-finite algebras, we refer the reader to [**63**, Section 5]. Unfortunately, derived equivalent algebras may have different Frobenius parts that are not derived equivalent (see [**62**]).

For Frobenius-finite algebras, a special type of stable equivalences can always be lifted to derived equivalences. They are the so-called stable equivalences of Morita type introduced by

Broué (see [**23**]). Recall that two finite-dimensional algebras A and B over a field is said to be *stably equivalent of Morita type* if there are bimodules ${}_AM_B$ and ${}_BN_A$ such that

(1) M and N all are projective and finitely generated as one-sided modules; and
(2) $M \otimes_B N \simeq A \oplus P$ as A-A-bimodules for some projective A-A-bimodule P, and $N \otimes_A M \simeq B \oplus Q$ as B-B-bimodules for some projective B-B-bimodule Q.

Note that $P = 0$ if and only $Q = 0$. In this case, we have a Morita equivalence (see Theorem 3.6). Clearly, the bimodule M in the definition induces an equivalence between the stable module categories of A and B. Thus the notion of stable equivalences of Morita type is a kind of combinations of Morita equivalences with stable equivalences.

In general, the two notions of derived equivalences and stable equivalences of Morita type are independent. That is, two derived equivalent algebras may not be stably equivalent of Morita type. Conversely, two stably equivalent algebras of Morita type may not be derived equivalent. However, almost ν-stable derived equivalences between algebras A and B always introduce stable equivalences between them (see [**59**]).

Definition 5.1. A stable equivalence $\Phi : A\text{-}\underline{\mathrm{mod}} \to B\text{-}\underline{\mathrm{mod}}$ between finite-dimensional algebras A and B is lifted to a derived equivalence between A an B if there is an almost ν-stable derived equivalence $F : \mathscr{D}^{\mathrm{b}}(A\text{-mod}) \to \mathscr{D}^{\mathrm{b}}(B\text{-mod})$ such that the induced stable functor $\bar{F}$ is naturally isomorphic to Φ.

The following result shows that one can always get a derived equivalence from a stable equivalence of Morita type between Frobenius-finite algebras.

Theorem 5.2 [**63**]. *Let A and B be finite-dimensional k-algebras over an algebraically closed field k and without non-zero semisimple direct summands. If A is Frobenius-finite, then every individual stable equivalence of Morita type between A and B gives rise to a derived equivalence between A and B.*

The proof of Theorem 5.2 is reduced to the following technical result which provides an induction procedure to lift stable equivalences of Morita type to derived equivalences (see Section 6.1).

Theorem 5.3 [**63**]. *Let A and B be finite-dimensional algebras over an algebraically closed field and without non-zero semisimple direct summands. Let e and f be projectively ν-stable idempotent elements in A and B, respectively. Suppose that $\Phi : A\text{-}\underline{\mathrm{mod}} \to B\text{-}\underline{\mathrm{mod}}$ is a stable equivalence of Morita type, satisfying the two conditions:*

(1) For each simple A-module S with $e \cdot S = 0$, $\Phi(S)$ is isomorphic in B-$\underline{\mathrm{mod}}$ to a simple module S' with $f \cdot S' = 0$.
(2) For each simple B-module T with $f \cdot T = 0$, $\Phi^{-1}(T)$ is isomorphic in A-$\underline{\mathrm{mod}}$ to a simple module T' with $e \cdot T' = 0$.

If the stable equivalence $\Phi_1 : eAe\text{-}\underline{\mathrm{mod}} \to fBf\text{-}\underline{\mathrm{mod}}$, induced from Φ, lifts to a derived equivalence between eAe and fBf, then Φ lifts to an iterated almost ν-stable derived equivalence between A and B.

Here, an *iterated* almost ν-stable derived equivalence between A and B is a finite zigzag sequence of almost ν-stable derived equivalences, starting from A and ending at B.

Roughly speaking, Theorem 5.3 means that, for a stable equivalence Φ of Morita type between A and B, if it sends some simple modules to simples, then we may throw away these simple modules and consider the induced stable equivalence of Morita type between corner algebras eAe and fBf, where simple modules belonging to e and f are not sent to simple modules by Φ and Φ^{-1}, respectively. If this stable equivalence between eAe and fBf can be lifted to a derived equivalence, then so does the original stable equivalence Φ.

Applying Theorem 5.3 to Frobenius parts of algebras, we can show that the two conditions in Theorem 5.3 are automatically fulfilled. So we have a simply formulated corollary which also indicates the idea of the proof of Theorem 5.2.

