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SURVEY ARTICLE

Constructions of derived equivalences for algebras and rings

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Abstract In this article, we shall survey some aspects of our recent (or related) constructions of derived equivalences for algebras and rings.

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1 Introduction

As is known, derived categories (or more generally, triangulated categories) and derived equivalences, introduced by Grothendieck and Verdier in [57], have connections with many branches of mathematics and physics. Rickard's Morita theory for derived categories of rings (see [52,54]) and Keller's Morita theory for derived categories of differential graded algebras (see [41]) provide powerful tools to understand derived module categories and equivalences of rings and graded rings. However, the following fundamental problem in the study of derived categories and equivalences still remains:

How can we construct derived equivalences for algebras and rings?

By Rickard's theory, this problem is reduced to constructing all tilting complexes and understanding their endomorphism rings. However, the latter is a very hard task and still not yet solved completely.

Related to the construction problem, it seems that the following three cases are worthy to be considered:

(1) Given an algebra or ring R, to find a class of rings that are derived equivalent to R.

(2) Given two rings R and S, to decide whether they are derived equivalent.

(3) Given a derived equivalence, to find a new derived equivalence based

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on the given one.

Case (1) is somewhat equivalent to getting the endomorphism rings of a class of tilting complexes. In this case, even one knows all tilting complexes over a ring, the next tough challenge encountered is how to work out their endomorphism rings. Here, we just mention a couple of discussions in this direction. For self-injective algebras with radical-square zero, every tilting complex is a shift of a free module (Dong Yang told the author that this seems to be a folk tale, but no proof has been yet published). For representation-finite, standard self-injective algebras, a derived classification is provided in [3]. For preprojective algebras, a complete description of tilting complexes is given in [1]. Partial answers to (1) are gotten for many classes of algebras, namely, one may construct some tilting complexes and determine their endomorphism rings. In this direction, see, for example, [32,33,43,60]. Regrettably, we cannot pursue all references here.

Case (2) is also a difficult question, let us just mention one open problem related to it, namely, the famous Broué Abelian Defect Group Conjecture in the modular representation theory of finite groups, which says that the module categories of a block algebra A of a finite group algebra and its Brauer correspondent B have equivalent derived categories if their common defect group is abelian (see [55]). This conjecture is considered as one of the hardest problems in the representation theory of finite groups. Though this conjecture is verified for many cases (see, for example, [22,47]), it seems far away from being solved completely. This may reflect a little flavor of difficulty of deciding whether two given algebras are derived equivalent. For further information on the developments of this conjecture, we refer the reader to [56] and the home page of Rickard: http://www.maths.bris.ac.uk/~majc/.

Related to case (3), Rickard used tensor products and trivial extensions to produce new derived equivalences in [52,53], Barot and Lenzing employed one-point extensions to transfer certain a derived equivalence to a new one in [8]. Up to the present time, however, it seems that not much is available for constructing new derived equivalences based on given ones.

In this article, we shall survey some of recent constructions of derived equivalences related to the last two cases. Here, the main idea is to construct derived equivalences from certain exact sequences, or by passing to quotient algebras or pullback algebras of derived equivalent algebras. After fixing some notation and recalling Rickard's theorem on derived equivalences for rings in Section 2, we survey, in Section 3, some developments on constructions of derived equivalences of the endomorphism rings of objects involved in a class of short exact sequences, including almost split sequences and certain triangles. In this section, a class of Yoneda algebras is introduced. In Section 4, we give some methods of constructing derived equivalences from given ones, including methods of getting derived equivalences from almost ν-stable derived equivalences and of passing to quotient algebras. In Section 5, we first recall the definitions of Frobenius type and stable equivalences of Morita type, and then show a result which gives a way to get derived equivalences from stable equivalences of Morita type for Frobenius-finite algebras. In the last section, we mention some new constructions given by Ladkani and by Hu-Xi. To limit the length of this article, the latter is illustrated only by an example, and the details will appear in a forthcoming preprint [39].

2 Preliminaries

In this section, we first fix some notation, and then recall some basic facts on derived equivalences of rings.

Let $\mathcal{C}$ be an additive category. Given two morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ in $\mathcal{C}$, we write fg for their composite which is a morphism from X to Z. The convenience of this writing is that $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is always a left $\mathrm{End}_{\mathcal{C}}(X)$- and right $\mathrm{End}_{\mathcal{C}}(Y)$-bimodule. But for two functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$ of categories, their composite is denoted by GF. For an object X in $\mathcal{C}$, we write $\mathrm{add}(X)$ for the full subcategory of $\mathcal{C}$ consisting of all direct summands of finite sums of copies of X.

