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# Derived equivalences and stable equivalences of Morita type, II

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**Abstract.** We consider the question of lifting stable equivalences of Morita type to derived equivalences. One motivation comes from an approach to Broué's abelian defect group conjecture. Another motivation is a conjecture by Auslander and Reiten on stable equivalences preserving the number of non-projective simple modules. A machinery is presented to construct lifts for a large class of algebras, including Frobenius-finite algebras introduced in this paper. In particular, every stable equivalence of Morita type between Frobenius-finite algebras over an algebraically closed field can be lifted to a derived equivalence. Consequently, the Auslander–Reiten conjecture is true for stable equivalences of Morita type between Frobenius-finite algebras. Examples of Frobenius-finite algebras are abundant, including representation-finite algebras, Auslander algebras, cluster-tilted algebras and certain Frobenius extensions. As a byproduct of our methods, we show that, for a Nakayama-stable idempotent element $e$ in an algebra $A$ over an algebraically closed field, each tilting complex over $eAe$ can be extended to a tilting complex over $A$ that induces an almost $\nu$-stable derived equivalence studied in the first paper of this series. Moreover, the machinery is applicable to verify Broué's abelian defect group conjecture for several cases mentioned in the literature.

## 1. Introduction

Derived and stable equivalences of algebras (or categories) are two kinds of fundamental equivalences both in the representation theory of algebras and groups and in the theory of triangulated categories. They preserve many significant algebraic, geometric or numeric properties, and provide surprising and useful new applications to as well as connections with other fields (see [8], [40], [41] and [48]). But what are the interrelations between these two classes of equivalences? Rickard

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showed in [40], [43] (see also [25]) that, for self-injective algebras, derived equivalences imply stable equivalences of Morita type which form a significant class of stable equivalences and have much better homological invariants. For instance, the global, finitistic, dominant and representation dimensions all are invariants (see [8], [21], [50]). Conversely, Asashiba's work [3] together with a recent work by Dugas in [14] shows that, for standard representation-finite self-injective algebras, every stable equivalence lifts to a derived equivalence. For general algebras, however, little is known about their relationship. That is, one does not know any methods to construct such an equivalence from the other for arbitrary algebras. In [21], we started discussing this kind of questions and gave a general method to construct stable equivalences of Morita type from almost $\nu$-stable derived equivalences, a class of derived equivalences defined in [21] (see Section 2.2 for definition). This generalizes the above-mentioned result of Rickard.

In a general context, the converse is how to get a derived equivalence from a known stable equivalence of Morita type. This is even less known. There are some stable equivalences of Morita type that cannot be lifted to derived equivalences, even in the self-injective case (see [8], Section 5A). So, our concern is the following.

**Main question.** *Given a stable equivalence of Morita type between arbitrary finite-dimensional algebras $A$ and $B$ over a field, under which conditions can we construct a derived equivalence therefrom between $A$ and $B$?*

This is of interest due to two major conjectures. One is *Broué's abelian defect group conjecture*, which says that the module categories of a block of a finite group algebra and its Brauer correspondent have equivalent derived categories if their defect groups are abelian. Note that block algebras are self-injective. So, by Rickard's result, Broué's abelian defect group conjecture would predict actually a stable equivalence of Morita type, while the latter arise fairly often in the modular representation theory of finite groups. For instance, it occurs very often as a restriction functor in Green correspondences. To be able to lift stable to derived equivalences is important in one approach, due to Rouquier [48], to Broué's abelian defect group conjecture: given two block algebras $A$ and $B$, to prove that $A$ and $B$ are derived equivalent, it is enough to find another algebra $C$ such that $B$ and $C$ are derived equivalent, and that there is a stable equivalence of Morita type between $A$ and $C$, which sends simple $A$-modules to simple $C$-modules, or can be lifted to a derived equivalence. Then $A$ and $B$ are derived equivalent by Linckelmann's result in [27], Theorem 2.1, or by composite of two derived equivalences.

The other conjecture is the *Auslander–Reiten conjecture* (or Alperin–Auslander conjecture referred in [48]) on stable equivalences, which states that two stably equivalent algebras have the same number of non-isomorphic non-projective simple modules (see, for instance, Conjecture (5), p. 409 in [4], or Conjecture 2.5 in [48]). For finite-dimensional algebras over an algebraically closed field, Martínez-Villa reduced the conjecture to self-injective algebras [32], and proved the conjecture for representation-finite algebras [31]. For weakly symmetric algebras of domestic type, the conjecture was verified in [52]. However, this conjecture is still open, even for stable equivalences of Morita type. Our main question is related to the conjecture in the following way: if two algebras are derived equivalent, then they have the same number of non-isomorphic simple modules (see [39], [24]), while it is known that stable equivalences of Morita type preserve projective simple modules. Thus the Auslander–Reiten conjecture is true for those stable equivalences of Morita type that can be lifted to derived equivalences. For block algebras, Broué's abelian defect group conjecture implies the Auslander–Reiten conjecture (see [48]). For some equivalent formulations of the Auslander–Reiten conjecture in terms of stable Hochschild homology and Higman ideal, we refer the reader to [30].

In this paper, we shall provide several answers to the main question. Our methods developed here are different from those in [3], [20], and can be used to re-verify Broué's abelian defect group conjecture in some cases (see Section 6). Moreover, these methods much simplify previous approaches in the literature, for instance in Asashiba's work [3], Müller–Shaps's work [34], Koshitani–Müller's work [26], and in Okuyama's examples [37].

We introduce a large class of algebras, called Frobenius-finite algebras, and then show that every stable equivalence of Morita type between Frobenius-finite algebras lifts to a derived equivalence (see Subsection 2.2 for definitions). Roughly speaking, a Frobenius part of a finite-dimensional algebra $A$ is the largest algebra of the form $eAe$ with $e$ an idempotent element such that $\mathrm{add}(Ae)$ is stable under the Nakayama functor of $A$. This was introduced first in the paper [31] by Martínez-Villa. An algebra is said to be *Frobenius-finite* if its Frobenius part is a representation-finite algebra. Examples of Frobenius-finite algebras are abundant and capture many interesting classes of algebras, for instance, representation-finite algebras, Auslander algebras and cluster-tilted algebras. Also, they can be constructed from triangular matrix algebras, Auslander–Yoneda algebras and Frobenius extensions (for more details and examples see Section 5.1).

**Theorem 1.1.** *Let $k$ be an algebraically closed field. Suppose that $A$ and $B$ are finite-dimensional $k$-algebras without nonzero semisimple direct summands. If $A$ is Frobenius-finite, then each individual stable equivalence of Morita type between $A$ and $B$ lifts to an iterated almost $\nu$-stable derived equivalence.*

In particular, the Auslander–Reiten conjecture holds true for stable equivalences of Morita type between Frobenius-finite algebras over an algebraically closed field. Of course, this also follows from [32], [31]. But Theorem 1.1 provides another approach to the conjecture in this case, and shows that Frobenius-finite algebras shares many common algebraical and numerical invariants of derived and stable equivalences. Moreover, Theorem 1.1 not only extends a result of Asashiba in [3] (in a different direction) to a much wider context, namely every stable equivalence of Morita type between *arbitrary* representation-finite (not necessarily self-injective) algebras lifts to a derived equivalence, but also provides a method to construct derived equivalences between algebras and their subalgebras because, under some mild conditions, each stable equivalence of Morita type can be realized as a Frobenius extension of algebras by Corollary 5.1 in [15].

The next result, Theorem 1.2, is the technical main result, providing a method to prove Theorem 1.1 and a general approach to lifting stable equivalences of Morita type to derived equivalences. Recall that an idempotent element $e$ of an algebra $A$

is said to be *$\nu$-stable* if $\mathrm{add}(\nu_A Ae) = \mathrm{add}(Ae)$, where $\nu_A$ is the Nakayama functor of $A$.

**Theorem 1.2.** *Let $A$ and $B$ be finite-dimensional algebras over a field. Suppose that $A$ and $B$ have no nonzero semisimple direct summands and that $A/\mathrm{rad}(A)$ and $B/\mathrm{rad}(B)$ are separable. Let $e$ and $f$ be $\nu$-stable idempotent elements in $A$ and $B$, respectively, and let $\Phi\colon A\text{-}\underline{\mathrm{mod}} \to B\text{-}\underline{\mathrm{mod}}$ be a stable equivalence of Morita type, satisfying the following two conditions:*

(1) *For each simple $A$-module $S$ with $e \cdot S = 0$, $\Phi(S)$ is isomorphic in $B$-$\underline{\mathrm{mod}}$ to a simple module $S'$ with $f \cdot S' = 0$;*

(2) *For each simple $B$-module $T$ with $f \cdot T = 0$, $\Phi^{-1}(T)$ is isomorphic in $A$-$\underline{\mathrm{mod}}$ to a simple module $T'$ with $e \cdot T' = 0$.*

*If the stable equivalence $\Phi_1\colon eAe\text{-}\underline{\mathrm{mod}} \to fBf\text{-}\underline{\mathrm{mod}}$, induced from $\Phi$, lifts to a derived equivalence between $eAe$ and $fBf$, then $\Phi$ lifts to an iterated almost $\nu$-stable derived equivalence between $A$ and $B$.*

Note that if a stable equivalence of Morita type preserves all simple modules (that is, $e = 0$ and $f = 0$ in Theorem 1.2), then it is a Morita equivalence. This was first proved by Linckelmann for self-injective algebras in [27], and then extended to arbitrary algebras by Liu in [28]. Theorem 1.2 deals with a general situation where a stable equivalence of Morita type may not preserve all simple modules, and can be used as kind of an inductive step to lift stable equivalences. Compared with the method in the literature (for instance, [37], [26], [34]), our inductive method has an advantage: each step reduces the number of simple modules, and therefore one may work very possibly with representation-finite algebras after some steps, and then apply Theorem 1.1. We shall use this technique to reprove some known cases where Brouré's abelian defect group conjecture holds true (see Section 6).

As an immediate consequence of Theorem 1.2, we state the following corollary, reducing the lifting problem between given algebras to the one between Frobenius parts. This also explains our strategy of the proof of Theorem 1.1.

**Corollary 1.3.** *Let $A$ and $B$ be finite-dimensional $k$-algebras over a field and without nonzero semisimple direct summands such that $A/\mathrm{rad}(A)$ and $B/\mathrm{rad}(B)$ are separable. Suppose that $\Phi$ is a stable equivalence of Morita type between $A$ and $B$, and that $\Phi_1$ is the restricted stable equivalence of $\Phi$ between the Frobenius parts $\Delta_A$ and $\Delta_B$. If $\Phi_1$ lifts to a derived equivalence between $\Delta_A$ and $\Delta_B$, then $\Phi$ lifts to an iterated almost $\nu$-stable derived equivalence between $A$ and $B$.*

The contents of the paper are outlined as follows. In Section 2, we fix notation and collect some basic facts needed in our later proofs. In Section 3, we first begin by reviewing aspects of stable equivalences of Morita type, and then discuss relationships of stable equivalences of Morita type between algebras and their Frobenius parts which play a prominent role in studying the main question. In Sections 4 and 5, we prove the main results, Theorem 1.2 and Theorem 1.1, respectively. While proving Theorems 1.1 and 1.2, we also obtain Proposition 4.1, which extends a result of Miyachi (see Theorem 4.11 in [33]). In Section 6, we illustrate the procedure of lifting stable equivalences of Morita type to derived equivalences discussed in the paper by two examples from modular representation theory of finite groups. This shows that our results can be applied to verify Broué's abelian defect group conjecture for some cases. We end this section by a few questions for further investigation suggested by the main results in the paper.

## 2. Preliminaries

In this section, we shall recall basic definitions and facts required in our proofs.

### 2.1. Derived and stable equivalences

Throughout this paper, unless specified otherwise, all algebras will be finite-dimensional algebras over a fixed field $k$. All modules will be finitely generated unitary left modules.

Let $\mathcal{C}$ be an additive category.

For two morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ in $\mathcal{C}$, the composite of $f$ with $g$ is written as $fg$, which is a morphism from $X$ to $Z$. But for two functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$ of categories, their composite is denoted by $G \circ F$ or simply by $GF$, which is a functor from $\mathcal{C}$ to $\mathcal{E}$. For an object $X$ in $\mathcal{C}$, we denote by $\mathrm{add}(X)$ the full subcategory of $\mathcal{C}$ consisting of all direct summands of finite direct sums of copies of $X$.

By $\mathscr{C}(C)$ we denote the category of complexes $X^\bullet = (X^i, d_X^i)$ over $\mathcal{C}$, where $X^i$ is an object in $\mathcal{C}$ and the differential $d_X^i\colon X^i \to X^{i+1}$ is a morphism in $\mathcal{C}$ with $d_X^i d_X^{i+1} = 0$ for each $i \in \mathbb{Z}$; and by $\mathscr{K}(\mathcal{C})$ the homotopy category of $\mathcal{C}$. When $\mathcal{C}$ is an abelian category, we denote the derived category of $\mathcal{C}$ by $\mathscr{D}(\mathcal{C})$. The full subcategories of $\mathscr{K}(\mathcal{C})$ and $\mathscr{D}(\mathcal{C})$ consisting of bounded complexes over $\mathcal{C}$ are denoted by $\mathscr{K}^{\mathrm{b}}(\mathcal{C})$ and $\mathscr{D}^{\mathrm{b}}(\mathcal{C})$, respectively.

Let $A$ be an algebra. Then we denote by $A$-mod the category of all $A$-modules, and by $A$-proj (respectively, $A$-inj) the full subcategory of $A$-mod consisting of projective (respectively, injective) modules. As usual, $D$ denotes the $k$-duality $\mathrm{Hom}_k(-,k)$ and $(-)^*$ the $A$-duality $\mathrm{Hom}_A(-,A)$ from $A$-mod to $A^{\mathrm{op}}$-mod. We denote by $\nu_A$ the Nakayama functor $D\mathrm{Hom}_A(-,A)$ which gives rise to an equivalence from $A$-proj to $A$-inj with $\nu_A^{-1} = \mathrm{Hom}_A(DA,-)$.

By $A$-$\underline{\mathrm{mod}}$ we denote the stable module category of $A$, in which the morphism set of two modules $X$ and $Y$ is denoted by $\underline{\mathrm{Hom}}_A(X,Y)$. Given two algebras $A$ and $B$, if there is an equivalence $F\colon A\text{-}\underline{\mathrm{mod}} \to B\text{-}\underline{\mathrm{mod}}$, then we say that $F$ is a *stable equivalence* between $A$ and $B$, or that $A$ and $B$ are *stably equivalent* via $F$.

As usual, we simply write $\mathscr{K}^{\mathrm{b}}(A)$ and $\mathscr{D}^{\mathrm{b}}(A)$ for $\mathscr{K}^{\mathrm{b}}(A\text{-mod})$ and $\mathscr{D}^{\mathrm{b}}(A\text{-mod})$, respectively. It is well known that $\mathscr{K}^{\mathrm{b}}(A)$ and $\mathscr{D}^{\mathrm{b}}(A)$ are triangulated categories. For a complex $X^\bullet$ in $\mathscr{K}(A)$ or $\mathscr{D}(A)$, we denote by $X^\bullet[1]$ the lift of $X^\bullet$. It is obtained from $X^\bullet$ by shifting $X^\bullet$ to the left by 1 degree.

For $X \in A$-mod, we denote by $P(X)$ (respectively, $I(X)$) the projective cover (respectively, injective envelope) of $X$. As usual, the syzygy and co-syzygy of $X$ are denoted by $\Omega(X)$ and $\Omega^{-1}(X)$, respectively. The socle and top, denoted by $\mathrm{soc}(X)$

and top$(X)$, are the largest semisimple submodule and the largest semisimple quotient module of $X$, respectively.

A homomorphism $f\colon X \to Y$ of $A$-modules is called a *radical map* if, for any module $Z$ and homomorphisms $h : Z \to X$ and $g\colon Y \to Z$, the composite $hfg$ is not an isomorphism. A complex over $A$-mod is called a *radical* complex if all of its differential maps are radical. Every complex over $A$-mod is isomorphic in the homotopy category $\mathscr{K}(A)$ to a radical complex. Moreover, if two radical complex $X^\bullet$ and $Y^\bullet$ are isomorphic in $\mathscr{K}(A)$, then $X^\bullet$ and $Y^\bullet$ are isomorphic in $\mathscr{C}(A)$.

Two algebras $A$ and $B$ are said to be *derived equivalent* if their derived categories $\mathscr{D}^{\mathrm{b}}(A)$ and $\mathscr{D}^{\mathrm{b}}(B)$ are equivalent as triangulated categories. A triangle equivalence $F\colon \mathscr{D}^{\mathrm{b}}(A) \to \mathscr{D}^{\mathrm{b}}(B)$ is called a *derived equivalence* between $A$ and $B$.

In [39], Rickard showed that two algebras $A$ and $B$ are derived equivalent if and only if there is a complex $T^\bullet$ in $\mathscr{K}^{\mathrm{b}}(A\text{-proj})$ satisfying

(1) $\mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A)}(T^\bullet, T^\bullet[n]) = 0$ for all $n \neq 0$,

(2) $\mathrm{add}(T^\bullet)$ generates $\mathscr{K}^{\mathrm{b}}(A\text{-proj})$ as a triangulated category, and

(3) $B \simeq \mathrm{End}_{\mathscr{K}^{\mathrm{b}}(A)}(T^\bullet)$.

A complex in $\mathscr{K}^{\mathrm{b}}(A\text{-proj})$ satisfying the above two conditions (1) and (2) is called a *tilting complex* over $A$. It is known that, given a derived equivalence $F$ between $A$ and $B$, there is a unique (up to isomorphism) tilting complex $T^\bullet$ over $A$ such that $F(T^\bullet) \simeq B$. This complex $T^\bullet$ is called a tilting complex *associated* to $F$.

Recall that a complex $\Delta^\bullet$ in $\mathscr{D}^{\mathrm{b}}(B \otimes_k A^{\mathrm{op}})$ is called a *two-sided tilting complex* provided that there is another complex $\Theta^\bullet$ in $\mathscr{D}^{\mathrm{b}}(A \otimes_k B^{\mathrm{op}})$ such that $\Delta^\bullet \otimes^{\mathbf{L}}_A \Theta^\bullet \simeq B$ in $\mathscr{D}^{\mathrm{b}}(B \otimes_k B^{\mathrm{op}})$ and $\Theta^\bullet \otimes^{\mathbf{L}}_B \Delta^\bullet \simeq A$ in $\mathscr{D}^{\mathrm{b}}(A \otimes_k A^{\mathrm{op}})$. In this case, the functor $\Delta^\bullet \otimes^{\mathbf{L}}_A - : \mathscr{D}^{\mathrm{b}}(A) \to \mathscr{D}^{\mathrm{b}}(B)$ is a derived equivalence. A derived equivalence of this form is said to be *standard*. For basic facts on the derived functor $- \otimes^{\mathbf{L}} -$, we refer the reader to [49].

### 2.2. Almost $\nu$-stable derived equivalences

In [21], almost $\nu$-stable derived equivalences were introduced. Recall that a derived equivalence $F\colon \mathscr{D}^{\mathrm{b}}(A) \to \mathscr{D}^{\mathrm{b}}(B)$ is called an *almost $\nu$-stable derived equivalence* if the following two conditions are satisfied:

(1) The tilting complex $T^\bullet = (T^i, d^i)_{i\in\mathbb{Z}}$ associated to $F$ has zero terms in all positive degrees, that is, $T^i = 0$ for all $i > 0$. In this case, the tilting complex $\bar{T}^\bullet$ associated to the quasi-inverse $G$ of $F$ has zero terms in all negative degrees, that is, $\bar{T}^i = 0$ for all $i < 0$ (see Lemma 2.1 in [21]).

(2) $\mathrm{add}(\bigoplus_{i<0} T^i) = \mathrm{add}(\bigoplus_{i<0} \nu_A T^i)$ and $\mathrm{add}(\bigoplus_{i>0} \bar{T}^i) = \mathrm{add}(\bigoplus_{i>0} \nu_B \bar{T}^i)$.

As was shown in [21], each almost $\nu$-stable derived equivalence between $A$ and $B$ induces a stable equivalence of Morita type between $A$ and $B$. Thus $A$ and $B$ share many common invariants of both derived and stable equivalences.

For the convenience of the reader, we briefly recall the construction of the stable equivalence in [21].

Suppose that $A$ and $B$ are two algebras over a field and $F\colon \mathscr{D}^{\rm b}(A) \to \mathscr{D}^{\rm b}(B)$ is an almost $\nu$-stable derived equivalence. By Lemma 3.1 in [21], for each $X \in A$-mod, one can fix a radical complex $\bar{Q}_X^\bullet \simeq F(X)$ in $\mathscr{D}^{\rm b}(B)$:

$$0 \longrightarrow \bar{Q}_X^0 \longrightarrow \bar{Q}_X^1 \longrightarrow \cdots \longrightarrow \bar{Q}_X^n \longrightarrow 0$$

with $\bar{Q}_X^i$ projective for all $i > 0$. Moreover, the complex of this form is unique up to isomorphism in $\mathscr{C}^b(B)$. For $X, Y$ in $A$-mod, there is an isomorphism

$$\phi : \mathrm{Hom}_A(X,Y) \longrightarrow \mathrm{Hom}_{\mathscr{D}^{\rm b}(B)}(\bar{Q}_X^\bullet, \bar{Q}_Y^\bullet).$$

Then a functor $\bar{F}\colon A\text{-}\underline{\mathrm{mod}} \to B\text{-}\underline{\mathrm{mod}}$, called the *stable functor* of $F$, was defined in [21] as follows: for each $X$ in $A$-mod, we set

$$\bar{F}(X) := \bar{Q}_X^0.$$

For any morphism $f\colon X \to Y$ in $A$-mod, we denote by $\underline{f}$ its image in $\underline{\mathrm{Hom}}_A(X,Y)$. By Lemma 2.2 in [21], the map $(f)\phi$ in $\mathrm{Hom}_{\mathscr{D}^{\rm b}(B)}(\bar{Q}_X^\bullet, \bar{Q}_Y^\bullet)$ can be presented by a chain map $g^\bullet = (g^i)_{i\in\mathbb{Z}}$. Then we define

$$\bar{F}\colon \underline{\mathrm{Hom}}_A(X,Y) \longrightarrow \underline{\mathrm{Hom}}_B(\bar{F}(X), \bar{F}(Y)), \quad \underline{f} \mapsto \underline{g}^0.$$

It was shown in [21] that $\bar{F}\colon A\text{-}\underline{\mathrm{mod}} \to B\text{-}\underline{\mathrm{mod}}$ is indeed a well-defined functor fitting into the following commutative diagram (up to isomorphism):

$$(\star)\qquad \begin{array}{ccccc} A\text{-}\underline{\mathrm{mod}} & \xrightarrow{\Sigma_A} & \mathscr{D}^{\rm b}(A)/\mathscr{K}^{\rm b}(A\text{-proj}) & \xleftarrow{\text{can.}} & \mathscr{D}^{\rm b}(A) \\ \downarrow{\scriptstyle \bar{F}} & & \downarrow{\scriptstyle F'} & & \downarrow{\scriptstyle F} \\ B\text{-}\underline{\mathrm{mod}} & \xrightarrow{\Sigma_B} & \mathscr{D}^{\rm b}(B)/\mathscr{K}^{\rm b}(B\text{-proj}) & \xleftarrow{\text{can.}} & \mathscr{D}^{\rm b}(B) \end{array}$$

where $\mathscr{D}^{\rm b}(A)/\mathscr{K}^{\rm b}(A\text{-proj})$ is a Verdier quotient, the functor

$$\Sigma_A : \ A\text{-}\underline{\mathrm{mod}} \longrightarrow \mathscr{D}^{\rm b}(A)/\mathscr{K}^{\rm b}(A\text{-proj})$$

is induced by the canonical embedding $A\text{-mod} \to \mathscr{D}^{\rm b}(A)$, and $F'$ is the triangle equivalence which is uniquely determined (up to isomorphism) by the commutative square on the right-hand side of the above diagram $(\star)$.

Note that if two almost $\nu$-stable derived equivalences are naturally isomorphic, then so are their stable functors.

If $A$ is self-injective, then it was shown in [25], [40] that the functor $\Sigma_A$ is a triangle equivalence. Let $\eta_A$ be the composite

$$\mathscr{D}^{\rm b}(A) \xrightarrow{\text{can.}} \mathscr{D}^{\rm b}(A)/\mathscr{K}^{\rm b}(A\text{-proj}) \xrightarrow{\Sigma_A^{-1}} A\text{-}\underline{\mathrm{mod}}.$$

Then, for a derived equivalence $F$ between two self-injective algebras $A$ and $B$, there is a equivalence functor $\Phi_F\colon A\text{-}\underline{\mathrm{mod}} \to B\text{-}\underline{\mathrm{mod}}$, uniquely determined (up to isomorphism), such that the diagram

$$(\star\star)\qquad \begin{tikzcd} \mathscr{D}^{\mathrm{b}}(A) \arrow[r, "F"] \arrow[d, "\eta_A"'] & \mathscr{D}^{\mathrm{b}}(B) \arrow[d, "\eta_B"] \\ A\text{-}\underline{\mathrm{mod}} \arrow[r, "\Phi_F"] & B\text{-}\underline{\mathrm{mod}} \end{tikzcd}$$

is commutative up to isomorphism. In this case, we say that the stable equivalence $\Phi_F$ is *induced by the derived equivalence* $F$ or $\Phi_F$ *lifts to a derived equivalence.*

In general, a derived equivalence does not give rise to a stable equivalence, nor conversely. However, if a derived equivalence $F$ is almost $\nu$-stable, then its stable functor $\bar{F}$ is a stable equivalence (see Theorem 3.7 in [21]). So we introduce the following definition.

**Definition 2.1.** If a stable equivalence $\Phi$ between arbitrary algebras is isomorphic to the stable functor $\bar{F}$ of an almost $\nu$-stable derived equivalence $F$, then we say that the stable equivalence $\Phi$ is *induced by the almost $\nu$-stable derived equivalence* $F$, or $\Phi$ *lifts to the almost $\nu$-stable derived equivalence* $F$. If a stable equivalence $\Phi$ can be written as a composite $\Phi \simeq \Phi_1 \circ \Phi_2 \circ \cdots \circ \Phi_m$ of stable equivalences with $\Phi_i$, or $\Phi_i^{-1}$ induced by an almost $\nu$-stable derived equivalence for all $i$, then we say that $\Phi$ is induced by an iterated almost $\nu$-stable derived equivalence, or $\Phi$ lifts to an iterated almost $\nu$-stable derived equivalence (see [19]).