Corollary 5.4. *Let A and B be finite-dimensional algebras over an algebraically closed field and without non-zero semisimple direct summands. Suppose that Φ is a stable equivalence of Morita type between A and B. Let Ψ be the induced stable equivalence of Φ between the Frobenius parts Δ_A and Δ_B. If Ψ lifts to a derived equivalence between Δ_A and Δ_B, then Φ lifts to an iterated almost ν-stable derived equivalence between A and B.*

For applications of the results in this section to Broué's abelian defect group conjecture, we refer to Section 6.

To be able to lift stable to derived equivalences is also important in dealing with a conjecture of Auslander and Reiten on stable equivalences, which states that two stably equivalent algebras have the same numbers of non-isomorphic non-projective simple modules (see, for instance [**8**, Conjecture 5, p. 409; **116**, Conjecture 2.5]). For finite-dimensional algebras over an algebraically closed field, Martínez–Villa proved the conjecture for representation-finite algebras [**86**] and reduced the conjecture to self-injective algebras [**87**]. For weakly symmetric algebras of domestic type, and for special biserial algebras, the conjecture was verified in [**126**] and in [**4**], respectively. In general, however, this conjecture is still open, even for stable equivalences of Morita type. Since derived equivalences preserve the number of non-isomorphic simple modules and since stable equivalences of Morita type preserve the number of non-isomorphic, projective simple modules, it follows that the Auslander–Reiten conjecture is true for those stable equivalences of Morita type that can be lifted to derived equivalences.

5.2. *Extending tilting complexes over corner algebras to algebras themselves*

Related to a ν-stable idempotent element $e \in A$ (that is, $\nu(Ae) \simeq Ae$), one can also lift tilting complexes over eAe to a tilting complex over A. This was first observed by Miyachi in [**90**, Theorem 4.11] for symmetric algebras (that is, ${}_AA_A \simeq {}_AD(A)_A$ as bimodules). We state the following generalization of Miyachi's result (see [**63**]).

Proposition 5.5. *Let A be a finite-dimensional algebra over a field, and let e be a ν-stable idempotent element in A. Suppose that $Q^\bullet$ is a complex in $\mathscr{K}^{\mathrm{b}}(\mathrm{add}(Ae))$ with $Q^i = 0$ for all $i > 0$ such that*

(1) *$eQ^\bullet$ is a tilting complex over eAe, and*
(2) *$\mathrm{End}_{\mathscr{K}^{\mathrm{b}}(eAe)}(eQ^\bullet)$ is a self-injective algebra.*

Then there exists a bounded complex $P^\bullet$ of projective A-modules such that $Q^\bullet \oplus P^\bullet$ is a tilting complex over A and induces an almost ν-stable derived equivalence between A and $\mathrm{End}_{\mathscr{K}^{\mathrm{b}}(A)}(Q^\bullet \oplus P^\bullet)$.

Remark that if the ground field is algebraically closed, or the algebra eAe is symmetric, then Condition (2) in Proposition 5.5 can be dropped because derived equivalences preserve both symmetric algebras over any field [**110**, Corollary 5.3] and self-injective algebras over an algebraically closed field [**3**]. But it is unknown whether derived equivalences preserve self-injective algebras over an arbitrary field.

Thus one can construct derived equivalences between symmetric algebras by extending derived equivalences between their corner algebras. Note that corner algebras eAe are always symmetric if algebras A are symmetric.

6. *Applications*

The constructions of derived equivalences in the previous sections can be used to understand some homological aspects of algebras, or conjectures in the representation theory of algebras and finite groups, such as Broué's abelian defect group conjecture and the finitistic dimension conjecture. Moreover, derived equivalences of algebras and rings can also provide reduction formulas for calculation of algebraic K-groups K_n in algebraic K-theory of rings.

6.1. *An approach to Broue's abelian defect group conjecture*

Stable equivalences of Morita type arise fairly often in the modular representation theory of finite groups. For instance, they appear very often as restriction functors in Green correspondences.

We first observe that the methods developed in Section 5.1 on lifting stable equivalences of Morita type to derived equivalences can be used to approach Broué's abelian defect group conjecture for many cases studied in the literature, for instance, in [**79**, **93**, **98**]. The approach can be described as follows.