A sequence of morphisms d_X^i between objects X^i in $\mathcal{C}$:

$$\cdots \to X^{i-1} \xrightarrow{d_X^{i-1}} X^i \xrightarrow{d_X^i} X^{i+1} \xrightarrow{d_X^{i+1}} X^{i+2} \to \cdots,$$

with $d_X^i d_X^{i+1} = 0$ for all $i \in \mathbb{Z}$, is called a *complex* over $\mathcal{C}$, and denoted by $X^\bullet = (X^i, d_X^i)$. The category of all complexes over $\mathcal{C}$ with the usual complex maps of degree zero is denoted by $\mathscr{C}(\mathcal{C})$. The homotopy and derived categories of complexes over $\mathcal{C}$ are denoted by $\mathscr{K}(\mathcal{C})$ and $\mathscr{D}(\mathcal{C})$, respectively. The full subcategory of $\mathscr{C}(\mathcal{C})$ consisting of bounded complexes over $\mathcal{C}$ is denoted by $\mathscr{C}^{\mathrm{b}}(\mathcal{C})$. Similarly, $\mathscr{K}^{\mathrm{b}}(\mathcal{C})$ and $\mathscr{D}^{\mathrm{b}}(\mathcal{C})$ denote the full subcategories consisting of bounded complexes in $\mathscr{K}(\mathcal{C})$ and $\mathscr{D}(\mathcal{C})$, respectively. As usual, the i-th shift functor of complexes is denoted by $[i]$.

Let $\mathcal{D}$ be a full subcategory of $\mathcal{C}$ and X be an object of $\mathcal{C}$. A morphism $f\colon D \to X$ in $\mathcal{C}$ is called a *right $\mathcal{D}$-approximation* of X (in the sense of Auslander-Smalø) if $D \in \mathcal{D}$ and, for any morphism $g\colon D' \to X$ with $D' \in \mathcal{D}$, there is a morphism $g'\colon D' \to D$ such that $g = g'f$. Similarly, there is defined a left $\mathcal{D}$-approximation of X.

Let A be a ring with identity. By an A-module we mean a left A-module. We denote by A-Mod the category of all A-modules, by A-mod the category of all finitely presented A-modules, and by A-proj (resp., A-inj) the category of finitely generated projective (resp., injective) A-modules.

Let X be an A-module. If $f\colon P \to X$ is a projective cover of X with P projective, then the kernel of f is called a *syzygy* of X, denoted by $\Omega(X)$. Dually, if $g\colon X \to I$ is an injective envelope with I injective, then the cokernel of g is called a *co-syzygy* of X, denoted by $\Omega^{-1}(X)$. Note that a syzygy or a co-syzygy of an A-module X (if it exists) is determined, up to isomorphism, uniquely by X. Hence, we may speak of the syzygy and the co-syzygy of a module.

It is well known that $\mathscr{K}(A\text{-Mod})$, $\mathscr{K}^{\mathrm{b}}(A\text{-Mod})$, $\mathscr{D}(A\text{-Mod})$, and $\mathscr{D}^{\mathrm{b}}(A\text{-Mod})$

all are triangulated categories. For simplicity, if A is a ring or algebra, we also write $\mathscr{C}(A)$ and $\mathscr{K}(A)$ for $\mathscr{C}(A\text{-Mod})$ and $\mathscr{K}(A\text{-Mod})$, respectively. Similarly, we will do the same abbreviations for other categories.

For further information on triangulated categories, we refer the reader to [30,46].

2.1 Rickard's Theorem for derived equivalences

In this section, we briefly recall a beautiful result of Rickard which describes the derived equivalences for algebras and rings. This result is a basis for the most of our discussion.

Recall that two rings R and S with identity are said to be *derived equivalent* if the derived categories $\mathscr{D}(R\text{-Mod})$ and $\mathscr{D}(S\text{-Mod})$ are equivalent as triangulated categories.

It is known that $\mathscr{D}^{\rm b}(R\text{-Mod})$ and $\mathscr{D}^{\rm b}(S\text{-Mod})$ are triangle equivalent if and only if so are $\mathscr{D}(R\text{-Mod})$ and $\mathscr{D}(S\text{-Mod})$.

The following is a useful description of derived equivalences by Rickard.

