ON THE SEMI-SIMPLICITY OF CYCLOTOMIC TEMPERLEY-LIEB ALGEBRAS

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Abstract. In [7], a class of associative algebras called cyclotomic Temperley-Lieb algebras over a commutative ring was introduced. In this note, we provide a necessary and sufficient condition for a cyclotomic Temperley-Lieb algebra to be semi-simple.

1. Introduction

The Temperley-Lieb algebras were first introduced in [9] in order to study the single bond transfer matrices for the Ising model and for the Potts model. Jones [4] defined a trace function on a Temperley-Lieb algebra so that he could construct Jones polynomial of a link when the trace is non-degenerate. It is known that the trace is non-degenerate if the Temperley-Lieb algebra is semi-simple. So it is an interesting question to provide a criterium for a Temperley-Lieb algebra to be semi-simple. In [10, §5], Westbury computed explicitly the determinants of Gram matrices associated to all “cell modules” via Tchebychev polynomials. This implies that a Temperley-Lieb algebra is semi-simple if and only if such polynomials do not take values zero for the parameters.

As a generalization of a Temperley-Lieb algebra, the cyclotomic Temperley-Lieb algebra $TL_{m,n}(\delta)$ of type $G(m,1,n)$ was introduced in [7]. It is proved in [7] that $TL_{m,n}(\delta)$ is a cellular algebra in the sense of [2]. Thus $TL_{m,n}(\delta)$ is semi-simple if and only if all of its “cell modules” are pairwise non-isomorphic irreducible. In order to describe a cell module to be irreducible, Rui and Xi computed the determinants of Gram matrices of certain cell modules [7, 8.1]. In general, it is hard to compute the determinants for all cell modules.

In this note, we shall consider the semi-simplicity of cyclotomic Temperley-Lieb algebras, this is an analog question considered in [8] (see [1] for the case $m = 1$). Following [5], we study two functors $F$ and $G$ between certain categories in section 3. Via these functors and [7, 8.1], we can show Theorem 4.6, the main result of this paper, which says that the semi-simplicity of a cyclotomic Temperley-Lieb algebra can be determined by generalized Tchebychev polynomials and the parameters $\tilde{\delta}_i, 1 \leq i \leq m$.

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2. Cyclotomic Temperley-Lieb algebras

In this section, we recall some of results on the cyclotomic Temperley-Lieb algebras in [7]. Throughout the paper, we fix two natural numbers \(m\) and \(n\).

A labelled Temperley-Lieb diagram (or labelled TL-diagram) \(D\) of type \(G(m, 1, n)\) is a Temperley-Lieb diagram with \(2n\) vertices and \(n\) arcs. Each arc is labelled by an element in \(\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}\), which will be considered as the number of dots on it. The following are two special labelled TL-diagrams.

\[
E_i = \begin{array}{c|c|c|c|c}
1 & \cdots & i & i+1 & n \\
1 & i & i+1 & \cdots & n
\end{array}, \quad T_i = \begin{array}{c|c|c|c|c}
1 & i & \cdots & \cdots & n \\
1 & i & \cdots & \cdots & n
\end{array}
\]

An arc in a labelled TL-diagram \(D\) is horizontal if both of its endpoints are in the same row of \(D\). Otherwise, it is vertical. A dot will be replaced by \(m-1\) dots if it moves from one endpoint of a horizontal arc to another. A dot in a vertical arc can move freely from one endpoint to another.

Given a horizontal arc \(\{i, j\}\) of \(D\) with \(i < j\). We say \(i\) (resp. \(j\)) the left (resp. right) endpoint of the arc. For a horizontal (resp. vertical) arc, we always assume that the dots on this arc concentrate on the left endpoint (resp. the endpoint on the top row of the labelled TL-diagram \(D\)).

Suppose an arc \(l_1\) joins another arc \(l_2\) with a common endpoint \(j\). A dot can move from \(l_1\) to \(l_2\). We always assume that a dot on the endpoint \(j \in l_1\) can be replaced by a dot on \(j \in l_2\).

Given two labelled TL-diagrams \(D_1\) and \(D_2\) of type \(G(m, 1, n)\). Following [7], we define a new labelled TL-diagram \(D_1 \circ D_2\) as follows: First, compose \(D_1\) and \(D_2\) in the same way as was done for the Temperley-Lieb algebra to get a new diagram \(P\). Second, applying the rule for the movement of dots to relabel each arc of \(P\). We get a new labelled TL-diagram, and this is defined to be \(D_1 \circ D_2\). Let \(n(i, D_1, D_2)\) be the number of the relabelled closed cycles on which there are \(\bar{i}\) dots.

**Definition 2.1.** [7, 3.3] Let \(R\) be a commutative ring containing 1 and \(\delta_0, \ldots, \delta_{m-1}\). Put \(\delta = (\delta_0, \ldots, \delta_{m-1})\). A cyclotomic Temperley-Lieb algebra \(TL_{m,n}(\delta)\) is an associative algebra over \(R\) with a basis consisting of all labelled TL-diagrams of type \(G(m, 1, n)\), and the multiplication is given by \(D_1 \cdot D_2 = \prod_{i=0}^{m-1} \delta_i^{n(i, D_1, D_2)} D_1 \circ D_2\).

It was shown in [7] that \(TL_{m,n}(\delta)\) can be defined by generators and relations. For the details we refer to [7, 2.1].
In the remaining part of this section, we recall some results on the representations of $TL_{m,n}(\delta)$. First, we give the notion of a cellular algebra in [2], which depends on the existence of certain basis. There is also a basis-free definition of cellular algebras, for this we refer to [6].