Assumption: Let $\Phi : A\text{-}\underline{\mathrm{mod}} \to B\text{-}\underline{\mathrm{mod}}$ be a stable equivalence of Morita type between k-algebras A and B without non-zero semisimple direct summands. Again, we assume that k is an algebraically closed field. For each simple A-module V, we choose a primitive idempotent element $e \in A$ such that $eV \neq 0$. Let $\mathcal{S}(A)$ be a complete set of non-isomorphic simple A-modules.

Step 1: If there is a simple A-module V such that $\Phi(V)$ is a simple B-module, then we set

$$\sigma := \{V \in \mathcal{S}_A \mid \Phi(V) \text{ is non-simple}\} \text{ and } \sigma' := \mathcal{S}_B \backslash \Phi(\mathcal{S}_A \backslash \sigma).$$

Case (i): σ is empty. Then, by a result of Liu (see [**84**] for general algebras and [**83**] for self-injective algebras), which says that a stable equivalence F of Morita type is indeed a Morita equivalence if F sends all simples to simples, we know that Φ lifts to a Morita equivalence, and therefore our procedure terminates.

Case (ii): Both σ and σ' are non-empty. By [**63**, Lemma 3.4], the functor Φ restricts to a stable equivalence Φ_1 of Morita type between $e_\sigma A e_\sigma$ and $e_{\sigma'} B e_{\sigma'}$. Moreover, the idempotent elements e_σ and $e_{\sigma'}$ are projectively ν-stable and the algebras $e_\sigma A e_\sigma$ and $e_{\sigma'} B e_{\sigma'}$ are self-injective with fewer simple modules. So, to lift Φ to a derived equivalence, it is enough to lift Φ_1 to a derived equivalence by Theorem 5.3.

Step 2: If there is a stable equivalence $\Xi : e_{\sigma'} B e_{\sigma'}\text{-}\underline{\mathrm{mod}} \to C\text{-}\underline{\mathrm{mod}}$ of Morita type between the algebra $e_{\sigma'} B e_{\sigma'}$ and another algebra C (to be found independently), such that the stable equivalence Ξ is induced by a derived equivalence and the composition $\Xi \circ \Phi_1$ sends some (not necessarily all) simple $e_\sigma A e_\sigma$-modules to simple C-modules, then we go back to Step 1. Once we arrive at representation-finite algebras in the procedure, Theorem 5.2 can be applied. This then implies that Φ_1 lifts to a derived equivalence, and therefore so does the given Φ.

Remark that this procedure is somewhat similar to, but different from the method of Okuyama in [**98**]. In the procedure, the numbers of simple modules over the resulting algebras after each step decrease. This means we may often arrive at representation-finite algebras. Moreover, to pursue Step 2, we only require that $\Xi \circ \Phi_1$ sends some simple modules to simple modules and do not require that $\Xi \circ \Phi_1$ sends *all* simples to simples, while the latter is needed in [**98**] and other approaches (for example, see [**116**]).

Now, we explain how the above procedure works by an example. It was proved in [**93**] that Broué's abelian defect group conjecture is true for the faithful 3-blocks of defect 2 of $4.M_{22}$, the non-split central extension of the sporadic simple group M_{22} by a cyclic group of order 4. Here we give a short proof of the result in [**93**] and avoid many technical calculations.

The two block algebras B_+ and b_+ have five simple modules, respectively. We may label the simple B_+-modules by $56a, 56b, 64, 160a, 160b$, and the simple b_+-modules by $1a, 1b, 2, 1c$ and $1d$. There is a stable equivalence

$$\Phi : B_+\text{-}\underline{\mathrm{mod}} \longrightarrow b_+\text{-}\underline{\mathrm{mod}}$$

of Morita type (see [**93**]) such that

$$\Phi(56a) = \Omega^{-1}(1a), \Phi(56b) = \Omega(1b), \Phi(160a) = 1c, \Phi(160b) = 1d,$$

while $\Phi(64)$ is not sent to simple b_+-module by the functor Φ.