Theorem 2.1 [52] *For two rings A and B with identity, the following are equivalent*:

(a) $\mathscr{D}^{\rm b}(A\text{-Mod})$ *and* $\mathscr{D}^{\rm b}(B\text{-Mod})$ *are equivalent as triangulated categories*;

(b) $\mathscr{K}^{\rm b}(A\text{-proj})$ *and* $\mathscr{K}^{\rm b}(B\text{-proj})$ *are equivalent as triangulated categories*;

(c) $B \simeq \mathrm{End}_{\mathscr{K}^{\rm b}(A\text{-proj})}(T^\bullet)$, *where* $T^\bullet$ *is a complex in* $\mathscr{K}^{\rm b}(A\text{-proj})$ *satisfying*

(1) $T^\bullet$ *is self-orthogonal in* $\mathscr{K}^{\rm b}(A\text{-proj})$, *that is,* $\mathrm{Hom}_{\mathscr{K}^{\rm b}(A\text{-proj})}(T^\bullet, T^\bullet[i]) = 0$ *for all integers* $i \neq 0$, *and*

(2) $\mathrm{add}(T^\bullet)$ *generates* $\mathscr{K}^{\rm b}(A\text{-proj})$ *as a triangulated category.*

A complex $T^\bullet$ in $\mathscr{K}^{\rm b}(A\text{-proj})$ satisfying conditions (1) and (2) in Theorem 2.1 is called a *tilting complex* over A. Given a derived equivalence F between A and B, there is a unique (up to isomorphism) tilting complex $T^\bullet$ over A such that $FT^\bullet = B$. This complex $T^\bullet$ is called a tilting complex *associated to* F.

For Artin algebras, derived equivalences can be reformulated in terms of finitely generated modules: Two Artin R-algebras A and B are derived equivalent if and only if $\mathscr{D}^{\rm b}(A\text{-mod})$ and $\mathscr{D}^{\rm b}(B\text{-mod})$ are equivalent as triangulated categories.

To get derived equivalences, one may use tilting modules. Recall that a module T over a ring A is called a *tilting module* if

(1) T has a finite projective resolution $P^\bullet \to T$:

$$0 \to P^n \to \cdots \to P^0 \to T \to 0,$$

where each P^i is a finitely generated projective A-module;

(2) $\mathrm{Ext}_A^i(T,T) = 0$ for all $i > 0$; and

(3) there is an exact sequence

$$0 \to A \to T_0 \to \cdots \to T_m \to 0$$

of A-modules with each T_i in add(T).

Clearly, the projective resolution $P^\bullet$ of a tilting A-module T is a tilting complex over A. Thus, each tilting module supplies a derived equivalence between A and $\mathrm{End}_A(T)$ via the tilting module T.

There is a class of special tilting modules which were first discussed by Brenner-Butler [12]. Recall that an A-module T is then called a *BB-tilting* module (after the names of Brenner and Butler) if it is of the form $T = P \oplus \tau^-(S)$, where τ is the Auslander-Reiten translation, S is a simple, non-injective module such that

$$\mathrm{Hom}_A(D(A), S) = 0 = \mathrm{Ext}^1_A(S, S),$$

and P is the direct sum of all indecomposable projective modules which are not isomorphic to the projective cover of S. In case that S is projective, the BB-tilting module is called an *APR-tilting* module (after the name of Auslander, Platzeck and Reiten). For a more general definition of n-BB-tilting modules, we refer the reader to [35, Section 4].

2.2 Some invariants of derived equivalences

Though derived equivalent algebras may have significant differences in algebraic structures, they can still have many common features in other aspects. Here, we shall list some of invariants of derived equivalences. A property $\mathscr{P}$ is said to be *invariant* under derived equivalences provided that if a ring (or an algebra) A has the property $\mathscr{P}$, then so do all rings (or algebras) B which are derived equivalent to A.

The following theorem shows a few invariants of derived equivalences.

Theorem 2.2 *The following are invariants of derived equivalences between rings.*

(1) *The Hochschild* (*co-*)*homology and cyclic homology, in particular, the centers of rings* (*see* [41,54]).

(2) *The number of non-isomorphic simple modules if we are restricted to Artin algebras.*

(3) *Finiteness of global* (*or finitistic*) *dimensions* (*see* [30,40,50]).

(4) *The Cartan determinants, and the characteristic polynomials of Coxeter matrices if the Cartan matrices of Artin algebras are invertible* (*see* [31, Lemma 4.1]; *for a detailed proof, see* [59, Proposition 6.8.9]).

(5) *Algebraic K-groups* (*see* [26]).

(6) *Self-injectivity of algebras over an algebraically closed field* (*resp., symmetry of algebras over an arbitrary field*) (*see* [2,52]).

