Algebraic K-theory of endomorphism rings

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Abstract

We establish formulas for computation of the higher algebraic K-groups of the endomorphism rings of objects linked by a morphism in an additive category. Let $\mathcal{C}$ be an additive category, and let $Y \to X$ be a covariant morphism of objects in $\mathcal{C}$. Then $K_n\big(\mathrm{End}_{\mathcal{C}}(X\oplus Y)\big) \simeq K_n\big(\mathrm{End}_{\mathcal{C},Y}(X)\big)\oplus K_n\big(\mathrm{End}_{\mathcal{C}}(Y)\big)$ for all $1 \le n \in \mathbb{N}$, where $\mathrm{End}_{\mathcal{C},Y}(X)$ is the quotient ring of the endomorphism ring $\mathrm{End}_{\mathcal{C}}(X)$ of X modulo the ideal generated by all those endomorphisms of X which factorize through Y. Moreover, let R be a ring with identity, and let e be an idempotent element in R. If $J := ReR$ is homological and ${}_RJ$ has a finite projective resolution by finitely generated projective R-modules, then $K_n(R) \simeq K_n(R/J)\oplus K_n(eRe)$ for all $n \in \mathbb{N}$. This reduces calculations of the higher algebraic K-groups of R to those of the quotient ring R/J and the corner ring eRe, and can be applied to a large variety of rings: Standardly stratified rings, hereditary orders, affine cellular algebras and extended affine Hecke algebras of type $\tilde{A}$.

1 Introduction

Algebraic K-theory collects elaborate invariants for rings. One of the most fundamental and important questions in this theory is to understand and calculate these invariants: algebraic K-groups K_n of rings. Unfortunately, this question is so hard that, up to now, only a few rings have gotten their algebraic K-groups satisfactorily calculated (for example, see [20, 23] for details), though general, abstract algebraic K-theories have been explosively developed in the last a few decades. Thus, it becomes more reasonable to look at relationship between algebraic K-groups of different rings linked by certain equivalences, homomorphisms, or functors between their relevant categories (for example, see [10, 26, 18]). In this way, one may compute the algebraic K-groups of a ring through those of another ring. Hopefully, this might lead to some knowledge on comprehensive understanding of higher algebraic K-groups K_n for rings.

This paper is a continuation of the project in this direction started in [29] where the techniques of derived equivalences developed in the representation theory of algebras were used to calculate the algebraic K-groups of rings. More precisely, we employed $\mathcal{D}$-split sequences introduced in [13] to give formulas for computation of the higher algebraic K-groups of a class of rings including many maximal orders in noncommutative algebraic geometry and in arithmetical representations (see [4, 5, 21]). One of the key techniques used in [29] is to embed a given ring into a big ring which is projective as a module over the given ring. This method is powerful for the rings considered there. But, for a general ring, we do not know the existence of such an embedding. For instance, let A be a commutative ring with I an ideal in A, we do not know how to embed the matrix ring $R := \begin{pmatrix} A & I \\ I & A \end{pmatrix}$ into a ring S such that ${}_RS$ is a finitely generated projective module and that the algebraic K-groups of R can be computed through those of S. Rings of matrix form are of importance because, for instance, they are the essential ingredients in the study of canonical singularities and minimal model program for orders over surfaces (see [4, 5]), and of Hecke orders for integral representations of finite groups (see [22, 7]). Thus, it would be interesting to know the K-theory of this kind of rings. This motivates us to consider the following general question (see [29]):

Question. *Suppose that I and J are two arbitrary ideals in a ring R with identity. For the ring $S := \begin{pmatrix} R & I \\ J & R \end{pmatrix}$, can one give a formula for $K_n(S)$ in terms of K_n-groups of quotient rings of R by ideals produced from I and J ?*

In this paper, we will consider, more generally, algebraic K-theory of the endomorphism rings of objects in a additive category, which are linked by a morphism. As a byproduct of our consideration, we get partial answers to the above-mentioned question. Our idea in this paper is again to use representation-theoretic methods for developing general results for calculations of the higher algebraic K-groups of rings which include particularly some rings mentioned above and cover also many other interesting classes of rings such as standardly stratified rings in the representation theory of semisimple Lie algebras and algebraic groups (see [7]), hereditary orders (see [21, 22]), affine cellular algebras and extended affine Hecke algebras of type $\tilde{A}$ in quantum groups and cell theory (see [16,

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14]). Two key ingredients of our proofs in this paper are recollements of triangulated categories in [2] and the Thomason-Waldhausen Localization Theorem which is due to Thomason [26, 1.9.8, 1.8.2] based on the work of Waldhausen [27].

Before stating our main results precisely, we first introduce the notion of covariant morphisms in an additive category (see Section 3 for more details).

Let $\mathcal{C}$ be an additive category, and let X,Y be objects in $\mathcal{C}$. A morphism $\lambda : Y \to X$ in $\mathcal{C}$ is said to be X-*covariant* if the induced map $\mathrm{Hom}_{\mathcal{C}}(X,\lambda) : \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X,X)$ is a split monomorphism of $\mathrm{End}_{\mathcal{C}}(X)$-modules; and *covariant* if the induced map $\mathrm{Hom}_{\mathcal{C}}(X,\lambda) : \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X,X)$ is injective and the induced map $\mathrm{Hom}_{\mathcal{C}}(Y,\lambda) : \mathrm{Hom}_{\mathcal{C}}(Y,Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(Y,X)$ is a split epimorphism of $\mathrm{End}_{\mathcal{C}}(Y)$-modules. For example, if $\mathcal{C}$ is the module category of a unitary ring R and if X is an R-module, then, for every submodule Y of X with $\mathrm{Hom}_R(Y,X/Y) = 0$, the inclusion map is covariant, and for an idempotent ideal I in R, the inclusion from I into R is also covariant. Note that covariant homomorphisms arise also from Auslander-Reiten sequences and GV-ideals.

Let $\mathrm{End}_{\mathcal{C},Y}(X)$ denote the quotient ring of the endomorphism ring $\mathrm{End}_{\mathcal{C}}(X)$ of the object X modulo the ideal generated by all those endomorphisms of X which factorize through the object Y. For example, if I is an idempitent ideal in a ring R with identity, then $\mathrm{End}_{R,I}(R) \simeq R/I$.

Recall that an ideal J in a ring R is said to be *homological* if $\mathrm{Tor}_j^R(R/J,R/J) = 0$ for all $j > 0$, and that an R-module M has a *finite-type resolution* if it has a finite projective resolution by finitely generated projective R-modules, that is, there is an exact sequence $0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$ for some $n \in \mathbb{N}$ such that all R-modules P_j are projective and finitely generated.

Throughout this paper, we denote by $K_n(R)$ the n-th algebraic K-group of a ring R in the sense of Quillen.

Our main results in this paper are the following theorems in which Theorem 1.1 is, in some sense, a replacement of the excision theorem of algebraic K-theory of rings with idempotent ideals. That is, it establishes a relationship between algebraic K-groups of rings linked by a special surjection.

Theorem 1.1. *Let R be a ring with identity, and I an ideal of R.*

(1) *If $I^2 = I$, then the K-theory space of $\mathrm{End}_R(R\oplus I)$ is homotopy equivalent to the product of the K-theory spaces of R/I and $\mathrm{End}_R(I)$, and therefore $K_n\big(\mathrm{End}_R(R\oplus I)\big) \simeq K_n(R/I)\oplus K_n\big(\mathrm{End}_R(I)\big)$ for all $n\in\mathbb{N}$. In particular, if the idempotent ideal I is projective and finitely generated as a left R-module, then $K_n(R) \simeq K_n(R/I)\oplus K_n\big(\mathrm{End}_R(I)\big)$ for all $n \in \mathbb{N}$.*

(2) *If $I = ReR$ for $e^2 = e \in R$ such that I is homological and has a finite-type resolution as a left R-module, then the K-theory space of R is homotopy equivalent to the product of the K-theory spaces of eRe and R/I, and therefore $K_n(R) \simeq K_n\big(eRe\big)\oplus K_n(R/I)$ for all $n \in \mathbb{N}$.*

In Theorem 1.1, if we assume instead all conditions for right R-modules (for example, in Theorem 1.1 (1), assume that I_R is a finitely generated projective right R-module), then the conclusion is still true because $K_n(R) \simeq K_n(R^{\mathrm{op}})$ for all $n \in \mathbb{N}$, where R^{op} is the opposite ring of R.

One cannot replace "projective" in the second statement of Theorem 1.1 (1) with "of finite projective dimension". Also, the condition that I is idempotent in Theorem 1.1 (1) cannot be dropped (see the examples in the last section).

As a consequence of Theorem 1.1, we have the following result.

Theorem 1.2. *Let $\mathcal{C}$ be an additive category and $f : Y \to X$ a morphism of objects in $\mathcal{C}$.*

(1) *If f is covariant, then the K-theory space of $\mathrm{End}_{\mathcal{C}}(X\oplus Y)$ is homotopy equivalent to the product of the K-theory spaces of $\mathrm{End}_{\mathcal{C},Y}(X)$ and $\mathrm{End}_{\mathcal{C}}(Y)$. In particular, $K_n\big(\mathrm{End}_{\mathcal{C}}(X\oplus Y)\big) \simeq K_n\big(\mathrm{End}_{\mathcal{C},Y}(X)\big)\oplus K_n\big(\mathrm{End}_{\mathcal{C}}(Y)\big)$ for all $n \in \mathbb{N}$.*

(2) *If f is X-covariant, then the K-theory space of $\mathrm{End}_{\mathcal{C}}(X\oplus Y)$ is homotopy equivalent to the product of the K-theory spaces of $\mathrm{End}_{\mathcal{C}}(X)$ and $\mathrm{End}_{\mathcal{C},X}(Y)$. In particular, $K_n\big(\mathrm{End}_{\mathcal{C}}(X\oplus Y)\big) \simeq K_n\big(\mathrm{End}_{\mathcal{C}}(X)\big)\oplus K_n\big(\mathrm{End}_{\mathcal{C},X}(Y)\big)$ for all $n \in \mathbb{N}$.*

For the dual statement of Theorem 1.2, we refer the reader to Theorem 3.8 in Section 3.

As is known, the excision theorem in algebraic K-theory of rings gives a relationship of algebraic K-groups for rings linked by a surjective ring homomorphism (see [25]). Similarly, Theorem 1.2 describes a relationship of the algebraic K-groups of the endomorphism rings of objects linked by a morphism in an additive category.

Clearly, we can apply Theorem 1.1 to standardly stratified rings (see Section 4 for definition) and get a reduction formula for algebraic K-groups of this class of rings. It is worth noting that ideals with the property mentioned in Theorem 1.1 (1) occur also frequently in matrix subrings.

As an application of Theorem 1.1, we have the following corollary which provides a partial answer to the above question and extends [29, Theorem 1.1 (1)].

Corollary 1.3. *Let R be a ring with identity, and let J and I_{ij} with $1 \leq i < j \leq n$ be arbitrary ideals of R such that $I_{i\,j+1}J \subseteq I_{i\,j}$, $JI_{i\,j} \subseteq I_{i+1\,j}$ and $I_{ij}I_{jk} \subseteq I_{ik}$ for $j < k \leq n$. Define a ring*

$$S := \begin{pmatrix} R & I_{12} & \cdots & \cdots & I_{1n} \\ J & R & \ddots & \ddots & \vdots \\ J^2 & J & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & R & I_{n-1\,n} \\ J^{n-1} & \cdots & J^2 & J & R \end{pmatrix}_{n\times n}.$$

If ${}_RJ$ is projective and finitely generated, then

$$K_*(S) \simeq K_*(R) \oplus \bigoplus_{j=1}^{n-1} K_*(R/I_{j\,j+1}J)$$

for all $* \in \mathbb{N}$.

Note that rings of the form in Corollary 1.3 not only cover some of tiled orders, Hecke orders, and minimal model program for orders over surfaces (see [21, 22, 4, 5]), but also occur in commutative rings (see [29, Section 7]) and stratification of derived module categories arising from infinitely generated tilting modules over tame hereditary algebras (see [6]).

Theorem 1.1 can also be applied to affine cellular algebras (see [14]) and reduces their algebraic K-theory to the one of affine commutative rings. In particular, we have the following corollary about the algebraic K-groups of extended affine Hecke algebras of type $\tilde{A}$ (for definition, see Section 4.3).