Since Φ sends the simple modules $160a$ and $160b$ to simple modules, we can use Step 1 and consider $\sigma := \{56a, 56b, 64\}$ and $\sigma' := \{1a, 1b, 2\}$. Then Φ induces a stable equivalence of Morita type

$$\Phi_1 : e_\sigma B_+ e_\sigma\text{-}\underline{\mathrm{mod}} \longrightarrow e_{\sigma'} b_+ e_{\sigma'}\text{-}\underline{\mathrm{mod}}.$$

By [**98**], there is an algebra C with three simple modules, labeled by $1a, 1b$ and 2, and a derived equivalence between $e_{\sigma'} b_+ e_{\sigma'}$ and C, inducing a stable equivalence $\Xi : e_{\sigma'} B e_{\sigma'}\text{-}\underline{\mathrm{mod}} \to C\text{-}\underline{\mathrm{mod}}$ of Morita type, such that $\Xi\Phi_1(64) \simeq 1b$. But $\Xi\Phi_1(56a)$ and $\Xi\Phi_1(56b)$ are not simple. Let $\sigma_1 := \{56a, 56b\}$ and $\sigma'_1 := \{1a, 2\}$. Then the composition $\Xi\Phi_1$ restricts to a stable equivalence of Morita type

$$\Phi_2 : e_{\sigma_1} B_+ e_{\sigma_1}\text{-}\underline{\mathrm{mod}} \longrightarrow e_{\sigma'_1} C e_{\sigma'_1}\text{-}\underline{\mathrm{mod}}.$$

Note that the Cartan matrix of $e_{\sigma'_1} C e_{\sigma'_1}$ is $\left[\begin{smallmatrix} 2 & 1 \\ 1 & 3 \end{smallmatrix}\right]$. It is easy to check that a symmetric algebra with this Cartan matrix is always representation-finite. Thus Φ_2 lifts to a derived equivalence by Theorem 5.2, and consequently Φ lifts to a derived equivalence.

The whole procedure can be illustrated by the commutative diagram

$$\begin{tikzcd}
B_+\text{-}\underline{\mathrm{mod}} \arrow[r, "\Phi"] & b_+\text{-}\underline{\mathrm{mod}} & \\
e_\sigma B_+ e_\sigma\text{-}\underline{\mathrm{mod}} \arrow[u, "\lambda"] \arrow[r, "\Phi_1"] & e_{\sigma'} b_+ e_{\sigma'}\text{-}\underline{\mathrm{mod}} \arrow[u, "\lambda"] \arrow[r, "\Xi"] & C\text{-}\underline{\mathrm{mod}} \\
e_{\sigma_1} B_+ e_{\sigma_1}\text{-}\underline{\mathrm{mod}} \arrow[u, "\lambda"] \arrow[rr, "\Phi_2"] & & e_{\sigma'_1} C e_{\sigma'_1}\text{-}\underline{\mathrm{mod}} \arrow[u, "\lambda"]
\end{tikzcd}$$

with Φ_2 lifting to a derived equivalence, where λ is a functor of the form $Ae \otimes_{eAe} -$ with e an idempotent element in an algebra A.

Finally, we point out that the procedure can also be used to simplify proofs of results in [**79**, **98**] (see [**63**] for details).

6.2. *Finitistic dimension conjecture*

Homological dimensions are important invariants of algebras and modules. Though derived equivalences do not always preserve these invariants, they still can provide a powerful tool to understand some homological properties of algebras. For example, the global and finitistic dimensions of algebras may change under derived equivalences, but one still can use derived categories and derived equivalences to find reduction techniques for estimation of these homological dimensions of algebras. In this section, we illustrate this philosophy.

Let A be a finite-dimensional algebra over a field. The finitistic dimension, denoted by $\mathrm{fin.dim}(A)$, is defined as

$$\mathrm{fin.dim}(A) = \sup\{\mathrm{pd}_A(M) \mid M \in A\text{-mod}, \mathrm{pd}_A(M) < \infty\},$$

where $\mathrm{pd}_A(M)$ means the projective dimension of the module ${}_AM$.

Finitistic dimension conjecture: $\mathrm{fin.dim}(A) < \infty$ for any finite-dimensional algebra A over a field.

This conjecture was initially a question by Rosenberg and Zelinsky, published by Bass in the paper [**12**] in 1960. Since then the conjecture has attracted attentions of many algebraists in the last five decades. Among them is Maurice Auslander who 'is considered to be one of the founders of the modern aspects of the representation theory of Artin algebras' (see [**107**, p. 501]). The finitistic dimension conjecture has intimate connections with the Nakayama conjecture, generalized Nakayama conjecture and Gorenstein symmetry conjecture (see [**8**] for more conjectures). Unfortunately, all conjectures mentioned here are open.

For finite-dimensional algebras A and B over a field, if they are derived equivalent via a tilting module, then Happel showed that the global (or finitistic) dimension of A is finite if and only if the global (or finitistic) dimension of B is finite (see [**47**, **49**]). This was extended to coherent rings in [**102**].