Thus, to understand some properties of a given algebra (or mathematical object), it may be convenient to pass to its derived equivalent algebras (or mathematical objects) which may be easy to handle. For example, to understand properties of weighted projective lines $\mathbb{X}$, Lenzing and Meltzer used Ringel's tubular algebras because the derived category of coherent sheaves over

$\mathbb{X}$ and the derived module category of tubular algebras are equivalent (see [44]). Another example is the well-known work of Beilinson who reduced the study of derived category of coherent sheaves over $\mathbb{P}^n$ to the one of a triangular matrix algebras (see [10]). For computation formulas of algebraic K-groups of matrix subrings, we refer to [58] where derived equivalences induced from $\mathcal{D}$-split sequences (see the next section) are used.

3 Short exact sequences and derived equivalences

In this section, we shall show that each short exact sequence or triangle leads to a derived equivalence of two rings which are defined by the objects in the sequence.

First, we recall the definition of relative split sequences from [35]. Suppose that $\mathcal{C}$ is an additive category and $\mathcal{D}$ is a full subcategory of $\mathcal{C}$. A sequence

$$X \xrightarrow{f} M \xrightarrow{g} Y$$

of morphisms between objects in $\mathcal{C}$ is said to be $\mathcal{D}$-*split* if the following three conditions are satisfied:

(1) $M \in \mathcal{D}$;

(2) f is a left $\mathcal{D}$-approximation of X, and g is a right $\mathcal{D}$-approximation of Y; and

(3) f is a kernel of g, and g is a cokernel of f.

Canonical examples of relative split sequences are almost split sequences, introduced by Auslander and Reiten. They are the most important sequences in the representation theory of Artin algebras (see, for example, [5] for more details). For an almost split sequence $0 \to X \to M \to Y \to 0$ in A-mod with A an Artin algebra, we take $\mathcal{C} = A$-mod and $\mathcal{D} = \mathrm{add}(M)$. Then the sequence $X \to M \to Y$ is a $\mathcal{D}$-split sequence in $\mathcal{C}$.

We have the following relationship between derived equivalences and relative split sequences.

Theorem 3.1 [35] *Let $\mathcal{C}$ be an additive category and M an object in $\mathcal{C}$. Suppose that $X \to M' \to Y$ is an $\mathrm{add}(M)$-split sequence in $\mathcal{C}$. Then the endomorphism rings $\mathrm{End}_{\mathcal{C}}(M \oplus X)$ and $\mathrm{End}_{\mathcal{C}}(M \oplus Y)$ are derived equivalent.*

In fact, the above derived equivalence between the endomorphism rings is given by a tilting module of projective dimension at most 1. Applying the above result to almost split sequences, we get a more substantial conclusion which reveals a close relation between BB-tilting modules and almost split sequences.

Corollary 3.2 *If $0 \to X \to M \to Y \to 0$ is an almost split sequence in A-mod, then $\mathrm{End}_A(X \oplus M)$ is derived equivalent to $\mathrm{End}_A(M \oplus Y)$ via a BB-tilting module.*

Since triangles in a triangulated category are a natural generalization of exact sequences, one may ask what happens with the above result for triangles.

Unfortunately, if we replace the almost split sequence by an Auslander-Reiten triangle in Corollary 3.2, then the result is no longer true in general. To get a more general statement, two generalizations of Theorem 3.1 are done in different directions. One is to use subalgebras of the endomorphism algebras, and the other is to pass to quotient algebras of the endomorphism algebras.

The first case is carried out by Yiping Chen [21] for exact sequences in abelian categories, and is then extended to triangles in triangulated categories by Shengyong Pan [48]. Recently, another two generalizations are given in [19] for additive categories and in [49] for a class of Beilinson-Green algebras. Here, we do not pursue these constructions and refer the reader to the original papers for details.

In the following, we summarize some constructions of derived equivalences via quotient algebras, these constructions deal with the so-called Auslander-Yoneda algebras. Let us now recall some relevant definitions from [36].

Let Φ be a subset of $\mathbb{N} := \{0, 1, 2, \dots\}$. Following [36], we call Φ an *admissible* subset of $\mathbb{N}$ if $0 \in \Phi$ and, for any $a, b, c \in \Phi$ with $a+b+c \in \Phi$, we have $a+b \in \Phi$ if and only if $b + c \in \Phi$.

For example, $\{0, 1, \dots, n\}$ and $n\mathbb{N} := \{nx \mid x \in \mathbb{N}\}$ are admissible for any n, and any subset $\{0, a, b\}$ of $\mathbb{N}$ is admissible. But the set $\{0, a, 2a, 2a+b\} \setminus \{a+b\}$ is not admissible for any nonzero natural numbers a and b.