Corollary 1.4. *Let k be a field of characteristic 0 and $q \in k$ such that $\sum_{w\in W_0} q^{\ell(w)} \neq 0$, where W_0 is the symmetric group of n letters and $\ell(w)$ is the usual length function on W_0. For the extended affine Hecke algebra $\mathscr{H}_k(n,q)$ of type $\widetilde{A}_{n-1}$, we have*

$$K_*(\mathscr{H}_k(n,q)) \simeq \bigoplus_{\mathbf{c}} K_*(R_{\mathbf{c}})$$

for $* \in \mathbb{N}$, where $\mathbf{c}$ runs over all two-sided cells of the extended affine Weyl groups W of type $\widetilde{A}_{n-1}$, and $R_{\mathbf{c}}$ stands for the representation ring associated with $\mathbf{c}$.

Note that the ring $R_{\mathbf{c}}$ is a tensor product of rings of the form $\mathbb{Z}[X_1,\cdots,X_s,X_{s+1},X_{s+1}^{-1}]$ with s a suitable natural number (see [14] for details). So the K-theory of $R_{\mathbf{c}}$ is closely related to that of $\mathbb{Z}$ by the Fundamental Theorem in K-theory.

The contents of this paper are outlined as follows. In Section 2, we provide necessary materials needed in our proofs. For instance, we recall the Thomason-Waldhausen Localization Theorem and the notion of recollements of triangulated categories. In Section 3, we introduce the notion of covariant and contravariant morphisms in an additive category, provide some of its basic properties, and prove the main results, Theorem 1.1 and Theorem 1.2. In Section 4, we apply our results to give formulas for calculations of algebraic K-groups K_n of some classes of rings, including standardly stratified rings, matrix subrings, quantum Schur algebras, affine cellular algebras, extended affine Hecke algebras of type $\tilde{A}$, and skew group rings. This also proves Corollaries 1.3 and 1.4. In the last section, we exhibit a few examples to demonstrate that some conditions of our results cannot be removed or weakened. Also, two open questions are proposed at the end of this section.

2 Preliminaries

Given a ring R with identity, we denote by R-Mod the category of all left R-modules, and by R-mod the category of all finitely generated left R-modules. As usual, by A-proj we denote the category of all finitely generated projective left R-modules. The complex, homotopy and derived categories of R-Mod are denoted by $\mathscr{C}(R\text{-Mod}), \mathscr{K}(R\text{-Mod})$ and $\mathscr{D}(R\text{-Mod})$, respectively.

The category R-mod with short exact sequences is an exact category in the sense of Quillen (see [19]), we denote its K-theory by $G_*(R)$. As usual, we denote by $K_*(R)$ the K-theory of R-proj with split exact sequences. If R is left noetherian and has finite global dimension, then $K_*(R) \simeq G_*(R)$ for all $* \in \mathbb{N}$. In general, even for finite dimensional algebras over a field, the G-theory and K-theory are not isomorphic, though the former is reduced to the one of the endomorphism rings of simple modules.

2.1 Waldhausen categories

Now we recall some elementary notion about the K-theory of small Waldhausen categories (see [27, 26]).

By *a category with cofibrations* we mean a category $\mathcal{C}$ with a zero object 0, together with a chosen class $\mathrm{co}(\mathcal{C})$ of morphisms in $\mathcal{C}$ satisfying the following three axioms:

(1) Any isomorphism in $\mathcal{C}$ is a morphism in $\mathrm{co}(\mathcal{C})$,

(2) For any object A in $\mathcal{C}$, the unique morphism $0 \to A$ is in $\mathrm{co}(\mathcal{C})$,

(3) If $X \to Y$ is a morphism in $\mathrm{co}(\mathcal{C})$, and $X \to Z$ is a morphism in $\mathcal{C}$, then the push-out $Y \cup_X Z$ exists in $\mathcal{C}$, and the canonical morphism $Z \to Y \cup_X Z$ is in $\mathrm{co}(\mathcal{C})$. In particular. finite coproducts exist in $\mathcal{C}$.

A morphism in $\mathrm{co}(\mathcal{C})$ is called a *cofibration*.

A category $\mathcal{C}$ with cofibrations is called a *Waldhausen category* if $\mathcal{C}$ admits a class $\mathrm{w}(\mathcal{C})$ of morphisms satisfying the two axioms:

(1) Any isomorphism in $\mathcal{C}$ is a morphism in $\mathrm{w}(\mathcal{C})$.

(2) Given a commutative diagram

$$\begin{array}{ccccc} B & \longleftarrow & A & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ B' & \longleftarrow & A' & \longrightarrow & C' \end{array}$$

in $\mathcal{C}$ with two morphisms $A \to B$ and $A' \to B'$ being cofibrations, and with $B \to B'$, $A \to A'$ and $C \to C'$ being in $\mathrm{w}(\mathcal{C})$, then the induced morphism $B \cup_A C \longrightarrow B' \cup_{A'} C'$ is in $\mathrm{w}(\mathcal{C})$.

The morphisms in $\mathrm{w}(\mathcal{C})$ are called *weak equivalences*. Thus a Waldhausen category consists of the triple data: A category, cofibrations and weak equivalences.

A functor between Waldhausen categories is called an *exact* functor if it preserves zero, cofibrations, weak equivalence classes and the pushouts along the cofibrations.

A typical example of Waldhausen categories can be obtained from complexes of modules over rings in the following manner:

Let R be a ring with identity. Let $\mathscr{C}^b(R\text{-proj})$ be the small category consisting of all bounded complexes of finitely generated projective R-modules. This is a Waldhausen category. That is, the weak equivalences are the homotopy equivalences, and the cofibrations are the degreewise split monomorphisms. By just inverting the weak equivalences, we get the derived category of $\mathscr{C}^b(R\text{-proj})$, which is the homotopy category $\mathscr{K}^b(R\text{-proj})$ of $\mathscr{C}^b(R\text{-proj})$.

For a small Waldhausen category $\mathcal{C}$, a K-theory $K_n(\mathcal{C})$ was defined in [27]. In particular, for the small, Waldhausen category $\mathscr{C}^b(R\text{-proj})$, it is shown by a theorem of Gillet-Waldhausen that its K-theory is the same as the K-theory of R defined by Quillen. That is, $K_n(R) \simeq K_n(\mathscr{C}^b(R\text{-proj}))$ for all $n \geq 0$.

In this paper, we always assume that all Waldhausen categories considered are of this type, that is, they are full subcategories of the category of chain complexes over some abelian category.

The following result, which is called the Thomason-Waldhausen Localization Theorem in the literature, says that we can get an exact sequence of K-groups of Waldhausen categories from a short exact sequence of their derived categories, which is induced from exact functors between the given Waldhausen categories (for example, see [26], [18, Theorem 2.3]).

Theorem 2.1. *(Thomason-Waldhausen Localization Theorem) Let $\mathcal{R}, \mathcal{S}$ and $\mathcal{T}$ be small, Waldhausen categories. Suppose $\mathcal{R} \to \mathcal{S} \to \mathcal{T}$ are exact functors of Waldhausen categories. Suppose further that*

(i) the induced triangulated functors of derived categories $\mathscr{D}(\mathcal{R}) \longrightarrow \mathscr{D}(\mathcal{S}) \longrightarrow \mathscr{D}(\mathcal{T})$ compose to zero.

(ii) The functor $\varphi : \mathscr{D}(\mathcal{R}) \longrightarrow \mathscr{D}(\mathcal{S})$ is fully faithful.

(iii) If x and x' are objects of $\mathscr{D}(\mathcal{S})$, and the direct sum $x \oplus x'$ is isomorphic in $\mathscr{D}(\mathcal{S})$ to $\varphi(z)$ for some $z \in \mathscr{D}(\mathcal{R})$, then x, x' are isomorphic to $\varphi(y), \varphi(y')$ for some $y, y' \in \mathscr{D}(\mathcal{R})$.

(iv) The natural map $\mathscr{D}(\mathcal{S})/\mathscr{D}(\mathcal{R}) \longrightarrow \mathscr{D}(\mathcal{T})$ is an equivalence of categories.

Then the sequence of spectra $K(\mathcal{R}) \to K(\mathcal{S}) \to K(\mathcal{T})$ is a homotopy fibration, and therefore there is a long exact sequence of K-theory

$$\cdots \longrightarrow K_{n+1}(\mathcal{T}) \longrightarrow K_n(\mathcal{R}) \longrightarrow K_n(\mathcal{S}) \longrightarrow K_n(\mathcal{T}) \longrightarrow K_{n-1}(\mathcal{R}) \longrightarrow$$

$$\cdots \longrightarrow K_0(\mathcal{S}) \longrightarrow K_0(\mathcal{R}) \longrightarrow K_0(\mathcal{T})$$

for all $n \in \mathbb{N}$.

2.2 Recollements

Another notion needed in our proofs is recollements which were introduced by Beilinson, Bernstein and Deligne (see [2]) to study the behaves of triangulated category of perverse sheaves over certain geometric objects.

Let $\mathcal{D}$ be a triangulated categories with a shift functor denoted by [1].

Let $\mathcal{D}'$ and $\mathcal{D}''$ be triangulated categories. We say that $\mathcal{D}$ is a *recollement* of $\mathcal{D}'$ and $\mathcal{D}''$ if there are six triangle functors as in the following diagram

$$\begin{tikzcd} \mathcal{D}'' \arrow[r, "i_*=i_!"] & \mathcal{D} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^!=j^*"] & \mathcal{D}' \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

such that

(1) $(i^*, i_*), (i_!, i^!), (j_!, j^!)$ and (j^*, j_*) are adjoint pairs;

(2) i_*, j_* and $j_!$ are fully faithful functors;

(3) $i^! j_* = 0$ (and thus also $j^! i_! = 0$ and $i^* j_! = 0$); and

(4) for each object $X \in \mathcal{D}$, there are two canonical triangles in $\mathcal{D}$:

$$i_! i^!(X) \longrightarrow X \longrightarrow j_* j^*(X) \longrightarrow i_! i^!(X)[1],$$

$$j_! j^!(X) \longrightarrow X \longrightarrow i_* i^*(X) \longrightarrow j_! j^!(X)[1],$$

where $i_! i^!(X) \to X$ and $j_! j^!(X) \to X$ are counit adjunction maps, and where $X \to j_* j^*(X)$ and $X \to i_* i^*(X)$ are unit adjunction maps.

It is know from the definition of recollements that the Verdier quotients of $\mathcal{D}$ by the images of the triangle functors $j_!$ and i_* are equivalent to $\mathcal{D}''$ and $\mathcal{D}'$, respectively.

A typical example of recollements occurs in the following situation. Let R be a ring with an idempotent ideal $J = ReR$ for $e^2 = e \in R$. Suppose that J is a stratifying ideal of R (for definition, see Section 4), then there is a recollement:

$$\begin{tikzcd} \mathscr{D}(R/J\text{-Mod}) \arrow[r, "D(\lambda_*)"] & \mathscr{D}(R\text{-Mod}) \arrow[l, bend right, "R/J\otimes^{\mathbb{L}}_R -"'] \arrow[l, bend left, "\mathbb{R}\mathrm{Hom}_R(R/J,-)"] \arrow[r, "eR\otimes^{\mathbb{L}}_R -"] & \mathscr{D}(eRe\text{-Mod}) \arrow[l, bend right, "Re\otimes^{\mathbb{L}}_{eRe} -"'] \arrow[l, bend left, "\mathbb{R}\mathrm{Hom}_{eRe}(eR,-)"] \end{tikzcd}$$

where $\lambda_* : R/J\text{-Mod} \longrightarrow R\text{-Mod}$ is the restriction functor, $Re \otimes^{\mathbb{L}}_{eRe} -$ is the total left-derived functor of $Re \otimes_{eRe} -$, and $\mathbb{R}\mathrm{Hom}_{eRe}(eR, -)$ is the total right-derived functor of $\mathrm{Hom}_{eRe}(eR, -)$. For more details, we refer the reader to [7].

Note that $\mathscr{D}(R\text{-Mod})$ is a triangulated category with small coproducts.

Finally, we point out the following homological fact which is needed in our proofs.