THEOREM 6.1. *Let A be a left coherent rings* (*that is, rings in which every finitely generated left ideal is finitely presented*) *and $T^\bullet$ be a tilting complex over A with $n+1$ non-zero terms. Then* $\mathrm{fin.dim}(A) - n \leqslant \mathrm{fin.dim}(\mathrm{End}_{\mathscr{D}^{\mathrm{b}}(A)}(T^\bullet)) \leqslant \mathrm{fin.dim}(A) + n$.

For a ring, the finitistic dimension is defined as the supremum of projective dimensions of those finitely presented left modules that have finite projective dimensions. This definition coincides with the one for finite-dimensional algebras.

Recollements of triangulated categories introduced in [**15**] can be regarded as a generalization of derived equivalences. They are 'exact sequences' of triangulated categories.

DEFINITION 6.2. Let $\mathcal{D}, \mathcal{D}'$ and $\mathcal{D}''$ be triangulated categories. We say that $\mathcal{D}$ *is a recollement of $\mathcal{D}'$ and $\mathcal{D}''$* if there are six triangle functors displayed in the diagram

$$\begin{tikzcd} \mathcal{D}'' \arrow[r, "i_*=i_!"] & \mathcal{D} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^!=j^*"] & \mathcal{D}' \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

satisfying the four conditions:

(1) $(i^*, i_*), (i_!, i^!), (j_!, j^!)$ and (j^*, j_*) are adjoint pairs.
(2) i_*, j_* and $j_!$ are fully faithful.
(3) $i^!j_* = 0$ (and thus also $j^!i_! = 0$ and $i^*j_! = 0$).
(4) Each object $X \in \mathcal{D}$ is endowed with the two triangles in $\mathcal{D}$:

$$i_!i^!(X) \longrightarrow X \longrightarrow j_*j^*(X) \longrightarrow i_!i^!(X)[1],$$
$$j_!j^!(X) \longrightarrow X \longrightarrow i_*i^*(X) \longrightarrow j_!j^!(X)[1]$$

in which morphisms are given by counits and units of the above-mentioned adjoint pairs of functors.

If one of the triangulated categories $\mathcal{D}'$ and $\mathcal{D}''$ is zero, then we come back to the notion of triangle equivalences.

The finitistic dimension conjecture has a reduction by recollements.

THEOREM 6.3 [49]. *If there is a recollement*

$$\begin{tikzcd} \mathscr{D}^{\mathrm{b}}(B\text{-mod}) \arrow[r] & \mathscr{D}^{\mathrm{b}}(A\text{-mod}) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}^{\mathrm{b}}(C\text{-mod}), \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

of the derived module categories of finite-dimensional algebras A, B and C over a field, then $\mathrm{fin.dim}(A) < \infty$ *if and only if both* $\mathrm{fin.dim}(B) < \infty$ *and* $\mathrm{fin.dim}(C) < \infty$.

This reduction is further refined in [30], where relations among the finitistic dimensions of three algebras in a recollement are described. To state this refinement precisely, we recall the following definitions from [30].

For a complex $X^\bullet$ in $\mathscr{C}^b(A\text{-Proj})$, we define

$$\sup(X^\bullet) := \max\{i \in \mathbb{Z} \mid H^i(X^\bullet) \neq 0\},$$

$$\inf(X^\bullet) := \min\{i \in \mathbb{Z} \mid H^i(X^\bullet) \neq 0\},$$

$$\mathrm{w}(X^\bullet) := \begin{cases} 0 & \text{if } X^\bullet \text{ is acyclic,} \\ \sup(X^\bullet) - \inf(X^\bullet) + \mathrm{pd}_A\left(\mathrm{Coker}(d_X^{\inf(X^\bullet)-1})\right) & \text{otherwise.} \end{cases}$$

Clearly, $0 \leqslant w(P^\bullet) < \infty$. We call $w(P^\bullet)$ the *homological width* of $X^\bullet$. Similarly, one defines the *homological cowidth* of a complex $Y^\bullet$ in $\mathscr{C}^b(A\text{-Inj})$, denoted by $cw(Y^\bullet)$:

$$cw(Y^\bullet) := \begin{cases} 0 & \text{if } Y^\bullet \text{ is acyclic,} \\ \sup(Y^\bullet) - \inf(Y^\bullet) + \mathrm{id}_A\left(\mathrm{Ker}(d_{Y^\bullet}^{\sup(Y^\bullet)})\right) & \text{otherwise,} \end{cases}$$

where $\mathrm{id}_A(M)$ stands for the injective dimension of an A-module M. It is not difficult to extend the definition of homological width and cowidth to any complexes which are quasi-isomorphic to complexes in $\mathscr{C}^b(A\text{-Proj})$ and $\mathscr{C}^b(A\text{-Inj})$, respectively.