Actually, the stable functor $\bar{F}$ and the induced equivalence functor $\Phi_F$ are compatible with each other when our consideration is restricted to self-injective algebras. In fact, let $F\colon \mathscr{D}^{\mathrm{b}}(A) \to \mathscr{D}^{\mathrm{b}}(B)$ be a derived equivalence between two self-injective algebras $A$ and $B$. Then, by the above diagrams $(\star)$ and $(\star\star)$, if the tilting complex associated to $F$ has no nonzero terms in positive degrees, then $F$ is an almost $\nu$-stable derived equivalence and the stable functor $\bar{F}$ is isomorphic to the functor $\Phi_F$ defined above. If the tilting complex $T^\bullet$ associated to $F$ has nonzero terms in positive degrees, then $F$ can be written as a composite $F \simeq F_1 \circ F_2^{-1}$ such that both $F_1$ and $F_2$ are almost $\nu$-stable derived equivalences, and thus $\Phi_F \simeq \Phi_{F_1} \circ \Phi_{F_2}^{-1} \simeq \bar{F}_1 \circ \bar{F}_2^{-1}$. Here we can take $F_2$ to be $[m]$ for which $T^\bullet[-m]$ has no nonzero terms in positive degrees. This shows that $\Phi_F$ lifts to an iterated almost $\nu$-stable derived equivalence.

If a derived equivalence $F$ is standard and almost $\nu$-stable, then the stable equivalence $\bar{F}$ is of Morita type (see the proof of Theorem 5.3 in [21]). This is compatible with (and generalizes) Corollary 5.5 in [41] of Rickard: $\Phi_F$ is a stable equivalence of Morita type provided that $F$ is a standard derived equivalence between two self-injective algebras.

**Remark 2.2.** For algebras with separable semisimple quotients, if a stable equivalence of Morita type between them is induced by an almost $\nu$-stable derived equivalence, then it is also induced by an almost $\nu$-stable, standard derived equivalence.

In fact, suppose that $\Phi$ is such a stable equivalence of Morita type. Then, by the proof of Theorem 5.3 in [21], there is an almost $\nu$-stable, standard derived equivalence $F$ such that $\bar{F}(X) \simeq \Phi(X)$ for all modules $X$. Hence $\Phi \circ \bar{F}^{-1}$ lifts to a Morita equivalence by Proposition 3.5 below. Thus $\Phi \simeq (\Phi \circ \bar{F}^{-1}) \circ \bar{F}$ is induced by the composite of a Morita equivalence with an almost $\nu$-stable, standard derived equivalence, and therefore Remark 2.2 follows.

### 2.3. Frobenius parts and $\nu$-stable idempotent elements

In this subsection, we recall the definition of Frobenius parts of algebras from [31], which is related to the Nakayama functor, and collect some basic facts on idempotent elements.

Let $A$ be an algebra, and let $e$ be an idempotent element in $A$. It is well known that $Ae \otimes_{eAe} - : eAe\text{-mod} \to A\text{-mod}$ is a full embedding and induces another full embedding

$$\lambda : eAe\text{-}\underline{\mathrm{mod}} \longrightarrow A\text{-}\underline{\mathrm{mod}}$$

of stable module categories. Further, there is another functor $eA \otimes_A - : A\text{-mod} \to eAe\text{-mod}$ such that the functors $Ae \otimes_{eAe} -$ and $eA \otimes_A -$ induce mutually inverse equivalences between $\mathrm{add}(Ae)$ and $eAe$-proj. Moreover, the functor $eA \otimes_A -$ induces a triangle equivalence between the homotopy categories $\mathscr{K}^{\mathrm{b}}(\mathrm{add}(Ae))$ and $\mathscr{K}^{\mathrm{b}}(eAe\text{-proj})$. In particular, if $P \in \mathrm{add}(Ae)$, then $Ae \otimes_{eAe} eA \otimes_A P \simeq P$ as $A$-modules.

For an $A$-module $X$, we define $\Delta_e(X) := Ae \otimes_{eAe} eX$ and denote by $P(X)$ the projective cover of $X$.

**Lemma 2.3.** *Let $A$ be an algebra and $e$ an idempotent element in $A$. If $S$ is a simple $A$-module with $eS \neq 0$, then*

(1) *$\Delta_e(S)$ is isomorphic to a quotient module of $P(S)$ and $e \cdot \mathrm{rad}(\Delta_e(S)) = 0$,*

(2) *if $e \cdot \mathrm{rad}(P(S)) \neq 0$, then $\Delta_e(S)$ is not projective.*

*Proof.* (1) Applying $Ae \otimes_{eAe} eA \otimes_A -$ to the epimorphism $P(S) \to S$, we obtain an epimorphism $Ae \otimes_{eAe} eP(S) \to \Delta_e(S)$. Since $eS \neq 0$, the projective cover $P(S)$ of $S$ is in $\mathrm{add}(Ae)$, and therefore $Ae \otimes_{eAe} eP(S) \simeq P(S)$ by the equivalence between $\mathrm{add}(Ae)$ and $eAe$-proj. Hence $\Delta_e(S)$ is isomorphic to a quotient module of $P(S)$. Thus $\Delta_e(S)$ has $S$ as a single top. Applying $eA \otimes_A -$ to the short exact sequence $0 \to \mathrm{rad}(\Delta_e(S)) \to \Delta_e(S) \to S \to 0$, we have another short exact sequence

$$0 \longrightarrow e \cdot \mathrm{rad}(\Delta_e(S)) \longrightarrow e \cdot \Delta_e(S) \xrightarrow{h} eS \longrightarrow 0.$$

Since $e \cdot \Delta_e(S) \simeq eAe \otimes_{eAe} eS \simeq eS$, the homomorphism $h$ must be an isomorphism, and therefore $e \cdot \mathrm{rad}(\Delta_e(S)) = 0$.

(2) Suppose contrarily that $\Delta_e(S)$ is projective. Then the epimorphism $P(S) \to \Delta_e(S)$ splits. This forces $\Delta_e(S) \simeq P(S)$. By assumption, $e \cdot \mathrm{rad}(P(S)) \neq 0$, while $e \cdot \mathrm{rad}(\Delta_e(S)) = 0$. This is a contradiction. $\square$

We say that an idempotent element $e$ in $A$ is *$\nu$-stable* provided $\mathrm{add}(\nu_A Ae) = \mathrm{add}(Ae)$. That is, for each indecomposable direct summand $P$ of $Ae$, the corresponding injective module $\nu_A P$ is still a direct summand of $Ae$. Clearly, the module $Ae$ is projective-injective. Note that the notion of $\nu$-stable idempotent elements is left-right symmetric, although it is defined by using left modules. In fact, $\mathrm{add}\big(\nu_A(Ae)\big) = \mathrm{add}(Ae)$ if and only if $\mathrm{add}(eA) = \mathrm{add}\big(\nu_{A^{\mathrm{op}}}(eA)\big)$ because $D(\nu_A Ae) \simeq DD(eA) \simeq eA$ and $D(Ae) \simeq \nu_{A^{\mathrm{op}}}(eA)$. Moreover, we have the following.

**Lemma 2.4.** *Let $e$ be a $\nu$-stable idempotent element in $A$. Then*

(1) $\mathrm{add}(\mathrm{top}(Ae)) = \mathrm{add}(\mathrm{soc}(Ae))$.

(2) *If* $\mathrm{add}(Ae) \cap \mathrm{add}\big(A(1-e)\big) = \{0\}$*, then* $\mathrm{soc}(eA)$ *is an ideal of $A$. Moreover,* $\mathrm{soc}(Ae) = \mathrm{soc}(eA)$.

*Proof.* (1) Since $\mathrm{top}(Ae) = \mathrm{soc}(\nu_A(Ae))$, the statement (1) follows from the definition of $\nu$-stable idempotent elements.

(2) By our assumption, it follows from Section 9.2 in [13] that $\mathrm{soc}(Ae)$ is an ideal of $A$. It follows from (1) that $(1-e)\mathrm{soc}(Ae) = 0$. Thus $\mathrm{soc}(Ae) = \big((1-e)\cdot\mathrm{soc}(Ae)\big) \oplus \big(e\cdot\mathrm{soc}(Ae)\big) = e\cdot\mathrm{soc}(Ae) \subseteq eA$. Moreover, for each $r \in \mathrm{rad}(A)$, the left $A$-module homomorphism $\phi_r\colon A \to A, x \mapsto xr$ is a radical map. The restriction of $\phi_r$ to any indecomposable direct summand $X$ of $Ae$ cannot be injective. Otherwise, the restriction $\phi_r|_X$ splits since the module $X$ is injective, and $\phi_r$ is not a radical map. This is a contradiction. Hence $\mathrm{soc}(X) \subseteq \mathrm{Ker}(\phi_r)$ and $\mathrm{soc}(Ae) \subseteq \mathrm{Ker}(\phi_r)$. This means $\mathrm{soc}(Ae)\cdot r = 0$. Consequently $\mathrm{soc}(Ae) \subseteq \mathrm{soc}(eA)$. The duality $\mathrm{Hom}_A(-, A)$ takes $Ae$ to $eA$, and $A(1-e)$ to $(1-e)A$. This implies $\mathrm{add}(eA) \cap \mathrm{add}\big((1-e)A\big) = \{0\}$. Similarly, $\mathrm{soc}(eA) \subseteq \mathrm{soc}(Ae)$, and therefore $\mathrm{soc}(eA) = \mathrm{soc}(Ae)$. □

The following definition was essentially introduced in [31].

**Definition 2.5.** (1) An $A$-module $X$ is said to be *$\nu$-stably projective* if $\nu_A^i X$ is projective for all $i \geqslant 0$. The full subcategory of all $\nu$-stably projective $A$-modules is denoted by $A$-stp and called the *Nakayama-stable category* of $A$.

(2) If $e$ is an idempotent element in $A$ such that $\mathrm{add}(Ae) = A$-stp, then the algebra $eAe$ is called a *Frobenius part* of $A$, or an *associated self-injective algebra* of $A$.

By definition, Frobenius part of an algebra is unique up to Morita equivalence, so we may speak of the Frobenius part of an algebra. Since the trivial module $\{0\}$ is always $\nu$-stably projective and algebras are allowed to be $\{0\}$, the Frobenius part of an algebra always exists. Clearly, the Nakayama-stable category $A$-stp of $A$ is closed under taking direct summands and finite direct sums.

The two notions of $\nu$-stably projective modules and $\nu$-stable idempotent elements are closely related. Actually, we have the following lemma.

**Lemma 2.6.** *Let $A$ be an algebra. Then the following hold:*

(1) *If $e$ is a $\nu$-stable idempotent element in $A$, then* $\mathrm{add}(Ae) \subseteq A$-stp.

(2) *If $e$ is an idempotent element in $A$ such that* $\mathrm{add}(Ae) = A\text{-stp}$, *then $e$ is $\nu$-stable.*

(3) *There exists a $\nu$-stable idempotent element $e$ in $A$ such that* $\mathrm{add}(Ae) = A\text{-stp}$.

(4) *All modules in* $A$-stp *are projective-injective.*

*Proof.* (1) Let $P \in \mathrm{add}(Ae)$. Then, by definition, $\nu_A P \in \mathrm{add}(\nu_A Ae) = \mathrm{add}(Ae)$, and consequently $\nu_A^i P \in \mathrm{add}(Ae)$ for all $i > 0$. Hence $P$ is a $\nu$-stably projective $A$-module, that is, $P \in A$-stp.

(2) Since $Ae \in A$-stp, the $A$-module $\nu_A^i(Ae)$ is projective for all $i > 0$. This implies that $\nu_A Ae$ is projective and $\nu_A^i(\nu_A Ae)$ is projective for all $i > 0$. Hence $\nu_A Ae \in A\text{-stp} = \mathrm{add}(Ae)$ and $\mathrm{add}(\nu_A Ae) \subseteq \mathrm{add}(Ae)$. Since $\nu_A$ is an equivalence from $A$-proj to $A$-inj, the categories $\mathrm{add}(\nu_A Ae)$ and $\mathrm{add}(Ae)$ have the same number of isomorphism classes of indecomposable objects. Hence $\mathrm{add}(\nu_A Ae) = \mathrm{add}(Ae)$, that is, $e$ is $\nu$-stable.

(3) Since $A$-stp is a full subcategory of $A$-proj, there is an idempotent element $e$ in $A$ such that $\mathrm{add}(Ae) = A$-stp. The statement (3) then follows from (2).

(4) By definition, all modules in $A$-stp are projective. By (3), there is a $\nu$-stable idempotent element $e \in A$ such that $\mathrm{add}(Ae) = A$-stp. This implies that all modules in $A$-stp are in $\mathrm{add}(Ae) = \mathrm{add}(\nu_A Ae)$, and consequently they are injective. □

**Lemma 2.7.** *Let $A$ be an algebra and $e$ an idempotent element in $A$.*

(1) *For $Y \in \mathrm{add}(Ae)$ and $X \in A$-mod, there is an isomorphism induced by the functor*

$$eA \otimes_A - : \mathrm{Hom}_A(Y, X) \longrightarrow \mathrm{Hom}_{eAe}(eY, eX).$$

(2) *There is a natural isomorphism $e(\nu_A Y) \simeq \nu_{eAe}(eY)$ for all $Y \in \mathrm{add}(Ae)$.*

(3) *If $e$ is $\nu$-stable, then $eAe$ is a self-injective algebra.*

(4) *Suppose that $e$ is $\nu$-stable. If the algebra $A$ does not have nonzero semisimple direct summands, then neither does the algebra $eAe$.*

*Proof.* (1) It is enough to check on $Y = Ae$. But this case is clear (see, for example, Proposition 2.1, p. 33 in [4]).

(2) follows from (1) and the following isomorphisms:

$$\begin{aligned}\nu_{eAe}(eY) &= D\mathrm{Hom}_{eAe}(eY, eAe) \simeq D\mathrm{Hom}_A(Y, Ae)\\ &\simeq D(Y^* \otimes_A Ae) \simeq \mathrm{Hom}_A(Ae, D(Y^*)) \simeq e(\nu_A Y).\end{aligned}$$

(3) follows immediately from (2) (see also [32]).

(4) Since the functor $eA \otimes_A - : \mathrm{add}(Ae) \to eAe$-proj is an equivalence, each indecomposable projective $eAe$-module is isomorphic to $eY$ for some indecomposable $A$-module $Y$ in $\mathrm{add}(Ae)$. By definition, $\mathrm{add}(Ae) = \mathrm{add}(\nu_A Ae)$. This means that $Y$ is projective-injective and $\mathrm{soc}(Y) \in \mathrm{add}(\mathrm{top}(Ae))$. Since $A$ has no nonzero semisimple direct summands, the module $Y$ is not simple. Thus $Y$ has at least two composition factors in $\mathrm{add}(\mathrm{top}(Ae))$, and consequently $eY$ has at least two composition factors. Hence $eY$ is not simple. This implies that the algebra $eAe$ has no nonzero semisimple direct summands. □

The following lemma is easy. But for the convenience of the reader, we include here a proof.

**Lemma 2.8.** *Let $A$ be an algebra, and let $M$ be an $A$-module which is a generator for $A$-mod, that is, $\mathrm{add}({}_AA) \subseteq \mathrm{add}(M)$. Then, for an $A$-module $X$, $\mathrm{Hom}_A(M,X)$ is a projective $\mathrm{End}_A(M)$-module if and only if $X \in \mathrm{add}(M)$.*

*Proof.* Clearly, if $X \in \mathrm{add}(M)$, then $\mathrm{Hom}_A(M,X)$ is a projective $\mathrm{End}_A(M)$-module. Now, suppose that $\mathrm{Hom}_A(M,X)$ is projective for an $A$-module $X$. Without loss of generality, we may assume that $A$ is a basic algebra. Then ${}_AA$ is a direct summand of $M$, that is, $M \simeq A \oplus N$ for some $A$-module $N$. Since $\mathrm{Hom}_A(M,X)$ is a projective $\mathrm{End}_A(M)$-module, there is some $M_X \in \mathrm{add}(M)$ such that $\mathrm{Hom}_A(M,M_X) \simeq \mathrm{Hom}_A(M,X)$ as $\mathrm{End}_A(M)$-modules. By Yoneda isomorphism, there is an $A$-module homomorphism $f\colon M_X \to X$ such that $\mathrm{Hom}_A(M,f)$ is an isomorphism, that is, $\mathrm{Hom}_A(A,f) \oplus \mathrm{Hom}_A(N,f)$ is an isomorphism. This implies that $\mathrm{Hom}_A({}_AA,f)$ is an isomorphism, and therefore so is $f$. □

Finally, we point out the following elementary facts on Nakayama functors, which we employ in our proofs without references.

**Remark 2.9.** (1) For any $A$-module $M$ and projective $A$-module $P'$, there is a natural isomorphism: $D\mathrm{Hom}_A(P',M) \simeq \mathrm{Hom}_A(M,\nu_AP')$. More generally, for any $P^\bullet \in \mathscr{K}^{\mathrm{b}}(A\text{-proj})$ and $X^\bullet \in \mathscr{K}^{\mathrm{b}}(A)$, there is an isomorphism of $k$-spaces:

$$D\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A)}(P^\bullet,X^\bullet) \simeq \mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A)}(X^\bullet,\nu_AP^\bullet).$$

(2) Let $M$ be a fixed generator for $A$-mod, and let $\Lambda := \mathrm{End}_A(M)$. Then, for each projective $A$-module $P'$, there is a natural isomorphism $\nu_\Lambda\mathrm{Hom}_A(M,P') \simeq \mathrm{Hom}_A(M,\nu_AP')$ of $\Lambda$-modules.

## 3. Stable equivalences of Morita type

In this section, we shall first collect some basic properties of stable equivalences of Morita type which were first introduced by Broué (see [7], [8]) in modular representation theory of finite groups, and then give sufficient conditions for lifting stable to Morita equivalences, a very special class of derived equivalences. The key result in this section is Proposition 3.5 that will be applied in Section 4 to the proof of the main result, Theorem 1.2.

### 3.1. Basic facts on stable equivalences of Morita type

Let $A$ and $B$ be algebras over a field $k$. Following [8], we say that two bimodules ${}_AM_B$ and ${}_BN_A$ define a *stable equivalence of Morita type* between $A$ and $B$ if the following conditions hold:

(1) The one-sided modules ${}_AM, M_B, {}_BN$ and $N_A$ all are projective;

(2) $M \otimes_B N \simeq A \oplus P$ as $A$-$A$-bimodules for some projective $A$-$A$-bimodule $P$, and $N \otimes_A M \simeq B \oplus Q$ as $B$-$B$-bimodules for some projective $B$-$B$-bimodule $Q$.

In this case, we have two exact functors $T_M = M\otimes_B - : B\text{-mod} \to A\text{-mod}$ and $T_N = {}_BN\otimes_A - : A\text{-mod} \to B\text{-mod}$. Analogously, the bimodules $P$ and $Q$ define two exact functors $T_P$ and $T_Q$, respectively. Note that the images of $T_P$ and $T_Q$ consist of projective modules. Moreover, the functor $T_N$ induces an equivalence $\Phi_N\colon A\text{-}\underline{\text{mod}} \to B\text{-}\underline{\text{mod}}$ of stable module categories, and is called a *stable equivalence of Morita type*. Similarly, we have a stable equivalence $\Phi_M$ of stable module categories, which is a quasi-inverse of $\Phi_N$.

Note that $P = 0$ if and only if $Q = 0$. In this situation, we come back to the notion of Morita equivalences.

**Lemma 3.1.** *Let $A$ and $B$ be algebras without nonzero semisimple direct summands. Suppose that ${}_AM_B$ and ${}_BN_A$ are two bimodules without nonzero projective direct summands and defining a stable equivalence of Morita type between $A$ and $B$. Write ${}_AM\otimes_B N_A \simeq A\oplus P$ and ${}_BN\otimes_A M_B \simeq B\oplus Q$ as bimodules. Then the following hold:*

(1) *$(M\otimes_B -, N\otimes_A -)$ and $(N\otimes_A -, M\otimes_B -)$ are adjoint pairs of functors.*

(2) *$\mathrm{add}(\nu_A P) = \mathrm{add}({}_AP)$ and $\mathrm{add}(\nu_B Q) = \mathrm{add}({}_BQ)$. Thus ${}_AP \in A\text{-stp}$ and ${}_BQ \in B\text{-stp}$.*

(3) *$N\otimes_A P \in \mathrm{add}({}_BQ)$, and $M\otimes_B Q \in \mathrm{add}({}_AP)$.*

(4) *For each indecomposable $A$-module $X \notin \mathrm{add}({}_AP)$, the $B$-module $N\otimes_A X$ is the direct sum of an indecomposable module $\bar{X} \notin \mathrm{add}({}_BQ)$ and a module $X' \in \mathrm{add}({}_BQ)$.*

(5) *If $S$ is a simple $A$-module with $\mathrm{Hom}_A({}_AP, S) = 0$, then $N\otimes_A S$ is simple with $\mathrm{Hom}_B({}_BQ, N\otimes_A S) = 0$.*

(6) *Suppose that $A/\mathrm{rad}(A)$ and $B/\mathrm{rad}(B)$ are separable. If $S$ is a simple $A$-module with $\mathrm{Hom}_A({}_AP, S) \neq 0$, then $N\otimes_A S$ is not simple, but indecomposable with both $\mathrm{soc}(N\otimes_A S)$ and $\mathrm{top}(N\otimes_A S)$ in $\mathrm{add}(\mathrm{top}({}_BQ))$.*

*Proof.* Some of these statements are proved or implied in different papers (see, for example, Sections 5.3-5.6 in [53], and the references therein). For the convenience of the reader, we include here some details.

(1) This follows from Lemma 4.1 in [11] (see also [15] and [29] for algebras with the separability condition).

(2) We first show the following:

(a) For an $A$-module $X$, $P\otimes_A X \in \mathrm{add}({}_AP)$.

In fact, taking a surjective homomorphism $({}_AA)^n \to X$, we get a surjective map $P\otimes_A A^n \to P\otimes_A X$. Since ${}_AP\otimes_A X$ is projective for all $A$-modules $X$, $P\otimes_A X$ is a direct summand of ${}_AP^n$.

(b) For any $A$-module $X$, $\nu_B(N\otimes_A X) \simeq N\otimes_A(\nu_A X)$. Similarly, for any $B$-module $Y$, $\nu_A(M\otimes_B Y) \simeq M\otimes_B(\nu_B Y)$.

Indeed, there are the following isomorphisms of $B$-modules:

$$(\dagger)\qquad \begin{aligned} \nu_B(N\otimes_A X) &= D\mathrm{Hom}_B(N\otimes_A X, B)\simeq D\mathrm{Hom}_A(X, M\otimes_B B)\quad \text{(by (1))}\\ &\simeq D\mathrm{Hom}_A(X, A\otimes_A M)\\ &\simeq D(\mathrm{Hom}_A(X,A)\otimes_A M)\quad (\text{because } {}_AM \text{ is projective})\\ &\simeq \mathrm{Hom}_A(M,\nu_A X)\quad \text{(by adjointness)}\\ &\simeq \mathrm{Hom}_B(B, N\otimes_A \nu_A X)\quad \text{(by (1))}\\ &\simeq N\otimes_A (\nu_A X).\end{aligned}$$

Similarly, for a $B$-module $Y$, $\nu_A(M\otimes_B Y)\simeq M\otimes_B(\nu_B Y)$.

Thus $\nu_A(M\otimes_B N\otimes_A A)\simeq M\otimes_B N\otimes_A(\nu_A A)$, and consequently $\nu_A A\oplus \nu_A P\simeq (A\oplus P)\otimes_A(\nu_A A)$. Hence $\nu_A P\simeq P\otimes_A(\nu_A A)\in \mathrm{add}({}_AP)$, and therefore $\mathrm{add}(\nu_A P)\subseteq \mathrm{add}({}_AP)$. Since $\nu_A$ is an equivalence from $A$-proj to $A$-inj, we deduce $\mathrm{add}({}_AP)=\mathrm{add}(\nu_A P)$ just by counting the number of indecomposable direct summands of ${}_AP$ and $\nu_A P$. Similarly, $\mathrm{add}({}_BQ)=\mathrm{add}(\nu_B Q)$. This proves (2).

(3) It follows from $N\otimes_A(A\oplus P)\simeq N\otimes_A M\otimes_B N\simeq (B\oplus Q)\otimes_B N$ that $N\otimes_A P\simeq Q\otimes_B N$ as bimodules. In particular, ${}_BN\otimes_A P$ is isomorphic to ${}_BQ\otimes_B N\in\mathrm{add}({}_BQ)$. Hence $N\otimes_A P\in\mathrm{add}({}_BQ)$. Similarly, $M\otimes_B Q\in\mathrm{add}({}_AP)$.

(4) Suppose that $X$ is an indecomposable $A$-module and $X\notin\mathrm{add}({}_AP)$. Let $N\otimes_A X=\bar{X}\oplus X'$ be a decomposition of $N\otimes_A X$ such that $X'\in\mathrm{add}({}_BQ)$ and $\bar{X}$ has no nonzero direct summands in $\mathrm{add}({}_BQ)$. If $\bar{X}=0$, then $N\otimes_A X\in\mathrm{add}({}_BQ)$ and consequently $X\oplus P\otimes_A X\simeq M\otimes_B(N\otimes_A X)\in\mathrm{add}({}_AP)$ by (3). This contradicts to $X\notin\mathrm{add}({}_AP)$. Hence $\bar{X}\neq 0$. Suppose $\bar{X}=Y_1\oplus Y_2$ with $Y_i\neq 0$ for $i=1,2$. Then $M\otimes_B Y_i\notin\mathrm{add}({}_AP)$ for $i=1,2$. It follows that both $M\otimes_B Y_1$ and $M\otimes_B Y_2$ have indecomposable direct summands which are not in $\mathrm{add}({}_AP)$. However, due to $X\oplus P\otimes_A X\simeq M\otimes_B N\otimes_A X\simeq M\otimes_B Y_1\oplus M\otimes_B Y_2\oplus M\otimes_B X'$, we know that $X$ is the only indecomposable direct summand of $X\oplus P\otimes_A X$ with $X\notin\mathrm{add}({}_AP)$. This is a contradiction and shows that $\bar{X}$ must be indecomposable.