**Definition 2.2.** [2, 1.1] An associative $R$–algebra $A$ is called a **cellular algebra** with cell datum $(I, M, C, i)$ if the following conditions are satisfied:

(C1) The finite set $I$ is partially ordered. Associated with each $\lambda \in I$ there is a finite set $M(\lambda)$. The algebra $A$ has an $R$–basis $C^\lambda_{S,T}$ where $(S, T)$ runs through all elements of $M(\lambda) \times M(\lambda)$ for all $\lambda \in I$.

(C2) The map $i$ is an $R$–linear anti–automorphism of $A$ with $i^2 = id$ which sends $C^\lambda_{S,T}$ to $C^\lambda_{T,S}$.

(C3) For each $\lambda \in I$ and $S, T \in M(\lambda)$ and each $a \in A$ the product $aC^\lambda_{S,T}$ can be written as

$$aC^\lambda_{S,T} = \sum_{U \in M(\lambda)} r_a(U, S)C^\lambda_{U,T} + r',$$

where $r'$ belongs to $A^{<\lambda}$ consisting of all $R$-linear combination of basis elements with upper index $\mu$ strictly smaller than $\lambda$, and the coefficients $r_a(U, S) \in R$ do not depend on $T$.

For each $\lambda \in I$, one can define a cell module $\Delta(\lambda)$ and a symmetric, associative bilinear form $\Phi_\lambda : \Delta(\lambda) \otimes_R \Delta(\lambda) \to R$ in the following way (see [2, §2]): As an $R$-module, $\Delta(\lambda)$ has an $R$-basis

$$\{C^\lambda_S \mid S \in M(\lambda)\},$$

the module structure is given by

$$aC^\lambda_S = \sum_{U \in M(\lambda)} r_a(U, S)C^\lambda_U.$$  \hfill (2.1)

The bilinear form $\Phi_\lambda$ is defined by

$$\Phi_\lambda(C^\lambda_S, C^\lambda_T)C^\lambda_U = C^\lambda_{U,S}C^\lambda_{T,V} (\mod A^{<\lambda}),$$

where $U$ and $V$ are arbitrary elements in $M(\lambda)$.

Let $\text{rad}\Delta(\lambda) = \{c \in \Delta(\lambda) \mid \Phi_\lambda(c, c') = 0 \text{ for all } c' \in \Delta(\lambda)\}$. Then $\text{rad}\Delta(\lambda)$ is a submodule of $\Delta(\lambda)$. Put $L(\lambda) = \Delta(\lambda)/\text{rad}\Delta(\lambda)$. Then either $L(\lambda) = 0$ or $L(\lambda)$ is irreducible [2, 3.2]. We will need the following result next section.

**Lemma 2.3.** $\text{rad}\Delta(\lambda)$ is annihilated by $A^{\leq \lambda}$.

**Proof.** Let $a = C^\mu_{S_l,T_l} \in A^{\leq \lambda}$ and $C^\lambda_S \in \text{rad}\Delta(\lambda)$. If $\mu < \lambda$, then $aC^\lambda_S = 0$ in $\Delta(\lambda)$. If $\mu = \lambda$, then we still have $aC^\lambda_S = 0$ since $r_a(S_l, S) = \Phi_\lambda(C^\lambda_{T_l}, C^\lambda_S)$ and $C^\lambda_S \in \text{rad}\Delta(\lambda)$. \hfill $\Box$

From now on, we assume that $R$ is a splitting field of $x^m - 1$. Then $x^m - 1 = \prod_{i=1}^m(x - u_i)$ for some $u_i \in R$, $1 \leq i \leq m$. Let $G_{m,n}$ be the $R$-subalgebra of $TL_{m,n}(\delta)$ generated by $T_1, T_2, \cdots, T_n$. Let $\Lambda(m, n) = \{(i_1, i_2, \cdots, i_n) \mid 1 \leq i_j \leq m\}$. Define $i \leq j$ if $i_k \geq j_k$ for all $1 \leq k \leq n$. Then $(\Lambda(m, n), \leq)$ is a poset. For any $i \in \Lambda(m, n)$, set $C^4_{i,1} = \prod_{j=1}^n \prod_{l=i_j+1}^{i_{j+1}} (t_j - u_l)$. 
Lemma 2.4. The set \( \{ C_{i,1} | i \in \Lambda(m, n) \} \) is a cellular basis of \( G_{m,n} \).

The cell module for \( i \in \Lambda(m, n) \) with respect to the above cellular basis will be denoted by \( \Delta(i) \).

An \((n,k)\)-labelled parenthesis graph is a graph consisting of \( n \) vertices \( \{1, 2, \ldots, n\} \) and \( k \) horizontal arcs (hence \( 2k \leq n \) and there are \( n-2k \) free vertices which do not belong to any arc) such that

1. there are at most \( m-1 \) dots on each arc,
2. there are no arcs \( \{i,j\} \) and \( \{q,l\} \) satisfying \( i < q < j < l \)
3. there is no arc \( \{i,j\} \) and free vertex \( q \) such that \( i < q < j \).

Let \( P(n,k) \) be the set of all \((n,k)\)-labelled parenthesis graphs. A labelled TL-diagram \( D \) with \( k \) horizontal arcs can be determined by a triple pair \( (v_1, v_2, x) \), \( x \in G_{m,n-2k} \) and \( v_1, v_2 \in P(n,k) \) (see [7, §5]) and vice versa. Such a \( D \) will be denoted by \( v_1 \otimes v_2 \otimes x \). In this case, we define \( \text{top}(D) = v_1 \) and \( \text{bot}(D) = v_2 \).

Let \( \Lambda_{m,n} = \{(k,i) | 0 \leq k \leq \lfloor n/2 \rfloor, i \in \Lambda(m,n-2k)\} \). For any \((k,i),(l,j) \in \Lambda_{m,n}\), say \((k,i) \leq (l,j)\) if either \( k > l \) or \( k = l \) and \( i \leq j \). Then \( \Lambda_{m,n} \) is a poset. For \( v_1, v_2 \in P(n,k) \) and \( i \in \Lambda(m,n-2k) \), define \( C_{v_1,v_2}^{(k,i)} = v_1 \otimes v_2 \otimes C_{i,1} \).