THEOREM 6.4 [30]. *Let R_1, R_2 and R_3 be rings. Suppose that there is a recollement among the derived module categories $\mathscr{D}(R_3)$, $\mathscr{D}(R_2)$ and $\mathscr{D}(R_1)$ of R_3, R_2 and R_1 :*

$$\begin{tikzcd} \mathscr{D}(R_1) \arrow[r, "i_*"] & \mathscr{D}(R_2) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^!"] & \mathscr{D}(R_3) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

Then the following hold true:

(1) *If $j_!$ restricts to a functor $\mathscr{D}^{\mathrm{b}}(R_3) \to \mathscr{D}^{\mathrm{b}}(R_2)$ of bounded derived module categories, then* $\mathrm{fin.dim}(R_3) \leqslant \mathrm{fin.dim}(R_2) + cw(j^!(\mathrm{Hom}_{\mathbb{Z}}(R_2, \mathbb{Q}/\mathbb{Z})))$.
(2) *Suppose that $i_*(R_1)$ is isomorphic in $\mathscr{D}(R_2)$ to a bounded complex of finitely generated projective R_2-modules. Then*
  (a) $\mathrm{fin.dim}(R_1) \leqslant \mathrm{fin.dim}(R_2) + w(i^*(R_2))$.
  (b) $\mathrm{fin.dim}(R_2) \leqslant \mathrm{fin.dim}(R_1) + \mathrm{fin.dim}(R_3) + w(i_*(R_1)) + w(j_!(R_3)) + 1$.

Theorem 6.4 extends Happel's result in [49] and shows how the finitistic dimensions of three rings in a recollement are related by the homological cowidth or width of specially fixed complexes.

Homological dimensions can also be studied through recollements of abelian categories (see [104]).

As to constructions of recollements of triangulated categories, a lot of methods is known. For example, see [**13**, **15**, **36**, **68**, **97**, **121**] and the references therein. But it seems to be difficult to get recollements of derived module categories of rings. Recently, some efforts in this direction are made (see [**25**, **26**, **36**]).

The validity of the finitistic dimension conjecture for an algebra A implies the validity of the Nakayama conjecture for A, which says that A is a self-injective algebra whenever its dominant dimension is infinite. This was proposed by Nakayama in 1958 and becomes now a central conjecture in the representation theory of finite-dimensional algebras (see, for instance, [**8**, Conjecture (8), p. 410]). We refer the reader to [**120**] for further information on dominant dimensions and the conjecture.

At the end of this section, we mention the following conjecture on dominant dimensions of derived equivalent algebras.

Conjecture 6.5. Suppose that A and B are finite-dimensional, derived equivalent algebras. Then the dominant dimension of A is infinite if and only if so is the dominant dimension of B.

For partial answers to this conjecture, we refer to [**28**]. Stimulated by Theorem 6.4, we also mention the following open

Question. Are there any relations for dominant dimensions of three algebras in a recollement of derived module categories?

6.3. *Algebraic K-theory of matrix subrings*

One of the interesting and hard problems in algebraic K-theory of rings is the calculation of higher algebraic K-groups K_n. In this section, we shall provide reduction formulas for computation of the K_n-groups of a class of rings by passing to derived equivalent algebras. The key idea behind these computation is that derived equivalences of rings preserve the K-theory and G-theory (see Theorem 3.8(5) or [**41**]). In the literature, there are many papers dealing with K-groups K_n by exploiting excision, Mayer–Vietoris exact sequences or other related sequences (for example, see [**46**, **119**, **122**]), but there are few works using derived equivalences to calculate algebraic K-groups. In the following, we will survey some results in this direction.

Let R be a ring. We denote by $K_*(R)$ the series of algebraic K-groups of R in the sense of Quillen for $* \in \{0, 1, 2, \dots, \}$ (see [**105**, **115**]). The algebraic K-theory of matrix-like rings has been of interest since a long time. In [**18**], Berrick and Keating showed the following result.