(5) By (1) and Lemma 3.2 in [51], together with the proof of Lemma 4.5 in [51], we have $P\simeq P^*$ as $A$-$A$-bimodules. Remark that this was proved in Proposition 3.4 of [15] with the condition that $A/\mathrm{rad}(A)$ and $B/\mathrm{rad}(B)$ are separable. If $\mathrm{Hom}_A(P,S)=0$, then $P\otimes_A S\simeq P^*\otimes_A S\simeq\mathrm{Hom}_A(P,S)=0$. Thus $M\otimes_B N\otimes_A S\simeq S\oplus P\otimes_A S=S$. Let $\ell(X)$ stand for the length of a composition series of $X$. Since the functor $N\otimes_A -$ is exact and faithful (due to $\mathrm{add}(A_A)=\mathrm{add}(N_A)$), we have $\ell(N\otimes_A X)\geqslant\ell(X)$ for all $A$-modules $X$. Similarly, $\ell(M\otimes_B Y)\geqslant\ell(Y)$ for all $B$-modules $Y$. Consequently, $1=\ell(S)=\ell(M\otimes_B N\otimes_A S)\geqslant\ell(N\otimes_A S)\geqslant\ell(S)=1$. This implies that $N\otimes_A S$ is a simple $B$-module. Finally, $\mathrm{Hom}_B({}_BQ, N\otimes_A S)\simeq\mathrm{Hom}_A(M\otimes_B Q,S)=0$ by (1) and (3).

(6) Let $e$ and $f$ be idempotent elements in $A$ and $B$, respectively, such that $\mathrm{add}({}_AAe)=\mathrm{add}({}_AP)$, $\mathrm{add}(Ae)\cap\mathrm{add}\big(A(1-e)\big)=\{0\}$, $\mathrm{add}({}_BBf)=\mathrm{add}({}_BQ)$ and $\mathrm{add}(Bf)\cap\mathrm{add}\big(B(1-f)\big)=\{0\}$. Then $e$ and $f$ are $\nu$-stable idempotents, and the modules $eA_A$ and ${}_BBf$ are projective-injective. Consequently, the $B$-$A$-bimodule $Bf\otimes_k eA$ is also projective-injective and

$$\mathrm{add}\big((B\otimes_k A)(f\otimes e)\big)\cap\mathrm{add}\big((B\otimes_k A)(1-f\otimes e)\big)=\{0\}.$$

By Lemma 2.4 (2), $\mathrm{soc}(Ae) = \mathrm{soc}(eA)$, and $\mathrm{soc}(eA_A)$, $\mathrm{soc}({}_BBf)$ and $\mathrm{soc}(Bf \otimes_k eA)$ are ideals of $A$, $B$ and $B \otimes_k A^{\mathrm{op}}$, respectively. Since $A/\mathrm{rad}(A)$ and $B/\mathrm{rad}(B)$ are separable, $\mathrm{soc}({}_BBf \otimes_k eA_A) = \mathrm{soc}(Bf) \otimes_k \mathrm{soc}(eA)$. By assumption, the bimodule $N$ has no nonzero projective direct summands. Particularly, $N$ has no nonzero direct summands in $\mathrm{add}(Bf \otimes_k eA)$. This is equivalent to $\mathrm{soc}(Bf \otimes_k eA)N = 0$ by [13], Section 9.2. That is,

$$\mathrm{soc}(Bf)N\mathrm{soc}(eA) = 0.$$

As $N_A$ is projective, we have $N \otimes_A \mathrm{soc}(eA) \simeq N\mathrm{soc}(eA)$. Thus

$$\mathrm{soc}(Bf)(N \otimes_A \mathrm{soc}(eA)) \simeq \mathrm{soc}(Bf)(N\mathrm{soc}(eA)) = \mathrm{soc}(Bf)N\mathrm{soc}(eA) = 0.$$

This means that the $B$-module $N \otimes_A \mathrm{soc}(eA)$ has no nonzero direct summands in $\mathrm{add}({}_BQ)$.

Now let $S$ be a simple $A$-module with $\mathrm{Hom}_A(P, S) \neq 0$. Then $S \in \mathrm{add}(\mathrm{top}(Ae)) = \mathrm{add}(\mathrm{soc}(Ae))$. Since $\mathrm{soc}(Ae) = \mathrm{soc}(eA)$, $S \in \mathrm{add}({}_A\mathrm{soc}(eA))$, and consequently the $B$-module $N \otimes_A S$ has no nonzero direct summands in $\mathrm{add}({}_BQ)$. Thus $S \notin \mathrm{add}({}_AP)$ and $N \otimes_A S$ is indecomposable by (4). Further, we show that $N \otimes_A S$ is not simple. Suppose contrarily that $N \otimes_A S$ is simple. Then $M \otimes_B (N \otimes_A S)$ must be indecomposable by the above discussion with replacing $N$ by $M$. However, the isomorphism $M \otimes_B (N \otimes_A S) \simeq S \oplus P \otimes_A S$ implies that $P \otimes_A S = 0$ and $\mathrm{Hom}_A(P, S) \simeq \mathrm{Hom}_A({}_AP, {}_AA) \otimes_A S \simeq P \otimes_A S = 0$, a contradiction. Hence the $B$-module $N \otimes_A S$ is indecomposable and not simple. Since $\mathrm{Hom}_A({}_AP, S) \neq 0$, there is a sequence $P \xrightarrow{f} S \xrightarrow{g} \nu_A P$ of homomorphisms with $f$ surjective and $g$ injective. Applying the exact functor $N \otimes_A -$ to this sequence, we get a new sequence

$$N \otimes_A P \xrightarrow{N \otimes_A f} N \otimes_A S \xrightarrow{N \otimes_A g} N \otimes_A \nu_A P$$

with $N \otimes_A f$ surjective and $N \otimes_A g$ injective. By (2) and (3), both $\mathrm{soc}(N \otimes_A S)$ and $\mathrm{top}(N \otimes_A S)$ lie in $\mathrm{add}(\mathrm{top}({}_BQ))$. $\square$

With all assumptions in Lemma 3.1(6), for a given simple $A$-module $S$, $N \otimes_A S$ is not simple if and only if $P(S)$ lies in $\mathrm{add}({}_AP)$. Thus, such simple $A$-modules are entirely determined by the top of $P$.

**Remark 3.2.** Suppose that $A$ is a finite-dimensional $k$-algebra over a field $k$.

(1) The following are equivalent:

(a) $A/\mathrm{rad}(A)$ is a separable algebra over $k$.

(b) The center of $\mathrm{End}_A(S)$ is a separable extension of $k$ for any simple $A$-module $S$.

(2) If $A$ satisfies the separability condition (that is, $A/\mathrm{rad}(A)$ is separable), then so do its quotient algebras and the algebras of the form $eAe$ with $e^2 = e \in A$.

(3) The separability condition on $A$ does not seem to be a strong restriction and can be satisfied actually by many interesting classes of algebras. For instance,

- $A$ is an algebra over a perfect field (for example, over a finite field, an algebraically closed field, or a field of characteristic zero).
- $A$ is given by quiver with relations.
- $A$ is the group algebra $kG$ of a finite group $G$ (see Lemma 1.28, p. 183, in [35]).

### 3.2. Stable equivalences of Morita type at different levels

We say that a stable equivalence $\Phi\colon A\text{-}\underline{\mathrm{mod}} \to B\text{-}\underline{\mathrm{mod}}$ of Morita type *lifts to a Morita equivalence* if there is a Morita equivalence $F\colon A\text{-mod} \to B\text{-mod}$ such that the diagram

$$\begin{tikzcd} A\text{-mod} \arrow[r, "F"] \arrow[d, "\text{can.}"] & B\text{-mod} \arrow[d, "\text{can.}"] \\ A\text{-}\underline{\mathrm{mod}} \arrow[r, "\Phi"] & B\text{-}\underline{\mathrm{mod}} \end{tikzcd}$$

of functors is commutative up to isomorphism, where the vertical functors are the canonical ones.

The following proposition collects conditions for stable equivalences of Morita type to be lifted to Morita equivalences.

**Proposition 3.3.** *Let $A$ and $B$ be algebras without nonzero semisimple direct summands. Suppose that ${}_AM_B$ and ${}_BN_A$ are two bimodules without projective direct summands and defining a stable equivalence of Morita type between $A$ and $B$. Write ${}_AM\otimes_B N_A \simeq A\oplus P$ and ${}_BN\otimes_A M_B \simeq B\oplus Q$ as bimodules. Then the following are equivalent:*

(1) $N\otimes_A -: A\text{-mod} \to B\text{-mod}$ *is an equivalence, that is,* $P = 0 = Q$.

(2) $N\otimes_A S$ *is a simple $B$-module for every simple $A$-module $S$.*

*If $A/\mathrm{rad}(A)$ and $B/\mathrm{rad}(B)$ are separable, then* (1) *and* (2) *are equivalent to each of the following:*

(3) *The stable equivalence $\Phi_N$ induced by $N\otimes_A -$ lifts to a Morita equivalence.*

(4) $N\otimes_A S$ *is isomorphic in $B$-$\underline{\mathrm{mod}}$ to a simple $B$-module for each simple $A$-module $S$.*

*Proof.* (1) $\Rightarrow$ (2) is trivial, since $N\otimes_A -$ is a Morita equivalence in case $P = 0 = Q$.

(2) $\Rightarrow$ (1) was first proved by Linckelmann in [27] for self-injective algebras, and then extended to arbitrary algebras by Liu in [28] under the condition that the ground field is splitting for both $A$ and $B$. Here, we give a proof that is independent of the ground field. Suppose contrarily $P \neq 0$. Let $\{S_1, \cdots, S_m\}$ be a complete set of non-isomorphic simple $A$-modules in $\mathrm{add}(\mathrm{top}({}_AP))$. Then, since ${}_AP$ is a projective-injective module and $A$ (as a bimodule) has no nonzero semisimple direct summands, the indecomposable direct summands of ${}_AP$ cannot be simple, and consequently all $S_i$ are not projective and $S_i \notin \mathrm{add}({}_AP)$. Thus, from $S_i\oplus P\otimes_A S_i \simeq M\otimes_B N\otimes_A S_i$ it follows that $N\otimes_A S_i \not\simeq N\otimes_A S_j$ as $B$-modules

whenever $i \neq j$. By Lemma 3.1 (1), we get the following isomorphisms:

$$\begin{aligned}\mathrm{End}_A(S_i) \oplus \mathrm{Hom}_A(P \otimes_A S_i, \textstyle\bigoplus_{j=1}^m S_j) &\simeq \mathrm{Hom}_A(S_i \oplus P \otimes_A S_i, \textstyle\bigoplus_{j=1}^m S_j)\\ &\simeq \mathrm{Hom}_A(M \otimes_B N \otimes_A S_i, \textstyle\bigoplus_{j=1}^m S_j)\\ &\simeq \mathrm{Hom}_B(N \otimes_A S_i, \textstyle\bigoplus_{j=1}^m N \otimes_A S_j)\\ &\simeq \mathrm{End}_B(N \otimes_A S_i) \simeq \underline{\mathrm{End}}_B(N \otimes_A S_i)\\ &\simeq \underline{\mathrm{End}}_A(S_i) \simeq \mathrm{End}_A(S_i).\end{aligned}$$

This implies $\mathrm{Hom}_A(P \otimes_A S_i, \bigoplus_{j=1}^m S_j) = 0$. However, the $A$-module $P \otimes_A S_i$ belongs to $\mathrm{add}({}_AP)$ and is not zero since

$$P \otimes_A S_i \simeq P^* \otimes_A S_i \simeq \mathrm{Hom}_A({}_AP, S_i) \neq 0.$$

This yields $\mathrm{Hom}_A(P \otimes_A S_i, \bigoplus_{j=1}^m S_j) \neq 0$, a contradiction. Thus $P = 0$, and therefore $Q = 0$.

Note that (1) $\Rightarrow$ (3) $\Rightarrow$ (4) is obvious.

(4) $\Rightarrow$ (2) Assume that $A/\mathrm{rad}(A)$ and $B/\mathrm{rad}(B)$ are separable algebras. According to Lemma 3.1 (5), it is enough to show $\mathrm{Hom}_A({}_AP, S) = 0$ for all simple $A$-modules $S$. In fact, let $S$ be an arbitrary simple $A$-module. Then, if $S$ is projective, then it cannot be in $\mathrm{add}({}_AP)$. Otherwise, $S$ would be projective-injective and $A$ would have a nonzero semisimple block, contradicting to our assumption. Hence $\mathrm{Hom}_A({}_AP, S) = 0$. Now suppose that $S$ is not projective. Then it follows from Lemma 3.1 (6) that $\mathrm{Hom}_A(P, S) = 0$ since $\Phi_N(S)$ is isomorphic to a simple $B$-module in $B$-$\underline{\mathrm{mod}}$ by (4). $\square$

Now we recall a result on stable equivalences of Morita type from Theorem 1.2 in [11]. Let $A$ and $B$ be two algebras without nonzero semisimple direct summands, and let ${}_AM_B$ and ${}_BN_A$ be two bimodules without projective direct summands and defining a stable equivalence of Morita type between $A$ and $B$. If $e$ and $f$ are idempotent elements in $A$ and $B$, respectively, such that $M \otimes_B Ne \in \mathrm{add}(Ae)$ and $\mathrm{add}(Bf) = \mathrm{add}(Ne)$, then the bimodules $eMf$ and $fNe$ define a stable equivalence of Morita type between $eAe$ and $fBf$, that is, the diagram

$$\begin{tikzcd} A\text{-}\underline{\mathrm{mod}} \arrow[r, "\Phi_N"] & B\text{-}\underline{\mathrm{mod}} \\ eAe\text{-}\underline{\mathrm{mod}} \arrow[u, "\lambda"] \arrow[r, "\Phi_{fNe}"] & fBf\text{-}\underline{\mathrm{mod}} \arrow[u, "\lambda"'] \end{tikzcd}$$

is commutative up to isomorphism, where $\lambda$ is defined in Section 2.3.

**Lemma 3.4.** *Let $A$ and $B$ be algebras without nonzero semisimple direct summands such that $A/\mathrm{rad}(A)$ and $B/\mathrm{rad}(B)$ are separable. Suppose that $e$ and $f$ are idempotent elements in $A$ and $B$, respectively. Let $\Phi\colon A$-$\underline{\mathrm{mod}} \to B$-$\underline{\mathrm{mod}}$ be a stable equivalence of Morita type such that the following conditions hold:*

(1) *For each simple $A$-module $S$ with $e\cdot S=0$, the $B$-module $\Phi(S)$ is isomorphic in $B$-$\underline{\mathrm{mod}}$ to a simple module $T'$ with $f\cdot T'=0$.*

(2) *For each simple $B$-module $T$ with $f\cdot T=0$, the $A$-module $\Phi^{-1}(T)$ is isomorphic in $A$-$\underline{\mathrm{mod}}$ to a simple module $S'$ with $e\cdot S'=0$.*

*Then there is, up to isomorphism, a unique stable equivalence $\Phi_1\colon eAe$-$\underline{\mathrm{mod}}\to fBf$-$\underline{\mathrm{mod}}$ of Morita type such that the following diagram of functors*

$$(\diamond)\qquad \begin{tikzcd} A\text{-}\underline{\mathrm{mod}} \arrow[r, "\Phi"] & B\text{-}\underline{\mathrm{mod}} \\ eAe\text{-}\underline{\mathrm{mod}} \arrow[u, "\lambda"] \arrow[r, "\Phi_1"] & fBf\text{-}\underline{\mathrm{mod}} \arrow[u, "\lambda"'] \end{tikzcd}$$

*is commutative up to isomorphism.*

*Proof.* We may assume that the stable equivalence $\Phi$ of Morita type between $A$ and $B$ is defined by bimodules ${}_AM_B$ and ${}_BN_A$ without nonzero projective direct summands, that is, $\Phi\simeq\Phi_N$, induced by the functor ${}_BN\otimes_A -$. By the assumption (1) and Lemma 3.1 (6), $\mathrm{Hom}_A({}_AP,S)=0$ for all simple $A$-modules $S$ with $e\cdot S=0$. This implies ${}_AP\in\mathrm{add}(Ae)$, and consequently $M\otimes_B Ne\simeq Ae\oplus Pe\in\mathrm{add}(Ae)$. Now, for each simple $B$-module $T$ with $f\cdot T=0$, it follows from the assumption (2) that $\mathrm{Hom}_A(Ae, M\otimes_B T)=0$. This is equivalent to $\mathrm{Hom}_B(N\otimes_A Ae,T)=0$ by Lemma 3.1 (1). Hence $Ne\simeq N\otimes_A Ae\in\mathrm{add}(Bf)$. Similarly, ${}_BQ\in\mathrm{add}(Bf)$ and $M\otimes_B Bf\in\mathrm{add}(Ae)$, and consequently $Bf\in\mathrm{add}(N\otimes_A M\otimes Bf)\subseteq\mathrm{add}(N\otimes_A Ae)=\mathrm{add}(Ne)$. Therefore $\mathrm{add}(Ne)=\mathrm{add}(Bf)$. Using Theorem 1.2 in [11], we get the desired commutative diagram $(\diamond)$. Note that the functor $\Phi_1$ is uniquely determined up to natural isomorphism because $\lambda$ is a full embedding. $\square$

The next proposition shows that a stable equivalence of Morita type lifts to a Morita equivalence if so does its restricted stable equivalence.

**Proposition 3.5.** *Let $A$ and $B$ be two algebras without nonzero semisimple direct summands such that $A/\mathrm{rad}(A)$ and $B/\mathrm{rad}(B)$ are separable, and let $e$ and $f$ be idempotent elements in $A$ and $B$, respectively. Suppose that there is a commutative (up to isomorphism) diagram*

$$(\diamond)\qquad \begin{tikzcd} A\text{-}\underline{\mathrm{mod}} \arrow[r, "\Phi"] & B\text{-}\underline{\mathrm{mod}} \\ eAe\text{-}\underline{\mathrm{mod}} \arrow[u, "\lambda"] \arrow[r, "\Phi_1"] & fBf\text{-}\underline{\mathrm{mod}} \arrow[u, "\lambda"'] \end{tikzcd}$$

*with $\Phi$ and $\Phi_1$ being stable equivalences of Morita type, and satisfying the following conditions:*

(1) *For each simple $A$-module $S$ with $e\cdot S=0$, the $B$-module $\Phi(S)$ is isomorphic in $B$-$\underline{\mathrm{mod}}$ to a simple $B$-module.*

(2) *For each simple $B$-module $T$ with $f \cdot T = 0$, the $A$-module $\Phi^{-1}(T)$ is isomorphic in $A$-$\underline{\mathrm{mod}}$ to a simple $A$-module.*

*Suppose that $\Phi_1$ lifts to a Morita equivalence. Then $\Phi$ lifts to a Morita equivalence.*

*Proof.* We can assume $e \neq 0$ and $f \neq 0$. Otherwise there is nothing to prove. Suppose that ${}_AM_B$ and ${}_BN_A$ are bimodules without nonzero projective direct summands and defining a stable equivalence of Morita type between $A$ and $B$ such that $\Phi$ is induced by $N \otimes_A -$. Assume that $M \otimes_B N \simeq A \oplus P$ and $N \otimes_A M \simeq B \oplus Q$ as bimodules. We shall prove $P = 0$.

Assume contrarily $P \neq 0$. Let $S$ be a simple $A$-module with $\mathrm{Hom}_A({}_AP, S) \neq 0$. Then $S$ cannot be projective. Otherwise, $S$ would be a direct summand of ${}_AP$ which is projective-injective, and $A$ would have a semisimple direct summand. We shall prove that $N \otimes_A S$ is isomorphic to a simple $B$-module $T$. This will lead to a contradiction by Lemma 3.1 (6).

First, we claim $eS \neq 0$. Otherwise, it would follow from the assumption (1) that $\Phi(S)$ is isomorphic to a simple $B$-module, leading to a contradiction by Lemma 3.1 (6). Hence $eS \neq 0$ and $P(S) \in \mathrm{add}(Ae)$. This implies that each indecomposable direct summand of $P$ is in $\mathrm{add}(Ae)$ since we can choose a simple module $S$ for each of such summands so that $\mathrm{Hom}_A(P, S) \neq 0$. Consequently, ${}_AP \in \mathrm{add}(Ae)$. Similarly, ${}_BQ \in \mathrm{add}(Bf)$. Since $\Phi_1$ lifts to a Morita equivalence, the module $\Phi_1(eS)$ is isomorphic in $fBf$-$\underline{\mathrm{mod}}$ to a simple $fBf$-module $fT$ with $T$ a simple $B$-module. Set $\Delta_e(S) := Ae \otimes_{eAe} eS$ and $\Delta_f(T) := Bf \otimes_{fBf} fT$. From the diagram $(\diamond)$, we get an isomorphism in $B$-$\underline{\mathrm{mod}}$

$$(*) \quad N \otimes_A \Delta_e(S) \simeq \Delta_f(T).$$

Now, we claim that $N \otimes_A \Delta_e(S)$ and $\Delta_f(T)$ are actually isomorphic in $B$-mod. To prove this, it suffices to show that $N \otimes_A \Delta_e(S)$ is indecomposable and non-projective.

In fact, it follows from $\mathrm{Hom}_A(P, S) \neq 0$ and Lemma 3.1 (2) that $P(S)$ is a direct summand of the projective-injective module $P$. Thus $\mathrm{soc}(P(S)) \subseteq \mathrm{soc}({}_AP)$. Since $\mathrm{add}(\nu_A P) = \mathrm{add}({}_AP)$ by Lemma 3.1 (2), $\mathrm{add}(\mathrm{soc}({}_AP)) = \mathrm{add}(\mathrm{top}({}_AP))$. Hence

$$\mathrm{soc}(P(S)) \in \mathrm{add}(\mathrm{top}({}_AP)) \subseteq \mathrm{add}(\mathrm{top}(Ae)).$$

Consequently $e \cdot \mathrm{soc}(P(S)) \neq 0$. On the other hand, as a direct summand of $P$, $P(S)$ is not simple because $A$ has no nonzero semisimple direct summands. Thus $\mathrm{soc}(P(S)) \subseteq \mathrm{rad}(P(S))$ and $e \cdot \mathrm{rad}(P(S)) \neq 0$. By Lemma 2.3, the $A$-module $\Delta_e(S)$, which is a quotient module of $P(S)$, is not projective. This implies that neither $N \otimes_A \Delta_e(S)$ is projective.

By Lemma 3.1 (4), to prove that $N \otimes_A \Delta_e(S)$ is indecomposable, we have to show that $N \otimes_A \Delta_e(S)$ has no nonzero direct summands in $\mathrm{add}({}_BQ)$. Suppose contrarily that $Q_1 \in \mathrm{add}({}_BQ)$ is an indecomposable direct summand of $N \otimes_A \Delta_e(S)$. We consider the exact sequence

$$(**) \quad 0 \longrightarrow N \otimes_A \mathrm{rad}(\Delta_e(S)) \longrightarrow N \otimes_A \Delta_e(S) \longrightarrow N \otimes_A S \longrightarrow 0$$

and show $\mathrm{Hom}_A(N \otimes_A \mathrm{rad}(\Delta_e(S)), Q_1) \neq 0$. Otherwise, it follows from the exact sequence $(**)$ that the direct summand $Q_1$ of $N \otimes_A \Delta_e(S)$ has to be a direct summand of $N \otimes_A S$ which is indecomposable by Lemma 3.1 (6). Thus $N \otimes_A S \simeq Q_1$ and it is projective. However, since $S$ is not projective, the module $N \otimes_A S$ cannot be projective. So, we have a contradiction which shows $\mathrm{Hom}_A(N \otimes_A \mathrm{rad}(\Delta_e(S)), Q_1) \neq 0$. Thanks to the formula $\mathrm{Hom}_A(\nu_A^{-1} Y, X) \simeq D\mathrm{Hom}_A(X, Y)$ for any $A$-module $X$ and any injective $A$-module $Y$ (see Remark 2.9 (1)), we have

$$\begin{aligned}\mathrm{Hom}_A\big(\nu_A^{-1}(M \otimes_B Q_1), \mathrm{rad}(\Delta_e(S))\big) &\simeq D\mathrm{Hom}_A(\mathrm{rad}(\Delta_e(S)), M \otimes_B Q_1) \\ &\simeq D\mathrm{Hom}_B(N \otimes_A \mathrm{rad}(\Delta_e(S)), Q_1) \neq 0.\end{aligned}$$

By Lemma 3.1 (2)-(3), $\nu_A^{-1}(M \otimes_B Q_1) \in \mathrm{add}(P)$, $\mathrm{Hom}_A({}_AP, \mathrm{rad}(\Delta_e(S))) \neq 0$ and

$$e \cdot \mathrm{rad}(\Delta_e(S)) \simeq \mathrm{Hom}_A(Ae, \mathrm{rad}(\Delta_e(S))) \neq 0.$$

This contradicts to Lemma 2.3 (1) and shows that $N \otimes_A \Delta_e(S)$ has no nonzero direct summands in $\mathrm{add}({}_BQ)$, and therefore it is indecomposable.

Thus $N \otimes_A \Delta_e(S) \simeq \Delta_f(T)$ in $B$-mod. From the exact sequence $(**)$, we deduce that $N \otimes_A S$ is isomorphic to a quotient module of $\Delta_f(T)$. By Lemma 3.1 (6), $\mathrm{soc}({}_BN \otimes_A S \in \mathrm{add}(\mathrm{top}({}_BQ))$. Since ${}_BQ \in \mathrm{add}(Bf)$, we have $\mathrm{soc}(N \otimes_A S) \in \mathrm{add}(\mathrm{top}(Bf))$. However, it follows from Lemma 2.3 (1) that $f \cdot \mathrm{rad}(\Delta_f(T)) = 0$ and $\mathrm{top}(\Delta_f(T))$ is isomorphic to $T$. This means that $\mathrm{rad}(\Delta_f(T))$ does not have composition factors in $\mathrm{add}(\mathrm{top}(Bf))$ and that $T$ is the only quotient module of $\Delta_f(T)$ with $\mathrm{soc}(T) \in \mathrm{add}(\mathrm{top}(Bf))$. Thus $N \otimes_A S \simeq T$. But this contradicts to Lemma 3.1 (6) and shows $P = 0$, and therefore $N \otimes_A -$ is a Morita equivalence between $A$-mod and $B$-mod. □

## 4. From stable equivalences of Morita type to derived equivalences

In this section, we shall prove the main result, Theorem 1.2. A key idea of the proof is to extend a tilting complex over $eAe$ with $e$ an idempotent element in $A$ to a tilting complex over $A$, see Proposition 4.1. This generalizes a result in [33]. With the help of Proposition 4.1, we get another crucial ingredient, Proposition 4.5, of the proof of Theorem 1.2. A special, but useful consequence of Theorem 1.2 is Corollary 4.7, which reduces the lifting problem for algebras to that for their Frobenius parts and will be used in the proof of Theorem 1.1 in Section 5.

### 4.1. Extending derived equivalences

Let $A$ be an algebra over a field $k$, and let $e$ be a $\nu$-stable idempotent element in $A$. In this subsection, we shall show that a tilting complex over $eAe$ can be extended to a tilting complex over $A$ which defines an almost $\nu$-stable derived equivalence.

First, we fix some terminology on approximations.