Lemma 2.2. *Let R be a ring with identity, and let $J = ReR$ for $e^2 = e \in R$. Suppose that M is an R-module with the following two properties:*

(1) $\mathrm{Tor}^R_j(R/J, M) = 0$ *for all* $j \geq 0$*, and*

(2) *M has a finite-type resolution, that is, there is an exact sequence* $0 \longrightarrow P'_n \xrightarrow{\varepsilon'_n} \cdots \longrightarrow P'_1 \xrightarrow{\varepsilon'_1} P'_0 \xrightarrow{\varepsilon'_0} M \to 0$ *with all P'_j finitely generated projective R-modules.*

Then there is an exact sequence of R-modules:

$$0 \to P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

such that all P_j lie in $\mathrm{add}(Re)$.

Proof. This result is known for modules over Artin algebras, where one may use minimal projective resolutions (see [1]). For general rings, projective covers of modules may not exist. For convenience of the reader, we include here a proof.

Given such a sequence in (2), we define $K_i' = \mathrm{Ker}(\varepsilon_{i-1}')$ for $1 \le i \le n$. Then K_i' is finitely generated.

It follows from $\mathrm{Tor}_0^R(R/J,M) = 0$ that $JM = M$. Since the trace of Re in the module M is just JM and since M is finitely generated, there is a finite index set I_0 and a surjective homomorphism $P_0 := \bigoplus_{i \in I_0} Re \xrightarrow{\varepsilon_0} M$. We define $K_1 = \mathrm{Ker}(\varepsilon_0)$. Then, by Schanuel's Lemma, we have $K_1 \oplus P_0' \simeq K_1' \oplus P_0$, and therefore K_1 is finitely generated. It follows from $\mathrm{Tor}_1^R(R/J,M) = 0$ that the sequence

$$0 \longrightarrow K_1/JK_1 \longrightarrow P_0/JP_0 \longrightarrow M/JM \longrightarrow 0$$

is exact. This means that $JK_1 = K_1$ because $JP_0 = P_0$. Observe that $\mathrm{Tor}_1^R(R/J,K_1) = \mathrm{Tor}_2^R(R/J,M) = 0$. So, for K_1, we can do the similar procedure as we did above and get a surjective homomorphism $P_1 := \bigoplus_{j \in I_1} Re \xrightarrow{\varepsilon_1} K_1$ with I_1 a finite set, such that $K_2 := \mathrm{Ker}(\varepsilon_1)$ is finitely generated with $JK_2 = K_2$ and $\mathrm{Tor}_1^R(R/J,K_2) = 0$. Hence, by using the generalized Schanuel's Lemma, we can iterate this procedure. Since the projective dimension of M is finite, we must stop after n steps and reach at a desired sequence mentioned in the lemma. $\square$

Remark. The above proof shows that for an R-module M, the condition (1) is equivalent to

(2') There is a projective resolution $\cdots \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$ such that $P_j \in \mathrm{Add}(Re)$, where $\mathrm{Add}(M)$ is the full subcategory of R-Mod consisting of all those R-modules which are direct summands of direct sums of copies of M.

Thus $J = ReR$ is homological if and only if such a sequence (2') for ${}_RJ$ exists.

3 Proofs of the main results

In this section, we introduce the notion of covariant and contravariant morphisms, and prove the main results, Theorems 1.1 and 1.2.

Let R be a ring with identity and X an R-module. A submodule Y of X is a *trace* in X if $\mathrm{Hom}_R(Y, X/Y) = 0$, and a *weak trace* in X if the inclusion from Y to X induces an isomorphism $\mathrm{Hom}_R(Y,Y) \to \mathrm{Hom}_R(Y,X)$ of abelian groups. For example, every idempotent ideal of R is a trace of the regular R-module ${}_RR$, and every GV-ideal J of R is a weak trace of ${}_RR$ (see [29, Section 7] for definition). Also, the socle of any finite dimensional algebra A over a field is a weak trace of ${}_AA$. In particular, the socle of the ring $R := \mathbb{Q}[X]/(X^2)$ is a weak trace in R, but not a trace in R.

In general, for any R-modules X and Y, there is a recipe for getting weak trace submodules of X. Let $t_Y(X)$ be the sum of all images of homomorphisms from Y to X of R-modules. Then $t_Y(X)$ is a weak trace of X.

We denote by $\mathrm{End}_{R,Y}(X)$ the quotient ring of the endomorphism ring $\mathrm{End}_R(X)$ of X modulo the ideal generated by those endomorphisms of X which factorize through the module Y. Note that this ideal consists actually of all those endomorphisms $f : X \to X$ such that there is an R-module $Z \in \mathrm{add}(Y)$ and homomorphisms $g : X \to Z$ and $h : Z \to X$ with $f = gh$, where $\mathrm{add}(Y)$ is the full subcategory of R-Mod consisting of all modules which are direct summands of direct sums of finitely many copies of Y. For instance, if I is an idempotent ideal in R, then $\mathrm{End}_{R,I}(R) \simeq R/I$.

Motivated by weak trace submodules, we introduce the following notion.

Definition 3.1. *Let $\mathcal{C}$ be an additive category. A morphism $\lambda : Y \to X$ of objects in $\mathcal{C}$ is said to be covariant if*

(1) *the induced map* $\mathrm{Hom}_{\mathcal{C}}(X,\lambda) : \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{C}}(X,X)$ *is injective, and*

(2) *the induced map* $\mathrm{Hom}_{\mathcal{C}}(Y,\lambda) : \mathrm{Hom}_{\mathcal{C}}(Y,Y) \to \mathrm{Hom}_{\mathcal{C}}(Y,X)$ *is a split epimorphism of* $\mathrm{End}_{\mathcal{C}}(Y)$*-modules.*

Dually, a morphism $\beta : N \to M$ in $\mathcal{C}$ is said to be contravariant if

$(1')$ $\mathrm{Hom}_{\mathcal{C}}(\beta,N) : \mathrm{Hom}_{\mathcal{C}}(M,N) \to \mathrm{Hom}_{\mathcal{C}}(N,N)$ *is injective, and*

$(2')$ $\mathrm{Hom}_{\mathcal{C}}(\beta,M) : \mathrm{Hom}_{\mathcal{C}}(M,M) \to \mathrm{Hom}_{\mathcal{C}}(N,M)$ *is a split epimorphism of right* $\mathrm{End}_{\mathcal{C}}(M)$*-modules*

Clearly, if Y is a weak trace submodule of an R-module X, then the inclusion map is a covariant homomorphism. Another example of covariant homomorphisms is the following: If $0 \to Z \to Y \xrightarrow{g} X \to 0$ is an Auslander-Reiten sequence in R-mod with $\mathrm{Hom}_R(Y,Z) = 0$, then the homomorphism g is covariant.

For covariant morphisms, we have the following properties.

Lemma 3.2. *Let $\mathcal{C}$ be an additive category, and let $\lambda : Y \to X$ be a covariant morphism of objects in $\mathcal{C}$. We define $\Lambda := \mathrm{End}_{\mathcal{C}}(X \oplus Y)$, and let e_Y be the idempotent element of Λ corresponding to the projection onto Y. Then*

(1) ${}_\Lambda \Lambda e_Y \Lambda$ is a finitely generated projective Λ-module.

(2) The composition map $\mu : \mathrm{Hom}_{\mathcal{C}}(X,Y) \otimes_{\mathrm{End}_{\mathcal{C}}(Y)} \mathrm{Hom}_{\mathcal{C}}(Y,X) \to \mathrm{End}_{\mathcal{C}}(X)$ is injective. Thus the cokernel of μ is isomorphic to $\mathrm{End}_{\mathcal{C},Y}(X)$.

To prove this lemma, we use the following observation.

Lemma 3.3. *Let S be a ring with identity, and let e be an idempotent element in S. Then ${}_S SeS$ (respectively, SeS_S) is projective and finitely generated if and only if $eS(1-e)$ (respectively, $(1-e)Se$) is projective and finitely generated as an eSe-module (respectively, a right eSe-module), and the multiplication map*

$$\mu : (1-e)Se \otimes_{eSe} eS(1-e) \longrightarrow (1-e)S(1-e)$$

is injective.

Proof. Suppose that $eS(1-e)$ is a finitely generated projective eSe-module and that the multiplication map $(1-e)Se \otimes_{eSe} eS(1-e) \xrightarrow{\mu} (1-e)S(1-e)$ is injective. Then it is easy to see that the multiplication map $Se \otimes_{eSe} eS \longrightarrow SeS$ is an isomorphism of S-S-bimodules. Since $eS = eSe \oplus eS(1-e)$, we know that ${}_S SeS$ is projective and finitely generated.

Conversely, suppose that ${}_S SeS$ is projective and finitely generated. One the one hand, since ${}_S SeS$ is projective, we can show that the multiplication map $\mu : Se \otimes_{eSe} eS \to SeS$ is injective (see [9, Statement 7]). This implies that the map $\mu : (1-e)Se \otimes_{eSe} eS(1-e) \longrightarrow (1-e)S(1-e)$ is injective. On the other hand, since ${}_S SeS$ is finitely generated, there is a finite subset $\{x_i \mid i \in I\}$ of S such that the map $\bigoplus_{i \in I} Se \longrightarrow SeS$, defined by $(a_i)_{i \in I} \mapsto \sum_{i \in I} a_i x_i$, is surjective. This shows that ${}_S SeS$ is a direct summand of a direct sum of finitely many copies of Se. Thus eS is a direct summand of a free eSe-module of finite rank. This implies that the eSe-module $eS(1-e)$ is projective and finitely generated.

The same arguments applies to the right module SeS_S. $\square$

Proof of Lemma 3.2. Clearly, $\Lambda = \begin{pmatrix} \mathrm{End}_{\mathcal{C}}(X) & \mathrm{Hom}_{\mathcal{C}}(X,Y) \\ \mathrm{Hom}_{\mathcal{C}}(Y,X) & \mathrm{End}_{\mathcal{C}}(Y) \end{pmatrix}$. Let $e := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $f := 1-e$. Thus $e = e_Y$, $e\Lambda e \simeq \mathrm{End}_{\mathcal{C}}(Y)$, $f\Lambda f \simeq \mathrm{End}_{\mathcal{C}}(X)$, $f\Lambda e \simeq \mathrm{Hom}_{\mathcal{C}}(X,Y)$ and $e\Lambda f \simeq \mathrm{Hom}_{\mathcal{C}}(Y,X)$, where the left $\mathrm{End}_{\mathcal{C}}(Y)$-module structure of $\mathrm{Hom}_{\mathcal{C}}(Y,X)$ is induced from the right $\mathrm{End}_{\mathcal{C}}(Y)$-module structure of Y. In the following we will often use these identifications, as was done in [28]. Since λ is a covariant homomorphism, the induced map

$$\lambda^* = \mathrm{Hom}_{\mathcal{C}}(Y,\lambda) : \mathrm{Hom}_{\mathcal{C}}(Y,Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(Y,X)$$

is a split epimorphism of $\mathrm{End}_{\mathcal{C}}(Y)$-modules. Thus there is a homomorphism $\gamma : \mathrm{Hom}_{\mathcal{C}}(Y,X) \to \mathrm{Hom}_{\mathcal{C}}(Y,Y)$ such that $\gamma\lambda^* = id$. This means that $\mathrm{Hom}_{\mathcal{C}}(Y,X)$ is a direct summand of the regular $\mathrm{End}_{\mathcal{C}}(Y)$-module. Thus $e\Lambda f$ is projective and finitely generated as a left $e\Lambda e$-module since a direct summand of a finitely generated module is finitely generated.

Now we show that the multiplication map $f\Lambda e \otimes_{e\Lambda e} e\Lambda f \longrightarrow f\Lambda f$ is injective. This is equivalent to showing that the composition map

$$\mu : \quad \mathrm{Hom}_{\mathcal{C}}(X,Y) \otimes_{\mathrm{End}_{\mathcal{C}}(Y)} \mathrm{Hom}_{\mathcal{C}}(Y,X) \longrightarrow \mathrm{End}_{\mathcal{C}}(X),$$

given by $x \otimes f \mapsto xf$ for $x \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ and $f \in \mathrm{Hom}_{\mathcal{C}}(Y,X)$, is injective. However, the injectivity of μ follows from the injectivity of $\mathrm{Hom}_{\mathcal{C}}(X,\lambda) : \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X,X)$ together with the following commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(X,Y) \otimes_{\mathrm{End}_{\mathcal{C}}(Y)} \mathrm{Hom}_{\mathcal{C}}(Y,Y) & \xrightarrow[\mu']{\simeq} & \mathrm{Hom}_{\mathcal{C}}(X,Y) \\ \big\downarrow{\scriptstyle \mathrm{Hom}_{\mathcal{C}}(X,Y)\otimes\lambda^*} & & \big\downarrow{\scriptstyle \mathrm{Hom}_{\mathcal{C}}(X,\lambda)} \\ \mathrm{Hom}_{\mathcal{C}}(X,Y) \otimes_{\mathrm{End}_{\mathcal{C}}(Y)} \mathrm{Hom}_{\mathcal{C}}(Y,X) & \xrightarrow{\mu} & \mathrm{Hom}_{\mathcal{C}}(X,X) \end{array}$$

since the bottom μ is a composite of three injective maps, that is, $\mu = \big(\mathrm{Hom}_{\mathcal{C}}(X,Y)\otimes\gamma\big)\mu'\,\mathrm{Hom}_{\mathcal{C}}(X,\lambda)$. Here, we use the identity $\gamma\lambda^* = id$. Thus, by Lemma 3.3, we see that ${}_\Lambda\Lambda e\Lambda$ is a finitely generated projective Λ-module.