Proposition 2.5. [7, 5.3] Let \( R \) be a splitting field of \( x^m - 1 \). The set \( \{ C_{v_1,v_2}^{(k,i)} | (k,i) \in \Lambda_{n,m}, v_1, v_2 \in P(n,k) \} \) is a cellular basis of \( TL_{m,n}(\delta) \).

Let \( \Delta(k,i) \) be the cell module with respect to the cellular basis given in Proposition 2.5. Then
\[
\Delta(k,i) \cong V(n,k) \otimes_R v_0 \otimes_R \Delta(i)
\]
where \( V(n,k) \) is the free \( R \)-module generated by \( P(n,k) \) and \( v_0 \) is a fix element in \( P(n,k) \). The following theorem is known as branching rule for the cell module \( \Delta(k,i) \).

Proposition 2.6. [7, 7.1] Suppose \( chR \nmid m \). For \( i = (i_1, i_2, \cdots, i_{n-2k}) \in \Lambda(m,n-2k) \), define \( i_0 = (i_1, i_2, \cdots, i_{n-2k-1}) \in \Lambda(m,n-2k+1) \) and \( 1 \cup j = (i_1, i_2, \cdots, i_{n-2k}, j) \in \Lambda(m,n-2k+1) \). Then there is a short exact sequence
\[
0 \longrightarrow \Delta(k,i_0) \longrightarrow \Delta(k,i) \downarrow \longrightarrow \bigoplus_{j=1}^{m} \Delta(k-1, i \cup j) \longrightarrow 0,
\]
where we denote by \( M \downarrow \) the restriction of a \( TL_{m,n}(\delta) \)-module \( M \) to a \( TL_{m,n-1}(\delta) \)-module.

Proof. It is proved in [7, 7.1] that
\[
0 \longrightarrow \Delta(k,i_0) \longrightarrow \Delta(k,i) \downarrow \longrightarrow V(n-1,k-1) \otimes_R v_0 \otimes_R \Delta(i) \otimes_R R(t_{n-2k+1}) \longrightarrow 0.
\]
Since \( chR \nmid m \), \( R(t_{n-2k+1}) \) is semi-simple. Therefore, \( R(t_{n-2k+1}) \cong \bigoplus_{j=1}^{m} \Delta(j) \), where \( \Delta(j) \) is the cell module of \( R(t_{n-2k+1}) \) with respect to the cellular basis given in Lemma 2.4 (the case
By direct computation, we have
\[ \Delta(i) \otimes_R \Delta(j) \cong \Delta(i \cup j). \]

By (2.2), we have (2.3). \qed

As \( G_{m,n} \)-module, \( \Delta(0, i) \cong \Delta(i) \). Note that a cellular algebra is semi-simple if and only if all of its cell modules are pairwise non-isomorphic irreducible [2]. Therefore, that \( TL_{m,n}(\delta) \) is semi-simple implies all \( \Delta(i) \) are pairwise non-isomorphic irreducible. So, \( G_{m,n} \) is semi-simple which is equivalent to the fact \( \text{ch}R \nmid m \). Moreover, \( u_i \neq u_j \) for any \( i \neq j \), \( 1 \leq i, j \leq m \).

In the sub-sequel, we assume \( \text{ch}R \nmid m \), \( u_i = \xi^i \), \( 1 \leq i \leq m \) where \( \xi \) is a primitive \( m \)-th root of unity. The reason is that the semi-simplicity of \( G_{m,n} \) is necessary for \( TL_{m,n}(\delta) \) to be semi-simple.

For the latter use, we need another construction of the cell modules as follows. Let \( J_{m,n}^k \) (resp. \( J_{m,n}^k \)) be the free \( R \)-submodule of \( TL_{m,n} \) generated by labelled TL-diagrams with \( l \) horizontal arcs such that \( l \geq k \) (resp. \( l > k \)). Let \( I_{m,n}^k(\delta) \) be the submodule of \( J_{m,n}^k / J_{m,n}^k \) generated by the coset of \( v \otimes v_0 \otimes x \), with \( v \in P(n, k) \), \( x \in G_{m,n-2k} \), and \( v_0 = \text{top}(E_{n-2k+1} \cdots E_{n-1}) \in P(n, k) \). Then \( I_{m,n}^k(\delta) \) is a right \( G_{m,n-2k} \)-module in which \( x \in G_{m,n-2k} \) acts on the free vertices of \( \text{bot}(D) \) of, \( D \in I_{m,n}^k(\delta) \). In the following we give an example to illustrate the action.

By the construction of cell modules, we have
\[ \Delta(k, i) \cong I_{m,n}^k(\delta) \otimes_{G_{m,n-2k}} \Delta(i) \]

Moreover, \( \{ v \otimes v_0 \otimes_{G_{m,n-2k}} C^{11} \mid v \in P(n, k) \} \) is a free \( R \)-basis of \( \Delta(k, i) \).

3. Restriction and induction

In this section, we assume that there is at least one non-zero parameter, say \( \delta_i \). Otherwise \( \bar{\delta}_j = 0 \) for \( 1 \leq j \leq m \) (see (4.1) for the definition of \( \bar{\delta}_j \)). By [7, 8.1], \( TL_{m,n}(\delta) \) is not semi-simple.