Clearly, the definition of admissible sets in $\mathbb{N}$ can be extend to the one in $\mathbb{Z}$ (or in a monoid). For the purpose of our applications in this note, we restrict us just to subsets of $\mathbb{N}$. Admissible subsets Φ of $\mathbb{N}$ can be used to define associative algebras with identity as follows.

Let $\mathcal{T}$ be a triangulated k-category with k a comutative ring, and let X be an object in $\mathcal{T}$. We define

$$R(\mathcal{T}, \Phi, X) := \bigoplus_{i\in\Phi} \mathrm{Hom}_{\mathcal{T}}(X, X[i]).$$

The multiplication on $R(\mathcal{T}, \Phi, X)$ is given by

$$(f_i)_{i\in\Phi} \cdot (g_j)_{j\in\Phi} = (h_l)_{l\in\Phi},$$

where

$$h_l = \sum_{u,v\in\Phi,\, u+v=l} f_u(g_v[u]).$$

The admissibility of Φ ensures that $R(\mathcal{T}, \Phi, X)$ is an associative algebra. The condition $0 \in \Phi$ guarantees that this algebra has identity. Following [36], the algebra $R(\mathcal{T}, \Phi, X)$ is called the Φ-*Auslander-Yoneda algebra* of X.

Let us mention a few cases of this kind of algebras. Let $\mathcal{T}$ be $\mathscr{D}(A)$ with A a ring, and let X be an A-module, If $\Phi = \mathbb{N}$, then $R(\mathcal{T}, \mathbb{N}, X)$ is just the Yoneda algebra $\mathrm{Ext}^*_A(X)$ of X with concatenation of exact sequences as its multiplication. If $\Phi = \{0, a\}$, then $R(\mathcal{T}, \Phi, X)$ is precisely the trivial extension of $\mathrm{End}_A(X)$ by the bimodule $\mathrm{Ext}^a_A(X, X)$. If $\Phi = \{0, 1, \dots, n\}$, then $R(\mathcal{T}, \Phi, X)$ is the quotient algebra of the Yoneda algebra $\mathrm{Ext}^*_A(X)$ of X by the ideal

$\oplus_{i>n}\mathrm{Ext}_A^i(X)$. If $\Phi = n\mathbb{N}$, then $R(\mathcal{T},\Phi,X)$ is just the n-th Veronese algebra of $\mathrm{Ext}_A^*(X)$. For some properties of Veronese algebras of graded algebras in commutative algebra, we refer to [6].

Clearly, for each admissible subset Φ, one may define a Φ-orbit category $\mathcal{T}/\Phi$ of $\mathcal{T}$ in a natural way: The objects of $\mathcal{T}/\Phi$ are the same as the objects of $\mathcal{T}$, the Hom-set of two objects X and Y is defined as

$$\mathrm{Hom}_{\mathcal{T}/\Phi}(X,Y) := \bigoplus_{i\in\Phi}\mathrm{Hom}_{\mathcal{T}}(X,Y[i]),$$

and the composition of morphisms is defined in obvious way. Now, one has to check the associativity of the composition (see [36] for details). Here, there is an open question: For which admissible sets Φ are the orbit categories $\mathcal{T}/\Phi$ triangulated?

In the following, we assume that $\mathcal{T}$ is a triangulated k-category with k a field and M is an object in $\mathcal{T}$. Furthermore, we fix an admissible subset Φ of $\mathbb{N}$ and a triangle in $\mathcal{T}$:

$$X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \to X[1].$$

We define $\widetilde{\alpha}$ to be the diagonal morphism

$$\mathrm{diag}(\alpha,1)\colon M_1\oplus M \to X\oplus M,$$

and $\widetilde{\beta}$ the skew-diagonal morphism

$$\mathrm{skewdiag}(1,\beta)\colon M_1\oplus M \to M\oplus Y.$$

Recall that a morphism $f\colon X \to D$ in $\mathcal{T}$ is called a *left* $(\mathrm{add}(M),\Phi)$-*approximation* of X if $D\in\mathrm{add}(M)$ and for any $i\in\Phi$ and any morphism $h\colon X\to D'[i]$ with $D'\in\mathrm{add}(M)$, there is a morphism $h'\colon D\to D'[i]$ such that $h = fh'$. Similarly, we have the definition of right $(\mathrm{add}(M),\Phi)$-approximations. So an $(\mathrm{add}(M),\Phi)$-approximation of X must be an $\mathrm{add}(M)$-approximation of X in the sense of Auslander-Smalø.

The following result tells us how to get derived equivalences for Auslander-Yoneda algebras from triangles in a triangulated category.