Now the second statement of Lemma 3.2 is also clear. $\square$

Dually, for contravariant morphisms, we have the following statement.

Lemma 3.4. *Let $\mathcal{C}$ be an additive category, and let $\lambda : Y \to X$ be a contravariant morphism of objects in $\mathcal{C}$. We define $\Lambda := \mathrm{End}_{\mathcal{C}}(X\oplus Y)$, and let e_X be the idempotent element of Λ corresponding to the projection onto X. Then*

(1) *$\Lambda e_X\Lambda_\Lambda$ is a finitely generated projective right Λ-module.*

(2) *The composition map $\mu : \mathrm{Hom}_{\mathcal{C}}(Y,X)\otimes_{\mathrm{End}_{\mathcal{C}}(X)}\mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{End}_{\mathcal{C}}(Y)$ is injective. Thus the cokernel of μ is isomorphic to $\mathrm{End}_{\mathcal{C},X}(Y)$.*

For convenience, we introduce the following definition of X-covariant morphisms. Observe that the condition in this definition strengthens only the first and does not involve the second condition in the definition of covariant or contravariant morphisms.

Definition 3.5. *A morphism $f : Y \to X$ in an additive category $\mathcal{C}$ is said to be*

(1) *X-covariant if the induced map $Hom_{\mathcal{C}}(X,f)$ is a split monomorphism of $\mathrm{End}_{\mathcal{C}}(X)$-modules.*

(2) *Y-contravariant if the induced map $Hom_{\mathcal{C}}(f,Y)$ is a split monomorphism of right $\mathrm{End}_{\mathcal{C}}(Y)$-modules.*

Clearly, if $f : Y \to X$ is covariant, then the inclusion from $\mathrm{Ker}(f)$ into Y is Y-covariant. Dually, if $f : Y \to X$ is contravariant, then the canonical surjection from X to $\mathrm{Coker}(f)$ is X-contravariant. For X-covariant and Y-contravariant morphisms, we have the following properties.

Lemma 3.6. *Let $\mathcal{C}$ be an additive category, and let $\lambda : Y \to X$ be a morphism of objects in $\mathcal{C}$. We define $\Lambda := \mathrm{End}_{\mathcal{C}}(X\oplus Y)$, and let e_X and e_Y be the idempotent elements of Λ corresponding to the projection onto X and Y, respectively.*

(1) *If λ is X-covariant, then ${}_\Lambda\Lambda e_X\Lambda$ is a finitely generated projective Λ-module. In this case, $\Lambda/\Lambda e_X\Lambda \simeq \mathrm{End}_{\mathcal{C},X}(Y)$.*

(2) *If λ is Y-contravariant, then $\Lambda e_Y\Lambda_\Lambda$ is a finitely generated projective Λ-module. In this case, $\Lambda/\Lambda e_Y\Lambda \simeq \mathrm{End}_{\mathcal{C},Y}(X)$.*

Proof. (1) The proof is similar to that of Lemma 3.2. Here, we only outline its main points.

Since $\lambda^* = \mathrm{Hom}_{\mathcal{C}}(X,\lambda) : \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X,X)$ is a split monomorphism of $\mathrm{End}_{\mathcal{C}}(X)$-modules, we see that

(a) $e_X\Lambda e_Y$ is a finitely generated projective $e_X\Lambda e_X$-module, and

(b) $\mathrm{Hom}_{\mathcal{C}}(Y,X)\otimes\lambda^* : \mathrm{Hom}_{\mathcal{C}}(Y,X)\otimes_{\mathrm{End}_{\mathcal{C}}(X)}\mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(Y,X)\otimes_{\mathrm{End}_{\mathcal{C}}(X)}\mathrm{Hom}_{\mathcal{C}}(X,X)$ is a split monomorphism.

To see that the multiplication map $\mu : e_Y\Lambda e_X\otimes_{e_X\Lambda e_X} e_X\Lambda e_Y \longrightarrow e_Y\Lambda e_Y$ is injective, we consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(Y,X)\otimes_{\mathrm{End}_{\mathcal{C}}(X)}\mathrm{Hom}_{\mathcal{C}}(X,Y) \arrow[r] \arrow[d, "\mathrm{Hom}_{\mathcal{C}}(Y,X)\otimes\lambda^*"] & \mathrm{Hom}_{\mathcal{C}}(Y,Y) \arrow[d, "\mathrm{Hom}_{\mathcal{C}}(Y,\lambda)"] \\
\mathrm{Hom}_{\mathcal{C}}(Y,X)\otimes_{\mathrm{End}_{\mathcal{C}}(X)}\mathrm{Hom}_{\mathcal{C}}(X,X) \arrow[r, "\simeq"] & \mathrm{Hom}_{\mathcal{C}}(Y,X)
\end{tikzcd}$$

where the horizontal maps are composition maps. This implies that μ is injective, and therefore $\Lambda/\Lambda e_X\Lambda \simeq \mathrm{End}_{\mathcal{C},X}(Y)$. Now (1) follows immediately from Lemma 3.3.

(2) The proof is left to the reader. $\square$

Proof of Theorem 1.1.

(1) We first show that (1) follows from (2) and Lemma 3.2.

Assume that I is an idempotent ideal of R. Then $\mathrm{Hom}_R(I,R/I) = 0$ and I is a trace of ${}_RR$. Note that $\mathrm{End}_{R,I}(R) \simeq R/I$. Thus, by Lemma 3.2, the first statement of (1) follows from (2) immediately.

Now assume further that ${}_RI$ is projective and finitely generated. Then the R-module $R\oplus I$ is a progenerator for R-Mod, and therefore R and $\Lambda := \mathrm{End}_R({}_RR\oplus I)$ are Morita equivalent. Hence, by the first statement of (1), $K_n(R) \simeq K_n(\Lambda) \simeq K_n\big(\mathrm{End}_R(I)\big)\oplus K_n(R/I)$ for all $n\in\mathbb{N}$. This finishes the proof of Theorem 1.1 (1).

(2) Now we prove (2). This is precisely the following proposition.

Proposition 3.7. *Let R be a ring with identity, and let $e^2 = e \in R$ such that $J := ReR$ is homological and ${}_RJ$ has a finite-type resolution. Then the K-theory space of R is homotopy equivalent to the product of the K-theory spaces of eRe and R/J, and therefore*

$$K_n(R) \simeq K_n\big(\mathrm{End}_R(eRe)\big) \oplus K_n(R/J)$$

for all $n \in \mathbb{N}$.

Proof. Recall that $\mathscr{C}^b(R\text{-proj})$ is the category consisting of all bounded complexes of finitely generated projective R-modules. This is a Waldhausen category. That is, the weak equivalences are the homotopy equivalences, and the cofibrations are the degreewise split monomorphisms. By just inverting the weak equivalences, we get the derived category of $\mathscr{C}^b(R\text{-proj})$, which is $\mathscr{K}^b(R\text{-proj})$.

Let $\mathscr{D}^c(R)$ be the full subcategory of $\mathscr{D}(R\text{-Mod})$ consisting of all compact objects in $\mathscr{D}(R\text{-Mod})$. Then $\mathscr{D}^c(R)$ is a triangulated subcategory. Recall that a complex $X^\bullet \in \mathscr{D}(R\text{-Mod})$ is said to be *compact* if $\mathrm{Hom}_{\mathscr{D}(R-\mathrm{Mod})}(X^\bullet, -)$ commutes with coproducts in $\mathscr{D}(R\text{-Mod})$. It is shown in [18, Corollary 4.4] that $\mathscr{D}^c(R)$ consists of objects which are isomorphic in $\mathscr{D}(R\text{-Mod})$ to bounded chain complexes of finitely generated, projective R-modules. Thus, any finite direct sum of compact objects is compact, any direct summand of a compact object is compact, and $\mathscr{K}^b(R\text{-proj})$ is equivalent to $\mathscr{D}^c(R)$ as triangulated categories.

Note that if $J := ReR$ is homological and ${}_RJ$ admits a finite-type resolution, that is, there is an exact sequence

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_0 \longrightarrow {}_RReR \longrightarrow 0$$

with all P_j finitely generated projective R-modules, then it follows from Lemma 2.2 that we may assume $P_j \in \mathrm{add}(Re)$ for all j. Thus $eP_j \in \mathrm{add}({}_{eRe}eRe)$ and the eRe-module eR has a finite-type resolution. Therefore eR is a compact object in $\mathscr{D}(eRe\text{-Mod})$. Clearly, $R/J \in \mathscr{D}^c(R/J)$. Now, we consider the small Waldhausen categories $\mathscr{C}^b(R\text{-proj})$, $\mathscr{C}^b(eRe\text{-proj})$ and $\mathscr{C}^b(R/J\text{-proj})$, and the functors

$$Re \otimes_{eRe} - : \mathscr{C}^b(eRe\text{-proj}) \longrightarrow \mathscr{C}^b(R\text{-proj}), \quad (R/J) \otimes_R - : \mathscr{C}^b(R\text{-proj}) \longrightarrow \mathscr{C}^b(R/J\text{-proj}).$$

Since the exact structure of these categories is the degreewise split short exact sequences, we see that the two functors are exact. Moreover, it is well known (for example, see [7]) that we have a recollement

$$\begin{tikzcd} \mathscr{D}(R/J\text{-Mod}) \arrow[r, "D(\lambda_*)"] & \mathscr{D}(R\text{-Mod}) \arrow[l, bend right, "R/J\otimes^{\mathbb{L}}_R-"'] \arrow[l, bend left] \arrow[r, "eR\otimes^{\mathbb{L}}_R-"] & \mathscr{D}(eRe\text{-Mod}) \arrow[l, bend right, "Re\otimes^{\mathbb{L}}_{eRe}-"'] \arrow[l, bend left] \end{tikzcd},$$

where $\lambda_* : R/J\text{-Mod} \longrightarrow R\text{-Mod}$ is the restriction functor. Note that R/J and eR are compact objects in $\mathscr{D}(R\text{-Mod})$ and $\mathscr{D}(eRe\text{-Mod})$, respectively. Thus the exact functors $D(\lambda_*)$ and $eR \otimes^{\mathbf{L}}_R -$ preserve compact objects. It is known that, for a recollement, the functors $R/J \otimes^{\mathbb{L}}_R -$ and $Re \otimes^{\mathbb{L}}_{eRe} -$ always preserves compact objects. Thus, from the above recollement we can get the following "half recollement" for the subcategories of compacts objects:

$$\begin{tikzcd} \mathscr{D}^c(R/J) \arrow[r, "D(\lambda_*)"] & \mathscr{D}^c(R) \arrow[l, bend right, "R/J\otimes^{\mathbb{L}}_R-"'] \arrow[r, "eR\otimes^{\mathbb{L}}_R-"] & \mathscr{D}^c(eRe) \arrow[l, bend right, "Re\otimes^{\mathbb{L}}_{eRe}-"'] \end{tikzcd}$$

Note that $\mathscr{D}^c(R)$ may not have small coproducts in general. This half recollement implies the following commutative diagram of triangle functors:

$$\begin{tikzcd} \mathscr{D}^c(R/J) & \mathscr{D}^c(R) \arrow[l, "R/J\otimes^{\mathbb{L}}_R-"'] & \mathscr{D}^c(eRe) \arrow[l, "Re\otimes^{\mathbb{L}}_{eRe}-"'] \\ \mathscr{K}^b(R/J\text{-proj}) \arrow[u, "\simeq"'] & \mathscr{K}^b(R\text{-proj}) \arrow[l, "R/J\otimes_R-"'] \arrow[u, "\simeq"'] & \mathscr{K}^b(eRe\text{-proj}) \arrow[l, "Re\otimes_{eRe}-"'] \arrow[u, "\simeq"'] \end{tikzcd}$$

This shows that the two functors in the top row are induced from the exact functors in the bottom row. Moreover, we have the following properties:

(1) Clearly, it follows from the half-recollement that the composition of the two functors $Re\otimes^{\mathbb{L}}_{eRe}-$ and $R/J\otimes^{\mathbb{L}}_{R}-$ is zero, that the functor $Re\otimes^{\mathbb{L}}_{eRe}-$ is fully faithful, and that the natural map $\mathscr{D}^c(R)/\mathscr{D}^c(eRe)\longrightarrow\mathscr{D}^c(R/J)$ is an equivalence of categories. The latter follows actually from a general known fact: If $F:\mathcal{C}\to\mathcal{D}$ is a triangle functor which admits a fully faithful right adjoint functor $G:\mathcal{D}\to\mathcal{C}$, then F induces uniquely a triangle equivalence between $\mathcal{C}/\mathrm{Ker}(F)$ and $\mathcal{D}$, where $\mathrm{Ker}(F)$ stands for the full subcategory of $\mathcal{C}$ consisting of all those objects x such that $F(x)=0$.