**Lemma 3.1.** Suppose \( \delta_i \neq 0 \). Let \( e = \delta_i^{-1}T^{11}_nE_{n-1} \in TL_{m,n}(\delta) \). Then \( e^2 = e \), and \( cTL_{m,n}(\delta)e \cong TL_{m,n-2}(\delta) \).
Proof. Each element in \( eTL_{m,n}(\delta_i)e \) is a linear combination of the labelled TL-diagrams \( D \) in which \( \text{top}(D) \) (resp. \( \text{bot}(D) \)) contains a horizontal arc \( \{n-1,n\} \) where there are \( i \) (resp. 0) dots. Let \( D^0 \) be the labelled TL-diagram obtained from \( D \) by removing the horizontal arc \( \{n-1,n\} \) on \( \text{top}(D) \) and \( \text{bot}(D) \). By the definition of the product of two labelled TL-diagrams in Definition 2.1, one can verify easily that the \( R \)-linear isomorphism \( \phi : eTL_{m,n}(\delta_i)e \to TL_{m,n-2} \) with \( \phi(D) = \delta_i D^0 \), is an algebraic isomorphism. \( \square \)

Now we may use the idempotent \( e \) to define two functors \( F \) and \( G \) as follows.

**Definition 3.2.** Let \( F : TL_{m,n}(\delta)\text{-mod} \to TL_{m,n-2}(\delta)\text{-mod} \) with \( F(M) = eM \) and \( G : TL_{m,n-2}(\delta)\text{-mod} \to TL_{m,n}(\delta)\text{-mod} \) with \( G(M) = TL_{m,n}(\delta)e \otimes_{TL_{m,n-2}(\delta)} M \).

**Proposition 3.3.** Assume \( i \in \Lambda(m, n-2k) \).

a) If \( \varphi \) is a non-zero \( TL_{m,n-2}(\delta)\text{-homomorphism} \), then \( G(\varphi) \neq 0 \).

b) \( FG \) is an identity functor.

c) \( G(\Delta(k-1, i)) = \Delta(k, i), \ G(\Delta(k-1, i) \downarrow) = \Delta(k, i) \downarrow \).

d) \( F(\Delta(k, i)) = \Delta(k-1, i), \ F(\Delta(k, i) \downarrow) = \Delta(k-1, i) \downarrow \).

Proof. (a) and (b) follows from a general result in [3, 6.2]. (d) follows from (c) and (b) by applying the functor \( F \) on both side of (c).

Let \( v_0 = \text{top}(E_{n-2k+1}E_{n-2k+3} \cdots E_{n-1}) \in P(n, k) \). We claim, as \( TL_{m,n}(\delta)\text{-modules} \),

\[
I^k_{m,n}(\delta) \cong TL_{m,n}(\delta)e \otimes_{TL_{m,n-2}(\delta)} I^{k-1}_{m,n-2}(\delta) \tag{3.1}
\]

In fact, let \( l = n - 2k \). Then \( \epsilon = T^l_{i+l+1}T^l_{i+l+2} \cdots T^l_{i+n-3}E_{l+3}E_{l+3} \cdots E_{n-3} \in I^{k-1}_{m,n-2}(\delta) \), that is,

\[
\epsilon = \begin{array}{ccccccc}
1 & l & l+1 & l+2 & n-3 & n-2 \\
& \cdots & & (i) & & \\
1 & l & l+1 & l+2 & n-3 & n-2
\end{array}
\]

Suppose \( D_1e \otimes D_2 \in TL_{m,n}(\delta)e \otimes_{TL_{m,n-2}(\delta)} I^{k-1}_{m,n-2}(\delta) \). Then \( D_2 \cdot \epsilon = \delta_i^{k-1}D_2, \ eD_2 = D_2e \) and \( D_1e \otimes D_2 = \delta_1^{l-k}D_1e \otimes D_2e = \delta_i^{l-k}D_1D_2e \otimes e \).

where \( D_2^0 \) can be obtained from \( D_2 \) by adding two horizontal arcs \( \{n-1,n\} \) on the top and bottom row of \( D_2 \). Obviously, \( D_1D_2^0 \in I^k_{m,n}(\delta) \). Therefore, any element in \( TL_{m,n}(\delta)e \otimes_{TL_{m,n-2}(\delta)} I^{k-1}_{m,n-2}(\delta) \) can be expressed as a linear combination of the element \( D_3e \otimes \epsilon \) with \( D_3 = D_1D_2^0 \). Define the \( R \)-linear map \( \alpha : TL_{m,n}(\delta)e \otimes_{TL_{m,n-2}(\delta)} I^{k-1}_{m,n-2}(\delta) \to I^k_{m,n}(\delta) \) with \( \alpha(D_3e \otimes \epsilon) = D_3 \).
Then α is an epimorphism. If $D_3 = 0$, then either $0 = D_3 \in TL_{m,n}(\delta)$ or $\text{bot}(D_3)$ contains at least one extra arc, say $(i', i' + 1)$, $i' \leq n - 2k - 1$, in which there are $s$ dots. So,

$$D_3 e \otimes \epsilon = \delta_{i}^{-1} D_3 \tau_{i'}^{-s} E_{i'} \tau_{i'}^{s} e \otimes \epsilon = \delta_{i}^{-1} D_3 \otimes \tau_{i'}^{-s} E_{i'} \tau_{i'}^{s} = \delta_{i}^{-1} D_3 \otimes 0 = 0.$$ 

Therefore, α is injective. By (3.1) and (2.2),

$$G(\Delta(k - 1, i)) = TL_{m,n}(\delta)e \otimes_{TL_{m,n-2}(\delta)} (I_{m,n-2}(\delta) \otimes G_{m,n-2k} \Delta(i))$$

$$\cong (TL_{m,n}(\delta)e \otimes_{TL_{m,n-2}(\delta)} I_{m,n-2}(\delta)) \otimes G_{m,n-2k} \Delta(i)$$

$$\cong I_{m,n}^{k}(\delta) \otimes G_{m,n-2k} \Delta(i)$$

$$= \Delta(k,i).$$

This completes the proof of the first isomorphism given in (c). The second isomorphism can be proved similarly.