Let $\mathcal{C}$ be a category, $\mathcal{D}$ be a full subcategory of $\mathcal{C}$, and $X$ be an object in $\mathcal{C}$. A morphism $f\colon D \to X$ in $\mathcal{C}$ is called a *right $\mathcal{D}$-approximation* of $X$ if $D \in \mathcal{D}$

and the induced map $\mathrm{Hom}_{\mathcal{C}}(-,f)\colon \mathrm{Hom}_{\mathcal{C}}(D',D) \to \mathrm{Hom}_{\mathcal{C}}(D',X)$ is surjective for every object $D' \in \mathcal{D}$. A morphism $f\colon X \to Y$ in $\mathcal{C}$ is said to be *right minimal* if any morphism $g\colon X \to X$ with $gf = f$ is an automorphism. A minimal right $\mathcal{D}$-approximation of $X$ is a right $\mathcal{D}$-approximation of $X$, which is right minimal. Dually, there is the notion of a *left $\mathcal{D}$-approximation* and a *minimal left $\mathcal{D}$-approximation*. The subcategory $\mathcal{D}$ is said to be *functorially finite* in $\mathcal{C}$ if every object in $\mathcal{C}$ has a right and left $\mathcal{D}$-approximation.

The following proposition extends Theorem 4.11 in [33], where algebras are assumed to be symmetric, that is, ${}_AA_A \simeq {}_AD(A)_A$ as bimodules. If $A$ is symmetric, then so is $eAe$ for $e^2 = e \in A$.

**Proposition 4.1.** *Let $A$ be an arbitrary algebra, and let $e$ be a $\nu$-stable idempotent element in $A$. Suppose that $Q^\bullet$ is a complex in $\mathscr{K}^{\mathrm{b}}(\mathrm{add}(Ae))$ with $Q^i = 0$ for all $i > 0$ such that*

(1) *$eQ^\bullet$ is a tilting complex over $eAe$, and*

(2) *$\mathrm{End}_{\mathscr{K}^{\mathrm{b}}(eAe)}(eQ^\bullet)$ is self-injective.*

*Then there exists a bounded complex $P^\bullet$ of projective $A$-modules such that $Q^\bullet \oplus P^\bullet$ is a tilting complex over $A$ and induces an almost $\nu$-stable derived equivalence between $A$ and $\mathrm{End}_{\mathscr{K}^{\mathrm{b}}(A)}(Q^\bullet \oplus P^\bullet)$.*

Remark that if the ground field $k$ is algebraically closed, or the algebra $eAe$ is symmetric, then the condition (2) in Proposition 4.1 can be dropped because derived equivalences preserve both symmetric algebras over any field (see Corollary 5.3 in [41]) and self-injective algebras over an algebraically closed field (see [1]). But it is unknown whether derived equivalences preserve self-injective algebras over an arbitrary field.

*Proof.* For convenience, we shall abbreviate $\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A)}(-,-)$ to $\mathrm{Hom}(-,-)$ in the proof. Assume that $Q^\bullet$ is of the following form:

$$0 \longrightarrow Q^{-n} \longrightarrow \cdots \longrightarrow Q^{-1} \longrightarrow Q^0 \longrightarrow 0$$

for some fixed natural number $n$.

Since both $\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A)}(Q^\bullet, X^\bullet)$ and $\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A)}(X^\bullet, Q^\bullet)$ are finite-dimensional for each $X^\bullet \in \mathscr{K}^{\mathrm{b}}(A)$, we take a basis for each space, form their direct sums and get right and left $\mathrm{add}(Q^\bullet)$-approximations of $X^\bullet$ by diagonal projection and injection, respectively. This means that $\mathrm{add}(Q^\bullet)$ is a functorially finite subcategory in $\mathscr{K}^{\mathrm{b}}(A)$. Thus, there is a minimal right $\mathrm{add}(Q^\bullet)$-approximation $f_n : Q_n^\bullet \to A[n]$. The following construction is standard. Let $P_n^\bullet := A[n]$. We define inductively a complex $P_i^\bullet$ for each $i \leqslant n$ by taking the following distinguished triangle in $\mathscr{K}^{\mathrm{b}}(A\text{-proj})$

$$(\star) \qquad P_{i-1}^\bullet \longrightarrow Q_i^\bullet \xrightarrow{f_i} P_i^\bullet \longrightarrow P_{i-1}^\bullet[1],$$

where $f_i$ is a minimal right $\mathrm{add}(Q^\bullet)$-approximation of $P_i^\bullet$ and where $P_{i-1}^\bullet[1]$ is a radical complex isomorphic in $\mathscr{K}^{\mathrm{b}}(A\text{-proj})$ to the mapping cone of $f_i$. In the following, we shall prove that $Q^\bullet \oplus P_0^\bullet$ is a tilting complex over $A$ and induces an almost $\nu$-stable derived equivalence.

By definition, $\operatorname{add}(Q^\bullet \oplus P_0^\bullet)$ generates $\mathscr{K}^{\mathrm{b}}(A\text{-proj})$. It remains to show

$$\operatorname{Hom}(Q^\bullet \oplus P_0^\bullet, Q^\bullet[m] \oplus P_0^\bullet[m]) = 0$$

for all $m \neq 0$. We shall prove this by four claims.

(a) $\operatorname{Hom}(Q^\bullet, Q^\bullet[m]) = 0$ for all $m \neq 0$.

In fact, it follows from the equivalence $eA \otimes_A - : \operatorname{add}(Ae) \to \operatorname{add}(eAe\text{-proj})$ that the functor $eA \otimes_A -$ induces a triangle equivalence $\mathscr{K}^{\mathrm{b}}(\operatorname{add}(Ae)) \to \mathscr{K}^{\mathrm{b}}(eAe\text{-proj})$. Since $eQ^\bullet$ is a tilting complex over $eAe$, $\operatorname{Hom}(eQ^\bullet, eQ^\bullet[m]) = 0$ for all $m \neq 0$. Therefore, for the complex $Q^\bullet \in \mathscr{K}^{\mathrm{b}}(\operatorname{add}(Ae))$, $\operatorname{Hom}(Q^\bullet, Q^\bullet[m]) = 0$ for all $m \neq 0$.

(b) $\operatorname{Hom}(Q^\bullet, P_0^\bullet[m]) = 0$ for all $m \neq 0$.

Indeed, applying $\operatorname{Hom}(Q^\bullet, -)$ to the triangle $(\star)$, we obtain a long exact sequence

$$(\star\star) \qquad \cdots \longrightarrow \operatorname{Hom}(Q^\bullet, P_{i-1}^\bullet[m]) \longrightarrow \operatorname{Hom}(Q^\bullet, Q_i^\bullet[m])$$
$$\longrightarrow \operatorname{Hom}(Q^\bullet, P_i^\bullet[m]) \longrightarrow \operatorname{Hom}(Q^\bullet, P_{i-1}^\bullet[m+1]) \longrightarrow \cdots$$

for each integer $i \leqslant n$. Since $\operatorname{Hom}(Q^\bullet, Q^\bullet[m]) = 0$ for all $m \neq 0$, one gets

$$\operatorname{Hom}(Q^\bullet, P_{i-1}^\bullet[m]) \simeq \operatorname{Hom}(Q^\bullet, P_i^\bullet[m-1])$$

for all $m < 0$. Thus, for all $m < 0$,

$$\begin{aligned}\operatorname{Hom}(Q^\bullet, P_0^\bullet[m]) &\simeq \operatorname{Hom}(Q^\bullet, P_1^\bullet[m-1]) \simeq \cdots \simeq \operatorname{Hom}(Q^\bullet, P_n^\bullet[m-n]) \\ &\simeq \operatorname{Hom}(Q^\bullet, A[m]) = 0.\end{aligned}$$

To prove $\operatorname{Hom}(Q^\bullet, P_0^\bullet[m]) = 0$ for $m > 0$, we shall show by induction on $i$ that

$$\operatorname{Hom}(Q^\bullet, P_i^\bullet[m]) = 0$$

for all $m > 0$ and all $i \leqslant n$.

If $i = n$, then $\operatorname{Hom}(Q^\bullet, P_n^\bullet[m]) = 0$ for all $m > 0$. Now, we assume inductively that $\operatorname{Hom}(Q^\bullet, P_j^\bullet[m]) = 0$ for all $m > 0$ and all $i \leqslant j \leqslant n$, and want to show $\operatorname{Hom}(Q^\bullet, P_{i-1}^\bullet[m]) = 0$ for all $m > 0$. Since $f_i$ is a right $\operatorname{add}(Q^\bullet)$-approximation of $P_i^\bullet$, the induced map $\operatorname{Hom}(Q^\bullet, f_i)$ is surjective. Thus $\operatorname{Hom}(Q^\bullet, P_{i-1}^\bullet[1]) = 0$ by (a). The long exact sequence $(\star\star)$, together with (a) and the induction hypothesis, yields $\operatorname{Hom}(Q^\bullet, P_{i-1}^\bullet[m]) = 0$ for all $m > 1$. Thus $\operatorname{Hom}(Q^\bullet, P_i^\bullet[m]) = 0$ for all $m > 0$ and all $i \leqslant n$. Particularly, for all $m > 0$, $\operatorname{Hom}(Q^\bullet, P_0^\bullet[m]) = 0$. This completes the proof of (b).

(c) $\operatorname{Hom}(P_0^\bullet, Q^\bullet[m]) = 0$ for all $m \neq 0$.

To prove (c), let $\Delta := \operatorname{End}_{\mathscr{K}^{\mathrm{b}}(eAe)}(eQ^\bullet)$, and let $G : \mathscr{D}^{\mathrm{b}}(eAe) \to \mathscr{D}^{\mathrm{b}}(\Delta)$ be the derived equivalence induced by the tilting complex $eQ^\bullet$. Then $G(eQ^\bullet)$ is isomorphic to $\Delta$. Since $\Delta$ is self-injective by assumption, $\operatorname{add}(\nu_\Delta \Delta) = \operatorname{add}({}_\Delta\Delta)$, and consequently $\mathrm{add}(eQ^\bullet) = \mathrm{add}(\nu_{eAe}eQ^\bullet)$, or equivalently $\mathrm{add}(Q^\bullet) = \mathrm{add}(\nu_A Q^\bullet)$. Therefore, by (b), for all $m \neq 0$,

$$\mathrm{Hom}(P_0^\bullet, Q^\bullet[m]) \simeq D\mathrm{Hom}(\nu_A^{-1}Q^\bullet, P_0^\bullet[-m]) = 0.$$

(d) $\mathrm{Hom}(P_0^\bullet, P_0^\bullet[m]) = 0$ for all $m \neq 0$.

Indeed, $G(eAe)$ is isomorphic to a complex $V^\bullet$ in $\mathscr{K}^{\mathrm{b}}(\Delta\text{-proj})$ with $V^i = 0$ for all $i < 0$ (see, for instance, Lemma 2.1 in [21]) and $\mathrm{Hom}(Q^\bullet, P_0^\bullet[m]) = 0$ for all $m \neq 0$ by (b). Then $\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(eAe)}\big(eQ^\bullet,\ eP_0^\bullet[m]\big) = 0$ for all $m \neq 0$, and consequently $G(e(P_0^\bullet))$ is isomorphic in $\mathscr{D}^{\mathrm{b}}(\Delta)$ to a $\Delta$-module. Thus, for all $m > 0$,

$$\begin{aligned}\mathrm{Hom}(Ae, P_0^\bullet[m]) &\simeq \mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(eAe)}(eAe, e(P_0^\bullet)[m]) \simeq \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(eAe)}(eAe, e(P_0^\bullet)[m])\\ &\simeq \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(\Delta)}(V^\bullet, G(e(P_0^\bullet))[m]) = 0.\end{aligned}$$

By the construction of $P_0^\bullet$, all terms of $P_0^\bullet$ in nonzero degrees lie in $\mathrm{add}(Ae)$. Since $P_0^\bullet$ is a radical complex, $P_0^m = 0$ for all $m > 0$. Otherwise we would have $\mathrm{Hom}(Ae, P_0^\bullet[t]) \neq 0$ for the maximal positive integer $t$ with $P_0^t \neq 0$.

Applying $\mathrm{Hom}(P_0^\bullet, -)$ to the triangle $(\star)$, we have an exact sequence (for all $m$ and $i \leqslant n$)

$$\begin{aligned}\mathrm{Hom}(P_0^\bullet, Q_i^\bullet[m-1]) \longrightarrow \mathrm{Hom}(P_0^\bullet, P_i^\bullet[m-1]) &\longrightarrow \mathrm{Hom}(P_0^\bullet, P_{i-1}^\bullet[m])\\ &\longrightarrow \mathrm{Hom}(P_0^\bullet, Q_i^\bullet[m]).\end{aligned}$$

If $m < 0$, then $\mathrm{Hom}(P_0^\bullet, Q_i^\bullet[m-1]) = 0 = \mathrm{Hom}(P_0^\bullet, Q_i^\bullet[m]) = 0$. Thus $\mathrm{Hom}(P_0^\bullet, P_i^\bullet[m-1]) \simeq \mathrm{Hom}(P_0^\bullet, P_{i-1}^\bullet[m])$. Therefore, for $m < 0$,

$$\begin{aligned}\mathrm{Hom}(P_0^\bullet, P_0^\bullet[m]) &\simeq \mathrm{Hom}(P_0^\bullet, P_1^\bullet[m-1]) \simeq \cdots \simeq \mathrm{Hom}(P_0^\bullet, P_n^\bullet[m-n])\\ &= \mathrm{Hom}(P_0^\bullet, A[m]) = 0.\end{aligned}$$

Now, applying $\mathrm{Hom}(-, P_0^\bullet)$ to the triangle $(\star)$, we obtain an exact sequence (for all $m$ and $i \leqslant n$)

$$\begin{aligned}\mathrm{Hom}(Q_i^\bullet, P_0^\bullet[m]) \longrightarrow \mathrm{Hom}(P_{i-1}^\bullet, P_0^\bullet[m]) &\longrightarrow \mathrm{Hom}(P_i^\bullet, P_0^\bullet[m+1])\\ &\longrightarrow \mathrm{Hom}(Q_i^\bullet, P_0^\bullet[m+1]).\end{aligned}$$

If $m > 0$, then $\mathrm{Hom}(Q_i^\bullet, P_0^\bullet[m]) = 0 = \mathrm{Hom}(Q_i^\bullet, P_0^\bullet[m+1])$, and consequently

$$\mathrm{Hom}(P_{i-1}^\bullet, P_0^\bullet[m]) \simeq \mathrm{Hom}(P_i^\bullet, P_0^\bullet[m+1]).$$

Thus, for $m > 0$,

$$\begin{aligned}\mathrm{Hom}(P_0^\bullet, P_0^\bullet[m]) &\simeq \mathrm{Hom}(P_1^\bullet, P_0^\bullet[m+1]) \simeq \cdots \simeq \mathrm{Hom}(P_n^\bullet, P_0^\bullet[m+n])\\ &= \mathrm{Hom}(A, P_0^\bullet[m]) = 0.\end{aligned}$$

Hence $T^\bullet := Q^\bullet \oplus P_0^\bullet$ is a tilting complex over $A$ such that all of its terms in negative degrees are $\nu$-stably projective. Let $B := \mathrm{End}_{\mathscr{D}^{\mathrm{b}}(A)}(T^\bullet)$ and $F\colon \mathscr{D}^{\mathrm{b}}(A) \to \mathscr{D}^{\mathrm{b}}(B)$

be the derived equivalence induced by $T^\bullet$. Then $F(Q^\bullet)$ is isomorphic in $\mathscr{D}^{\mathrm{b}}(B)$ to the $B$-module $\mathrm{Hom}(T^\bullet, Q^\bullet)$ with $\mathrm{add}(\nu_B \mathrm{Hom}(T^\bullet, Q^\bullet)) = \mathrm{add}(\mathrm{Hom}(T^\bullet, Q^\bullet))$, since $\mathrm{add}(Q^\bullet) = \mathrm{add}(\nu_A Q^\bullet)$ and $F$ commutes with the Nakayama functor (see Lemma 2.3 in [21]). By the definition of $P_0^\bullet$, $F(A)$ is isomorphic to a complex $\bar{T}^\bullet$ with terms in $\mathrm{add}(\mathrm{Hom}(T^\bullet, Q^\bullet))$ for all positive degrees, and zero for all negative degrees. This implies that all terms of $\bar{T}^\bullet$ in positive degrees are $\nu$-stably projective. Thus, by Proposition 3.8 (2) in [21], the derived equivalence $F$ is almost $\nu$-stable. If we define $P^\bullet := P_0^\bullet$, then Proposition 4.1 follows. □

**Lemma 4.2.** *Keep the assumptions and notation as in Proposition 4.1. Let $B$ be the endomorphism algebra $\mathrm{End}_{\mathscr{K}^{\mathrm{b}}(A)}(Q^\bullet \oplus P^\bullet)$ of $Q^\bullet \oplus P^\bullet$, and let $f$ be the idempotent element in $B$ corresponding to the summand $Q^\bullet$. Then there exists a stable equivalence $\Phi\colon A\text{-}\underline{\mathrm{mod}} \to B\text{-}\underline{\mathrm{mod}}$ of Morita type, an idempotent element $e \in A$ and a stable equivalence $\Phi_1 : eAe\text{-}\underline{\mathrm{mod}} \to fBf\text{-}\underline{\mathrm{mod}}$ of Morita type, such that the following diagram of functors commutates*

$$(\blacklozenge)\qquad \begin{array}{ccc} A\text{-}\underline{\mathrm{mod}} & \xrightarrow{\Phi} & B\text{-}\underline{\mathrm{mod}} \\ \uparrow{\scriptstyle\lambda} & & \uparrow{\scriptstyle\lambda} \\ eAe\text{-}\underline{\mathrm{mod}} & \xrightarrow{\Phi_1} & fBf\text{-}\underline{\mathrm{mod}} \end{array}$$

*up to isomorphism, and that*

(1) *$\Phi$ is induced by an almost $\nu$-stable derived equivalence.*

(2) *$\Phi_1$ is induced by a derived equivalence $G$ with $G(eQ^\bullet) \simeq fBf$.*

(3) *For all simple $A$-modules $S$ with $e \cdot S = 0$, $\Phi(S)$ is isomorphic in $B$-$\underline{\mathrm{mod}}$ to a simple $B$-module $S'$ with $f \cdot S' = 0$.*

(4) *For all simple $B$-modules $T$ with $f \cdot T = 0$, $\Phi^{-1}(T)$ is isomorphic in $A$-$\underline{\mathrm{mod}}$ to a simple $A$-module $T'$ with $e \cdot T' = 0$.*

*Proof.* We first show the existence of the commutative diagram of functors and the statements (1) and (2).

By Proposition 4.1, there is an almost $\nu$-stable derived equivalence $F\colon \mathscr{D}^{\mathrm{b}}(A) \to \mathscr{D}^{\mathrm{b}}(B)$ such that $F(Q^\bullet \oplus P^\bullet) \simeq B$ and $F(Q^\bullet) \simeq Bf$. Since $eQ^\bullet$ is a tilting complex over $eAe$, $\mathrm{add}(eQ^\bullet)$ generates $\mathscr{K}^{\mathrm{b}}(eAe\text{-proj})$ as a triangulated category. Equivalently, $\mathrm{add}(Q^\bullet)$ generates $\mathscr{K}^{\mathrm{b}}(\mathrm{add}(Ae))$ as a triangulated category. Thus, the functor $F$ induces a triangle equivalence between $\mathscr{K}^{\mathrm{b}}(\mathrm{add}(Ae))$ and $\mathscr{K}^{\mathrm{b}}(\mathrm{add}(Bf))$.

By Corollary 3.5 in [41], there is a standard derived equivalence which agrees with $F$ on $\mathscr{K}^{\mathrm{b}}(A\text{-proj})$. So, we can assume that $F$ itself is a standard derived equivalence, that is, there are complexes $\Delta^\bullet \in \mathscr{D}^{\mathrm{b}}(B \otimes_k A^{\mathrm{op}})$ and $\Theta^\bullet \in \mathscr{D}^{\mathrm{b}}(A \otimes_k B^{\mathrm{op}})$ such that

$$\Delta^\bullet \otimes_A^{\mathbf{L}} \Theta^\bullet \simeq {}_BB_B,\ \Theta^\bullet \otimes_B^{\mathbf{L}} \Delta^\bullet \simeq {}_AA_A \ \text{ and } \ F = \Delta^\bullet \otimes_A^{\mathbf{L}} -.$$

By Lemma 5.2 in [21], we can further assume that the complex $\Delta^\bullet$ is of the following form:

$$(\ddagger)\qquad 0 \longrightarrow \Delta^0 \longrightarrow \Delta^1 \longrightarrow \cdots \longrightarrow \Delta^n \longrightarrow 0$$

such that $\Delta^i \in \operatorname{add}(Bf \otimes_k eA)$ for all $i > 0$ and $\Delta^0$ is projective as left and right modules, and that $\Theta^\bullet$ can be chosen to equal $\operatorname{Hom}^\bullet_B(\Delta^\bullet, {}_BB)$. Moreover,

$$\Delta^\bullet \otimes^\bullet_A \Theta^\bullet \simeq {}_BB_B \text{ in } \mathscr{K}^{\mathrm{b}}(B \otimes_k B^{\mathrm{op}}) \quad \text{and} \quad \Theta^\bullet \otimes^\bullet_B \Delta^\bullet \simeq {}_AA_A \text{ in } \mathscr{K}^{\mathrm{b}}(A \otimes_k A^{\mathrm{op}}),$$

where $\Delta^\bullet \otimes^\bullet_A \Theta^\bullet$ stands for the total complex of the double complex with $(i,j)$-term $\Delta^i \otimes_\Lambda \Theta^j$. Thus, the $n$-th term of $\Delta^\bullet \otimes^\bullet_A \Theta^\bullet$ is $\bigoplus_{p+q=n} \Delta^p \otimes_A \Theta^q = \bigoplus_{q\in\mathbb{Z}} \Delta^{n-q} \otimes_A \Theta^q$, the differential is given by $x \otimes y \mapsto x \otimes (y)d^q_\Theta + (-1)^q (x) d^{n-q}_\Delta \otimes y$ for $x \in \Delta^{n-q}$ and $y \in \Theta^q$.

In the following, we shall prove that $f\Delta^\bullet e$ is a two-sided tilting complex over $fBf \otimes_k (eAe)^{\mathrm{op}}$, defining a derived equivalence $f\Delta^\bullet e \otimes^\bullet_{eAe} - : \mathscr{D}^{\mathrm{b}}(eAe) \to \mathscr{D}^{\mathrm{b}}(fBf)$ with the associated tilting complex $eQ^\bullet$.

Since all terms of $\Delta^\bullet$ are projective as right $A$-modules, $F(X^\bullet) = \Delta^\bullet \otimes^{\mathbf{L}}_A X^\bullet \simeq \Delta^\bullet \otimes^\bullet_A X^\bullet$ for all $X^\bullet \in \mathscr{D}^{\mathrm{b}}(A)$. Hence $F(Ae) \simeq \Delta^\bullet \otimes^\bullet_A Ae = \Delta^\bullet \otimes_A Ae \simeq \Delta^\bullet e$ which is isomorphic in $\mathscr{D}^{\mathrm{b}}(B)$ to a complex $Y^\bullet$ in $\mathscr{K}^{\mathrm{b}}(Bf)$. This also means that $\Delta^\bullet e$ is isomorphic in $\mathscr{K}^{\mathrm{b}}(B\text{-proj})$ to the complex $Y^\bullet$ in $\mathscr{K}^{\mathrm{b}}(Bf)$. Let $f^\bullet : \Delta^\bullet e \to Y^\bullet$ be a chain map such that $f^\bullet$ is an isomorphism in $\mathscr{K}^{\mathrm{b}}(B\text{-proj})$. Then the mapping cone of $f^\bullet$ is isomorphic to zero in $\mathscr{K}^{\mathrm{b}}(B\text{-proj})$, that is, it is an exact sequence. In fact, this sequence is even split exact because its terms all are projective $B$-modules. Since $\Delta^i e \in \operatorname{add}(Bf)$ for $i > 0$, the split exactness implies $\Delta^0 e \in \operatorname{add}(Bf)$. Thus $\Delta^i e \in \operatorname{add}(Bf)$ for all integers $i$, and all terms of the complex $f\Delta^\bullet e$:

$$0 \longrightarrow f\Delta^0 e \longrightarrow f\Delta^1 e \longrightarrow \cdots \longrightarrow f\Delta^n e \longrightarrow 0$$

are projective as left $fBf$-modules. Similarly, $\Theta^i f \in \operatorname{add}(Ae)$ for all integer $i$, and all terms of the complex $e\Theta^\bullet f$ are projective as left $eAe$-modules.

Now, we show that $f\Delta^i e$ is projective as a right $eAe$-module for all $i$. Applying $\Delta^\bullet \otimes^\bullet_A -$ to the isomorphisms

$$\Theta^\bullet \otimes^\bullet_B \operatorname{Hom}^\bullet_A(\Theta^\bullet, A) \simeq \operatorname{Hom}^\bullet_B(\Delta^\bullet, \operatorname{Hom}^\bullet_A(\Theta^\bullet, A)) \simeq \operatorname{Hom}_A(A, A) \simeq A$$

in $\mathscr{K}^{\mathrm{b}}(A \otimes_k A^{\mathrm{op}})$, where $\operatorname{Hom}^\bullet_A(X^\bullet, Y^\bullet)$ denotes the total complex of the double complex with $(i,j)$-term $\operatorname{Hom}_A(X^{-i}, Y^j)$, we obtain $\Delta^\bullet \simeq \operatorname{Hom}^\bullet_A(\Theta^\bullet, A)$ in $\mathscr{K}^{\mathrm{b}}(B \otimes_k A^{\mathrm{op}})$. Further, the isomorphisms

$$\begin{aligned} f\Delta^\bullet &\simeq \operatorname{Hom}^\bullet_B(Bf, \Delta^\bullet) \simeq \operatorname{Hom}^\bullet_B(Bf, \operatorname{Hom}^\bullet_A(\Theta^\bullet, A)) \\ &\simeq \operatorname{Hom}^\bullet_A(\Theta^\bullet \otimes^\bullet_B Bf, A) \simeq \operatorname{Hom}^\bullet_A(\Theta^\bullet f, A) \end{aligned}$$

in $\mathscr{K}^{\mathrm{b}}(A^{\mathrm{op}})$ imply that all terms of $f\Delta^\bullet$ belong to $\operatorname{add}(eA)$, since all terms of $\Theta^\bullet f$ are in $\operatorname{add}(Ae)$. Hence the right $eAe$-module $f\Delta^i e$ is projective for all $i$. Similarly, we prove that the right $fBf$-module $e\Theta^i f$ is projective for all $i$.