(2) If x and x' are two objects of $\mathscr{D}^c(R)$, and the direct sum $x\oplus x'$ is isomorphic in $\mathscr{D}^c(R)$ to $Re\otimes^{\mathbb{L}}_{eRe}z$ for some $z\in\mathscr{D}^c(eRe)$, then x,x' are isomorphic to $Re\otimes^{\mathbb{L}}_{eRe}y$, $Re\otimes^{\mathbb{L}}_{eRe}y'$ for some $y,y'\in\mathscr{D}^c(eRe)$, respectively. That is, the image of the functor $Re\otimes^{\mathbb{L}}_{eRe}-:\mathscr{D}^c(eRe)\longrightarrow\mathscr{D}^c(R)$ is closed under direct summands.

Indeed, let $y:=eR\otimes^{\mathbb{L}}_{R}x$ and $y':=eR\otimes^{\mathbb{L}}_{R}x'$. Then it follows from

$$0=(R/J)\otimes^{\mathbb{L}}_{R}(Re\otimes^{\mathbb{L}}_{eRe}z)\simeq(R/J)\otimes^{\mathbb{L}}_{R}(x\oplus x')\simeq(R/J)\otimes^{\mathbb{L}}_{R}x\oplus(R/J)\otimes^{\mathbb{L}}_{R}x'$$

that $(R/J)\otimes^{\mathbb{L}}_{R}x\simeq 0\simeq(R/J)\otimes^{\mathbb{L}}_{R}x'$. Now, by the definition of recollements, there are two triangles in $D^c(R)$:

$$Re\otimes^{\mathbb{L}}_{eRe}y\longrightarrow x\longrightarrow D(\lambda_*)\big((R/J)\otimes^{\mathbb{L}}_{R}x\big)\longrightarrow Re\otimes^{\mathbb{L}}_{eRe}y[1],$$

$$Re\otimes^{\mathbb{L}}_{eRe}y'\longrightarrow x'\longrightarrow D(\lambda_*)\big((R/J)\otimes^{\mathbb{L}}_{R}x'\big)\longrightarrow Re\otimes^{\mathbb{L}}_{eRe}y'[1].$$

Since the third terms of the two triangles are isomorphic to zero, we get $x\simeq Re\otimes^{\mathbb{L}}_{eRe}y$ and $x'\simeq Re\otimes^{\mathbb{L}}_{eRe}y'$.

By (1) and (2), we have verified all conditions of Thomason-Waldhausen Localization Theorem 2.1 for $\mathscr{D}^c(R)$. This implies that all conditions of the Thomason-Waldhausen Localization Theorem for $\mathscr{K}^b(R\text{-proj})$ are satisfied, and therefore the sequence of the K-theory spaces: $K(R/J)\longleftarrow K(R)\longleftarrow K(eRe)$ is a homotopy fibration.

Let $\mathscr{P}^{<\infty}(R)$ be the full subcategory of R-mod consisting of all R-modules X with a finite-type resolution:

$$0\longrightarrow Q_m\longrightarrow\cdots\longrightarrow Q_1\longrightarrow Q_0\longrightarrow X\longrightarrow 0$$

for some $m\in\mathbb{N}$ such that all Q_j are finitely generated projective modules. By [19, Section 4, Corollary 2], we have $K(R)\simeq K\big(\mathscr{P}^{<\infty}(R)\big)$. So, in the following we identify $K(eRe)$ with $K\big(\mathscr{P}^{<\infty}(eRe)\big)$.

With this identification of K-theory spaces, now we show that the map $K(Re\otimes_{eRe}-):K(eRe)\longrightarrow K(R)$ is a homotopy-split injection, that is, there is a map $K(eR\otimes_R-):K(R)\to K(eRe)$ of K-theory spaces, such that the composite of $K(Re\otimes_{eRe}-)$ with $K(eR\otimes_R-)$ is homotopic to the identity map on $K(eRe)$.

Consider the following commutative diagrams among exact categories eRe-proj, R-proj and $\mathscr{P}^{<\infty}(eRe)$:

$$\begin{tikzcd}
eRe\text{-proj} \arrow[r, "Re\otimes_{eRe}-"] \arrow[dr, hook] & R\text{-proj} \arrow[d, "eR\otimes_R-"] \arrow[r, phantom, "\Longrightarrow"] & K(eRe) \arrow[r, "K(Re\otimes_{eRe}-)"] \arrow[dr, "\simeq", "\text{homo. equ}"'] & K(R) \arrow[d, "K(eR\otimes_R-)"] \arrow[r, phantom, "\Longrightarrow"] & K_n(eRe) \arrow[r, "K_n(Re\otimes_{eRe}-)"] \arrow[dr, "\simeq"] & K_n(R) \arrow[d, "K_n(eR\otimes_R-)"] \\
 & \mathscr{P}^{<\infty}(eRe) & & K(\mathscr{P}^{<\infty}(eRe)) & & K_n(\mathscr{P}^{<\infty}(eRe))
\end{tikzcd}$$

Note that the functor $eR\otimes_R-:R\text{-proj}\longrightarrow\mathscr{P}^{<\infty}(eRe)$ is well defined. Thus the map $K(Re\otimes_{eRe}-):K(eRe)\longrightarrow K(R)$ is a homotopy-split injection. By [17], the K-theory spaces of rings are always homotopy equivalent to CW-complexes. To conclude our statement, we cite the following result in [24, Corollary 7.15]:

For a homotopy fibration $X\xrightarrow{f}Y\longrightarrow Z$ with Z homotopy equivalent to a CW-complex, if the map f is homotopy-split injection, then Y is homotopy equivalent to the product of X and Y.

Thus, from this result we know that the K-theory space $K(R)$ of R is homotopy equivalent to the product of the K-theory spaces of eRe and R/J, and therefore

$$K_n(R)\simeq K_n(R/ReR)\oplus K_n(eRe)$$

for all $n\in\mathbb{N}$. This completes the proof of Proposition 3.7, and also the proof of Theorem 1.1. $\square$

Proof of Theorem 1.2.

(1) Now, we assume that $\lambda:Y\to X$ is a covariant morphism of objects in $\mathcal{C}$. In this case, we consider $\Lambda:=\mathrm{End}_{\mathcal{C}}(X\oplus Y)$, and let J be the ideal of Λ generated by the projection e from $X\oplus Y$ onto Y. Then $e\Lambda e\simeq\mathrm{End}_{\mathcal{C}}(Y)$ and Λ/J is isomorphic to the quotient ring of $\mathrm{End}_{\mathcal{C}}(X)$ modulo the ideal generated by those endomorphisms of X which factorize through the object Y, that is, $\Lambda/J\simeq\mathrm{End}_{\mathcal{C},Y}(X)$ by Lemma 3.2 (2). Since ${}_{\Lambda}J$ is projective and finitely generated by Lemma 3.2 (1), we can apply Theorem 1.1 to Λ and J. In this case, we see that the K-theory

space of $\mathrm{End}_{\mathcal{C}}(X\oplus Y)$ is homotopy equivalent to the product of the K-theory spaces of $\mathrm{End}_{\mathcal{C},Y}(X)$ and $\mathrm{End}_{\mathcal{C}}(Y)$, and get

$$K_n(\Lambda)\simeq K_n\big(\mathrm{End}_{\mathcal{C},Y}(X)\big)\oplus K_n\big(\mathrm{End}_{\mathcal{C}}(Y)\big)$$

for all $n\in\mathbb{N}$.

(2) Similarly, we may use Lemma 3.6 and Theorem 1.1 to show (2). □

Remarks. (1) If $I:=ReR$ is a finitely generated projective R-module for $e^2=e\in R$, then $\mathrm{End}_R(I)$ and eRe are Morita equivalent. In fact, it follows from the projectivity of ${}_RI$ that $Re\otimes_{eRe}eR\simeq ReR$ and that eR is projective and finitely generated as an eRe-module by Lemma 3.3. Thus we have $\mathrm{add}(Re)=\mathrm{add}({}_RReR)$, where $\mathrm{add}(Re)$ stands for the full subcategory of R-Mod consisting of all direct summands of direct sums of finitely many copies of Re. This means that $\mathrm{End}_R({}_RI)$ is Morita equivalent to eRe.

(2) Dually, we may define *weak trace factor modules*. Let X and Y be an R-modules. The module Y is called a week trace factor module of X if there is a surjective homomorphism $\pi: X\to Y$ of R-modules such that the induced map $\pi^*:\mathrm{Hom}_R(Y,Y)\longrightarrow\mathrm{Hom}_R(X,Y)$ is an isomorphism of abelian groups. Obviously, in this case, the map π is a contravariant homomorphism.

The dual of Theorem 1.2 can be stated as follows.

Theorem 3.8. *Let $\mathcal{C}$ be an additive category and $f: Y\to X$ a morphism of objects in $\mathcal{C}$.*

(1) *If f is contravariant, then the K-theory space of $\mathrm{End}_{\mathcal{C}}(X\oplus Y)$ is homotopy equivalent to the product of the K-theory spaces of $\mathrm{End}_{\mathcal{C}}(X)$ and $\mathrm{End}_{\mathcal{C},X}(Y)$. In particular,*

$$K_*\big(\mathrm{End}_{\mathcal{C}}(X\oplus Y)\big)\simeq K_*\big(\mathrm{End}_{\mathcal{C}}(X)\big)\oplus K_*\big(\mathrm{End}_{\mathcal{C},X}(Y)\big)$$

for all $*\in\mathbb{N}$.

(2) *If f is Y-contravariant, then the K-theory space of $\mathrm{End}_{\mathcal{C}}(X\oplus Y)$ is homotopy equivalent to the product of the K-theory spaces of $\mathrm{End}_{\mathcal{C},Y}(X)$ and $\mathrm{End}_{\mathcal{C}}(Y)$. In particular,*

$$K_*\big(\mathrm{End}_{\mathcal{C}}(X\oplus Y)\big)\simeq K_*\big(\mathrm{End}_{\mathcal{C},Y}(X)\big)\oplus K_*\big(\mathrm{End}_{\mathcal{C}}(Y)\big)$$

for all $*\in\mathbb{N}$.

The following is a consequence of Theorem 1.2 for $\mathcal{C}=R$-Mod.

Corollary 3.9. *Let R be a ring with identity, and let $\mathrm{soc}(R)$ be the socle of ${}_RR$. Then*

$$K_n\big(\mathrm{End}_R(R\oplus\mathrm{soc}(R))\big)\simeq K_n\big(\mathrm{End}_R(\mathrm{soc}(R))\big)\oplus K_n\big(R/\mathrm{soc}(R)\big)$$

for all $n\in\mathbb{N}$.

Proof. Recall that for an R-module M, the socle of M is the sum of all simple submodules of M. Thus $\mathrm{soc}(R)$ is a direct sum of minimal left ideals of R, and therefore it is actually an ideal in R. Since $\mathrm{soc}(R)$ is a weak trace submodule of ${}_RR$ by the definition of socles, we can apply Theorem 1.2 and get

$$K_n\big(\mathrm{End}_R(R\oplus\mathrm{soc}(R))\big)\simeq K_n\big(\mathrm{End}_R(\mathrm{soc}(R))\big)\oplus K_n(R/\mathrm{soc}(R))$$

for all $n\in\mathbb{N}$. □

For Auslander-Reiten sequences, we have the following result.