Theorem 3.3 [34] *For the above-given triangle in $\mathcal{T}$, if the following two conditions are satisfied*:

(i) *the morphism α is a left* $(\mathrm{add}(M),\Phi)$-*approximation of X and β is a right* $(\mathrm{add}(M),-\Phi)$-*approximation of Y, and*

(ii) $\mathrm{Hom}_{\mathcal{T}}(M,X[i]) = 0 = \mathrm{Hom}_{\mathcal{T}}(Y[-i],M)$ *for all* $0\neq i\in\Phi$,

then the quotient rings $R(\mathcal{T},\Phi,M\oplus X)/I$ and $R(\mathcal{T},\Phi,M\oplus Y)/J$ are derived equivalent, where I is the ideal of $R(\mathcal{T},\Phi,M\oplus X)$ consisting of all elements $(x_i)_{i\in\Phi}$ such that $x_i = 0$ for $0\neq i\in\Phi$, x_0 factorizes through $\mathrm{add}(M)$ *and $x_0\widetilde{\alpha} = 0$, and where J is the ideal of $R(\mathcal{T},\Phi,M\oplus Y)$ consisting of all elements $(y_i)_{i\in\Phi}$ such that $y_i = 0$ for $0\neq i\in\Phi$, y_0 factorizes through* $\mathrm{add}(M)$ *and* $\widetilde{\beta}y_0 = 0$.

Theorem 3.3 looks complicated, but it supplies a large class of derived equivalences by flexible choices of Φ. Moreover, there are many cases where both I and J vanish. For example, this happens when we deal with exact sequences in the module categories of rings.

Corollary 3.4 *Let A be an Artin algebra, and let $M \in A$-mod. If*

$$0 \to X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \to 0$$

is an exact sequence in A-mod such that α is a left $(\mathrm{add}(M), \Phi)$-approximation of X and β is a right $(\mathrm{add}(M), -\Phi)$-approximation of Y in $\mathscr{D}^{\mathrm{b}}(A\text{-mod})$, and that

$$\mathrm{Ext}^i_A(M, X) = 0 = \mathrm{Ext}^i_A(Y, M)$$

for all $0 \neq i \in \Phi$, then the Φ-Auslander-Yoneda algebras of $X \oplus M$ and $M \oplus Y$ are derived equivalent.

A special case of Corollary 3.4 is the following situation of self-injective algebras. So we re-obtain the result [36, Corollary 3.4].

Corollary 3.5 [36] *If A is a self-injective Artin algebra, then, for any admissible subset Φ of $\mathbb{N}$ and any integer i, the Φ-Auslander-Yoneda algebras of $A \oplus X$ and $A \oplus \Omega^i(X)$ are derived equivalent.*

There are two further generalizations of Theorem 3.3. One is to extend it to n-angulated categories which are more general than triangulated categories. This is carried out in [20]. The other is to introduce one or two auto-functors of $\mathcal{T}$ into the definition of Φ-Auslander-Yoneda algebras, so that the result can be applied in a more general context (see [36, Appendix A]). For example, the AR-translation on derived module categories of hereditary algebras can be covered. In this case, preprojective algebras, introduced by Gelfand and Ponomarev [27], are included (see, for example, [7]). For further details of these generalizations, the interested reader is referred to the original papers [20,36].

Finally, let us mention a recent construction given by Dugas [25], where the conditions for approximations are modified.

Suppose that $\mathcal{T}$ is an algebraic, Krull-Schmidt triangulated category with a suspension denoted by [1]. Let $\mathcal{D}$ be a full subcategory of $\mathcal{T}$. We denote by $\langle\mathcal{D}\rangle$ the full, additive subcategory of $\mathcal{T}$ generated by $\cup_{i\in\mathbb{Z}}\mathcal{D}[i]$. The following result was proved in [25].

Theorem 3.6 [25] *Suppose that $\mathcal{T}$ contains a triangle*

$$X \xrightarrow{f} M' \xrightarrow{g} Y \to X[1],$$

where $M' \in \langle M\rangle$ for some $M \in \mathcal{T}$, and

(a) *f is a left $\langle M\rangle$-approximation,*

(b) *g is a right $\langle M\rangle$-approximation.*

Then

(1) *$R(\mathcal{T}, \mathbb{Z}, X\oplus M)$ and $R(\mathcal{T}, \mathbb{Z}, M\oplus Y)$ are derived equivalent*;

(2) *for any $M''\in\langle M\rangle$ with $M'\in \mathrm{add}(M'')$, the rings*

$$\Lambda := \mathrm{End}_{\mathcal{T}}(M''\oplus X), \quad \Gamma := \mathrm{End}_{\mathcal{T}}(M''\oplus Y),$$

are derived equivalent.