Now we have the following isomorphisms in $\mathscr{D}^{\mathrm{b}}(fBf \otimes_k fBf^{\mathrm{op}})$:

$$\begin{aligned} f\Delta^\bullet e \otimes^{\mathbf{L}}_{eAe} e\Theta^\bullet f &\simeq f\Delta^\bullet e \otimes^\bullet_{eAe} e\Theta^\bullet f \\ &\simeq (fB \otimes^\bullet_B \Delta^\bullet \otimes^\bullet_A Ae) \otimes^\bullet_{eAe} (eA \otimes^\bullet_A \Theta^\bullet \otimes^\bullet_B Bf) \\ &\simeq fB \otimes^\bullet_B \Delta^\bullet \otimes^\bullet_A \left(Ae \otimes^\bullet_{eAe} eA \otimes^\bullet_A (\Theta^\bullet \otimes^\bullet_B Bf)\right) \\ &\simeq fB \otimes^\bullet_B \Delta^\bullet \otimes^\bullet_A \Theta^\bullet \otimes^\bullet_B Bf \quad (\text{because } \Theta^\bullet \otimes^\bullet_B Bf \in \mathscr{K}^{\mathrm{b}}(\operatorname{add}(Ae))) \\ &\simeq fB \otimes_B B \otimes_B Bf \simeq fBf. \end{aligned}$$

Similarly, $e\Theta^\bullet f\otimes^{\mathbf{L}}_{fBf} f\Delta^\bullet e\simeq eAe$ in $\mathscr{D}^{\rm b}(eAe\otimes_k eAe^{\rm op})$. Thus $f\Delta^\bullet e$ is a two-sided tilting complex and $f\Delta^\bullet e\otimes^{\mathbf{L}}_{eAe} -: \mathscr{D}^{\rm b}(eAe)\to\mathscr{D}^{\rm b}(fBf)$ is a derived equivalence. Furthermore, the following isomorphisms in $\mathscr{D}^{\rm b}(fBf)$:

$$f\Delta^\bullet e\otimes^{\mathbf{L}}_{eAe} eQ^\bullet\simeq f\Delta^\bullet e\otimes^\bullet_{eAe} eQ^\bullet\simeq f\Delta^\bullet\otimes^\bullet_A Q^\bullet\simeq fBf$$

show that $eQ^\bullet$ is the associated tilting complex to the functor $G:=f\Delta^\bullet e\otimes^{\mathbf{L}}_{eAe}-$.

Since $F=\Delta^\bullet\otimes^{\mathbf{L}}_A -$ is an almost $\nu$-stable, standard derived equivalence, it follows from Theorem 5.3 in [21] that $\Delta^0\otimes_A -$ induces a stable equivalence $\Phi$ of Morita type between $A$ and $B$ with the defining bimodules $\Delta^0$ and $\Theta^0$. Since $eAe$ and $fBf$ are self-injective algebras, the functor $G$ is clearly an almost $\nu$-stable derived equivalence, and therefore the functor $f\Delta^0 e\otimes_{eAe}-$ induces a stable equivalence $\Phi_1$ of Morita type between $eAe$ and $fBf$ with defining bimodules $e\Delta^0 f$ and $f\Theta^0 e$.

Due to $\Delta^0\otimes_A Ae=\Delta^0 e\in{\rm add}(Bf)$, the following isomorphisms hold in $B$-mod for each $eAe$-module $X$:

$$Bf\otimes_{fBf}(f\Delta^0 e\otimes_{eAe}X)\simeq\big(Bf\otimes_{fBf}fB\otimes_B(\Delta^0\otimes_A Ae)\big)\otimes_{eAe}X\simeq\Delta^0\otimes_A Ae\otimes_{eAe}X.$$

This implies that the functors $\Phi\lambda$ and $\lambda\Phi_1$ are naturally isomorphic, where the functor $\lambda$ was described in Section 2.3. Thus the diagram ($\blacklozenge$) exists and the statements (1) and (2) then follow by the definitions of $\Phi$ and $\Phi_1$.

(3) Since ${}_B\Delta^i\in{\rm add}(Bf)$ for all $i>0$, the term $\Theta^{-i}={\rm Hom}_B(\Delta^i,{}_BB)\in{\rm add}(fB)$ as a right $B$-module for all $i>0$. Now let $S$ be a simple $A$-module with $eS=0$, that is, $eA\otimes_A S=0$. Then, by the definition of $\Delta^\bullet$ and $\Theta^\bullet$, there is an isomorphism $\Theta^\bullet\otimes^\bullet_B\Delta^\bullet\otimes^\bullet_A S\simeq S$ in $\mathscr{D}^{\rm b}(A)$. Thus the following isomorphisms hold in $\mathscr{D}^{\rm b}(A)$:

$$\begin{aligned}
S&\simeq\Theta^\bullet\otimes^\bullet_B\Delta^\bullet\otimes^\bullet_A S\\
&\simeq\Theta^\bullet\otimes^\bullet_B(\Delta^0\otimes_A S)\quad(\Delta^i_A\in{\rm add}(eA)\text{ for all }i>0)\\
&\simeq\Theta^\bullet\otimes_B(\Delta^0\otimes_A S)\\
&\simeq(\Theta^\bullet\otimes_B\Delta^0)\otimes_A S\\
&\simeq\Theta^0\otimes_B\Delta^0\otimes_A S\quad(\Theta^i_B\in{\rm add}(fB)\text{ for all }i<0\text{ and }fB\otimes_B\Delta^0\in{\rm add}(eA)).
\end{aligned}$$

Similar to the proof of Lemma 3.1 (5), we can show that $\Phi(S)=\Delta^0\otimes_A S$ is a simple $B$-module. Moreover, since $fB\otimes_B\Delta^0\in{\rm add}(eA)$ and $eA\otimes_A S=0$, we have

$$f\cdot\Phi(S)\simeq fB\otimes_B\Delta^0\otimes_A S=0.$$

Hence (3) holds true.

(4) Using the two-sided tilting complex $\Theta^\bullet={\rm Hom}^\bullet_B(\Delta^\bullet,B)$, we proceed the proof of (4) similarly as we have done in (3). $\square$

Now, we state the dual version of Proposition 4.1 and Lemma 4.2, and leave their proofs to the interested reader.

**Proposition 4.3** (Dual version of Proposition 4.1). *Let $A$ be an arbitrary algebra, and let $e$ be a $\nu$-stable idempotent element in $A$. Suppose that $Q^\bullet$ is a complex in $\mathscr{K}^{\mathrm{b}}(\mathrm{add}(Ae))$ with $Q^i = 0$ for all $i < 0$ such that*

(1) *$eQ^\bullet$ is a tilting complex over $eAe$, and*

(2) *$\mathrm{End}_{\mathscr{K}^{\mathrm{b}}(eAe)}(eQ^\bullet)$ is self-injective.*

*Then there exists a complex $P^\bullet$ of $A$-modules such that $Q^\bullet \oplus P^\bullet$ is a tilting complex over $A$, and there exists an almost $\nu$-stable derived equivalence*

$$F\colon \mathscr{D}^{\mathrm{b}}(\mathrm{End}_{\mathscr{K}^{\mathrm{b}}(A)}(Q^\bullet \oplus P^\bullet)) \to \mathscr{D}^{\mathrm{b}}(A)$$

*such that $Q^\bullet \oplus P^\bullet$ is a tilting complex associated to the quasi-inverse of $F$.*

**Lemma 4.4** (Dual version of Lemma 4.2). *Keep the assumptions and notation as in* Proposition 4.3. *Let $B := \mathrm{End}_{\mathscr{K}^{\mathrm{b}}(A)}(Q^\bullet \oplus P^\bullet)$, and let $f$ be the idempotent element in $B$ corresponding to the summand $Q^\bullet$. Then there exists a stable equivalence $\Phi\colon A\text{-}\underline{\mathrm{mod}} \to B\text{-}\underline{\mathrm{mod}}$ of Morita type, an idempotent element $e \in A$ and a stable equivalence $\Phi_1 : eAe\text{-}\underline{\mathrm{mod}} \to fBf\text{-}\underline{\mathrm{mod}}$ of Morita type, such that the following diagram of functors commutates*

$$\begin{tikzcd} A\text{-}\underline{\mathrm{mod}} \arrow[r, "\Phi"] & B\text{-}\underline{\mathrm{mod}} \\ eAe\text{-}\underline{\mathrm{mod}} \arrow[u, "\lambda"] \arrow[r, "\Phi_1"] & fBf\text{-}\underline{\mathrm{mod}} \arrow[u, "\lambda"'] \end{tikzcd}$$

*up to isomorphism, and that*

(1) *$\Phi$ is induced by a quasi-inverse of the almost $\nu$-stable derived equivalence $F$ in Proposition 4.3.*

(2) *$\Phi_1$ is induced by a derived equivalence $G$ with $G(eQ^\bullet) \simeq fBf$.*

(3) *For all simple $A$-modules $S$ with $e \cdot S = 0$, $\Phi(S)$ is isomorphic in $B$-$\underline{\mathrm{mod}}$ to a simple $B$-module $S'$ with $f \cdot S' = 0$.*

(4) *For all simple $B$-modules $T$ with $f \cdot T = 0$, $\Phi^{-1}(T)$ is isomorphic in $A$-$\underline{\mathrm{mod}}$ to a simple $A$-module $T'$ with $e \cdot T' = 0$.*

In the following, we shall construct a Morita equivalence from a $\nu$-stable idempotent element together with a stable equivalence of Morita type induced by a derived equivalence.

**Proposition 4.5.** *Let $A$ be an algebra and $e$ be a $\nu$-stable idempotent element in $A$, and let $\Delta$ be a self-injective algebra. Suppose that $\Xi : eAe\text{-}\underline{\mathrm{mod}} \to \Delta\text{-}\underline{\mathrm{mod}}$ is a stable equivalence of Morita type induced by a derived equivalence. Then there exists another algebra $B$ (not necessarily isomorphic to $A$), a stable equivalence $\Phi\colon B\text{-}\underline{\mathrm{mod}} \to A\text{-}\underline{\mathrm{mod}}$ of Morita type, a $\nu$-stable idempotent element $f$ in $B$ and a stable equivalence $\Phi_1 : fBf\text{-}\underline{\mathrm{mod}} \to eAe\text{-}\underline{\mathrm{mod}}$ of Morita type with $\Xi \circ \Phi_1$ lifting to* *a Morita equivalence, such that the following diagram of functors:*

$$\begin{tikzcd}
B\text{-}\underline{\mathrm{mod}} \arrow[r, "\Phi"] & A\text{-}\underline{\mathrm{mod}} \\
fBf\text{-}\underline{\mathrm{mod}} \arrow[u, "\lambda"] \arrow[r, "\Phi_1"] \arrow[d, "\Xi\circ\Phi_1", "\text{Morita equiv.}"'] & eAe\text{-}\underline{\mathrm{mod}} \arrow[u, "\lambda"'] \arrow[d, equal] \\
\Delta\text{-}\underline{\mathrm{mod}} & eAe\text{-}\underline{\mathrm{mod}} \arrow[l, "\Xi"']
\end{tikzcd}$$

*commutes up to isomorphism, and that*

(1) $\Phi$ *is induced by an iterated almost $\nu$-stable derived equivalence.*

(2) $\Phi(T)$ *is isomorphic in $A$-$\underline{\mathrm{mod}}$ to a simple $A$-module $T'$ with $e \cdot T' = 0$ for all simple $B$-modules $T$ with $f \cdot T = 0$.*

(3) $\Phi^{-1}(S)$ *is isomorphic in $B$-$\underline{\mathrm{mod}}$ to a simple $B$-module $S'$ with $f \cdot S' = 0$ for all simple $A$-modules $S$ with $e \cdot S = 0$.*

*Proof.* Since $eAe$ and $\Delta$ are self-injective algebras, for each derived equivalence $G$ between $eAe$ and $\Delta$, the functor $[i] \circ G$ is almost $\nu$-stable for some $i < 0$ by Proposition 3.8 in [21]. Observe that the shift functor $[i] \simeq (\Delta[i]) \otimes^{\mathbf{L}}_{\Delta} -$ is a standard derived equivalence for all integers $i$. So, by Remark 2.2, we may suppose that the stable equivalence $\Xi$ is induced by a standard derived equivalence $F\colon \mathscr{D}^{\mathrm{b}}(eAe) \to \mathscr{D}^{\mathrm{b}}(\Delta)$ and that $[m] \circ F$ is an almost $\nu$-stable, standard derived equivalence for a negative integer $m$. Thus $\Xi$ can be written as a composite $\Xi = \Xi_2 \circ \Xi_1$ of stable equivalences $\Xi_1$ and $\Xi_2$ of Morita type such that $\Xi_1$ is induced by $[m] \circ F : \mathscr{D}^{\mathrm{b}}(eAe) \to \mathscr{D}^{\mathrm{b}}(\Delta)$ and $\Xi_2$ is induced by $[-m] : \mathscr{D}^{\mathrm{b}}(\Delta) \to \mathscr{D}^{\mathrm{b}}(\Delta)$.

Let $X^{\bullet}$ be a tilting complex over $eAe$ associated to $[m] \circ F$. Then $X^i = 0$ for all $i > 0$. Set $Q^{\bullet} := Ae \otimes^{\bullet}_{eAe} X^{\bullet}$. Then $Q^{\bullet}$ satisfies all conditions in Proposition 4.1 since $eQ^{\bullet} \simeq X^{\bullet}$ is a tilting complex over $eAe$ and $\mathrm{End}_{\mathscr{K}^{\mathrm{b}}(eAe)}(X^{\bullet}) \simeq \Delta$ is self-injective. Hence, by Lemma 4.2, there is an algebra $B'$ and a $\nu$-stable idempotent element $f'$ in $B'$, together with a commutative diagram (up to isomorphism) of functors:

$$\begin{tikzcd}
B'\text{-}\underline{\mathrm{mod}} & A\text{-}\underline{\mathrm{mod}} \arrow[l, "\Phi'"'] & \\
f'B'f'\text{-}\underline{\mathrm{mod}} \arrow[u, "\lambda"] & eAe\text{-}\underline{\mathrm{mod}} \arrow[u, "\lambda"] \arrow[l, "\Phi'_1"'] \arrow[r, "\Xi_1"] & \Delta\text{-}\underline{\mathrm{mod}} \\
\mathscr{D}^{\mathrm{b}}(f'B'f') \arrow[u, "\eta_{f'B'f'}"] & \mathscr{D}^{\mathrm{b}}(eAe) \arrow[u, "\eta_{eAe}"] \arrow[l, "G_1"'] \arrow[r, "{[m]\circ F}"] & \mathscr{D}^{\mathrm{b}}(\Delta) \arrow[u, "\eta_{\Delta}"]
\end{tikzcd}$$

such that $\Phi'$ is a stable equivalence of Morita type induced by a standard, almost $\nu$-stable derived equivalence, and that $G_1$ is a standard derived equivalence with $X^{\bullet}$ as an associated tilting complex. Thus $f'B'f'$ is a tilting complex associated to the derived equivalence $[m] \circ F \circ G_1^{-1} : \mathscr{D}^{\mathrm{b}}(f'B'f') \to \mathscr{D}^{\mathrm{b}}(\Delta)$. This means that $(f'B'f')[m]$ is a tilting complex associated to $F \circ G_1^{-1}$. Note that $f'B'f'$ is a self-injective algebra by Lemma 2.7 (3), and that the complex $B'f'[m]$ satisfies the assumptions of Proposition 4.3. By Lemma 4.4, there is an algebra $B$, a stable equivalence $\Phi''\colon B'\text{-}\underline{\mathrm{mod}} \to B\text{-}\underline{\mathrm{mod}}$, a $\nu$-stable idempotent element $f$ in $B$, and a stable equivalence $\Phi_1'' : f'B'f'\text{-}\underline{\mathrm{mod}} \to fBf\text{-}\underline{\mathrm{mod}}$, together with a commutative diagram of functors

$$\begin{tikzcd}
B\text{-}\underline{\mathrm{mod}} & B'\text{-}\underline{\mathrm{mod}} \arrow[l, "\Phi''"'] & \\
fBf\text{-}\underline{\mathrm{mod}} \arrow[u, "\lambda"] & f'B'f'\text{-}\underline{\mathrm{mod}} \arrow[u, "\lambda"] \arrow[l, "\Phi_1''"'] \arrow[r, "\Xi_2\Xi_1(\Phi_1')^{-1}"] & \Delta\text{-}\underline{\mathrm{mod}} \\
\mathscr{D}^{\mathrm{b}}(fBf) \arrow[u, "\eta_{fBf}"] & \mathscr{D}^{\mathrm{b}}(f'B'f') \arrow[u, "\eta_{f'B'f'}"] \arrow[l, "G_2"'] \arrow[r, "F\circ G_1^{-1}"] & \mathscr{D}^{\mathrm{b}}(\Delta) \arrow[u, "\eta_\Delta"]
\end{tikzcd}$$

up to isomorphism, such that $\Phi''$ is a stable equivalence of Morita type, the quasi-inverse $(\Phi'')^{-1}$ is induced by a standard, almost $\nu$-stable derived equivalence, and that $G_2$ is a standard derived equivalence with $(f'B'f')[m]$ as an associated tilting complex.

Now we define $\Phi := (\Phi')^{-1} \circ (\Phi'')^{-1}$ and $\Phi_1 := (\Phi_1')^{-1} \circ (\Phi_1'')^{-1}$. Then we get the following commutative diagram up to isomorphism:

$$(*)\qquad \begin{tikzcd}
B\text{-}\underline{\mathrm{mod}} \arrow[r, "\Phi"] & A\text{-}\underline{\mathrm{mod}} & \\
fBf\text{-}\underline{\mathrm{mod}} \arrow[u, "\lambda"] \arrow[r, "\Phi_1"] & eAe\text{-}\underline{\mathrm{mod}} \arrow[u, "\lambda"] \arrow[r, "\Xi_2\circ\Xi_1"] & \Delta\text{-}\underline{\mathrm{mod}} \\
\mathscr{D}^{\mathrm{b}}(fBf) \arrow[u, "\eta_{fBf}"] \arrow[r, "G_1^{-1}\circ G_2^{-1}"] & \mathscr{D}^{\mathrm{b}}(eAe) \arrow[u, "\eta_{eAe}"] \arrow[r, "F"] & \mathscr{D}^{\mathrm{b}}(\Delta) \arrow[u, "\eta_\Delta"]
\end{tikzcd}$$

One can check that $fBf$ is a tilting complex associated to $F\circ G_1^{-1}\circ G_2^{-1}$. Hence the derived equivalence $F\circ G_1^{-1}\circ G_2^{-1}$ is induced by a Morita equivalence. Consequently, the stable equivalence $\Xi \circ \Phi_1 \simeq \Xi_2 \circ \Xi_1 \circ \Phi_1$ lifts to a Morita equivalence. Thus (1) follows, while (2) and (3) follow easily from $(*)$ and Lemma 4.2 (3)-(4). $\square$

## 4.2. Proof of Theorem 1.2

*Proof of Theorem* 1.2. The assumptions (1) and (2) of Theorem 1.2 show that the stable equivalence $\Phi\colon A\text{-}\underline{\mathrm{mod}} \to B\text{-}\underline{\mathrm{mod}}$ satisfies the conditions in Lemma 3.4. Thus, by Lemma 3.4, there exists a stable equivalence $\Phi_1 : eAe\text{-}\underline{\mathrm{mod}} \to fBf\text{-}\underline{\mathrm{mod}}$

of Morita type such that the following diagram of functors:

$$(\dagger)\qquad \begin{tikzcd} A\text{-}\underline{\mathrm{mod}} \arrow[r, "\Phi"] & B\text{-}\underline{\mathrm{mod}} \\ eAe\text{-}\underline{\mathrm{mod}} \arrow[u, "\lambda"] \arrow[r, "\Phi_1"] & fBf\text{-}\underline{\mathrm{mod}} \arrow[u, "\lambda"] \end{tikzcd}$$

is commutative up to isomorphism. Note that $\Phi_1$ is uniquely determined up to isomorphism by the commutative diagram $(\dagger)$, since $\lambda$ is fully faithful.

Note that $e$ and $f$ are $\nu$-stable idempotents by assumption. It follows that both $eAe$ and $fBf$ is self-injective. Now, suppose that $\Phi_1 : eAe\text{-}\underline{\mathrm{mod}} \to fBf\text{-}\underline{\mathrm{mod}}$ lifts to a derived equivalence. We want to show that $\Phi$ is induced by an iterated almost $\nu$-stable derived equivalence.

In fact, let $\Delta := fBf$ and $\Xi := \Phi_1$. Then we apply Proposition 4.5 to the algebra $A$ with the $\nu$-stable idempotent element $e$. Thus there exists another algebra $B'$, a $\nu$-stable idempotent element $f'$ in $B'$, two stable equivalences $\Phi' : B'\text{-}\underline{\mathrm{mod}} \to A\text{-}\underline{\mathrm{mod}}$ and $\Phi'_1 : f'B'f'\text{-}\underline{\mathrm{mod}} \to eAe\text{-}\underline{\mathrm{mod}}$ of Morita type, and a commutative (up to isomorphism) diagram of functors

$$(\dagger\dagger)\qquad \begin{tikzcd} B'\text{-}\underline{\mathrm{mod}} \arrow[r, "\Phi'"] & A\text{-}\underline{\mathrm{mod}} \\ f'B'f'\text{-}\underline{\mathrm{mod}} \arrow[u, "\lambda"] \arrow[r, "\Phi'_1"] & eAe\text{-}\underline{\mathrm{mod}} \arrow[u, "\lambda"] \end{tikzcd}$$

such that

(a) $\Xi \circ \Phi'_1$ lifts to a Morita equivalence.

(b) $\Phi'$ is induced by an iterated almost $\nu$-stable derived equivalence from $B'$ to $A$.

(c) For any simple $B'$-module $S'$ with $f'\cdot S' = 0$, the module $\Phi'(S')$ is isomorphic to a simple $A$-module $S$ with $e \cdot S = 0$.

(d) For any simple $A$-module $S$ with $e\cdot S = 0$, the module ${\Phi'}^{-1}(S)$ is isomorphic to a simple $B'$-module $S'$ with $f' \cdot S' = 0$.

By splicing the two diagrams $(\dagger)$ and $(\dagger\dagger)$, one gets the following commutative (up to isomorphism) diagram

$$\begin{tikzcd} B'\text{-}\underline{\mathrm{mod}} \arrow[rr, "\Phi\circ\Phi'"] & & B\text{-}\underline{\mathrm{mod}} \\ f'B'f'\text{-}\underline{\mathrm{mod}} \arrow[u, "\lambda"] \arrow[rr, "\Phi_1\circ\Phi'_1"] & & fBf\text{-}\underline{\mathrm{mod}} \arrow[u, "\lambda"] \end{tikzcd}$$

such that $\Phi_1 \circ \Phi'_1 = \Xi \circ \Phi'_1$ lifts to a Morita equivalence.

Now, we show that $\Phi \circ \Phi'$ lifts to a Morita equivalence. Indeed, according to (c), for each simple $B'$-module $T'$ with $f' \cdot T' = 0$, the $A$-module $\Phi'(T')$ is isomorphic to a simple $A$-module $S$ with $e\dot{S} = 0$. Thus the assumption (1) in Theorem 1.2 implies that $\Phi \circ \Phi'(S')$ is isomorphic to a simple $B$-module $T$ with $f \cdot T = 0$. Similarly, according to (d) and the assumption (2) in Theorem 1.2, it follows that, for each simple $B$-module $T$ with $f \cdot T = 0$, the image $\Phi^{-1}(T)$ is a simple $A$-module $S$ with $e \cdot S = 0$, and therefore the image ${\Phi'}^{-1}\Phi^{-1}(T) \simeq {\Phi'}^{-1}(S)$ is isomorphic to a simple $B'$-module $T'$ with $f' \cdot T' = 0$. Hence the two conditions of Proposition 3.5 are satisfied by the stable equivalence $\Phi \circ \Phi'$. Since the restricted stable equivalence $\Phi_1 \circ \Phi'_1$ of $\Phi \circ \Phi'$ lifts to a Morita equivalence, it follows from Proposition 3.5 that the stable equivalence $\Phi \circ \Phi'$ lifts to a Morita equivalence.

Thus $\Phi \simeq (\Phi \circ \Phi') \circ {\Phi'}^{-1}$ is induced by an iterated almost $\nu$-stable derived equivalence. This finishes the proof of Theorem 1.2. □

Every stable equivalence of Morita type between algebras $A$ and $B$ can be restricted to a stable equivalence of Morita type between $eAe$ and $fBf$ for some $\nu$-stable idempotent elements $e \in A$ and $f \in B$. There are two typical ways to implement this point.

**Remark 4.6.** (i) For each algebra $A$, there is an associated self-injective algebra $\Delta_A$ (see Definition 2.5). Theorem 4.2 in [15] shows that if $A/\mathrm{rad}(A)$ and $B/\mathrm{rad}(B)$ are separable, then every stable equivalence of Morita type between $A$ and $B$ is restricted to a stable equivalence of Morita type between $\Delta_A$ and $\Delta_B$.

(ii) Under the setting of Lemma 3.1, let $e_0$ and $f_0$ be idempotent elements in $A$ and $B$, respectively, such that $\mathrm{add}(Ae_0) = \mathrm{add}({}_AP)$ and $\mathrm{add}(Bf_0) = \mathrm{add}({}_BQ)$. Then it follows from Lemma 3.1 (2) that the idempotent elements $e_0$ and $f_0$ are $\nu$-stable. By Lemma 3.1 (3) and Theorem 1.2 in [11], the given stable equivalence of Morita type between $A$ and $B$ in Lemma 3.1 can be restricted to a stable equivalence of Morita type between $e_0Ae_0$ and $f_0Bf_0$.

As a consequence of Theorem 1.2, we have the following.

**Corollary 4.7.** *Let $A$ and $B$ be algebras without nonzero semisimple direct summands such that $A/\mathrm{rad}(A)$ and $B/\mathrm{rad}(B)$ are separable. Suppose that $\Phi$ is a stable equivalence of Morita type between $A$ and $B$, with ${}_AP_A$ and ${}_BQ_B$ the bimodules belonging to $\Phi$, and that $\Phi_1$ is the restricted stable equivalence of $\Phi$ between $\mathrm{End}_A(P)$ and $\mathrm{End}_B(Q)$ or between the associated self-injective algebras $\Delta_A$ and $\Delta_B$. Then $\Phi$ lifts to an iterated almost $\nu$-stable derived equivalence between $A$ and $B$ if $\Phi_1$ lifts to a derived equivalence between*

(1) $\mathrm{End}_A(P)$ *and* $\mathrm{End}_B(Q)$, *or*

(2) $\Delta_A$ *and* $\Delta_B$.