Corollary 3.10. *Let A be an Artin algebra, and let $0\longrightarrow Z\xrightarrow{g}Y\xrightarrow{f}X\longrightarrow 0$ be an Auslander-Reiten sequence in A-mod. If $\mathrm{Hom}_A(Y,Z)=0$, then*

$$\begin{aligned}K_n\big(\mathrm{End}_A(X\oplus Y)\big)&\simeq K_n\big(\mathrm{End}_A(Y)\big)\oplus K_n\big(\mathrm{End}_A(X)/\mathrm{rad}(\mathrm{End}_A(X))\big)\\&\simeq K_n\big(\mathrm{End}_A(Y)\big)\oplus K_n\big(\mathrm{End}_A(Z)/\mathrm{rad}(\mathrm{End}_A(Z))\big)\\&\simeq K_n\big(\mathrm{End}_A(Y\oplus Z)\big)\end{aligned}$$

for all $n\in\mathbb{N}$.

Proof. For Auslander-Reiten sequences, we know from [13] that $\mathrm{End}_A(Y \oplus Z)$ and $\mathrm{End}_A(X \oplus Y)$ are derived equivalent, and therefore they have the isomorphic algebraic K-groups, that is, $K_n(\mathrm{End}_A(X \oplus Y)) \simeq K_n(\mathrm{End}_A(Y \oplus Z))$ for all $n \in \mathbb{N}$. Note that $\mathrm{Hom}_A(Y,Z) = 0$ if and only if the induced surjective map $\mathrm{Hom}_A(Y,Y) \to \mathrm{Hom}_A(Y,X)$ is an isomorphism of $\mathrm{End}_A(Y)$-modules. Since f is surjective, it follows also from $\mathrm{Hom}_A(Y,Z) = 0$ that $\mathrm{Hom}_A(X,Z) = 0$. Thus $f : Y \to X$ is a covariant homomorphism and, by Theorem 1.2, we have

$$K_n\big(\mathrm{End}_A(X \oplus Y)\big) \simeq K_n\big(\mathrm{End}_A(Y)\big) \oplus K_n\big(\mathrm{End}_{A,Y}(X)\big)$$

for all $n \in \mathbb{N}$.

By properties of Auslander-Reiten sequences, we see that $\mathrm{rad}(\mathrm{End}_A(X))$ is the image of the map $\mathrm{Hom}_A(X,f) : \mathrm{Hom}_A(X,Y) \to \mathrm{Hom}_A(X,X)$. Thus $\mathrm{End}_{A,Y}(X) \simeq \mathrm{End}_A(X)/\mathrm{rad}(\mathrm{End}_A(X))$ which is a division ring and isomorphic to $\mathrm{End}_A(Z)/\mathrm{rad}(\mathrm{End}_A(Z))$. Hence

$$K_n\big(\mathrm{End}_A(X \oplus Y)\big) \simeq K_n\big(\mathrm{End}_A(Y \oplus Z)\big) \simeq K_n\big(\mathrm{End}_A(Y)\big) \oplus K_n\big(\mathrm{End}_A(Z)/\mathrm{rad}(\mathrm{End}(Z))\big)$$

for all $n \in \mathbb{N}$. $\square$

Further applications of Theorems 1.1 and 1.2 will be discussed in the next section.

4 Applications

In this section, we deduce some consequences of Theorems 1.1 and 1.2.

4.1 Standardly stratified rings

First, we consider standardly stratified rings and finite dimensional quasi-hereditary algebras.

Standardly stratified and quasi-hereditary algebras were well defined for finite dimensional algebras or semiprimary rings in [7] and [9], respectively. Now let us formulate them for arbitrary rings.

Let R be a ring with identity. Recall that an ideal J of R is called a *stratifying ideal* if
(1) $J = ReR$ for some idempotent element $e \in R$,
(2) $Re \otimes_{eRe} eR \simeq ReR$, and
(3) $\mathrm{Tor}_j^{eRe}(Re, eR) = 0$ for $j \geq 1$.

Note that $J = ReR$ for $e^2 = e \in R$ is a stratifying ideal if and only if J is homological. In particular, if ${}_RJ = ReR$ is projective and finitely generated, then J is a stratifying ideal of R. In this case, J is called a *standardly stratifying* ideal of R. The ring R is called *standardly stratified* if there is a chain of ideals of R:

$$0 = J_{n+1} \subseteq J_n \subseteq \cdots \subseteq J_2 \subseteq J_1 = R$$

such that J_i/J_{i+1} is a standardly stratifying ideal in R/J_{i+1} for all i.

By this definition, every ring with identity is standardly stratified. But the most interesting case for us is that for rings we do have such a chain of length bigger than one.

For a standardly stratified ring R with a defining chain of ideals as above, there is an idempotent element $e \in R/J_{i+1}$ such that $(R/J_{i+1})e(R/J_{i+1}) = J_i/J_{i+1}$, we denote by $\Delta(i)$ the R-module $(R/J_{i+1})e$. All these modules $\Delta(i)$ are called the *standard modules* with respect to the chain of ideals of R. Note that standardly stratified rings may have infinite global dimension.

By definition, a *quasi-hereditary ring* is a standardly stratified ring R such that the endomorphism ring $\mathrm{End}_R(\Delta(i))$ of each standard module $\Delta(i)$ is a division ring. As in the case of finite dimensional algebras, one can show that every quasi-hereditary ring has finite global dimension.

Note that, by definition, the hereditary ring $\mathbb{Z}$ of integers is a standardly stratified ring, but it is not a quasi-hereditary ring. Thus, left hereditary rings may not be quasi-hereditary. This example shows the difference of quasi-hereditary rings defined in this paper from quasi-hereditary algebras (or rings) in the sense of Cline, Parshall and Scott [7] (or of Dlab and Ringel [9]). For further information on finite dimensional standardly stratified and quasi-hereditary algebras, we refer the reader to [7, 9] and the references therein.

Corollary 4.1. (1) *If R is a standardly stratified ring with the standard modules $\Delta(i)$ for $1 \leq i \leq n$, then*

$$K_*(R) \simeq \bigoplus_{j=1}^{n} K_*\big(\mathrm{End}_R(\Delta(j))\big)$$

for all $* \in \mathbb{Z}$.

(2) *If* A *is a finite-dimensional quasi-hereditary algebra over an algebraically closed field* k *with* n *non-isomorphic simple* A*-modules, then* $G_*(A) \simeq K_*(A) \simeq nK_*(k)$ *for all* $* \in \mathbb{N}$.

Proof. (1) follows from Theorem 1.1 inductively. (2) is a consequence of (1) since for a finite-dimensional quasi-hereditary algebra over an algebraically closed field k, we can refine a hereditary chain into a maximal chain, and in this case, the endomorphism ring of each standard module is isomorphic to the ground field k.

Note that (2) follows also from the fact that finite-dimensional quasi-hereditary algebras over an algebraically closed field k are noetherian and of finite global dimension. This implies that their G-theory and K-theory coincide. $\square$

4.2 Matrix subrings

In the following, we consider algebraic K-theory of matrix subrings some of which are used in noncommutative algebraic geometry (see [4]) and arithmetic representation theory (see [21, Chapter 39]). In our discussions below, the key idea is to find standardly stratifying ideals in those rings.

Corollary 4.2. *Let* R *and* S *be rings with identity, and let* ${}_RM_S$ *and* ${}_SN_R$ *be bimodule. Suppose that* $\varphi : M \otimes_S N \to R$ *and* $\psi : N \otimes_R M \to S$ *define a Morita context ring* $T := \begin{pmatrix} R & M \\ N & S \end{pmatrix}$. *If* φ *is injective, and* ${}_SN$ *is projective and finitely generated, then*

$$K_n(T) \simeq K_n(S) \oplus K_n\big(R/(M \cdot N)\big)$$

for all $n \in \mathbb{N}$, *where* $M \cdot N$ *stands for the image of* φ *in* R,

Proof. Let $e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Since φ is injective and ${}_SN$ is projective and finitely generated, it follows that $TeT = \begin{pmatrix} M \cdot N & M \\ N & S \end{pmatrix} \simeq Te \otimes_S N \oplus Te$, which is a finitely generated projective T-module, and that $T/TeT \simeq R/(M \cdot N)$. Thus, the corollary follows immediately from Theorem 1.1. $\square$

Remark. The statement in Corollary 4.2 appeared in [8, Theorem 1.2]. However, the proof in [8] seems to be wrong because the functor H in the proof is not well defined.

As a further corollary of Theorem 1.1, we consider the question mentioned in Introduction (see also [29]) and provide some partial answers. First of all, we mention the following consequence of Corollary 4.2, namely a result of Berrick and Keating in [3].

Corollary 4.3. [3] *Let* R_i *be a ring with identity for* $i = 1, 2$, *and let* M *be an* R_1-R_2*-bimodule. Then, for the triangular matrix ring*

$$S = \begin{pmatrix} R_1 & M \\ 0 & R_2 \end{pmatrix},$$

there is an isomorphism of K*-groups:* $K_n(S) \simeq K_n(R_1) \oplus K_n(R_2)$ *for* $n \in \mathbb{N}$.

In the next result, we consider slightly general matrix subrings. Here, under the assumption that ${}_RJ$ is an idempotent, projective and finitely generated ideal of a ring R, we extend the result [29, Proposition 5.3] for K_1 to a result for higher K-groups.

Corollary 4.4. *Let* R *be a ring with identity, and let* J *and* I_{ij} *with* $1 \le i < j \le n$ *be arbitrary ideals of* R *such that* $I_{i\,j+1}J \subseteq I_{i\,j}$, $JI_{i\,j} \subseteq I_{i+1\,j}$ *and* $I_{i\,j}I_{jk} \subseteq I_{ik}$ *for* $j < k \le n$. *Define a ring*

$$S := \begin{pmatrix} R & I_{12} & \cdots & \cdots & I_{1n} \\ J & R & \ddots & \ddots & \vdots \\ J^2 & J & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & R & I_{n-1\,n} \\ J^{n-1} & \cdots & J^2 & J & R \end{pmatrix}_{n \times n}.$$

If $_RJ$ is projective and finitely generated, then

$$K_*(S) \simeq K_*(R) \oplus \bigoplus_{j=1}^{n-1} K_*(R/I_{j\,j+1}J).$$

Proof. We use induction on n to prove this corollary.

Now let e_i be the idempotent element of S with 1_R at the (i,i)-entry and zero at all other entries, and $e := e_2 + \cdots + e_n$. As J is a projective R-module, we have $I_{ij} \otimes_R J \simeq I_{ij}J$. Thus

$$SeS = \begin{pmatrix} I_{12}J & I_{12} & \cdots & \cdots & I_{1n} \\ J & R & \ddots & \cdots & I_{2n} \\ J^2 & J & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & R & I_{n-1n} \\ J^{n-1} & J^{n-2} & \cdots & J & R \end{pmatrix} \simeq Se \oplus Se_2 \otimes_{e_2Se_2} J.$$

Here we identify R with e_2Se_2. Since $_RJ$ is projective and finitely generated, we infer that the S-module SeS is also projective and finitely generated. Clearly, S/SeS is isomorphic to $R/I_{12}J$. By Theorem 1.1 (2), we get $K_*(S) \simeq K_*(R/I_{12}J) \oplus K_*(eSe)$. By induction, we know that $K_*(eSe) \simeq K_*(R) \oplus \bigoplus_{j=2}^{n-1} K_*\big(R/I_{j\,j+1}J\big)$. Thus

$$K_*(S) \simeq K_*(R) \oplus \bigoplus_{j=1}^{n-1} K_*(R/I_{j\,j+1}J).$$

This finishes the proof. $\square$

As a consequence of Corollary 4.4, we can prove the following corollary.