This result is then applied to get derived equivalent, symmetric algebras.

Corollary 3.7 [25, Theorem 5.2] *Let A be a finite-dimensional, symmetric k-algebra, and let X, M be any complexes in $\mathscr{K}^{\mathrm{b}}(A\text{-proj})$. Then there exists a left $\langle M\rangle$-approximation $f\colon X\to M'$ of X in $\mathscr{K}(A)$. If Y is the mapping cone of f, then*

(1) *$R(\mathscr{K}(A), \mathbb{Z}, X\oplus M)$ and $R(\mathscr{K}(A), \mathbb{Z}, Y\oplus M)$ are derived equivalent, symmetric algebras*;

(2) *$\mathrm{End}_{\mathscr{K}(A)}(X\oplus M'')$ and $\mathrm{End}_{\mathscr{K}(A)}(Y\oplus M'')$ are derived equivalent, symmetric algebras for any $M''\in\langle M\rangle$ with $M'\in \mathrm{add}(M'')$.*

Note that the symmetry of the endomorphism algebras in the above corollary follows from a simple but an interesting observation (see [25, Proposition 5.1]): Let A be a finite-dimensional, symmetric k-algebra. Then for any $X^\bullet\in\mathscr{K}^{\mathrm{b}}(A\text{-proj})$, the rings $\mathrm{End}_{\mathscr{K}(A)}(X^\bullet)$ and $R(\mathscr{K}(A), \mathbb{Z}, X^\bullet)$ are finite-dimensional, symmetric k-algebras.

For applications to Calabi-Yau categories, one may also find them in [25].

4 Derived equivalences constructed from given ones

In this section, we give two methods to construct new derived equivalences from given ones. One is to form Φ-Auslander-Yoneda algebras, and the other is to form quotient algebras of derived equivalent algebras.

Throughout this section, we consider $\mathcal{T}=\mathscr{D}^{\mathrm{b}}(A\text{-mod})$ with A an Artin algebra.

To state our results, we first introduce a few terminologies.

Suppose that

$$F\colon \mathscr{D}^{\mathrm{b}}(A\text{-mod})\to\mathscr{D}^{\mathrm{b}}(B\text{-mod})$$

is a derived equivalence between two Artin algebras A and B, with the quasi-inverse functor G. Furthermore, suppose that

$$T^\bullet\colon \cdots\to 0\to T^{-n}\to\cdots\to T^{-1}\to T^0\to 0\to\cdots$$

is a radical tilting complex over A associated to F, and suppose that

$$\overline{T}^\bullet\colon \cdots\to 0\to \overline{T}^0\to \overline{T}^1\to\cdots\to \overline{T}^n\to 0\to\cdots$$

is a radical tilting complex over B associated to G. The functor F is called *almost ν-stable* if

$$\mathrm{add}\Big(\bigoplus_{i=-1}^{-n} T^i\Big) = \mathrm{add}\Big(\bigoplus_{i=-1}^{-n} \nu_A T^i\Big)$$

and

$$\mathrm{add}\Big(\bigoplus_{i=1}^{n}\overline{T}^{i}\Big)=\mathrm{add}\Big(\bigoplus_{i=1}^{n}\nu_B\overline{T}^{i}\Big),$$

where ν_A is the Nakayama functor for A.

Given an almost ν-stable functor F, it was shown in [37] that there is an equivalence functor $\overline{F}$, associated to F, between the stable module categories A-$\underline{\mathrm{mod}}$ and B-$\underline{\mathrm{mod}}$.

Theorem 4.1 [36] *Let A and B be two Artin algebras, and let*

$$\overline{F}\colon A\text{-}\underline{\mathrm{mod}}\to B\text{-}\underline{\mathrm{mod}}$$

be the stable equivalence induced by an almost ν-stable derived equivalence F between A and B. Suppose that X is an A-module, we set

$$M:=A\oplus X,\quad N:=B\oplus\overline{F}(X).$$

Let Φ be an admissible subset of $\mathbb{N}$. Then

(1) *the Φ-Auslander-Yoneda algebras $R(A\text{-mod},\Phi,M)$ and $R(B\text{-mod},\Phi,N)$ are derived equivalent*;

(2) *if Φ is finite, then there is an almost ν-stable derived equivalence between the Φ-Auslander-Yoneda algebras $R(A\text{-mod},\Phi,M)$ and $R(B\text{-mod},\Phi,N)$, and in particular, there is an almost ν-stable derived equivalence and a stable equivalence between $\mathrm{End}_A(M)$ and $\mathrm{End}_B(N)$.*

Recall that an Auslander algebra is by definition an algebra of the form $\mathrm{End}_B(X)$, where B is a representation-finite Artin algebra and X is the direct sum of all non-isomorphic indecomposable B-modules. Since Auslander algebras and Yoneda algebras are special classes of Φ-Auslander-Yoneda algebras, Theorem 4.1 supplies a lot of examples of derived equivalences between Auslander algebras, and between Yoneda algebras. For example, we have the following corollary for self-injective algebras.