**Definition 3.4.** For any $TL_{m,n}(\delta)$-modules $M$ and $N$, define

$$\langle M, N \rangle_n = \langle M, N \rangle_{TL_{m,n}(\delta)} = \dim_R \text{Hom}_{TL_{m,n}(\delta)}(M,N).$$

**Proposition 3.5.** Suppose $i \in \Lambda(m,n), j \in \Lambda(m,n - 2k)$ and $k_0 \in \mathbb{N}$. Then $\langle \Delta(k_0,i), \Delta(k + k_0,j) \rangle_{n + 2k_0} \neq 0$ if and only if $\langle \Delta(0,i), \Delta(k,j) \rangle_n \neq 0$.

**Proof.** "$\Leftarrow$" follows from Proposition 3.3(a) and (c) by applying $G$ repeatedly.

"$\Rightarrow$" Suppose $0 \neq \varphi \in \text{Hom}_{TL_{m,n+2k_0}(\delta)}(\Delta(k_0,i), \Delta(k + k_0,j))$ and $W = \varphi(\Delta(k_0,i))$. Let $e = \delta_{i}^{-1} T_{n+2k_0-1} E_{n+2k_0-1}$. We claim

(3.2) 

$$eW \neq 0.$$ 

Otherwise, we have $eW = 0$. Let $v_i = \text{top}(E_i) = \text{bot}(E_i)$. Then

$$E_1 = \delta_{i}^{-2} (v_1 \otimes v_{n+2k_0-1} \otimes id) \cdot T_{n+2k_0-1} E_{n+2k_0-1} T_{n+2k_0-1}^i \cdot (v_{n+2k_0-1} \otimes v_1 \otimes id).$$

So, $E_1 W = 0$ which implies $EW = 0$ with $E = E_1 E_3 \cdots E_{2k_0-1}$. On the other hand, Let $U_0 = \text{rad}\Delta(k_0, i)$. Then either $\Delta(k_0,i) = U_0$ or $\Delta(k_0,i)/U_0$ is irreducible [2, 3.2]. Let $m = (m, m, \cdots, m) \in \Lambda(m,n)$. Since $E \in TL_{m,n+2k_0}(\delta) \subset TL_{m,n+2k_0}(\delta)$, Lemma 2.3 shows $EU_0 = 0$. We have $W = \varphi(\Delta(k_0,i)) \cong \Delta(k_0,i)/U$. We claim $U \subset U_0$. Otherwise, $U + U_0 = \Delta(k_0,i)$ and hence $U/(U_0 \cap U) \cong \Delta(k_0,i)/U_0$ is irreducible. So, there is a composition series of $\Delta(k_0,i)$ such that the multiplicity of $L(k_0,i)$ is greater than 2, a contradiction.

Let $y = \text{top}(T_1^i T_3^i \cdots T_{2k_0-1}^i E)$. Then $v = y \otimes v_0 \otimes C_{1,1}^i \in \Delta(k_0,i)$ is a non-zero element, where $v_0$ is a fixed element in $P(n + 2k_0, k_0)$. Since $\delta_{i} \neq 0$, $T_1^i T_3^i \cdots T_{2k_0-1}^i E \cdot v = (\delta_{i})^{k_0} v \neq 0$, which implies $v \notin U$. Therefore, $T_1^i T_3^i \cdots T_{2k_0-1}^i E(v + U) = \delta_{i}^{k_0}(v + U) \neq 0 \mod U$, which contradicts to the fact $eW = 0$. This completes the proof of (3.2).

If $eW \neq 0$, then $F(\varphi) \neq 0$. Now, the result follows from induction and (3.2).
**Proposition 3.6.** Suppose $M$ is a $TL_{m,n}(\delta)$-module. Then $M \uparrow \cong G(M) \downarrow$, where $M \uparrow$ is the induced module of a $TL_{m,n}(\delta)$-module $M$ to $TL_{m,n+1}(\delta)$. In particular, for any $i \in \Lambda(m, n-2k)$, $\Delta(k, i) \uparrow \cong \Delta(k+1, i) \downarrow$.

*Proof.* Suppose $x \in TL_{m,n+1}(\delta)$. Add $(n+2)$-th vertex on top$(x)$ and bot$(x)$ to get a new labelled TL-diagram $D$ in which

1. the $(n+2)$-th vertex of top$(D)$ joins the vertex $j$ if $\{j, n+1\}$ is an arc in $x$. Here $n+1$ is the $(n+1)$-th vertex in bot$(x)$. Moreover, if there are $s$ dots on the arc $\{j, n+1\}$, so is the new arc $\{j, n+2\}$

2. $\{n+1, n+2\}$ is a horizontal arc in bot$(D)$ in which there is no dot.

We give two examples to illustrate the above definition.

![Diagram](image)

Define an $R$-linear map $\alpha : TL_{m,n+1}(\delta) \to TL_{m,n+2}(\delta)e$ with $\alpha(x) = D$. Obviously, $\alpha$ is an $R$-linear isomorphism. By the definition of the product of two labelled TL-diagrams, $\alpha$ is a left $TL_{m,n+1}(\delta)$-module and right $TL_{m,n}(\delta)$-module isomorphism. That is,

\begin{equation}
TL_{m,n+1}(\delta) \cong TL_{m,n+2}(\delta)e.
\end{equation}

For any $TL_{m,n}(\delta)$-module $M$,

\[ M \uparrow \cong TL_{m,n+1}(\delta) \otimes_{TL_{m,n}(\delta)} M \]

\[ \cong TL_{m,n+2}(\delta)e \otimes_{TL_{m,n}(\delta)} M \text{ by (3.3)} \]

\[ \cong G(M) \downarrow. \]

\[ \square \]

**Corollary 3.7.** Suppose $chR \uparrow m$. Assume $i = (i_1, i_2, \ldots, i_n) \in \Lambda(m, n)$. If $j = (i_1, i_2, \ldots, i_n, j) \in \Lambda(m, n+1)$, then $\langle \Delta(0, i) \uparrow, \Delta(0, j) \rangle_{n+1} \neq 0$.