Lemma 6.6 [**18**]. *If R_1 and R_2 are rings and M is an R_1-R_2-bimodule, then, for the triangular matrix ring*

$$S = \begin{pmatrix} R_1 & M \\ 0 & R_2 \end{pmatrix},$$

there is an isomorphism of K-groups: $K_n(S) \simeq K_n(R_1) \oplus K_n(R_2)$ for all integers $n \in \mathbb{N}$. Moreover, this isomorphism is induced from the canonical inclusion of $R_1 \oplus R_2$ into S.

For $n = 0$, this is classical. For $n = 1, 2$, this was already shown by Dennis and Geller in 1976.

For a matrix ring of the form

$$T = \begin{pmatrix} R & I & \cdots & I \\ R & R & \ddots & \vdots \\ \vdots & \vdots & \ddots & I \\ R & R & \cdots & R \end{pmatrix}_{n\times n},$$

where R is a ring and I is an ideal in R such that the R-modules ${}_RI$ and I_R are projective, Keating proved in [70] that there is an isomorphism of K-theory:

$$K_*(T) \simeq K_*(R) \oplus (n-1)K_*(R/I).$$

Note that this class of rings appears very often as tiled triangular orders or maximal orders (see [106]).

In [46], the authors furthered the above result and considered the following matrix ring S: Let I be an ideal of a $\mathbb{Z}_p$-algebra R with identity, where $\mathbb{Z}_p$ is the p-adic integers (or, equivalently, $\mathbb{Z}_p = \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$), and define

$$S = \begin{pmatrix} R & I^{t_{12}} & \cdots & I^{t_{1n}} \\ R & R & \ddots & \vdots \\ \vdots & \vdots & \ddots & I^{t_{n-1\,n}} \\ R & R & \cdots & R \end{pmatrix},$$

where t_{ij} are positive integers. Assume that S is a ring and R/I^n is a finite ring for all $n \geqslant 1$. If both ${}_RI$ and I_R are projective, it was proved in [46] that the isomorphism of algebraic K-groups holds:

$$K_*(S)(1/s) \simeq K_*(R)(1/s) \oplus (n-1)K_*(R/I)(1/s),$$

where s is any rational integer such that p divides s, and where $G(1/s)$ denotes the group $G \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{s}]$ for an abelian group G.

In fact, we can extend this result substantially to a more general situation without any homological restrictions on ideal I of R by using derived equivalences.

THEOREM 6.7 [125]. *Let R be a ring and $K_*(R)$ the $*$-th algebraic K-group of R with $* \in \mathbb{N}$.*

(1) *If I_{ij} is (not necessarily projective) an ideal of R for $1 \leqslant i < j \leqslant n$ such that*

$$S := \begin{pmatrix} R & I_{12} & I_{13} & \cdots & I_{1\,n} \\ R & R & I_{23} & \cdots & I_{2\,n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ R & R & \cdots & R & I_{n-1\,n} \\ R & R & \cdots & R & R \end{pmatrix}$$

is a ring, then

$$K_*(S) \simeq K_*(R) \oplus \bigoplus_{j=2}^{n} K_*(R/I_{j-1\,j})$$

for all $* \in \mathbb{N}$.

(2) *Suppose that R_i is a subring of R with the same identity such that $I_i \subseteq R_i$ is a right ideal of R_i and I_i is a left ideal of R for $2 \leqslant i \leqslant n$. Further, let I_{ij} be ideals of R. If*

$$T := \begin{pmatrix} R & I_2 & I_3 & \cdots & I_n \\ R & R_2 & I_3 & \cdots & I_n \\ R & I_{32} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & R_{n-1} & I_n \\ R & I_{n2} & \cdots & I_{n\,n-1} & R_n \end{pmatrix}$$

is a ring, then

$$K_*(T) \simeq K_*(R) \oplus \bigoplus_{j=2}^{n} K_*(R_j/I_j)$$

for all $* \in \mathbb{N}$.

Thus Theorem 6.7 reduces the calculation of K_n-groups of S and T to these of R and its quotients R/I_i (or $R/I_{i,i+1}$), and shows that the algebraic K-groups of S and T are independent of those $I_{i,j}$ with $|i-j| \geqslant 2$. The proof of Theorem 6.7 is based on the following proposition on derived equivalences (see [**125**, Lemma 3.1]).