*Proof.* Let ${}_AM_B$ and ${}_BN_A$ be two bimodules without nonzero projective direct summands and defining the stable equivalence $\Phi$ of Morita type between $A$ and $B$, that is, $\Phi = {}_BN \otimes_A -$, ${}_AM \otimes_B N_A \simeq A \oplus {}_AP_A$ and ${}_BN \otimes_A M_B \simeq B \oplus {}_BQ_B$ as bimodules. In the following, we shall check all conditions in Theorem 1.2 for the both cases in Corollary 4.7.

(1) Let $e_0$ and $f_0$ be idempotent elements in $A$ and $B$, respectively, such that $\mathrm{add}(Ae_0) = \mathrm{add}({}_AP)$ and $\mathrm{add}(Bf_0) = \mathrm{add}({}_BQ)$. Then both $e_0$ and $f_0$ are $\nu$-stable by Lemma 3.1 (2). Let $S$ be a simple $A$-module with $e_0S = 0$, that is, $\mathrm{Hom}_A({}_AP, S) = 0$. By Lemma 3.1 (5), the $B$-module $N \otimes_A S$, which is the image of $S$ under $\Phi$, is simple with $\mathrm{Hom}_B({}_BQ, N \otimes_A S) = 0$. Namely, the image $\Phi(S)$ is a simple module with $f_0 \cdot \Phi(S) = 0$. By a similar argument, we show that, for each simple $B$-module $S'$ with $f_0S' = 0$, the image $\Phi^{-1}(S')$ is a simple $A$-module with $e_0\Phi^{-1}(S') = 0$. Now, the first case of the corollary follows from Theorem 1.2.

(2) By definition, $\Delta_A = eAe$ for some idempotent element $e$ in $A$ with $\mathrm{add}(Ae) = A\text{-stp}$, and $\Delta_B = fBf$ for some idempotent element $f$ in $B$ with $\mathrm{add}(Bf) = B\text{-stp}$. We first show that $N \otimes_A Ae \in \mathrm{add}({}_BBf)$ and $M \otimes_B Bf \in \mathrm{add}({}_AAe)$. By the proof of Lemma 3.1 (2), we have

$$\nu_A^i(N \otimes_A Ae) \simeq N \otimes_A (\nu_A^i(Ae))$$

for all $i \geqslant 0$. Note that $\nu_A^i(Ae)$ is projective for all $i \geqslant 0$ since $Ae \in A\text{-stp}$. Thus $\nu_B^i(N \otimes_A Ae)$ is projective for all $i \geqslant 0$, that is, $N \otimes_A Ae \in B\text{-stp} = \mathrm{add}({}_BBf)$. Similarly, $M \otimes_B Bf \in \mathrm{add}({}_AAe)$.

Let $S$ be a simple $A$-module with $e \cdot S = 0$. Then ${}_AP \in A\text{-stp} = \mathrm{add}({}_AAe)$ by Lemma 3.1 (2), and consequently $\mathrm{Hom}_A(P, S) = 0$ and $\Phi(S) = N \otimes_A S$ is a simple $B$-module by Lemma 3.1 (5). Moreover, since $M \otimes_B Bf \in \mathrm{add}(Ae)$,

$$f \cdot \Phi(S) = \mathrm{Hom}_B(Bf, N \otimes_A S) \simeq \mathrm{Hom}_A(M \otimes_B Bf, S) = 0.$$

Similarly, for each simple $B$-module $V$ with $f \cdot V = 0$, the $A$-module $\Phi^{-1}(V)$ is simple with $e \cdot \Phi^{-1}(V) = 0$. Now, the second case of the corollary follows from Theorem 1.2. $\square$

In the next section we will describe a large class of algebras for which $\Phi_1$ can be lifted to a derived equivalence.

## 5. Frobenius-finite algebras. Proof of Theorem 1.1

In this section we shall introduce Frobenius-finite algebras and show that the constructions of Auslander–Yoneda algebras, triangular matrix algebras, cluster-tilted algebras and Frobenius extensions produce a large class of Frobenius-finite algebras. After these discussions, we prepare a combinatorial result, Lemma 5.6, on stable Auslander–Reiten quivers and then prove the main result, Theorem 1.1, by applying Corollary 4.7, Theorem 1.2 and results in Section 3.

### 5.1. Frobenius-finite algebras and examples

Given an algebra $A$ over a field, the associated self-injective algebra of $A$ exists and is unique up to Morita equivalence (see Section 2.3). Moreover, Corollary 4.7 shows that the associated self-injective algebra is of prominent importance in lifting stable equivalences of Morita type to derived equivalences. So, we make the following definition.

**Definition 5.1.** An algebra is said to be *Frobenius-finite* if its associated self-injective algebra is representation-finite, and *Frobenius-free* if its associated self-injective algebra is zero.

Similarly, one can define Frobenius-tame, Frobenius-wild, Frobenius-symmetric and Frobenius-domestic algebras. By *Frobenius type* we mean the representation type of the associated self-injective algebra.

Clearly, Frobenius-free and representation-finite algebras are Frobenius-finite. Moreover, the ubiquity of Frobenius-finite algebras is guaranteed by the next proposition.

Before we present methods to produce Frobenius-finite algebras, let us recall the definition of Auslander–Yoneda algebras introduced in [22]. A subset $\Theta$ of $\mathbb{N}$ is called an *admissible subset* if $0 \in \Theta$ and if, for any $l, m, n \in \Theta$ with $l+m+n \in \Theta$, we have $l+m \in \Theta$ if and only if $m+n \in \Theta$. There are a lot of admissible subsets of $\mathbb{N}$. For example, for each $n \in \mathbb{N}$, the subsets $\{xn \mid x \in \mathbb{N}\}$ and $\{0,1,2,\ldots,n\}$ of $\mathbb{N}$ are admissible. But not all subsets of $\mathbb{N}$ containing $\{0\}$ are admissible. A minimal nonexample is $\{0,1,2,4\}$.

Let $\Theta$ be an admissible subset of $\mathbb{N}$, and let $\mathcal{T}$ be a triangulated $k$-category. Then there is a bifunctor

$$\mathrm{E}^{\Theta}_{\mathcal{T}}(-,-): \mathcal{T} \times \mathcal{T} \longrightarrow k\text{-Mod}$$

$$(X, Y) \mapsto \mathrm{E}^{\Theta}_{\mathcal{T}}(X, Y) := \bigoplus_{i \in \Theta} \mathrm{Hom}_{\mathcal{T}}(X, Y[i])$$

with composition given in an obvious way (for details, see Subsection 3.1 in [22]). In particular, if $f \in \mathrm{Hom}_{\mathcal{T}}(X, Y[i])$ and $g \in \mathrm{Hom}_{\mathcal{T}}(Y, Z[j])$, then the composite $f \cdot g = f(g[i])$ if $i + j \in \Theta$, and $f \cdot g = 0$ otherwise. In this way, for each object $M \in \mathcal{T}$, we get an associative algebra $\mathrm{E}^{\Theta}_{\mathcal{T}}(M, M)$, denoted by $\mathrm{E}^{\Theta}_{\mathcal{T}}(M)$ and called the $\Theta$-*Auslander–Yoneda algebra* of $M$. If $\mathcal{T} = \mathscr{D}^{\mathrm{b}}(A)$ for an algebra $A$, then we write $\mathrm{E}^{\Theta}_{A}(X, Y)$ for $\mathrm{E}^{\Theta}_{\mathscr{D}^{\mathrm{b}}(A)}(X, Y)$, and $\mathrm{E}^{\Theta}_{A}(M)$ for $\mathrm{E}^{\Theta}_{\mathscr{D}^{\mathrm{b}}(A)}(M)$ with all $X, Y, M \in \mathscr{D}^{\mathrm{b}}(A)$.

The following proposition shows that there are plenty of Frobenius-finite algebras. Recall that an $A$-module $M$ is called a *generator* in $A$-mod if $\mathrm{add}(M)$ contains ${}_AA$; a *generator-cogenerator* in $A$-mod if $\mathrm{add}(M)$ contains both ${}_AA$ and ${}_AD(A)$; and a *torsionless* module if it is a submodule of a projective module. An algebra is called an *Auslander algebra* if it is of the form $\mathrm{End}_B(M)$ with $B$ a representation-finite algebra and $M$ a basic $B$-module with $\mathrm{add}(M) = B$-mod.

**Proposition 5.2.** (1) *Let $M$ be a generator-cogenerator over an algebra $A$. Then $\mathrm{End}_A(M)$ and $A$ have the same Frobenius type. In particular, $\mathrm{End}_A(M)$ is Frobenius-finite if and only if so is $A$. Consequently, Auslander algebras are Frobenius-finite.*

(2) *Let $M$ be a torsionless generator over an algebra $A$. Suppose that $\Theta$ is a finite admissible subset of $\mathbb{N}$ and $\mathrm{Ext}^i_A(M, A) = 0$ for all $0 \neq i \in \Theta$. Then $\mathrm{E}^{\Theta}_A(M)$ and $A$ have the same Frobenius type. In particular, if $A$ is a representation-finite self-injective algebra, then $\mathrm{E}^{\Theta}_A(A \oplus X)$ is Frobenius-finite for each $A$-module $X$ and for arbitrary finite admissible subset $\Theta$ of $\mathbb{N}$.*

(3) *If $A$ and $B$ are Frobenius-finite algebras and ${}_BM_A$ is a bimodule, then the triangular matrix algebra* $\left[\begin{smallmatrix} A & 0 \\ M & B \end{smallmatrix}\right]$ *is Frobenius-finite. More generally, if* $\{A_1, \ldots, A_m\}$ *is a family of Frobenius-finite algebras and if $M_{ij}$ is an $A_i$-$A_j$-bimodule for all $1 \leqslant j < i \leqslant m$, then the triangular matrix algebra of the form*

$$\begin{bmatrix} A_1 & & & \\ M_{21} & A_2 & & \\ \vdots & \vdots & \ddots & \\ M_{m1} & M_{m2} & \cdots & A_m \end{bmatrix}$$

*is Frobenius-finite.*

(4) *If $A = A_0 \oplus A_1 \oplus \cdots \oplus A_n$ is an $\mathbb{N}$-graded algebra with $A_0$ Frobenius-finite, then the Beilinson–Green algebra*

$$\Lambda_m := \begin{bmatrix} A_0 & & & \\ A_1 & A_0 & & \\ \vdots & \ddots & \ddots & \\ A_m & \cdots & A_1 & A_0 \end{bmatrix}$$

*is Frobenius-finite for all $1 \leqslant m \leqslant n$.*

Remark that the triangular matrix algebra of a graded algebra $A$ in (4) seems first to appear in the paper [16] by Edward L. Green in 1975. A special case of this kind of algebras appeared in the paper [5] by A. A. Beilinson in 1978, where he described the derived category of coherent sheaves over $\mathbb{P}^n$ as the one of this triangular matrix algebra. Perhaps it is more appropriate to name this triangular matrix algebra as the *Beilinson–Green* algebra of $A$.

*Proof.* (1) We set $\Lambda := \mathrm{End}_A(M)$. Since $M$ is a generator-cogenerator for $A$-mod, every indecomposable projective-injective $\Lambda$-module is of the form $\mathrm{Hom}_A(M, I)$ with $I$ an indecomposable injective $A$-module. Moreover, for each projective $A$-module $P'$, there is a natural isomorphism $\nu_\Lambda \mathrm{Hom}_A(M, P') \simeq \mathrm{Hom}_A(M, \nu_A P')$. This implies $\mathrm{Hom}_A(M, P') \in \Lambda$-stp for all $P' \in A$-stp. Now let $I$ be an indecomposable injective $A$-module such that $\mathrm{Hom}_A(M, I)$ lies in $\Lambda$-stp. Then it follows from $\nu_\Lambda^{-1}\mathrm{Hom}_A(M, I) \simeq \mathrm{Hom}_A(M, \nu_A^{-1} I)$ that $\mathrm{Hom}_A(M, \nu_A^{-1} I)$ lies in $\Lambda$-stp. Consequently, the $\Lambda$-module $\mathrm{Hom}_A(M, \nu_A^{-1} I)$ is injective, and therefore the $A$-module $\nu_A^{-1} I$ is projective-injective. Applying $\nu_\Lambda^{-1}$ repeatedly, one sees that $\nu_A^i I$ is projective-injective for all $i < 0$. This implies $I \in A$-stp. Hence the restriction of the functor $\mathrm{Hom}_A(M, -) : \mathrm{add}({}_AM) \to \Lambda$-proj gives rise to an equivalence between $A$-stp and $\Lambda$-stp. Consequently, the associated self-injective algebras $\Delta_A$ and $\Delta_\Lambda$ are Morita equivalent. Thus (1) follows.

(2) Set $\Lambda := \mathrm{E}_A^\Theta(M) = \bigoplus_{i \in \Theta} \Lambda_i$ with $\Lambda_i := \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A)}(M, M[i])$, and identify $\mathrm{Ext}_A^i(U, V)$ with $\mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A)}(U, V[i])$ for all $A$-modules $U, V$ and integers $i$. Then $\mathrm{rad}(\Lambda) = \mathrm{rad}(\Lambda_0) \oplus \Lambda_+$, where $\Lambda_+ := \bigoplus_{0 \neq i \in \Theta} \Lambda_i$.

We shall prove that $A$-stp and $\Lambda$-stp are equivalent. Let $Y$ be an indecomposable, non-projective direct summand of $M$. We claim that $\mathrm{E}_A^\Theta(M, Y)$ cannot be in

$\Lambda$-stp. Suppose contrarily $\mathrm{E}_A^{\Theta}(M,Y) \in \Lambda$-stp. Then the $\Lambda$-module $\mathrm{E}_A^{\Theta}(M,Y)$ must be indecomposable projective-injective. Now, we have to consider the following two cases:

(a) $\bigoplus_{0\neq i\in\Theta} \mathrm{Ext}_A^i(M,Y) = 0$. Since $Y$ is torsionless, there is an injective $A$-module homomorphism $f\colon Y \to A^n$. This induces another injective map

$$\mathrm{Hom}_A(M,f) : \mathrm{Hom}_A(M,Y) \to \mathrm{Hom}_A(M,A^n).$$

Thus, from the assumption $\mathrm{Ext}_A^i(M,A) = 0$ for all $0 \neq i \in \Theta$, it follows that $\mathrm{E}_A^{\Theta}(M,Y) = \mathrm{Hom}_A(M,Y)$, $\mathrm{E}_A^{\Theta}(M,A^n) = \mathrm{Hom}_A(M,A^n)$ and $\mathrm{E}_A^{\Theta}(M,f) = \mathrm{Hom}_A(M,f)$. This implies that $\mathrm{E}_A^{\Theta}(M,f) : \mathrm{E}_A^{\Theta}(M,Y) \to \mathrm{E}_A^{\Theta}(M,A^n)$ is an injective map and must splits. Thus $Y$ must be a direct summand of $A^n$, a contradiction.

(b) $\bigoplus_{0\neq i\in\Theta} \mathrm{Ext}_A^i(M,Y) \neq 0$. Let $m \neq 0$ be the maximal integer in $\Theta$ with $\mathrm{Ext}_A^m(M,Y) \neq 0$. Then $\Lambda_+\mathrm{Ext}_A^m(M,Y) = 0$, and consequently

$$\mathrm{rad}(\Lambda)\mathrm{soc}_{\Lambda_0}\big(\mathrm{Ext}_A^m(M,Y)\big) = 0.$$

This yields that $\mathrm{soc}_{\Lambda_0}\big(\mathrm{Ext}_A^m(M,Y)\big) = \Lambda\cdot\mathrm{soc}_{\Lambda_0}\big(\mathrm{Ext}_A^m(M,Y)\big)$ is a $\Lambda$-submodule of $\mathrm{soc}_{\Lambda}\big(\mathrm{E}_A^{\Theta}(M,Y)\big)$. Next, we show that $\mathrm{soc}_{\Lambda_0}\big(\mathrm{Hom}_A(M,Y)\big)$ is also a $\Lambda$-submodule of $\mathrm{soc}_{\Lambda}\big(\mathrm{E}_A^{\Theta}(M,Y)\big)$. Let $g\colon M \to Y$ be in $\mathrm{soc}_{\Lambda_0}\big(\mathrm{Hom}_A(M,Y)\big)$. Suppose $M = M_p \oplus X$ where $M_p$ is projective and $X$ does not contain any nonzero projective direct summands. Now, for each $x \in X$, there are indecomposable projective modules $P_j$, $1 \leqslant j \leqslant s$ and radical homomorphisms $h_j\colon P_j \to X$, such that $x = \sum_{j=1}^{s}(p_j)h_j$ for some $p_j \in P_j$ with $j = 1,\dots,s$. Since $M$ is a generator for $A$-mod, $P_j$ is isomorphic to a direct summand of $M$. Thus we get a map $\tilde{h}_j\colon M \to P_j \xrightarrow{h_j} X \hookrightarrow M$, such that $\tilde{h}_j \in \mathrm{rad}(\Lambda_0)$ for all $j$ and that the composite $\tilde{h}_j g$ has to be zero. This implies that the image of $x$ under $g$ is 0, and consequently the restriction of $g$ to $X$ is 0. Let $\pi\colon M \to M_p$ be the canonical projection. Then $g = \pi g'$ for some $g' : M_p \to Y$. For each $t : M \to M[i]$ in $\mathscr{D}^{\mathrm{b}}(A)$ with $0 \neq i \in \Theta$, $t \cdot g = t(g[i]) = t(\pi[i])(g'[i])$. Since $\mathrm{Ext}_A^i(M,A) = 0$, $\mathrm{Ext}_A^i(M,M_p) = 0$, and consequently $t(\pi[i]) = 0$. Hence $t\cdot g = 0$, $\Lambda_+\cdot\mathrm{soc}_{\Lambda_0}\big(\mathrm{Hom}_A(M,Y)\big) = 0$ and $\mathrm{rad}(\Lambda)\cdot\mathrm{soc}_{\Lambda_0}\big(\mathrm{Hom}_A(M,Y)\big) = 0$. Thus $\mathrm{soc}_{\Lambda_0}\big(\mathrm{Hom}_A(M,Y)\big) = \Lambda\cdot\mathrm{soc}_{\Lambda_0}\big(\mathrm{Hom}_A(M,Y)\big)$ is a $\Lambda$-submodule of $\mathrm{soc}_{\Lambda}\big(\mathrm{E}_A^{\Theta}(M,Y)\big)$. So the $\Lambda$-module

$$\mathrm{soc}_{\Lambda_0}\big(\mathrm{Hom}_A(M,Y)\big) \oplus \mathrm{soc}_{\Lambda_0}\big(\mathrm{Ext}_A^m(M,Y)\big)$$

is contained in $\mathrm{soc}_{\Lambda}\big(\mathrm{E}_A^{\Theta}(M,Y)\big)$. This shows that $\mathrm{soc}_{\Lambda}\big(\mathrm{E}_A^{\Theta}(M,Y)\big)$ cannot be simple, and therefore $\mathrm{E}_A^{\Theta}(M,Y)$ cannot be an indecomposable injective module. This is again a contradiction.

Thus, every indecomposable projective $\Lambda$-module in $\Lambda$-stp has to be of the form $\mathrm{E}_A^{\Theta}(M,P')$ for some indecomposable projective $A$-module $P'$. Suppose $\mathrm{E}_A^{\Theta}(M,P') \in \Lambda$-stp. We shall prove $P' \in A$-stp. In fact, $\nu_\Lambda\mathrm{E}_A^{\Theta}(M,P') \simeq \mathrm{E}_A^{\Theta}(M,\nu_A P')$, by Lemma 3.5 in [22], and therefore $\nu_\Lambda\mathrm{E}_A^{\Theta}(M,P') \in \Lambda$-stp. This means that there is an isomorphism $\mathrm{E}_A^{\Theta}(M,\nu_A P') \simeq \mathrm{E}_A^{\Theta}(M,U)$ for some indecomposable projective $A$-module $U$. Since $\mathrm{Ext}_A^i(M,A) = 0$ for all $0 \neq i \in \Theta$ and since $\nu_A P'$ is injective,

$\mathrm{Hom}_A(M, \nu_A P') = \mathrm{E}_A^{\Theta}(M, \nu_A P') \simeq \mathrm{E}_A^{\Theta}(M, U) = \mathrm{Hom}_A(M, U)$. Hence $\nu_A P' \simeq U$ is projective by Lemma 2.8. Repeating this argument, we see that $\nu_A^i P'$ is projective for all $i > 0$, that is, $P' \in A$-stp. Conversely, let $P'$ be an indecomposable module in $A$-stp. Then, due to the isomorphism $\nu_\Lambda \mathrm{E}_A^{\Theta}(M, P') \simeq \mathrm{E}_A^{\Theta}(M, \nu_A P')$, the $\Lambda$-module $\mathrm{E}_A^{\Theta}(M, P')$ belongs to $\Lambda$-stp. Thus the functor $\mathrm{E}_A^{\Theta}(M, -)$ induces an equivalence from $A$-stp to $\Lambda$-stp. Hence the associated self-injective algebras $\Delta_A$ and $\Delta_\Lambda$ are Morita equivalent, and (2) follows.

(3) Set $\Lambda := \left[\begin{smallmatrix} A & 0 \\ M & B \end{smallmatrix}\right]$. Then each $\Lambda$-module can be interpreted as a triple $({}_AX, {}_BY, f)$ with $X \in A$-mod, $Y \in B$-mod and $f\colon {}_BM \otimes_A X \to {}_BY$ a $B$-module homomorphism. Let $({}_AX, {}_BY, f)$ be an indecomposable $\Lambda$-module in $\Lambda$-stp. Then $({}_AX, {}_BY, f)$ is projective-injective with $\nu_\Lambda({}_AX, {}_BY, f) \in \Lambda$-stp. By Proposition 2.5, p. 76, in [4], there are two possibilities:

(i) ${}_BY = 0$ and ${}_AX$ is an indecomposable projective-injective $A$-module with $M \otimes_A X = 0$;

(ii) ${}_AX = 0$ and ${}_BY$ is an indecomposable projective-injective $B$-module with

$$\mathrm{Hom}_B(M, Y) = 0.$$

Now we assume (i). Then $\nu_\Lambda(X, 0, 0) \simeq (\nu_A X, 0, 0)$ is still in $\Lambda$-stp. This implies that $\nu_A^i X$ is projective-injective for all $i \geqslant 0$, and therefore $X \in A$-stp. Similarly, assuming (ii), then $Y \in B$-stp. Thus, we can assume that $\{(X_1, 0, 0), \ldots, (X_r, 0, 0), (0, Y_1, 0), \ldots, (0, Y_s, 0)\}$ is a complete set of non-isomorphic indecomposable modules in $\Lambda$-stp with both $X_i \in A$-stp and $Y_j \in B$-stp for all $i$ and $j$. Then the associated self-injective algebra

$$\Delta_\Lambda := \mathrm{End}_\Lambda\Big(\bigoplus_{i=1}^{r}(X_i, 0, 0) \oplus \bigoplus_{i=1}^{s}(0, Y_i, 0)\Big) \simeq \mathrm{End}_A\Big(\bigoplus_{i=1}^{r} X_i\Big) \times \mathrm{End}_B\Big(\bigoplus_{i=1}^{s} Y_i\Big)$$

is representation-finite if both $A$ and $B$ are Frobenius-finite. Note that $\Delta_\Lambda$ is of the form $e\Delta_A e \times f\Delta_B f$ for some idempotent element $e \in \Delta_A$ and some idempotent element $f \in \Delta_B$.

(4) This is an immediate consequence of (3). $\square$

Suppose that $B$ is a subalgebra of an algebra $A$ with the same identity. In this case, we say that $B \hookrightarrow A$ is an extension of algebras, and denote by $F$ the induction functor ${}_AA \otimes_B - : B$-mod $\to A$-mod and by $H$ the restriction functor ${}_B(-) : A$-mod $\to B$-mod. Observe that for any $k$-algebra $C$, the functor $F$ is also a functor from $B$-$C$-bimodules to $A$-$C$-bimodules and $H$ is also a functor from $A$-$C$-bimodules to $B$-$C$-bimodules.

**Proposition 5.3.** *Let $B \hookrightarrow A$ be a Frobenius extension of algebras, that is,* $\mathrm{Hom}_B({}_BA, -) \simeq A \otimes_B -$ *as functors from $B$-mod to $A$-mod.*

(1) *Suppose that the extension $B \hookrightarrow A$ splits, that is, the inclusion map $B \to A$ is a split monomorphism of $B$-$B$-bimodules. If $A$ is Frobenius-finite, then so is $B$.*

(2) *Suppose that the extension $B \hookrightarrow A$ is separable, that is, the multiplication map $A \otimes_B A \to A$ is a split epimorphism of $A$-$A$-bimodules. If $B$ is Frobenius-finite, then so is $A$.*

*Proof.* An extension $B \hookrightarrow A$ of algebras is a Frobenius extension if and only if ${}_BA$ is a finitely generated projective module and $\mathrm{Hom}_B({}_BA, B) \simeq A$ as $A$-$B$-bimodules (see, for example, [9], 40.21, p. 423). We first show that both $F$ and $H$ commutes with the Nakayama functors. In fact, for each $B$-module $X$, we have the following natural isomorphisms of $A$-modules:

$$\begin{aligned}
\nu_A(F(X)) &= D\mathrm{Hom}_A({}_AA \otimes_B X, {}_AA_A) \\
&\simeq D\mathrm{Hom}_B(X, {}_BA_A) \quad (F \text{ and } H \text{ form an adjoint pair}) \\
&\simeq D\mathrm{Hom}_B(X, {}_BB \otimes_B A_A) \\
&\simeq D\big(\mathrm{Hom}_B(X, B) \otimes_B A_A\big) \quad ({}_BA \text{ is projective}) \\
&\simeq \mathrm{Hom}_B({}_BA_A, {}_BD(X^*)) \quad (\text{adjunction}) \\
&\simeq A \otimes_B D(X^*) \quad (\text{Frobenius extension}) \\
&= A \otimes_B (\nu_B X) \simeq F(\nu_B X).
\end{aligned}$$

For each $A$-module $Y$, we have the following natural isomorphisms of $B$-modules:

$$\begin{aligned}
\nu_B(H(Y)) &= D\mathrm{Hom}_B({}_BA \otimes_A Y, {}_BB_B) \\
&\simeq D\mathrm{Hom}_A(Y, \mathrm{Hom}_B({}_BA, {}_BB_B)) \quad (\text{adjunction}) \\
&\simeq D\mathrm{Hom}_A(Y, {}_AA_B) \quad (\text{Frobenius extension}) \\
&= H(\nu_A Y).
\end{aligned}$$

Note that the functor $F$ takes projective $B$-modules to projective $A$-modules. For each projective $B$-module $P$ in $B$-stp, $\nu_A^i F(P) \simeq F(\nu_B^i P)$ is projective for all $i \geqslant 0$, that is, $F(P) \in A$-stp. Since ${}_BA$ is projective, the functor $H$ takes projective $A$-modules to projective $B$-modules. Thus, a similar argument shows $H(Q) \in B$-stp for all $Q \in A$-stp.