Corollary 4.5. *Let R be a ring with identity, and let r be a regular element of R with $Rr = rR$. If I_{ij} is an ideal of R for $1 \leq i < j \leq n$ such that $I_{ij+1}r \subseteq I_{i\,j}$, $rI_{ij} \subseteq I_{i+1\,j}$ and $I_{ij}I_{jk} \subseteq I_{ik}$ for $j < k \leq n$, then, for the matrix ring*

$$T := \begin{pmatrix} R & I_{12} & I_{13} & \cdots & I_{1\,n} \\ Rr & R & I_{23} & \cdots & I_{2\,n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ Rr^{n-2} & \cdots & Rr & R & I_{n-1\,n} \\ Rr^{n-1} & \cdots & Rr^2 & Rr & R \end{pmatrix},$$

we have

$$K_*(T) \simeq K_*(R) \oplus \bigoplus_{j=1}^{n-1} K_*(R/I_{j\,j+1}r)$$

for all $n \in \mathbb{N}$.

By a *regular element* we mean an element of R, which is not a zero-divisor of R.

As an immediate consequence of Corollary 4.5, we mention the following corollary for integral domain.

Corollary 4.6. *Let D be an integral domain, $x \in D$, and I an ideal of D. Then*

$$S := \begin{pmatrix} D & I & \cdots & I \\ Dx & \ddots & \ddots & \vdots \\ \vdots & \ddots & D & I \\ Dx^{n-1} & \cdots & Dx & D \end{pmatrix}_{n\times n}$$

is a ring, and

$$K_*(S) \simeq K_*(D) \oplus (n-1)K_*(D/Ix)$$

for all $* \in \mathbb{N}$.

Now, we point out the following result.

Proposition 4.7. *Let R be a commutative ring with identity, and let $x,y \in R$ such that $Rx+Ry=R$ and $Rx\cap Ry=Rxy$ (for example, R is a principle integral domain with x and y coprime in R). Suppose that y is invertible in an extension ring R' of R. Then, for the ring*

$$S := \begin{pmatrix} R & Rx & \cdots & Rx \\ Ry & \ddots & \ddots & \vdots \\ \vdots & \ddots & R & Rx \\ Ry & \cdots & Ry & R \end{pmatrix}_{n\times n},$$

we have

$$K_n(S) \simeq K_n(R)\oplus(n-1)K_n(R/Rx)\oplus(n-1)K_n(R/Ry)$$

for all $n\in\mathbb{N}$.

Proof. Let σ be the diagonal matrix with the $(1,1)$-entry y and all other diagonal entries 1. Then σ is invertible in $M_n(R')$, the n by n full matrix ring of R'. Let $B := \sigma S\sigma^{-1}$. Thus $S\simeq B$ and B is of the form

$$B := \begin{pmatrix} R & Rxy & Rxy & \cdots & Rxy \\ R & R & Rx & \cdots & Rx \\ R & Ry & R & \ddots & \vdots \\ \vdots & \vdots & \ddots & R & Rx \\ R & Ry & \cdots & Ry & R \end{pmatrix}_{n\times n}.$$

Define $A := M_n(R)$. Then B is a subring of A with the same identity. Moreover, ${}_BA$ is isomorphic to the direct sum of n copies of Be_1 where e_1 is the diagonal matrix $\mathrm{diag}(1,0,\cdots,0)$ of B. Thus ${}_BA$ is a finitely generated projective B-module. Hence, by [29, Lemma 3.1], B is derived equivalent to $\mathrm{End}_B(B\oplus A/B)$. Clearly, the latter is Morita equivalent to $\mathrm{End}_B(Be_1\oplus Q_2\oplus\cdots\oplus Q_n)$, where Q_j is given by the exact sequence

$$0\longrightarrow Be_j\longrightarrow Be_1\longrightarrow Q_j\longrightarrow 0,\ \ 2\le j\le n.$$

As in [29, Section 3], we can show that $\mathrm{End}_B(Be_1\oplus Q_2\oplus\cdots\oplus Q_n)$ is isomorphic to the following ring

$$C := \begin{pmatrix} R & R/Rxy & R/Rxy & \cdots & R/Rxy \\ 0 & R/Rxy & Rx/Rxy & \cdots & Rx/Rxy \\ 0 & Ry/Rxy & R/Rxy & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & Rx/Rxy \\ 0 & Ry/Rxy & \cdots & Ry/Rxy & R/Rxy \end{pmatrix}.$$

From the Chinese remainder theorem we know that $R/Rxy\simeq R/Rx\oplus R/Ry$ as rings. Moreover, it follows from the assumptions that the R/Rxy-bimodules Rx/Rxy and Ry/Rxy are isomorphic to R/Ry and R/Rx, respectively. Let D be the lower right corner $(n-1)\times(n-1)$-submatrix of C. Then the ring D is actually a direct sum of the following two rings:

$$D = \begin{pmatrix} R/Ry & R/Ry & \cdots & R/Ry \\ 0 & R/Ry & \ddots & \vdots \\ \vdots & \ddots & R/Ry & R/Ry \\ 0 & \cdots & 0 & R/Ry \end{pmatrix}_{n-1} \bigoplus \begin{pmatrix} R/Rx & 0 & \cdots & 0 \\ R/Rx & R/Rx & \ddots & \vdots \\ \vdots & \ddots & R/Ry & 0 \\ R/Rx & \cdots & R/Rx & R/Rx \end{pmatrix}_{n-1}.$$

Since derived equivalences preserve algebraic K_n-groups (see [10]), we have

$$K_n(S)\simeq K_n(C)\simeq K_n(R)\oplus K_n(D)\simeq K_n(R)\oplus(n-1)K_n(R/Rx)\oplus(n-1)K_n(R/Ry)$$

for all $n\in\mathbb{N}$. $\square$

Remark. For $n=2$, we can remove the conditions "$Rx+Ry=R$ and $Rx\cap Ry=Rxy$" in Proposition 4.7, and get $K_*(S)\simeq K_*(R)\oplus K_*(R/Rxy)$ for all $*\in\mathbb{N}$.

Related to calculation of algebraic K-groups of the rings in the proof of Proposition 4.7, the following result may be of interest.

Corollary 4.8. *Let R be a ring with identity, and let I and J be ideals in R with $JI=0$. If ${}_RI$ (or J_R) is projective and finitely generated, then, for the ring*

$$S:=\begin{pmatrix} R & I & \cdots & I \\ J & R & \ddots & \vdots \\ \vdots & \ddots & \ddots & I \\ J & \cdots & J & R \end{pmatrix}_{n\times n},$$

we have

$$K_*(S)\simeq nK_*(R)$$

for all $*\in\mathbb{N}$.

Proof. We assume that the R-module ${}_RI$ is projective and finitely generated. Let $e:=e_1\in S$. Then

$$SeS:=\begin{pmatrix} R & I & \cdots & I \\ J & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ J & 0 & \cdots & 0 \end{pmatrix}.$$

Since ${}_RI$ is projective, we have $J\otimes_R I\simeq JI=0$ and $Se\otimes_R I\simeq SeSe_j$ for $2\le j\le n$. Here we identify eSe with R. Since ${}_RI$ is projective and finitely generated, we know that ${}_SSeSe_j$ is projective and finitely generated for $j=2,\cdots,n$, and therefore the S-module ${}_SSeS\simeq Se\oplus SeSe_2\oplus\cdots\oplus SeSe_n$ is a finitely generated projective module. Thus, by Theorem 1.1 and induction on n, we have

$$K_*(S)\simeq nK_*(R)$$

for all $*\in\mathbb{N}$.

The proof for the case that J_R is projective and finitely generated can be done similarly. □

Remark. If R is an arbitrary ring with I,J ideals in R such that $IJ=JI=0$, then the ring S in Corollary 4.8 is the trivial extension of $R\times R\times\cdots\times R$ by the bimodule L, where

$$L:=\begin{pmatrix} 0 & I & \cdots & I \\ J & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & I \\ J & \cdots & J & 0 \end{pmatrix}.$$

Thus we always have $K_n(S)\simeq nK_n(R)\oplus K_n(S,L)$ for all $n\in\mathbb{N}$, where $K_n(S,L)$ is the n-th relative K-group of S with respect to the ideal L. This is due to the split epimorphism $K_n(S)\to K_n(S/L)$ of abelian groups, which is induced from the split surjection $S\to S/L$.

Observe that rings of the form in Corollaries 4.5, 4.6 or Proposition 4.7 occur in terminal orders over smooth projective surfaces (see [4]). For example, if we take D to be the power series ring $k[[z]]$ over a field k in one variable z, $I=zk[[z]]$ and $x=1$, then the ring S in Corollary 4.6 is related to the completion of a closed point in a quasi-projective surface. It would be interesting to know how K-theory or recollements could be applied in this situation.

4.3 Some special rings

In this section, we consider the algebraic K-theory of rings appearing in different areas.

4.3.1 Algebraic K-theory for affine cellular algebras

As a generalization of cellular algebras in the sense of Graham and Lehrer [11], affine cellular algebras were introduced in [14] to study the representation theory and homological properties of certain infinite dimensional algebras which include extended affine Hecke algebras of type $\tilde{A}$. We shall see that the K-theory of affine cellular algebras can be studied in local information. First, we recall the definition of affine cellular algebras from [14].

Let k be a noetherian integral domain. For two k-modules W and V, we denote the switch map by $\tau : W \otimes_k V \to V \otimes_K W, w \otimes v \mapsto v \otimes w$ with $w \in W$ and $v \in V$.

Definition 4.9. [14] *Let A be a unitary k-algebra with a k-involution i on A. A two-sided ideal J in A is called an* affine cell ideal *if and only if the following data are given and the following three conditions are satisfied:*

(1) *The ideal J is fixed by i, that is, $i(J) = J$.*

(2) *There exist a free k-module V of finite rank and a finitely generated commutative k-algebra B with identity and with a k-involution σ such that $\Delta := V \otimes_k B$ is an A-B-bimodule, where the right B-module structure is induced by that of the right regular B-module B_B.*

(3) *There is an A-A-bimodule isomorphism $\alpha : J \longrightarrow \Delta \otimes_B \Delta'$, where $\Delta' = B \otimes_k V$ is a B-A-bimodule with the left B-structure induced by ${}_BB$ and with the right A-structure via i, that is, $(b \otimes v)a := \tau(i(a)(v \otimes b))$ for $a \in A, b \in B$ and $v \in V$), such that the following diagram is commutative:*

$$\begin{array}{ccc} J & \xrightarrow{\alpha} & \Delta \otimes_B \Delta' \\ i \big\downarrow & & \big\downarrow v_1 \otimes b_1 \otimes_B b_2 \otimes v_2 \mapsto v_2 \otimes \sigma(b_2) \otimes_B \sigma(b_1) \otimes v_1 \\ J & \xrightarrow{\alpha} & \Delta \otimes_B \Delta' \end{array}$$

The algebra A (with the involution i) is called an affine cellular algebra *if and only if there is a k-module decomposition $A = J_1' \oplus J_2' \oplus \cdots \oplus J_n'$ (for some n) with $i(J_j') = J_j'$ for each j, such that setting $J_j = \oplus_{l=1}^{j} J_l'$ gives a chain of two-sided ideals of A:*

$$0 = J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_n = A$$

(each of them fixed by i) and for each j ($j = 1, \ldots, n$) the quotient $J_j' = J_j/J_{j-1}$ is an affine cell ideal of A/J_{j-1} (with respect to the involution induced by i on the quotient).

By definition, for each subquotient J_j/J_{j-1} of an affine cellular algebra A, there is a commutative algebra B_j and an A-B_j-bimodule $\Delta(j)$ such that J_j/J_{j-1} is an affine cell ideal in A/J_{j-1}. In this case, we say that B_j is *associated* with J_j/J_{j-1}, and $\Delta(j)$ is a *cell module*.

Proposition 4.10. *Let A be an affine cellular algebra with a cell chain $J_0 = 0 \subset J_1 \subset \cdots \subset J_n = A$ and the associated commutative rings B_j for $1 \le j \le n$. Suppose that each B_j satisfies $rad(B_j) = 0$ and that each J_j/J_{j-1} is idempotent and contains a non-zero idempotent element in A/J_{j-1}. Then*

$$K_*(A) \simeq \bigoplus_{j=1}^{n} K_*(B_j)$$

for all $* \in \mathbb{N}$.