PROPOSITION 6.8. *Let* $B \subseteq A$ *be an extension of rings with the same identity.*

(1) *If* $\mathrm{Ext}^1_B({}_BA, {}_BB) = 0$, *then the sequence*

$$0 \longrightarrow B \longrightarrow A \longrightarrow A/B \longrightarrow 0$$

is an $\mathrm{add}({}_BA)$*-split sequence in* B*-Mod. Thus* $\mathrm{End}_B({}_BB \oplus {}_BA)$ *and* $\mathrm{End}_B({}_BA \oplus (A/B))$ *are derived-equivalent.*

(2) *If* ${}_BA$ *is projective, then the above sequence is an* $\mathrm{add}({}_BA)$*-split sequence.*

(3) *Suppose* $\mathrm{Ext}^1_B({}_BA, {}_BA) = 0$. *If* ${}_BA$ *is finitely presented with* $\mathrm{pd}({}_BA) \leqslant 1$ *(for instance,* ${}_BA$ *is projective and finitely generated), then* $A \oplus (A/B)$ *is a tilting* B*-module of projective dimension at most* 1*. In particular,* $\mathrm{End}_B(A \oplus (A/B))$ *is derived equivalent to* B.

As a consequence of Proposition 6.8, we have the following derived equivalences between matrix subrings.

COROLLARY 6.9 [**32**, **125**]. *If* $I_2, \ldots, I_n$ *are ideals of a ring* R, *then the two rings*

$$\begin{pmatrix} R & I_2 & I_3 & \cdots & I_{n-1} & I_n \\ R & R & I_3 & \cdots & I_{n-1} & I_n \\ R & R & R & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & I_{n-1} & I_n \\ R & R & R & \cdots & R & I_n \\ 0 & 0 & 0 & \cdots & 0 & R \end{pmatrix} \text{ and } \begin{pmatrix} R & R/I_2 & R/I_3 & \cdots & R/I_{n-1} & R/I_n \\ 0 & R/I_2 & R/I_3 & \cdots & R/I_{n-1} & R/I_n \\ 0 & 0 & R/I_3 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & R/I_{n-1} & R/I_n \\ 0 & 0 & 0 & \cdots & R/I_{n-1} & R/I_n \\ 0 & 0 & 0 & \cdots & 0 & R/I_n \end{pmatrix}$$

are derived equivalent.

With the help of Proposition 6.8, we can prove that the rings S and T in Theorem 6.7 are derived equivalent to the matrix rings

$$B := \begin{pmatrix} R & I_{12} & I_{13} & \cdots & I_{1\,n-1} & I_{1\,n-1}/I_{1n} \\ R & R & I_{23} & \cdots & I_{2\,n-1} & I_{2\,n-1}/I_{2n} \\ R & R & R & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & I_{n-2\,n-1} & I_{n-2\,n-1}/I_{n-2\,n} \\ R & R & R & \cdots & R & R/I_{n-1\,n} \\ 0 & 0 & 0 & \cdots & 0 & R/I_{n-1\,n} \end{pmatrix} \text{ and}$$

$$C := \begin{pmatrix} R_2/I_2 & 0 & 0 & \cdots & 0 & 0 \\ R/I_2 & R & I_3 & I_4 & \cdots & I_n \\ R/I_{32} & R & R_3 & I_4 & \cdots & I_n \\ R/I_{42} & R & I_{43} & R_4 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & I_n \\ R/I_{n2} & R & I_{n3} & \cdots & I_{n\,n-1} & R_n \end{pmatrix}$$

with the usual matrix addition and multiplication, respectively. Thus Theorem 6.7 follows from Lemma 6.6 inductively.

For more details on the proof of Theorem 6.7 and further information on computation formulas for algebraic K-groups of other type of matrix subrings, we refer the reader to [**125**]. For applications of recollements to computation of algebraic K-groups of rings, we refer to [**27**, **29**].

Finally, we mention an open problem related to computation of algebraic K-groups of matrix subrings.

QUESTION. Given a ring R and an ideal I of R with $I^2 = 0$, how can we find a reduction formula for calculation of the algebraic K-groups $K_n(S)$? where

$$S := \begin{pmatrix} R & I \\ I & R \end{pmatrix}.$$

Note that there is a split surjective homomorphism from S to $R \times R$ with the kernel $J := \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$. Thus $K_n(S) \simeq K_n(R) \oplus K_n(R) \oplus K_n(S, J)$, where $K_n(S, J)$ is the nth relative algebraic K-group with respect to J. The problem is reduced to studying relations between $K_n(S, J)$ and $K_n(R/I)$.

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