Let $e$ and $f$ be idempotent elements in $A$ and $B$, respectively, such that $\mathrm{add}(Ae) = A$-stp and $\mathrm{add}(Bf) = B$-stp. Then, by definition, $eAe$ and $fBf$ are the Frobenius parts of $A$ and $B$, respectively.

There is an equivalence between $fBf$-mod and the full subcategory of $B$-mod, denoted by $\mathrm{mod}(Bf)$, consisting of $B$-modules $X$ that admit a projective presentation

$$P_1 \to P_0 \to X \to 0$$

with $P_i \in \mathrm{add}(Bf)$ for $i = 0, 1$. Similarly, the module category $eAe$-mod is equivalent to the full subcategory $\mathrm{mod}(Ae)$ of $A$-mod. For a $B$-module $X$ in $\mathrm{mod}(Bf)$, we take a presentation of $X$: $P_1 \to P_0 \to X \to 0$ with $P_0, P_1 \in \mathrm{add}(Bf) = B$-stp. Then the sequence $F(P_1) \to F(P_0) \to F(X) \to 0$ is exact, with $F(P_i) \in A\text{-stp} = \mathrm{add}(Ae)$. Thus $F(X) \in \mathrm{mod}(Ae)$ for all $X \in \mathrm{mod}(Bf)$. Since the restriction functor $H$ is exact, $H(Y)$ lies in $\mathrm{mod}(Bf)$ for all $A$-modules $Y$ in $\mathrm{mod}(Ae)$.

(1) Let $X \in \mathrm{mod}(Bf)$. Then the assumption (1) implies that $X$ is a direct summand of $HF(X)$. If $X$ is indecomposable, then $X$ is a direct summand of $H(Y)$ for some indecomposable direct summand $Y$ of $F(X) \in \mathrm{mod}(Ae)$. Thus, if $eAe$ is representation-finite, then $\mathrm{mod}(Ae)$ has finitely many isomorphism classes of indecomposable objects, and consequently so does $\mathrm{mod}(Bf)$. Hence $fBf$ is representation-finite.

(2) For each $A$-module $Y$ in $\mathrm{mod}(Ae)$, the assumption (2) guarantees that $Y$ is a direct summand of $FH(Y)$. Using the same arguments as in (1), we can prove that $eAe$ is representation-finite provided that $fBf$ is representation-finite. $\square$

Note that Frobenius extensions with the conditions (1) and (2) in Proposition 5.3 appear frequently in stable equivalences of Morita type. In fact, by Corollary 5.1 in [15], if $A$ and $B$ are algebras such that their semisimple quotients are separable and if at least one of them is indecomposable, then there is a $k$-algebra $\Lambda$, Morita equivalent to $A$, and an injective ring homomorphism $B \hookrightarrow \Lambda$ such that

$$_\Lambda\Lambda \otimes_B \Lambda_\Lambda \simeq {}_\Lambda\Lambda_\Lambda \oplus {}_\Lambda P_\Lambda \quad \text{and} \quad {}_B\Lambda_B \simeq {}_B B_B \oplus {}_B Q_B$$

with $P$ and $Q$ projective bimodules. This means that the extension $B \hookrightarrow \Lambda$ is a split, separable Frobenius extension.

Suppose that $A$ is a $k$-algebra and $G$ is a finite group together with a group homomorphism from $G$ to $\mathrm{Aut}(A)$, the group of automorphisms of the $k$-algebra $A$. Let $kG$ be the group algebra of $G$ over $k$. Then one may form the skew group algebra $A *_k G$ of $A$ by $G$ over $k$, that is, $A *_k G$ has the underlying $k$-space $A \otimes_k kG$ with the multiplication given by

$$(a \otimes g)(b \otimes h) := a(b)g \otimes gh \quad \text{for } a, b \in A, g, h \in G,$$

where $(b)g$ denotes the image of $b$ under $g$.

**Corollary 5.4.** *Let $A$ be a $k$-algebra, and let $A *_k G$ be the skew group algebra of $A$ by $G$ with $G$ a finite group. If the order of $G$ is invertible in $A$, then $A *_k G$ is Frobenius-finite if and only if so is $A$.*

*Proof.* Note that $A$ is a subalgebra of $A *_k G$. We just need to verify all conditions in Proposition 5.3. However, all of them follow from Theorem 1.1 in [38]. $\square$

We show now that cluster-tilted algebras are Frobenius-finite. Suppose that $H$ is a finite-dimensional hereditary algebra over an algebraically closed field. Let $\tau_D$ be the Auslander–Reiten translation functor on $\mathscr{D}^{\mathrm{b}}(H)$, and let $\mathcal{C} := \mathscr{D}^{\mathrm{b}}(H)/\langle \tau_D^{-1}[1]\rangle$ be the orbit category, which is a triangulated category with Auslander–Reiten translation $\tau_{\mathcal{C}}$. Let $\mathcal{S}$ be the class of objects in $\mathscr{D}^{\mathrm{b}}(H)$ consisting of all modules in $H$-mod and the objects $P[1]$, where $P$ runs over all modules in $H$-proj. The following facts are taken from Propositions 1.3 and 1.6 in [10].

(a) $\tau_D X$ and $\tau_{\mathcal{C}} X$ are isomorphic in $\mathcal{C}$ for each object $X$ in $\mathscr{D}^{\mathrm{b}}(H)$;

(b) Two objects $X$ and $Y$ in $\mathcal{S}$ are isomorphic in $\mathcal{C}$ if and only if they are isomorphic in $\mathscr{D}^{\mathrm{b}}(H)$;

(c) $\mathrm{Hom}_{\mathcal{C}}(X, Y) = \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(H)}(X, Y) \oplus \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(H)}(X, \tau_D^{-1} Y[1])$ for all $X, Y \in \mathcal{S}$. In particular, if $X$ is an $H$-module, then

$$\mathrm{End}_{\mathcal{C}}(X) = \mathrm{End}_{\mathscr{D}^{\mathrm{b}}(H)}(X) \ltimes \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(H)}(X, \tau_D^{-1} X[1]),$$

the trivial extension of $\mathrm{End}_{\mathscr{D}^{\mathrm{b}}(H)}(X)$ by the bimodule $\mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(H)}(X, \tau_D^{-1} X[1])$ (see Proposition 1.5 in [10]).

Recall that, given an algebra $A$ and an $A$-$A$-bimodule $M$, the *trivial extension* of $A$ by $M$, denoted by $A \ltimes M$, is the algebra with the underlying $k$-module $A \oplus M$ and the multiplication given by

$$(a, m)(a', m') := (aa', am' + ma') \quad \text{for } a, a' \in A,\ m, m' \in M.$$

If $M = DA$, then $A \ltimes DA$ is simply called the *trivial extension of* $A$, denoted by $\mathbb{T}(A)$.

For further information on cluster-tilted algebras, we refer to [10], [47].

If $T$ is a cluster-tilting object in $\mathcal{C}$, then its endomorphism algebra $\mathrm{End}_{\mathcal{C}}(T)$ is called a *cluster-tilted algebra*. If $T$ is a basic tilting $H$-module, then $\mathrm{End}_{\mathcal{C}}(T)$ is a cluster-tilted algebra, and all cluster-tilted algebras are of this form.

Let $\tau_H$ (respectively, $\tau_H^-$) be the Auslander–Reiten translation $D\mathrm{Tr}$ (respectively, $\mathrm{Tr}D$) of the algebra $H$. Recall that modules in $\mathrm{add}\{\tau_H^{-i}H | i \geqslant 0\}$ are called *preprojective* modules, and modules in $\mathrm{add}\{\tau_H^i D(H) \mid i \geqslant 0\}$ are called *preinjective* modules. For a hereditary algebra $H$, $\tau_H = D\mathrm{Ext}_H^1(-, A)$.

**Proposition 5.5.** *All cluster-tilted algebras over an algebraically closed field are Frobenius-finite.*

*Proof.* Let $A$ be a cluster-tilted algebra. Then, without loss of generality, we can assume $A = \mathrm{End}_{\mathcal{C}}(T)$, where $T$ is a basic tilting module over a finite-dimensional, connected, hereditary $k$-algebra $H$ with $k$ an algebraically closed field. If $H$ is of Dynkin type, then $A$ is representation-finite and, of course, Frobenius-finite.

From now on, we assume that $H$ is representation-infinite. Using a method similar to the one in the proof of Lemma 1 in [47], we deduce that the associated self-injective algebra of $A$ is isomorphic to $\mathrm{End}_{\mathcal{C}}(T')$ where $T'$ is a maximal direct summand of $T$ with $\tau_{\mathcal{C}}^2 T' \simeq T'$ in $\mathcal{C}$. By the above fact (a), the objects $\tau_D^2 T'$ and $T'$ are isomorphic in $\mathcal{C}$. Suppose that $T'$ has a decomposition $T' = U \oplus M \oplus E$ such that $U$ is preprojective, $M$ is regular and $E$ is preinjective. For each projective $H$-module $P$, we have an Auslander–Reiten triangle

$$\nu_H P[-1] \longrightarrow V \longrightarrow P \longrightarrow \nu_H P$$

in $\mathscr{D}^{\mathrm{b}}(H)$, which shows $\tau_D P = \nu_H P[-1]$. Thus $\tau_D^2 P$, which is just $\tau_D(\nu_H P)[-1]$, is isomorphic in $\mathcal{C}$ to $\nu_H P$ since $\mathcal{C}$ is the orbit category of $\mathscr{D}^{\mathrm{b}}(H)$ with respect to the auto-equivalence functor $\tau_D^{-1}[1]$. As $H$ is representation-infinite, the object $\tau_D^i(\nu_H P)$ for each $i > 0$ is isomorphic in $\mathscr{D}^{\mathrm{b}}(H)$ to $\tau_H^i(\nu_H P)$ which is a preinjective $H$-module. Hence $\tau_D^m P$ is isomorphic in $\mathcal{C}$ to a preinjective $H$-module for all $m \geqslant 2$. It follows that, for each preprojective $H$-module $V$, the object $\tau_D^n V$ is isomorphic in $\mathcal{C}$ to a preinjective module if $n$ is big enough. Applying $\tau_D$ to a regular (preinjective, respectively) $H$-module always results in a regular (preinjective, respectively) $H$-module. Thus, by applying $\tau_D^{2n}$ with $n$ large enough, $\tau_D^{2n} T' = \tau_D^{2n} U \oplus \tau_D^{2n} M \oplus \tau_D^{2n} E$ is isomorphic in $\mathcal{C}$ to an $H$-module $T''$ which has no preprojective direct summands. Hence $T'$ and $T''$ are isomorphic in $\mathcal{C}$. By the fact (b), $T'$ and $T''$ are isomorphic in $\mathscr{D}^{\mathrm{b}}(H)$, and therefore they are also isomorphic as $H$-modules. However, $T''$ has no preprojective direct summands. This forces $U = 0$. Dually, one can prove $E = 0$. Hence $T'$ is actually a regular $H$-module. In this case, $\tau_D^2 T'$ is just $\tau_H^2 T'$. By the fact (b) again, $\tau_H^2 T'$ and $T'$ are isomorphic in $\mathscr{D}^{\mathrm{b}}(H)$, and consequently $\tau_H^2 T' \simeq T'$ as $H$-modules.

If $H$ is wild, then there are not any $\tau_H$-periodic $H$-modules at all. Hence $T' = 0$ and $A$ is a Frobenius-free algebra. If $H$ is tame, then we have the following isomorphisms of algebras:

$$\begin{aligned}
\mathrm{End}_{\mathcal{C}}(T') &= \mathrm{End}_{\mathscr{D}^{\mathrm{b}}(H)}(T') \ltimes \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(H)}(T', \tau_D^{-1} T'[1]) \quad \text{(by the fact (c) above)}\\
&\simeq \mathrm{End}_H(T') \ltimes \mathrm{Ext}^1_H(T', \tau_H^{-1} T')\\
&\simeq \mathrm{End}_H(T') \ltimes D\mathrm{Hom}_H(\tau_H^{-1} T', \tau_H T') \quad \text{(by Auslander–Reiten formula)}\\
&\simeq \mathrm{End}_H(T') \ltimes D\mathrm{Hom}_H(T', \tau_H^2 T') \simeq \mathrm{End}_H(T') \ltimes D\mathrm{Hom}_H(T', T')\\
&= \mathrm{End}_H(T') \ltimes D\mathrm{End}_H(T') = \mathbb{T}(\mathrm{End}_H(T')),
\end{aligned}$$

where the Auslander–Reiten formula means the isomorphism $D\mathrm{Ext}^1_A(X, Y) \simeq \overline{\mathrm{Hom}}_A(Y, \tau_A(X))$ for all $X, Y \in A$-mod (see Proposition 4.6, p. 131, in [4]).

We claim that $\mathbb{T}(\mathrm{End}_H(T'))$ is representation-finite. Since $T$ is a tilting module over the tame hereditary algebra $H$, it must contain either an indecomposable preprojective or preinjective summand (see, for example, the proof of Lemma 3.1 in [18]). Thus there is an integer $n$ with $|n|$ minimal, such that $\tau_H^n T$ has a nonzero projective or injective direct summand. Assume that $\tau_H^n T \simeq He \oplus X$ for some idempotent $e$ in $H$ and that $X$ has no projective direct summands. Then $\tau_H X$ is a tilting $H/HeH$-module. Thus $\mathrm{End}_H(X) \simeq \mathrm{End}_H(\tau_H X)$ is a tilted algebra of Dynkin type (not necessarily connected), and consequently its trivial extension $\mathbb{T}(\mathrm{End}_H(X))$ is representation-finite (see [17], Chapter V). Since $T'$ is $\tau_H$-periodic, $\tau_H^n T'$ has to be a direct summand of $X$. Thus, $\mathrm{End}_H(T') \simeq \mathrm{End}_H(\tau_H^n T')$ is isomorphic to $f\mathrm{End}_H(X)f$ for some idempotent $f$ in $\mathrm{End}_H(X)$. Hence $\mathbb{T}(\mathrm{End}_H(T'))$ is isomorphic to $f\mathbb{T}(\mathrm{End}_H(X))f$, and therefore representation-finite. When $\tau_H^n T$ contains an injective direct summand, the proof can be proceeded similarly. $\square$

### 5.2. Proof of Theorem 1.1

Throughout this subsection, $k$ denotes an algebraically closed field. The main idea of the proof of Theorem 1.1 is to use Theorem 1.2 inductively. The following lemma is crucial to the induction procedure.

**Lemma 5.6.** *Let $A$ and $B$ be two representation-finite, self-injective $k$-algebras without nonzero semisimple direct summands. Suppose that $\Phi\colon A\text{-}\underline{\mathrm{mod}} \to B\text{-}\underline{\mathrm{mod}}$ is a stable equivalence of Morita type. Then there are a simple $A$-module $X$ and integers $r$ and $t$ such that $\tau^r \circ \Omega^t \circ \Phi(X)$ is isomorphic in $B\text{-}\underline{\mathrm{mod}}$ to a simple $B$-module, where $\tau$ and $\Omega$ stand for the Auslander–Reiten translation and Heller operator, respectively.*

*Proof.* Let $\Gamma_s(A)$ denote the stable Auslander–Reiten quiver of $A$ which has isomorphism classes of non-projective indecomposable $A$-modules as vertices and irreducible maps as arrows. Then $\Gamma_s(A)$ and $\Gamma_s(B)$ are isomorphic as translation quivers. By [29], we may assume that the algebras $A$ and $B$ are indecomposable. Then $\Gamma_s(A)$ and $\Gamma_s(B)$ are of the form $\mathbb{Z}\Delta/G$ for some Dynkin graph $\Delta = A_n, D_n (n \geqslant 4), E_n (n = 6, 7, 8)$ and a non-trivial admissible automorphism group $G$ of $\mathbb{Z}\Delta$ (see [44]). We fix an isomorphism $s_A : \mathbb{Z}\Delta/G \to \Gamma_s(A)$, and set

$$\pi_A : \mathbb{Z}\Delta \xrightarrow{\text{can}} \mathbb{Z}\Delta/G \xrightarrow{s_A} \Gamma_s(A).$$

Then $\pi_A$ is a covering map of translation quivers (see [44]). Now we fix some automorphisms of these translation quivers.

- The Heller operator $\Omega_A$ gives rise to an automorphism $\omega_A : \Gamma_s(A) \to \Gamma_s(A)$.
- The translation $\tau_A$ gives rise to an automorphism $\tau_A : \Gamma_s(A) \to \Gamma_s(A)$.

Similarly, we have:

- Two automorphisms $\omega_B$ and $\tau_B : \Gamma_s(B) \to \Gamma_s(B)$.
- The functor $\Phi$ induces an isomorphism $\phi : \Gamma_s(A) \to \Gamma_s(B)$.

Since the stable equivalence $\Phi$ is of Morita type, $\tau_A\phi = \phi\tau_B$ and $\omega_A\phi = \phi\omega_B$. Let $\pi_B := \pi_A\phi$. Then $\pi_B$ is also a covering map.

Let $\Delta$ be a Dynkin diagram of $n$ vertices. For the vertices of $\mathbb{Z}\Delta$, we use the coordinates $(s, t)$ with $1 \leqslant t \leqslant n$ as described in Fig. 1 of [6]. A vertex $(p, 1)$ with $p \in \mathbb{Z}$ is called a *bottom vertex*. The vertices $(p, n)$ in $\mathbb{Z}A_n$ and $(p, 5)$ in $\mathbb{Z}E_6$ with $p \in \mathbb{Z}$ are called *top vertices*.

By definition, $\tau_\Delta \colon (p, q) \mapsto (p-1, q)$ is a translation on $\mathbb{Z}\Delta$ and all homomorphisms of translation quivers commute with this translation. Clearly, $\tau_\Delta$ is an admissible automorphism of $\mathbb{Z}\Delta$. The automorphism $\omega_A$ can be lifted to an admissible automorphism $\omega_\Delta$ of $\mathbb{Z}\Delta$ such that $\pi_A\omega_A = \omega_\Delta\pi_A$. For instance, if $\Delta = A_n$, then $\omega_{A_n}(p, q) = (p + q - n, n + 1 - q)$ (see [23], Section 4). For $\Delta = E_6$, one may use a method in 4.4 of [23] and easily get $\omega_{E_6}(p, q) = (p + q - 6, 6 - q)$ for $q \neq 6$ and $\omega_{E_6}(p, 6) = (p - 6, 6)$. Note that the method in 4.4 of [23] does not depend on higher Auslander–Reiten theory and its main ingredients are actually the Auslander–Reiten formula and ordinary Auslander–Reiten theory. Thus, for $\Delta = A_n$ or $E_6$, the automorphism $\omega_\Delta$ interchanges top vertices and bottom vertices.

Let $\mathcal{S}_A$ and $\mathcal{S}_B$ be the complete sets of isomorphism classes of simple modules over $A$ and $B$, respectively. Define $\mathscr{C}_A := \{x \in \mathbb{Z}\Delta \mid (x)\pi_A \in \mathcal{S}_A\}$ and $\mathscr{C}_B := \{x \in \mathbb{Z}\Delta \mid (x)\pi_B \in \mathcal{S}_B\}$. Since $\pi_A$ and $\pi_B$ are covering maps, $\mathscr{C}_A$ and $\mathscr{C}_B$ are "configurations" on $\mathbb{Z}\Delta$ by Propositions 2.3 and 2.4 in [45]). For the precise definition of configurations, we refer the reader to [45]. Note that if $\mathscr{C}$ is a configuration on $\mathbb{Z}\Delta$, then so is the image $(\mathscr{C})g$ for any admissible automorphism $g$ of $\mathbb{Z}\Delta$. In particular, $(\mathscr{C})\omega_\Delta$ and $(\mathscr{C})\tau_\Delta$ are configurations for all configurations $\mathscr{C}$.

*Claim* 1. Each configuration $\mathscr{C}$ on $\mathbb{Z}A_n$ contains either a top vertex or a bottom vertex.

*Proof.* Recall from Proposition 2.6 in [45] that there is a bijection between the configurations on $\mathbb{Z}A_n$ and the partitions $\sigma$ of the vertices of the regular $n$-polygon such that the convex hulls of different parts of $\sigma$ are disjoint. For such a partition $\sigma$, either there is a part consisting of a single vertex, or there is a part containing two adjoint vertices. Due to the bijection of Proposition 2.6 in [45], we see that in the former case, the corresponding configuration contains a vertex $(i,n)$ for some integer $i$, and in the latter case, the corresponding configuration contains $(j,1)$ for some integer $j$. $\square$

*Claim* 2. Let $\mathscr{C}$ be a configuration on $\mathbb{Z}\Delta$ with $\Delta = A_n, D_n (n \geqslant 4), E_6, E_7$ or $E_8$. Then either $\mathscr{C}$ or $(\mathscr{C})\omega_\Delta$ contains a bottom vertex.

*Proof.* We verify the statement in several cases.

(a) $\Delta = A_n$. Since $\omega_{A_n}$ maps top vertices to bottom vertices, Claim 2 follows from Claim 1.

(b) $\Delta = D_n$. The statement for $\mathbb{Z}D_4$ follows directly from 7.6 in [6]. Suppose $n \geqslant 5$. For $m \leqslant n-2$, let $\psi_m : \mathbb{Z}A_m \to \mathbb{Z}D_n$ be the embedding defined in Section 6 of [46]. By definition, $\psi_m$ maps all top and bottom vertices of $\mathbb{Z}A_m$ to bottom vertices of $\mathbb{Z}D_n$. By the two propositions in Section 6 of [46], each configuration on $\mathbb{Z}D_n$ contains the image of some configuration on $\mathbb{Z}A_r$ under $\tau_{D_n}^t \psi_r$ for some $0 < r \leqslant n-2$ and $t \in \mathbb{Z}$. Together with Claim 1, this implies that each configuration on $\mathbb{Z}D_n$ with $n \geqslant 5$ contains at least one bottom vertex.

(c) $\Delta = E_6$. Note that $\omega_{E_6}$ maps top vertices to bottom vertices, and all the automorphisms of $\mathbb{Z}E_6$ are of the form $\tau_\Delta^s \omega_\Delta$ for some integer $s$ (see [44]). Thus, the claim for $E_6$ follows from the list of isomorphism classes of configurations given in Section 8 of [6]

(d) $\Delta = E_7$ or $E_8$. All the automorphisms of $\mathbb{Z}E_7$ and $\mathbb{Z}E_8$ are of the form $\tau_\Delta^s$ for some integer $s$. The claim then follows by checking on the list of isomorphism classes of configurations on $\mathbb{Z}E_7$ and $\mathbb{Z}E_8$ given in Section 8 of [6]. $\square$

By Claim 2, we can assume that $(\mathscr{C}_A)\omega_\Delta^a$ contains a bottom vertex $(r_1, 1)$ and that $(\mathscr{C}_B)\omega_\Delta^b$ contains a bottom vertex $(r_2, 1)$, where $a$ and $b$ are taken from $\{0, 1\}$. Let $x \in \mathscr{C}_A$ such that $(x)\omega_\Delta^a = (r_1, 1)$. Then $(x)\omega_\Delta^a \tau_\Delta^{(r_1 - r_2)} = (r_2, 1)$, and

$$y := (x)\omega_\Delta^{(a-b)} \tau_\Delta^{(r_1-r_2)} = (x)\omega_\Delta^a \tau_\Delta^{(r_1-r_2)} \omega_\Delta^{-b} = (r_2, 1)\omega_\Delta^{-b} \in \mathscr{C}_B$$

Let $r = r_1 - r_2$ and $t = a - b$. Then

$$(x)\pi_A \phi \omega_B^t \tau_B^r = (x)\pi_A \omega_A^t \tau_A^r \phi = (x)\omega_\Delta^t \tau_\Delta^r \pi_A \phi = (y)\pi_B.$$

Thus the simple $A$-module $X := (x)\pi_A$ is sent to the simple $B$-module $Y := (y)\pi_B$ by the functor $\tau_B^r \circ \Omega_B^t \circ \Phi$ in $B$-$\underline{\mathrm{mod}}$. This finishes the proof of Lemma 5.6. $\square$

It would be nice to have a homological proof of Lemma 5.6.

We have now accumulated all information necessary to prove the main result, Theorem 1.1.

*Proof of Theorem* 1.1. We first note that, for self-injective algebras, both $\tau$ and $\Omega$ are stable equivalences of Morita type, and can be lifted to standard derived equivalences. Actually, given a self-injective algebra $\Lambda$, the stable equivalences $\Omega$ and $\tau$ lift to the standard derived equivalences given by $\Lambda[-1] \otimes_\Lambda^{\mathbf{L}} -$ and $D(\Lambda)[-2] \otimes_\Lambda^{\mathbf{L}} -$, respectively. This in turn implies that $\Omega$ and $\tau$ are stable equivalences of Morita type.

Let $\Phi\colon A\text{-}\underline{\mathrm{mod}} \to B\text{-}\underline{\mathrm{mod}}$ be a stable equivalence of Morita type, and let $P$ and $Q$ be the bimodules belonging to $\Phi$. Suppose that $\Delta_A$ and $\Delta_B$ are the basic, associated self-injective algebras of $A$ and $B$, respectively. By the definition of associated self-injective algebras and Lemma 2.7 (4), the algebras $\Delta_A$ and $\Delta_B$ have no nonzero semisimple direct summands. It follows from Theorem 4.2 in [15] that $\Phi$ can be restricted to a stable equivalence $\Phi_1 : \Delta_A\text{-}\underline{\mathrm{mod}} \to \Delta_B\text{-}\underline{\mathrm{mod}}$ of Morita type. By Corollary 4.7, the stable equivalence $\Phi$ lifts to an almost $\nu$-stable derived equivalence provided that $\Phi_1$ lifts to a derived equivalence.

If $\Delta_A = 0$, then it follows from Lemma 3.1 (2) that $P = 0$ and $Q = 0$. Thus, by Proposition 3.3, $\Phi$ is a Morita equivalence between $A$ and $B$, and therefore Theorem 1.1 follows.