*Proof.* By Proposition 3.6, $\langle \Delta(0, i) \uparrow, \Delta(0, j) \rangle_{n+1} = \langle \Delta(1, i) \downarrow, \Delta(0, j) \rangle_{n+1}$. Now Proposition 2.6 implies that, for all $j = (i_1, i_2, \ldots, i_n, j), 1 \leq j \leq m, \langle \Delta(1, i) \downarrow, \Delta(0, j) \rangle_{n+1} \neq 0$. \[ \square \]

**Proposition 3.8.** Suppose $chR \uparrow m$ and $\langle \Delta(0, i), \Delta(k, j) \rangle_n \neq 0$ for $i \in \Lambda(m, n)$ and $j \in \Lambda(m, n-2k)$. 
(a) If \( i^0 = (i_1, i_2, \ldots, i_{n-1}) \in \Lambda(m, n-1) \), then \( \langle \Delta(0, i^0), \Delta(k, j) \downarrow \rangle \) \( \)_{n-1} \neq 0.

(b) Let \( j^0 = (j_1, j_2, \ldots, j_{n-2k-1}) \) and \( j^1 = (j_1, j_2, \ldots, j_{n-2k}, j_0), 1 \leq j_0 \leq m \). Then either

\( \langle \Delta(0, i^0), \Delta(k, j^0) \rangle \) \( \)_{n-1} \neq 0 or \( \langle \Delta(0, i^0), \Delta(k-1, j^1) \rangle \) \( \)_{n-1} \neq 0.

Proof. Since \( i^0 \in \Lambda(m, n-1) \), Corollary 3.7 implies \( \langle \Delta(0, i^0), \Delta(0, i) \rangle \) \( \)_{n} \neq 0. Since \( chR \uparrow m \), \( \Delta(i) \) is a simple \( G_{m,n} \)-module, forcing \( \Delta(0, i) \) to be an irreducible \( TL_{m,n}(\delta) \)-module. So, \( \langle \Delta(0, i^0), \Delta(k, j) \rangle \) \( \)_{n} \neq 0. Using Frobenius reciprocity, we get (a).

Let \( V = \Delta(k, j) \downarrow \). By Proposition 2.6, there is a submodule \( W \subset V \) such that \( W \cong \Delta(k, j^0) \).

Let \( 0 \neq S \) be the image of \( \Delta(0, i^0) \) in \( V \). Since \( \Delta(0, i^0) \) is irreducible, \( S \cong \Delta(0, i^0) \). If \( S \subset W \),

\( \langle \Delta(0, i^0), \Delta(k, j^0) \rangle \) \( \)_{n-1} \neq 0.

If \( S \not\subset W \), then \( S \cap W = 0 \). Thus, \( (S \oplus W)/W \cong S/(W \cap S) = S \) is an irreducible submodule of \( V/W \). By Proposition 2.6,

\[
V/W \cong \bigoplus_{j=1}^{m} \Delta(k-1, j \cup j).
\]

Hence there is a \( j^1 = (j_1, j_2, \ldots, j_{n-2k}, j_0) \in \Lambda(m, n-2k+1) \) such that \( (S \oplus W)/W \subset \Delta(k-1, j^1) \), forcing \( \langle \Delta(0, i^0), \Delta(k-1, j^1) \rangle \) \( \)_{n-1} \neq 0.

\[\square\]

4. Semi-simplicity of the cyclotomic Temperley-Lieb algebras

In this section, we shall give the necessary and sufficient conditions on the semi-simplicity of \( TL_{m,n}(\delta) \). The key is [7, 8.1]. First, let us recall some of results in [7].

Let \( u_i = \xi^i \) where \( \xi \) is a primitive \( m \)-th root of unity. For any \( i = (i_1, i_2, \cdots, i_{n-2}) \in \Lambda(m, n-2) \), let

\[
\Psi_i(n, 1) = \begin{pmatrix}
A & B_1 \\
B_1^T & A & B_2 \\
& B_2^T & A & B_3 \\
& & & \ddots & \vdots \\
& & & & \ddots & A & B_{n-2} \\
& & & & & B_{n-2}^T & A
\end{pmatrix},
\]

where \( B_j = (b_{st}) \) with \( b_{st} = u_{i_j}^{s-t} \) for \( 1 \leq s, t \leq m \), and \( B_i^T \) stands for the transpose of \( B_i \), and

\[
A = \begin{pmatrix}
\delta_0 & \delta_1 & \cdots & \delta_{m-1} \\
\delta_1 & \delta_2 & \cdots & \delta_0 \\
\vdots & \vdots & \ddots & \vdots \\
\delta_{m-1} & \delta_0 & \cdots & \delta_{m-2}
\end{pmatrix}.
\]
Let \( p(x) = \delta_0 x^{m-1} + \delta_1 x^{m-2} + \cdots + \delta_{m-1} \). Write

\[
\frac{p(x)}{x^m - 1} = \frac{\tilde{\delta}_1}{x - u_1} + \frac{\tilde{\delta}_2}{x - u_2} + \cdots + \frac{\tilde{\delta}_m}{x - u_m}.
\]

Then

\[
\tilde{\delta}_j = p(u_j)/\prod_{i \neq j}(u_j - u_i).
\]

Following [7], we partition \( i = (i_1, i_2, \ldots, i_{n-2}) \) into \( (i_{1,1}, i_{1,2}, \ldots, i_{1,j_1}, i_{2,1}, i_{2,2}, \ldots, i_{2,j_2}, \ldots, i_{r,j_r}) \) with \( j_1 + j_2 + \cdots + j_r = n - 2 \) such that \( m \) divides \( i_{p,q} + i_{p,q+1} \) for all \( p \) with \( 1 \leq q < j_p \) and that \( m \) does not divide \( i_{p,j_p} + i_{p+1,1} \) for all \( 1 \leq p < r \). Let

\[
P_{n}(x_1, \ldots, x_n) = \det \left( \begin{array}{ccc} x_1 & 1 & \cdot \cdot \cdot & \cdot \cdot \cdot & \cdot \\
1 & x_2 & 1 & \cdot \cdot \cdot & \cdot \\
 & \cdot & \cdot & \cdot & \cdot \\
 & & \cdot & \cdot & 1 \\
 & & & 1 & x_n \end{array} \right).
\]

We call \( P_n(x_1, x_2, \cdots, x_n) \) the \( n \)-th generalized Tchebychev polynomial. The following result was proved in [7, §8].