Proof. Under the assumptions of Proposition 4.10, we know from the proof of [14, Theorem 4.3] that each ideal J_j/J_{j-1} of A/J_{j-1} is generated by an idempotent element e_j and that, as an A/J_{j-1}-module, J_j/J_{j-1} is projective and isomorphic to a direct sum of finitely many copies of the cell module $\Delta(j)$. Moreover, it follows from the proof of [14, Theorem 4.3] that $\mathrm{add}({}_A\Delta(j)) = \mathrm{add}({}_A(A/J_{j-1})e)$. This implies that $\Delta(j)$ is a finitely generated A-module and that $e_j(A/J_{j-1})e_j$ is Morita equivalent to B_j. Thus we may inductively apply Theorem 1.1 (2) to get Proposition 4.10. $\square$

4.3.2 Algebraic K-theory for affine Hecke algebras and quantum Schur algebras

Let k be the Laurent polynomial ring $\mathbb{Z}[q,q^{-1}]$ in variable q over the ring $\mathbb{Z}$ of integers. Let (W,S) be a Coxeter system. For example, if W is the symmetric group on the letters $\{1,2,\cdots,n\}$ with $S:=\{s_i=(i,i+1)\mid i=1,2,\cdots,n-1\}\subseteq W$, then the Coxeter system is said to be of type A_{n-1}. The *Hecke algebra* of (W,S) over k, denoted by $\mathcal{H}_k(W,S)$, is a unitary associative algebra with a k-basis $\{T_w\mid w\in W\}$, subject to the following relations:

$$\begin{array}{ll}(T_s-q^2)(T_s+1)=0 & \text{if } s\in S,\\ T_wT_u=T_{wu} & \text{if } \ell(wu)=\ell(w)+\ell(u),\end{array}$$

where ℓ is the usual length function of W.

Let (W,S) be the Coxeter system of type $\tilde{A}_{n-1}$. Then the cyclic group $\mathbb{Z}/n\mathbb{Z}$ acts on W. Thus we may form the semiproduct $\widetilde{W}:=W\ltimes\mathbb{Z}/n\mathbb{Z}$, and define similarly the Hecke algebra over k of the extended Coxeter system $(\widetilde{W},S)$. This Hecke algebra is then called the *extended affine Hecke algebra* of type $\tilde{A}_{n-1}$, denoted by $\mathscr{H}_k(n,r)$. For more details about affine Hecke algebras we refer to [16].

We may apply Proposition 4.10 to the extended affine Hecke algebra of type $\widetilde{A}_n$ since this algebra was shown to be affine cellular in [14]. The proofs there imply the following corollary.

Corollary 4.11. *Let k be a field of characteristic 0 and $q\in k$ such that $\sum_{w\in W_0}q^{\ell(w)}\neq 0$, where W_0 is the symmetric group of n letters. For the extended affine Hecke algebra $\mathscr{H}_k(n,r)$, we have*

$$K_*(\mathscr{H}_k(n,r))\simeq\bigoplus_{\mathbf{c}}K_*(R_{\mathbf{c}})$$

where c runs over all two-sided cells of the extended affine Weyl groups $\tilde{W}$, and $R_{\mathbf{c}}$ stands for the representation ring associated with $\mathbf{c}$, which is isomorphic to a tensor product of rings of the form $\mathbb{Z}[X_1,X_2,\cdots,X_{s+1}]/(X_sX_{s+1}-1)$.

Now, we turn to quantum Schur algebras. Let (W,S) be the Coxeter system of type A_{n-1}, and $\mathcal{H}_q(n)$ be its Hecke algebra over k. Given a partition λ of n, one may define a Young subgroup W_λ of W, and an element $x_\lambda:=\sum_{w\in W_\lambda}q^{\ell(w)}T_w$. Suppose $r\leq n$. Let $\Lambda^+(n,r)$ be the set of partitions of n with at most r parts. The *quantum Schur algebra* $\mathcal{S}_q(n,r)$ is defined as

$$\mathcal{S}_q(n,r):=\mathrm{End}_{\mathcal{H}_q(n)}\big(\bigoplus_{\lambda\in\Lambda^+(n,r)}\mathcal{H}_q(n)x_\lambda\big).$$

Quantum Schur algebras have many nice homological properties, for example, they are (integral) quasi-hereditary algebras over k and their standard modules $\Delta(\lambda)$, indexed by $\Lambda^+(n,r)$, have the property that $\mathrm{End}_{\mathcal{S}_q(n,r)}(\Delta(\lambda))\simeq k$ for all $\lambda\in\Lambda^+(n,r)$. Thus, by Corollary 4.1, we have the following result.

Corollary 4.12. *For the quantum Schur algebra $\mathcal{S}_q(n,r)$, we have*

$$K_*(\mathcal{S}_q(n,r))\simeq mK_*(\mathbb{Z})\oplus mK_{*-1}(\mathbb{Z})$$

for all $*\in\mathbb{N}$, where m is the cardinality of the set $\Lambda^+(n,r)$.

Proof. Since the ring $\mathbb{Z}$ is noetherian and of finite global dimension, we know that

$$K_i(\mathbb{Z}[t,t^{-1}])\simeq K_i(\mathbb{Z})\oplus K_{i-1}(\mathbb{Z})$$

(see [19, Theorem 8]). Thus Corollary 4.12 follows immediately from Corollary 4.1. □

4.4 Algebraic K-theory for skew group rings

Let S be a ring with identity and suppose that G is a finite group of automorphisms of the ring S such that the order of G is a unit in S. Let $R=S*G$ be the skew group ring and $e:=\frac{1}{|G|}\sum_{g\in G}g$. Then $e^2=e$ and the ring $S^G:=\{s\in S\mid s^g=s \text{ for all } g\in G\}$ of invariants of G is isomorphic to $\mathrm{End}_R(Re)$. We write $R(S,G)$ for the ramification algebra R/ReR. The trace ideal of Re in R is ReR. If ${}_RReR$ (or ReR_R) is projective and finitely generated, then $\mathrm{End}_R(ReR)$ is Morita equivalent to S^G. Thus we have the following consequence of Theorem 1.1.

Corollary 4.13. *For any $n \in \mathbb{N}$, there hold:*

(1) $K_n\big(\mathrm{End}_R(R \oplus ReR)\big) \simeq K_n\big(R(S,G)\big) \oplus K_n\big(\mathrm{End}_R(ReR)\big)$.

(2) *If the R-module ${}_RReR$ or ReR_R is finitely generated and projective, then*

$$K_n(R) \simeq K_n\big(R(S,G)\big) \oplus K_n(S^G).$$

Note that the case $ReR = R$ was considered in [15] to compare the K_0-groups of R with those of S and S^G. In fact, in this case, the condition $|G|^{-1} \in S$ implies that R and S^G are Morita equivalent. The higher algebraic K-groups of S^G were discussed in [12] under some additional assumptions on both S and G.

Let us mention an example in which the second condition in Corollary 4.13 holds true. For instance, if S is a finite product of simple rings, and if G is a finite group acting as automorphisms of S such that the order of G is invertible in S, then the skew group ring R is also a finite product of simple rings. Thus ${}_RReR$ is projective.

5 Examples

The following examples illustrate how our results in this note can be used to compute algebraic K_n-groups of rings. They also show that some conditions on I in Theorem 1.1 cannot be omitted or weakened.

Example 1. Let k be a field, and let R be the ring $k[X]/(X^2)$. We denote by x the element $X + (X^2)$ in R. Then we may form the matrix ring

$$A := \begin{pmatrix} R & k \\ k & k \end{pmatrix}, \quad \begin{pmatrix} r+sx & a \\ b & c \end{pmatrix} \begin{pmatrix} r'+s'x & a' \\ b' & c' \end{pmatrix} = \begin{pmatrix} rr' + (rs' + sr' + ab')x & ra' + ac' \\ br' + cb' & cc' \end{pmatrix}$$

for $r, r', s, s', a, a', b, b', c, c' \in k$. One can check that this matrix ring is isomorphic to the quotient algebra of the path algebra of the quiver

$$\begin{tikzcd} 1 \bullet \arrow[r, bend left, "\alpha"] & \bullet 2 \arrow[l, bend left, "\beta"] \end{tikzcd}$$

modulo the ideal generated by $\beta\alpha$, and the latter is is a quasi-hereditary k-algebra of global dimension 2. Note also that $A \simeq \mathrm{End}_R(R \oplus k)$ and that $K_*(A) \simeq K_*(k) \oplus K_*(k)$ by Theorem 1.1. If k is a finite field, then we have a full knowledge of $K_*(A)$ by a result in [20].

Let $I = AeA$, where e is the idempotent of A corresponding to the vertex 1. Then I is an idempotent ideal in A, and ${}_AI$ is finitely generated, and has finite projective dimension, but not projective. Clearly, we have $K_n(eAe) \simeq K_n(R)$ and $K_n(A/AeA) \simeq K_n(k)$. Clearly, $K_0(A) \not\simeq K_0(A/I) \oplus K_0\big(\mathrm{End}_A(I)\big)$. Hence, if ${}_AI$ is not projective, then the second statement of Theorem 1.1 (1) may fail in general. Since $K_1(R) \simeq k \oplus\ k^{\times}$, we get $K_1(A) \not\simeq K_1(eAe) \oplus K_1(A/AeA)$. Note that in this example, the condition that $\mathrm{Tor}_j^R(A/I, A/I) = 0$ for all $j > 0$ fails, that is, the ideal I is not homological. Thus, in Theorem 1.1 (2), that ${}_RI$ is homological cannot be dropped. This example also shows that Corollary 4.2 may be false if ${}_SN$ is projective but φ is not injective.

If we modify this example slightly and just consider the algebra B given by the above quiver but with the relation $\beta\alpha\beta = 0$, then the ideal $I' = Be_1B$ is homological with infinite projective dimension as a left B-module. In this case, $K_*(B) \simeq K_*(R) \oplus K_*(k)$ by Corollary 4.1(1). But, since $\mathrm{End}_B(I') \simeq A$ and $B/I' \simeq k$, we cannot get $K_n(B) \simeq K_n(B/I') \oplus K_n(\mathrm{End}_B(I'))$. This shows that the projectivity of I in the second statement of Theorem 1.1 (1) cannot be relaxed to homological ideal.

Example 2. Note that for the triangular matrix ring $T := \begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$, if we take $I := \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix}$, then $I^2 = 0$ and ${}_TI$ is projective and finitely generated. In this case, we can see that $K_n(T) \simeq K_n(k) \oplus K_n(k)$ by Corollary 4.3. Thus $K_n(T/I) \oplus K_n(\mathrm{End}_T(I)) = K_n(k) \oplus K_n(k) \oplus K_n(k) \not\simeq K_n(T)$. Hence the condition $I^2 = I$ in Theorem 1.1 (1) cannot be removed.

Example 3. Let $p > 0$ be a prime integer, and let $\mathbb{Z}_p$ be the ring of p-adic-integers. We consider the ring

$$R := \begin{pmatrix} \mathbb{Z}_p & p\mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}.$$

Clearly, $p\mathbb{Z}_p$ is isomorphic to $\mathbb{Z}_p$ as left $\mathbb{Z}_p$-modules, and therefore is projective and finitely generated. Thus, by Corollary 4.6 (see also Remark to Proposition 4.7), we have

$$K_n(R) \simeq K_n(\mathbb{Z}_p) \oplus K_n(\mathbb{Z}_p/p^2\mathbb{Z}_p) \simeq K_n(\mathbb{Z}_p) \oplus K_n(\mathbb{Z}/p^2\mathbb{Z}).$$

Based on the results and examples in this note, we mention the following questions.

Open questions. (1) Let R be a ring with identity and I an ideal of R with $I^2 = 0$. We define a ring $S := \begin{pmatrix} R & I \\ I & R \end{pmatrix}$. How is the algebraic K-group $K_n(S)$ of S related to the K_n-groups of rings produced from R and the ideal I for $n \geq 2$?

Note that $K_i(S) = K_i(R) \oplus K_i(R)$ for $i = 0, 1$. This can be done by using Mayer-Vietoris sequences.

(2) Let R be a ring with identity and $e = e^2 \in R$. Suppose that ReR is homological and ${}_RReR$ possesses an infinite resolution by finitely generated projective R-modules. Does the following isomorphism hold true:

$$K_n(R) \simeq K_n(R/ReR) \oplus K_n(eRe)$$

for every $n \in \mathbb{N}$?

In the question (2), the canonical surjection $R \to R/ReR$ is a ring epimorphism. From the representation-theoretic point of view, the ring R/ReR has less simple modules than R does. There are ring epimorphisms for which the cardinality of simple modules may increase (see [6]). They arise from the so-called universal localizations. In another paper we shall establish a formula for higher algebraic K-groups of two rings linked by such a ring epimorphism.

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