Assume $\Delta_A \neq 0$. Then, by Lemma 5.6, there are integers $r$ and $s$ such that the functor $\tau^r\Omega^s\Phi_1 : \Delta_A\text{-}\underline{\mathrm{mod}} \to \Delta_B\text{-}\underline{\mathrm{mod}}$ sends some simple $\Delta_A$-module to a simple $\Delta_B$-module. In this case, we can choose an idempotent element $e$ in $\Delta_A$ such that, for a simple $\Delta_A$-module $S$, the $\Delta_B$-module $\tau^r\Omega^s\Phi_1(S)$ is isomorphic in $\Delta_B\text{-}\underline{\mathrm{mod}}$ to a non-simple module if and only if $e \cdot S \neq 0$. Note that $\tau^r\Omega^s\Phi_1$ is a stable equivalence of Morita type between $\Delta_A$ and $\Delta_B$. Let $P'$ and $Q'$ be the bimodules belongs to $\tau^r\Omega^s\Phi_1$. It follows from Lemma 3.1 (2) and (5)-(6) that $\Delta_A e \in \mathrm{add}(P')$ and $e$ is $\nu$-stable. Similarly, we can choose a $\nu$-stable idempotent element $f$ in $\Delta_B$ such that, for a simple $\Delta_B$-module $T$, the $\Delta_A$-module $(\tau^r\Omega^s\Phi_1)^{-1}(T)$ is isomorphic in $\Delta_A\text{-}\underline{\mathrm{mod}}$ to a non-simple module if and only if $f \cdot T \neq 0$. Then, by Lemma 3.4, the equivalence $\tau^r\Omega^s\Phi_1$ is restricted to a stable equivalence $\Phi_2\colon e\Delta_A e\text{-}\underline{\mathrm{mod}} \to f\Delta_B f\text{-}\underline{\mathrm{mod}}$ of Morita type.

If $\Delta_A$ has only one non-isomorphic simple module, then $e = 0$ and $\tau^r\Omega^s\Phi_1$ sends every simple $\Delta_A$-module to simple $\Delta_B$-module, and therefore it is a Morita equivalence by Proposition 3.3 (see also Theorem 2.1 in [28]). Now, suppose that $\Delta_A$ has at least two non-isomorphic simple modules and $e \neq 0$. Clearly, the number of non-isomorphic simple modules of the algebra $e\Delta_A e$ is less than the one of $\Delta_A$. Since $e\Delta_A e$ and $f\Delta_B f$ are again representation-finite, self-injective algebras without nonzero semisimple direct summands, we can assume, by induction, that $\Phi_2$ lifts to a derived equivalence. Thus, by Theorem 1.2, the stable equivalence $\tau^r\Omega^s\Phi_1$ lifts to a derived equivalence. Since both $\tau$ and $\Omega$ lift to derived equivalences between the self-injective algebras $\Delta_A$ and $\Delta_B$, $\Phi_1$ lifts to a derived equivalence. Hence $\Phi$ lifts to an almost $\nu$-stable derived equivalence. □

Since derived equivalences preserve the number of simple modules and since stable equivalences of Morita type between algebras without semisimple summands preserve the number of projective simples, the Auslander–Reiten conjecture is true for Frobenius-finite algebras over an algebraically closed field by Theorem 1.1. This also follows from [32].

Let $k$ be an algebraically closed field. For standard representation-finite self-injective $k$-algebras $A$ and $B$ not of type $(D_{3m}, s/3, 1)$ with $m \geqslant 2$ and $3 \nmid s$, Asashiba proved in [3] that each individual stable equivalence between $A$ and $B$ over an algebraically closed field lifts to a derived equivalence. His proof is done case by case, and depends on his derived equivalence classification of standard representation-finite self-injective algebras (see [2]). Recently, Dugas treats the case left by Asashiba, again using Asashiba's derived classification together with a technique of tilting mutations (see [14]). From the works of Asashiba and Dugas, it follows that any stable equivalence between standard representation-finite self-injective $k$-algebras lifts to a standard derived equivalence and is of Morita type. In Theorem 1.1, we consider instead stable equivalences of Morita type and handle all Frobenius-finite algebras: first, applying Corollary 4.7 to reduce the lifting problem for general algebras to the one for representation-finite self-injective algebras, and then using the technical Lemma 5.6 and the inductive Theorem 1.2 to complete the proof by induction on the number of simple modules. So, our proof is independent of Asashiba's derived equivalence classification of standard representation-finite self-injective algebras, and simplifies both Asashiba's work and Dugas' proof.

Now, we state the following generalization of Asashiba's main result in [3].

**Corollary 5.7.** *If $A$ and $B$ are arbitrary representation-finite algebras over an algebraically closed field and without nonzero semisimple direct summands, then every stable equivalence of Morita type between $A$ and $B$ can be lifted to an iterated almost $\nu$-stable derived equivalence.*

Recall that a finite-dimensional algebra over a field is called an *Auslander algebra* if it has global dimension at most 2 and dominant dimension at least 2. Algebras of global dimension at most 2 seem to be of great interest in representation theory because they are quasi-hereditary (see [12]) and every finite-dimensional algebra (up to Morita equivalence) can be obtained from an algebra of global dimension 2 by a universal localization (see [36]).

Since Auslander algebras and cluster-tilted algebras are Frobenius-finite by Propositions 5.2 (1) and 5.5, we have the following immediate consequence of Theorem 1.1.

**Corollary 5.8.** *If $A$ and $B$ are Auslander algebras or cluster-tilted algebras over an algebraically closed field and without nonzero semisimple direct summands, then each individual stable equivalence of Morita type between $A$ and $B$ lifts to a derived equivalence.*

Finally, we mention the following result on trivial extensions and tensor products of algebras.

**Corollary 5.9.** *Suppose that $A, B, R$ and $S$ are Frobenius-finite algebras over an algebraically closed field and without nonzero semisimple direct summands. If $(A, B)$ and $(R, S)$ are two pairs of stably equivalent algebras of Morita type, then*

(1) *the trivial extensions of $A$ and $B$ are derived equivalent;*

(2) *the tensor products $A \otimes_k R$ and $B \otimes_k S$ are derived equivalent.*

*Proof.* By [40], [41], derived equivalences are preserved under taking trivial extensions and tensor products. So the corollary follows immediately from Theorem 1.1. □

## 6. A machinery for lifting stable to derived equivalences

In this section we give an inductive procedure for lifting a class of stable equivalences of Morita type to derived equivalences. With this machinery we recheck some known cases for which Broué's abelian defect group conjecture holds true, and simplify proofs given in [34], [26]. The machinery works as well for all examples in [37].

Given a finite group $G$ and a block $A$ of the group algebra $kG$ with defect group $D$, there is a unique block $B$ of the group algebra $kN_G(D)$ with defect group $D$, where $N_G(D)$ stands for the normalizer of $D$ in $G$, such that the restriction from $\mathscr{D}^b(A)$ to $\mathscr{D}^b(B)$ is faithful. This is the well-known Brauer correspondence, which provides a bijection between blocks $A$ of $kG$ with defect group $D$ and blocks $B$ of $kN_G(D)$ with defect group $D$. Broué's abelian defect group conjecture [8] asserts that if $D$ is abelian, then $A$ and $B$ are derived equivalent. The conjecture is verified in many cases, but still wide open (see Rouquier's survey [48]).

Stable equivalences of Morita type can be achieved in many cases in modular representation theory of finite groups. For instance, in the case of the defect group having the trivial intersection property. To be able to lift stable equivalences of Morita type to derived equivalences is important for instance in one approach, due to Rouquier [48], to Broué's abelian defect group conjecture. The general idea is as follows: To show that two block algebras $A$ and $B$ are derived equivalent, one may start with a known stable equivalence of Morita type between them and try to lift this stable equivalence to a derived equivalence, or find independently another self-injective algebra $C$ and a derived equivalence from $B$ to $C$ such that the composite of the stable equivalence from $A$ to $B$ with the induced stable equivalence from $B$ to $C$ either can be lifted to a derived equivalence or sends all simple modules to simple modules. In the later case, one gets a Morita equivalence between $A$ and $C$ by Theorem 2.1 in [27]. Thus, in both cases, one can obtain a derived equivalence between $A$ and $B$. For further information on this approach to and progresses on Broué's abelian defect group conjecture, the reader is referred to [37], [42], [48].

Let $A$ be an algebra, and let $\mathcal{S}_A$ be a complete set of pairwise non-isomorphic simple $A$-modules. For each simple $A$-module $V \in \mathcal{S}_A$, we fix a primitive idempotent element $e_V$ in $A$ with $e_V \cdot V \neq 0$, such that the idempotent elements $\{e_V \mid V \in \mathcal{S}_A\}$ are pairwise orthogonal. Thus, for any nonempty subset $\sigma$ of $\mathcal{S}_A$, the element $e_\sigma := \sum_{V\in\sigma} e_V$ is an idempotent element in $A$.

Theorem 1.2 and the proof of Theorem 1.1 suggest an inductive method to check whether a stable equivalence of Morita type can be lifted to a derived equivalence. The procedure can be described as follows:

**Assumption.** Let $\Phi\colon A\text{-}\underline{\mathrm{mod}} \to B\text{-}\underline{\mathrm{mod}}$ be a stable equivalence of Morita type between algebras $A$ and $B$ without nonzero semisimple direct summands. Suppose that $A/\mathrm{rad}(A)$ and $B/\mathrm{rad}(B)$ are separable.

**Step 1:** If there is a simple $A$-module $V$ such that $\Phi(V)$ is a simple $B$-module, then we set

$$\sigma := \{V \in \mathcal{S}_A \mid \Phi(V) \text{ is non-simple}\} \text{ and } \sigma' := \mathcal{S}_B \backslash \Phi(\mathcal{S}_A \backslash \sigma).$$

Case (i): $\sigma$ is empty. Then $\Phi$ lifts to a Morita equivalence, and therefore our procedure terminates.

Case (ii): Both $\sigma$ and $\sigma'$ are nonempty. By Lemma 3.4, the functor $\Phi$ is restricted to a stable equivalence $\Phi_1$ of Morita type between $e_\sigma Ae_\sigma$ and $e_{\sigma'}Be_{\sigma'}$. Moreover, the idempotent elements $e_\sigma$ and $e_{\sigma'}$ are $\nu$-stable. In fact, by (5)-(6) of Lemma 3.1, for each $V$ in $\mathcal{S}_A$, the $B$-module $\Phi(V)$ is not simple if and only if $\mathrm{Hom}_A({}_AP, V) \neq 0$, or equivalently, $V \in \mathrm{add}(\mathrm{top}({}_AP))$, where $P$ is given by the definition of the stable equivalence $\Phi$ of Morita type. This implies $\mathrm{add}(Ae_\sigma) = \mathrm{add}({}_AP)$. It follows from Lemma 3.1 (2) that $e_\sigma$ is $\nu_A$-stable. Similarly, $e_{\sigma'}$ is $\nu_B$-stable. By Lemma 2.7 (3), the algebras $e_\sigma Ae_\sigma$ and $e_{\sigma'}Be_{\sigma'}$ are self-injective with fewer simple modules. So, to lift $\Phi$ to a derived equivalence, it is enough to lift $\Phi_1$ by Theorem 1.2.

**Step 2:** If there is a stable equivalence $\Xi : e_{\sigma'}Be_{\sigma'}\text{-}\underline{\mathrm{mod}} \to C\text{-}\underline{\mathrm{mod}}$ of Morita type between the algebra $e_{\sigma'}Be_{\sigma'}$ and another algebra $C$ (to be found independently), such that the stable equivalence is induced by a derived equivalence and the composite $\Xi \circ \Phi_1$ sends some (not necessarily all) simple $e_\sigma Ae_\sigma$-modules to simple $C$-modules, then we go back to Step 1 and consider the lifting problem for $\Xi \circ \Phi_1$. Once we arrive at representation-finite algebras in the procedure, Theorem 1.1 can be applied. This implies that $\Phi_1$ lifts to a derived equivalence, and therefore so does the given $\Phi$.

This procedure is somewhat similar to, but different from the method of Okuyama in [37]. In our procedure, Step 1 always reduces the number of simple modules and makes situations considered easier after each step if $\sigma$ is not the set of all simple modules. Particularly, one may often get representation-finite algebras, while the procedure in [37] does not change the number of simple modules and cannot get any representation-finite algebras if the procedure starts with representation-infinite algebras. If $\sigma$ is the whole set of all non-isomorphic simple modules, then Step 1 cannot run and does not give any help for lifting problem. In this case, one passes to Step 2. To do Step 2, one needs pieces of information independently. Nevertheless, Step 2 does not require that $\Xi\circ\Phi_1$ sends *all* simples to simples, while this is needed in [37] and other approaches.

In the following, we will illustrate the above procedure by examples.

**Example 1.** In [34], it was proved that Broué's abelian defect group conjecture is true for the faithful 3-blocks of defect 2 of $4.M_{22}$, which is the non-split central extension of the sporadic simple group $M_{22}$ by a cyclic group of order 4. Now we shall show that the procedure described above can be used to give a short proof of the conjecture in this case, which avoids many technical calculations, compared with the original proof in [34].

It is known that each of the two block algebras $B_+$ and $b_+$ has 5 simple modules. The simple $B_+$-modules are labeled by $56a, 56b, 64, 160a, 160b$, and the simple $b_+$-modules are labeled by $1a, 1b, 2, 1c$ and $1d$. There is a stable equivalence

$$\Phi\colon B_+\text{-}\underline{\mathrm{mod}} \longrightarrow b_+\text{-}\underline{\mathrm{mod}}$$

of Morita type (see [34]) such that

$$\Phi(56a) = \Omega^{-1}(1a), \quad \Phi(56b) = \Omega(1b), \quad \Phi(160a) = 1c, \quad \Phi(160b) = 1d,$$

and $\Phi(64)$ has the following Loewy structure:

$$\begin{bmatrix} 1b \\ 2 \\ 1a \end{bmatrix}.$$

For $x \in \{a,b,c,d\}$ and $\{y,y',y''\} = \{a,b,c,d\}\backslash\{x\}$, the Loewy structures of the projective $b_+$-modules are

$$P(1x) : \begin{bmatrix} 1x \\ 2 \\ 1y\ 1y'\ 1y'' \\ 2 \\ 1x \end{bmatrix}, \quad P(2) : \begin{bmatrix} 2 \\ 1a\ 1b\ 1c\ 1d \\ 2\ 2\ 2 \\ 1a\ 1b\ 1c\ 1d \\ 2 \end{bmatrix}.$$

Now, we use Steps 1 and 2 repeatedly and verify that the stable equivalence $\Phi$ lifts to a derived equivalence.

Note that $\Phi$ sends the simple module $160b$ to a simple module. So we can use Step 1. Let $\sigma = \{56a, 56b, 64\}$, and $\sigma' = \{1a, 1b, 2\}$. Then $\Phi$ can be restricted to a stable equivalence of Morita type

$$\Phi_1 : e_\sigma B_+ e_\sigma\text{-}\underline{\mathrm{mod}} \longrightarrow e_{\sigma'} b_+ e_{\sigma'}\text{-}\underline{\mathrm{mod}}.$$

The Loewy structures of the projective $e_{\sigma'} b_+ e_{\sigma'}$-modules $e_{\sigma'}P(1a)$ and $e_{\sigma'}P(1b)$ are

$$e_{\sigma'}P(1a) : \begin{bmatrix} 1a \\ 2 \\ 1b \\ 2 \\ 1a \end{bmatrix}, \quad \text{and} \quad e_{\sigma'}P(1b) : \begin{bmatrix} 1b \\ 2 \\ 1a \\ 2 \\ 1b \end{bmatrix}.$$

The images of the simple modules under $\Phi_1$ are

$$\Phi_1(56a) \simeq \begin{bmatrix} 1a \\ 2 \\ 1b \\ 2 \end{bmatrix}, \Phi_1(56b) \simeq \begin{bmatrix} 2 \\ 1a \\ 2 \\ 1b \end{bmatrix}, \quad \text{and} \quad \Phi_1(64) \simeq \begin{bmatrix} 1b \\ 2 \\ 1a \end{bmatrix}.$$

By [37], the idempotent $e = e_{1a} + e_{1b}$ defines a tilting complex $T^\bullet$ over $e_{\sigma'} b_+ e_{\sigma'}$. Now, setting $C := \mathrm{End}(T^\bullet)$ and labeling the simple $C$-modules by $1a, 1b$ and $2$, we see that the derived equivalence between $e_{\sigma'} b_+ e_{\sigma'}$ and $C$ induces a stable equivalence of Morita type $\Xi : e_{\sigma'} b_+ e_{\sigma'}\text{-}\underline{\mathrm{mod}} \to C\text{-}\underline{\mathrm{mod}}$ such that $\Xi(2) \simeq 2$, $\Xi(\left[\begin{smallmatrix} 1b \\ 2 \\ 1a \end{smallmatrix}\right]) \simeq 1b$, and $\Xi(\left[\begin{smallmatrix} 1a \\ 2 \\ 1b \end{smallmatrix}\right]) \simeq 1a$. Thus $\Xi\Phi_1(64) \simeq 1b$, $\Xi\Phi_1(56a) \simeq \left[\begin{smallmatrix} 1a \\ 2 \end{smallmatrix}\right]$ and $\Xi\Phi_1(56b) \simeq \left[\begin{smallmatrix} 2 \\ 1a \end{smallmatrix}\right]$. Let $\sigma_1 := \{56a, 56b\}$ and $\sigma_1' := \{1a, 2\}$. Then the composite $\Xi\Phi_1$ is restricted to a stable equivalence of Morita type

$$\Phi_2 : e_{\sigma_1} B_+ e_{\sigma_1}\text{-}\underline{\mathrm{mod}} \longrightarrow e_{\sigma_1'} C e_{\sigma_1'}\text{-}\underline{\mathrm{mod}}$$

such that $\Phi_2(56a) = \left[\begin{smallmatrix} 1a \\ 2 \end{smallmatrix}\right]$ and $\Phi_2(56b) = \left[\begin{smallmatrix} 2 \\ 1a \end{smallmatrix}\right]$. Note that the Cartan matrix of $e_{\sigma_1'} C e_{\sigma_1'}$ is $\left[\begin{smallmatrix} 2 & 1 \\ 1 & 3 \end{smallmatrix}\right]$. It is easy to check that a symmetric algebra with this Cartan matrix is always representation-finite. Thus $\Phi_2$ lifts to a derived equivalence by Theorem 1.1, and consequently $\Phi$ lifts to a derived equivalence. The whole procedure can be illustrated by the following commutative diagram

$$\begin{tikzcd}
B_+\text{-}\underline{\mathrm{mod}} \arrow[r, "\Phi"] & b_+\text{-}\underline{\mathrm{mod}} & \\
e_\sigma B_+ e_\sigma\text{-}\underline{\mathrm{mod}} \arrow[u, "\lambda"] \arrow[r, "\Phi_1"] & e_{\sigma'} b_+ e_{\sigma'}\text{-}\underline{\mathrm{mod}} \arrow[u, "\lambda"] \arrow[r, "\Xi"] & C\text{-}\underline{\mathrm{mod}} \\
e_{\sigma_1} B_+ e_{\sigma_1}\text{-}\underline{\mathrm{mod}} \arrow[u, "\lambda"] \arrow[rr, "\Phi_2"] & & e_{\sigma_1'} C e_{\sigma_1'}\text{-}\underline{\mathrm{mod}} \arrow[u, "\lambda"]
\end{tikzcd}$$

with $\Phi_2$ lifting to a derived equivalence.

**Example 2.** Let $G$ be the Harada–Norton simple group **HN**, and let $k$ be an algebraically closed field of characteristic 3. In [26], Broué's abelian defect group conjecture was verified for non-principal blocks of $kG$ with defect group $C_3 \times C_3$. In the following, we will show how our results can be applied to give another proof to the conjecture in this case. In fact, the two block algebras $A$ and $B$ have 7 non-isomorphic simple modules with $\mathcal{S}_A = \{1, 2, 3, 4, 5, 6, 7\}$ and $\mathcal{S}_B = \{9a, 9b, 9c, 9d, 18a, 18b, 18c\}$, and there is a stable equivalence $F\colon A\text{-}\underline{\mathrm{mod}} \to B\text{-}\underline{\mathrm{mod}}$ of Morita type such that

$$F(1) \simeq 9a, \quad F(2) \simeq 9b, \quad F(3) \simeq 9c, \quad F(4) \simeq \begin{array}{ccccc} & & 18a & & \\ & / & & \backslash & \\ 18b & & & & 18c \\ & \backslash & & / & \\ & & 18a & & \end{array},$$

$$F(5) \simeq \begin{array}{ccccc} & & 18c & & \\ & / & & \backslash & \\ 9a & & & & 9d \\ & \backslash & & / & \\ & & 18b & & \end{array}, \quad F(6) \simeq \begin{array}{ccccccc} & & 18a & & & & 9d \\ & / & & \backslash & & / & \\ 18c & & & & 18b & & \\ / & & \backslash & & / & & \\ 9d & & 18a & & & & \end{array}, \quad F(7) \simeq \begin{array}{ccccc} & & 18b & & \\ & / & & \backslash & \\ 9b & & & & 9c \\ & \backslash & & / & \\ & & 18c & & \end{array}.$$

The Loewy structures of the indecomposable projective $B$-modules $P(9d)$, $P(18a)$, $P(18b)$ and $P(18c)$ are as follows:

$$P(9d): \begin{bmatrix} 9d \\ 18b \\ 9c\ 18a \\ 18c \\ 9d \end{bmatrix}, \quad P(18a): \begin{array}{ccccc} & & 18a & & \\ & & & \backslash & \\ & 18b & & 18c & \\ & | & / & | & \\ 9b & 9c & 18a & 9a & 9d \\ & | & / & | & \\ & 18c & & 18b & \\ & & \backslash & & \\ & & 18a & & \end{array},$$

$$P(18b): \begin{bmatrix} 18b \\ 9b\ 18a\ 9c \\ 18c\ 18b\ 18c \\ 9a\ 18a\ 9d \\ 18b \end{bmatrix}, \quad P(18c): \begin{bmatrix} 18c \\ 9a\ 18a\ 9d \\ 18b\ 18c\ 18b \\ 9b\ 18a\ 9c \\ 18c \end{bmatrix}$$

Taking $\sigma = \{4,5,6,7\}$ and $\sigma' = \{9d, 18a, 18b, 18c\}$, we see from Step 1 that the functor $F$ is restricted to a stable equivalence of Morita type

$$F_1 : e_\sigma A e_\sigma\text{-}\underline{\mathrm{mod}} \longrightarrow e_{\sigma'} B e_{\sigma'}\text{-}\underline{\mathrm{mod}}$$

such that

$$F_1(4) \simeq \begin{array}{ccccc} & & 18a & & \\ & / & & \backslash & \\ 18b & & & & 18c \\ & \backslash & & / & \\ & & 18a & & \end{array}, \quad F_1(5) \simeq \begin{bmatrix} 18c \\ 9d \\ 18b \end{bmatrix},$$

$$F_1(6) \simeq \begin{array}{ccccccccc} & & & & 18a & & & & 9d \\ & & & / & & \backslash & & / & \\ & & 18c & & & & 18b & & \\ & / & & \backslash & & / & & & \\ 9d & & & & 18a & & & & \end{array}, \quad F_1(7) \simeq \begin{bmatrix} 18b \\ 18c \end{bmatrix}.$$

The idempotent element $e_{18a}$ in $B$ defines a tilting complex $T^\bullet$ over $e_{\sigma'} B e_{\sigma'}$ (see [37]). Set $C := \mathrm{End}(T^\bullet)$ and label simple $C$-modules by $9d, 18a, 18b$ and $18c$. Then the derived equivalence between $e_{\sigma'} B e_{\sigma'}$ and $C$ induces a stable equivalence of Morita type $\Xi : e_{\sigma'} B e_{\sigma'}\text{-}\underline{\mathrm{mod}} \to C\text{-}\underline{\mathrm{mod}}$ such that $\Xi(9d) \simeq 9d$, $\Xi(18b) \simeq 18b$, $\Xi(18c) \simeq 18c$, and $\Xi F_1(4) \simeq 18a$. Taking $\sigma_1 = \{5,6,7\}$ and $\sigma_1' = \{9d, 18b, 18c\}$, we see that the functor $\Xi F_1$ is restricted to a stable equivalence of Morita type

$$F_2 : e_{\sigma_1} A e_{\sigma_1}\text{-}\underline{\mathrm{mod}} \longrightarrow e_{\sigma_1'} C e_{\sigma_1'}\text{-}\underline{\mathrm{mod}}$$

such that $F_2(5) \simeq \left[\begin{smallmatrix} 18c \\ 9d \\ 18b \end{smallmatrix}\right]$, $F_2(6) \simeq \left[\begin{smallmatrix} 9d \\ 9d \end{smallmatrix}\right]$ and $F_2(7) \simeq \left[\begin{smallmatrix} 18b \\ 18c \end{smallmatrix}\right]$. Note that the Cartan matrix of $e_{\sigma_1'} C e_{\sigma_1'}$ is $\left[\begin{smallmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{smallmatrix}\right]$, where the columns are dimension vectors of the projective modules $e_{\sigma_1'} C e_{18b}$, $e_{\sigma_1'} C e_{18c}$ and $e_{\sigma_1'} C e_{9d}$, respectively. Then $F_2(5) \simeq \Omega^{-1}(18c)$. Thus, taking $\sigma_2 = \{6,7\}$ and $\sigma_2' = \{18b, 9d\}$, the functor $\Omega F_2$ can be restricted to a stable equivalence of Morita type

$$F_3 : e_{\sigma_2} A e_{\sigma_2}\text{-}\underline{\mathrm{mod}} \longrightarrow e_{\sigma_2'} C e_{\sigma_2'}\text{-}\underline{\mathrm{mod}}.$$

The Cartan matrix of $e_{\sigma_2'} C e_{\sigma_2'}$ is $\left[\begin{smallmatrix} 2 & 1 \\ 1 & 3 \end{smallmatrix}\right]$. This implies that $e_{\sigma_2'} C e_{\sigma_2'}$ is representation-finite and that $F_3$ lifts to a derived equivalence by Theorem 1.1. Hence $F$ lifts to a derived equivalence.

Finally, we point out that our methods work for all examples in [37] and can simplify Okuyama's proofs.

Let us end this section by mentioning the following questions suggested by our main results.

**Question 1.** Given a stable equivalence $\Phi$ of Morita type between two self-injective algebras such that $\Phi$ does not send any simple modules to simple modules, under which conditions can $\Phi$ be lifted to a derived equivalence?

**Question 2.** Find sufficient and necessary conditions for stable equivalences of Morita type between Frobenius-tame algebras to be lifted to derived equivalences.

**Question 3.** Find more methods to construct Frobenius-finite algebras, or sufficient conditions for algebras to be Frobenius-finite. For example, when is a cellular algebra Frobenius-finite? We guess that a cellular algebra is Frobenius-finite if and only if it is representation-finite.

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