**Proposition 4.1.** Keep the setup. Then

\[
\det \Psi_i(n, 1) = (-1)^{\frac{1}{2}m(m-1)(n-1)}m^{m(n-1)} \prod_{p=1}^{r} (\tilde{\delta}_{1-p,q} \prod_{q=1}^{p} \delta_{i_p, q}) P_{n}(\tilde{\delta}_{i_{p,1}}, \tilde{\delta}_{i_{p,2}}, \cdots, \tilde{\delta}_{i_{p,j_p}}).
\]

**Proposition 4.2.** Suppose \( i \in \Lambda(m, n), j \in \Lambda(m, n - 2) \). If \( \langle \Delta(0, i), \Delta(1, j) \rangle_n \neq 0 \), then \( \det \Psi_j(n, 1) = 0 \).

**Proof.** Since \( \langle \Delta(0, i), \Delta(1, j) \rangle_n \neq 0 \), there is a \( \varphi \in \text{Hom}_{\text{TL}_{m,n}(\delta)}(\Delta(0, i), \Delta(1, j)) \) such that \( \varphi(v) \neq 0 \) for some \( v \in \Delta(0, i) \). Consider an element

\[
T = \sum_{i=1}^{n-1} \sum_{s=0}^{m-1} T_i^s E_i T_i^s \in \text{TL}_{m,n}(\delta)
\]

We have \( T \varphi(v) = \varphi(Tv) = \varphi(0) = 0 \). Write

\[
\varphi(v) = \sum_{i=1}^{n-1} \sum_{s=0}^{m-1} a_{i,s} v_i^{(s)} \otimes v_0 \otimes C_{1,1}^d,
\]

where \( v_i^{(s)} = \text{top}(T_i^s E_i) \) and \( v_0 \) is a fixed element in \( P(n, 1) \). Since

\[
(v_1 \otimes v_1 \otimes C_{1,1}^d)(v_2 \otimes v_2 \otimes C_{1,1}^d) \equiv v_1 \otimes v_2 \otimes \phi^{(n,1)}_{v_1,v_2}(t_1, t_2, \ldots, t_n, 2)(C_{1,1}^d)^2 \pmod{\text{TL}^{(1,1)}_{n,m}},
\]

then by Proposition 4.1, we have

\[
\det \Psi_j(n, 1) = 0.
\]
for some elements $\phi_{v_i,v_j}^{(n)}(t_1,t_2,\ldots,t_{n-2})$ in $G_{m,n-2}$. By a direct computation, we have
\[ 0 = T\varphi(v) = \sum_{1 \leq i,j \leq n-1} \sum_{0 \leq s,t \leq m-1} \phi_{v_i,v_j}^{(n)}(u_{j_1},u_{j_2},\ldots,u_{j_n}) a_{j,t} v_i(s) \otimes v_0 \otimes C_{1,1}^j \]
Therefore, for all $i, s$, we have $\sum_{1 \leq j \leq n-1} \sum_{0 \leq t \leq m-1} \phi_{v_i,v_j}^{(n)}(u_{j_1},u_{j_2},\ldots,u_{j_n}) a_{j,t} = 0$.
Since $\varphi(v) \neq 0$, there is at least one of $a_{i,t} \neq 0$, which implies $\det \Psi_l(n,1) = 0$. □

**Proposition 4.3.** Suppose $R$ is a splitting field of $x^m - 1$ with $\text{ch} R \nmid m$. If $\det \Psi_l(1,1) \neq 0$ for all $2 \leq l \leq n$ and $i \in \Lambda(m,l-2)$, then $TL_{m,n}(\delta)$ is semi-simple.

**Proof.** It is proved in [7] that $TL_{m,n}(\delta)$ is a cellular algebra. Note that a cellular algebra is semi-simple if and only if all of its cell modules are pairwise non-isomorphic irreducible (see [2]). So, $TL_{m,n}(\delta)$ is not semi-simple if there is a cell module, say $\Delta(k_1,i)$, which is not irreducible. Thus, the length of $\Delta(k_1,i)$ is strictly greater than 1, and there is an irreducible proper submodule $D$ of $\Delta(k_1,i)$. Note that any simple module of a cellular algebra is the simple head of a cell module. Therefore, $D$ is the simple quotient of a cell module, say $\Delta(k_2,j)$. Since $D$ is a composition factor of $\Delta(k_1,i)$, it follows from Definition 2.2 and (2.1) that $(k_1,i) \leq (k_2,j)$. Moreover, $(k_1,i) \neq (k_2,j)$. Otherwise, $\Delta(k_1,i)$ would have a simple head $D$. So, the multiplicity of $D$ in $\Delta(k_1,i)$ is at least two, a contradiction. We have $\langle \Delta(k_2,j), \Delta(k_1,i) \rangle_n \neq 0$. Moreover, either $k_1 > k_2$ or $k_1 = k_2$ and $i < j$.

Suppose $k_1 > k_2$. Using Proposition 3.5, we can assume $j \in \Lambda(m,l), l = n - 2k_2$. Let $k = k_1 - k_2$. Then $\langle \Delta(0,j), \Delta(k,i) \rangle_l \neq 0$. Applying Proposition 3.8 repeatedly, we can assume $k = 1$. By Proposition 4.2, $\det \Psi_l(1,1) = 0$, a contradiction.
Suppose $k_1 = k_2$ and $i < j$. By Proposition 3.5, $\langle \Delta(0,j), \Delta(0,i) \rangle_{n-2k_1} \neq 0$, a contradiction since $\Delta(0,j) \not\cong \Delta(0,i)$ and both of them are irreducible.

Thus we have shown that under our assumption all cell modules are irreducible. It is clear that they are also pairwise non-isomorphic. Hence $TL_{m,n}(\delta)$ is semi-simple. □

**Lemma 4.4.** Suppose $\det \Psi_l(n,1) \neq 0$ for all $i \in \Lambda(m,n-2)$ with $m \geq 2$. Then $\delta_i \neq 0$ for any $i, 1 \leq i \leq m$.

**Proof.** Take $i = (m,m,\ldots,m) \in \Lambda(m,n-2)$. Then $i$ can be divided into one part with $j_1 = n - 2$. By Proposition 4.1, $\delta_i \neq 0, 1 \leq i \leq m - 1$ since they are the factors of $\det \Psi_l(n,1)$. Take $i = (1,1,\ldots,1) \in \Lambda(m,n-2)$. Then $i$ can be divided into either one part if $m = 2$ or $n-2$ parts if $m > 2$. By Proposition 4.1, $\delta_m \neq 0$ since it is a factor of $\det \Psi_l(n,1)$ in any case. □

It is proved in [7, 8.1] that $\det \Psi_l(n,1) \neq 0$ for all $i \in \Lambda(m,n-2)$ and $\text{ch} R \nmid m$ if $TL_{m,n}(\delta)$ is semi-simple. The following is the inverse of this result.

**Proposition 4.5.** Suppose $R$ is a splitting field of $x^m - 1$ with $\text{ch} R \nmid m$ and $m \geq 2$. If $\det \Psi_l(n,1) \neq 0$ for all $i \in \Lambda(m,n-2)$, then $TL_{m,n}(\delta)$ is semi-simple.
Proof. By Proposition 4.3, we need prove det $\Psi_i(l,1) \neq 0$ for all $2 \leq l \leq n, i \in \Lambda(m,l-2)$ under our assumption. If det $\Psi_i(l,1) = 0$ for some $l, l \neq n$ and $i \in \Lambda(m,l-2)$, then $P_{jp}(\bar{\delta}_{ip,1}, \bar{\delta}_{ip,2}, \ldots, \bar{\delta}_{ip,jp}) = 0$ for some $p, 1 \leq p \leq r$ by Proposition 4.1 and Lemma 4.4.

On the other hand, take $i_0 = (i_1, i_2, \ldots, i_{l-2}, a, a, \ldots, a) \in \Lambda(m, n-2)$ with $m \nmid (i_{l-2} + a)$. By Proposition 4.1, $P_{jp}(\bar{\delta}_{ip,1}, \bar{\delta}_{ip,2}, \ldots, \bar{\delta}_{ip,jp})$ must be a factor of det $\Psi_{i_0}(n,1)$ and hence det $\Psi_{i_0}(n,1) = 0$, a contradiction. □

Remark. The reason we assume $m \geq 2$ is that we need the fact that $i_{l-2}$ and $a$ cannot be in the same part. When $m = 1$, we cannot use the above argument. However, one can get a necessary and sufficient condition for $TL_{n,1}$ to be semi-simple [10, §5].

Together with [7, 8.1] and Proposition 4.5, we have the main result of this paper as follows. Note that Theorem 4.6 is not true if $m = 1$.

**Theorem 4.6.** Suppose $m \geq 2$. Let $R$ be a splitting field of $x^m - 1$, containing $1, \delta_0, \ldots, \delta_{m-1}$.

Then the following conditions are equivalent.

(a) $TL_{m,n}(\delta)$ is semi-simple.
(b) $TL_{m,n}(\delta)$ is split semi-simple.
(c) $\text{ch} R \nmid m$ and det $\Psi_i(n,1) \neq 0$ for all $i \in \Lambda(m,n-2)$.
(d) All cell modules $\Delta(k,i)$ with $(k,i) \in \Lambda_{n,m}$ are pairwise non-isomorphic irreducible.
(e) All cell modules $\Delta(k,i)$ with $(k,i) \in \Lambda_{n,m}, k \in \{0,1\}$ are pairwise non-isomorphic irreducible.

Proof. Since $TL_{m,n}(\delta)$ is a cellular algebra, (a), (b) and (d) are equivalent. By [7, 8.1], (c) and (e) are equivalent. By Proposition 4.5 and [7, 8.1], (a) and (c) are equivalent. □

The following Corollary follows immediately from [7, 8.1] and Proposition 4.5.

**Corollary 4.7.** Keep the setup. Then $TL_{m,n}(\delta)$ is semi-simple if and only if

(a) $\text{ch} R \nmid m$
(b) $P_l(\bar{\delta}_i) = \bar{\delta}_i \neq 0, 1 \leq i \leq m$.
(c) For any $(i_1, i_2, \ldots, i_l) \in \Lambda(m,l)$ with $m \mid (i_j + i_{j+1}), 1 \leq j \leq l-1, P_l(\bar{\delta}_{i_1}, \bar{\delta}_{i_2}, \ldots, \bar{\delta}_{i_l}) \neq 0, 2 \leq l \leq n$.

Remark. When $m = 1$, $\Lambda(m,n)$ contains only one element $(1,1,\ldots,1)$ which can be partitioned into one part. In this case, Corollary 4.7 is Westbury’s Theorem given in [10, §5].

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