Corollary 4.2 [36] (1) *For a self-injective Artin algebra A and an A-module Y, the Φ-Auslander-Yoneda algebras of $A\oplus Y$ and $A\oplus\Omega^i_A(Y)$ are derived equivalent for all $i\in\mathbb{Z}$, where Ω is the syzygy operator.*

(2) *Suppose that A and B are self-injective Artin algebras of finite representation type with ${}_AX$ and ${}_BY$ additive generators for A-mod and B-mod, respectively. If A and B are derived equivalent, then*

(i) *the Auslander algebras of A and B are both derived and stably equivalent*;

(ii) *the Yoneda algebra $\mathrm{Ext}^*_A(X)$ of X and the Yoneda algebra $\mathrm{Ext}^*_B(Y)$ of Y are derived equivalent.*

Note that for self-injective algebras, every derived equivalence (up to shift) is almost ν-stable, and the syzygy functor on stable categories is closely related to the auto-equivalence functor $K\otimes_A -$, where K is a kernel of the multiplication map $A \otimes A \to A$. This explains why Theorem 4.1 can be applied to self-injective algebras.

Another natural idea for getting derived equivalences from given ones is to pass to quotient algebras.

Suppose that A is an Artin algebra and I is an ideal in A. Let $\overline{A} := A/I$. Then the category $\overline{A}$-mod can be regarded as a full subcategory of A-mod. There is a canonical functor from A-mod to $\overline{A}$-mod which sends each $X \in A$-mod to $\overline{X} := X/IX$. This functor induces a functor $^{-}\colon \mathscr{C}(A) \to \mathscr{C}(\overline{A})$, which sends $X^\bullet$ to the quotient complex $\overline{X}^\bullet := X^\bullet/IX^\bullet$, where $IX^\bullet = (IX^i)_{i\in\mathbb{Z}}$ is a sub-complex of $X^\bullet$. The action of $^{-}$ on a chain map can be defined canonically. For each complex $X^\bullet$ of A-modules, we have a canonical exact sequence of complexes:

$$0 \to IX^\bullet \xrightarrow{i^\bullet} X^\bullet \xrightarrow{\pi^\bullet} \overline{X}^\bullet \to 0.$$

For a complex $Y^\bullet$ of $\overline{A}$-modules, this sequence induces another exact sequence:

$$0 \to \mathrm{Hom}_{\mathscr{C}(A)}(\overline{X}^\bullet, Y^\bullet) \xrightarrow{\pi^*} \mathrm{Hom}_{\mathscr{C}(A)}(X^\bullet, Y^\bullet) \xrightarrow{i^*} \mathrm{Hom}_{\mathscr{C}(A)}(IX^\bullet, Y^\bullet).$$

Since $Y^\bullet \in \mathscr{C}(\overline{A})$, the map i^* must be zero, and consequently, π^* is an isomorphism. Moreover, π^* actually induces an isomorphism between $\mathrm{Hom}_{\mathscr{K}(A)}(\overline{X}^\bullet, Y^\bullet)$ and $\mathrm{Hom}_{\mathscr{K}(A)}(X^\bullet, Y^\bullet)$.

Given arbitrary complexes $X^\bullet$ and $X'^\bullet$ of A-modules, we have a natural map

$$\eta\colon \mathrm{Hom}_{\mathscr{K}(A)}(X^\bullet, X'^\bullet) \to \mathrm{Hom}_{\mathscr{K}(A)}(\overline{X}^\bullet, \overline{X'}^\bullet),$$

which is the composite of

$$\pi^\bullet_*\colon \mathrm{Hom}_{\mathscr{K}(A)}(X^\bullet, X'^\bullet) \to \mathrm{Hom}_{\mathscr{K}(A)}(X^\bullet, \overline{X'}^\bullet)$$

with the map $(\pi^*)^{-1}$. In particular, if $X^\bullet = X'^\bullet$, then we get a homomorphism of algebras

$$\eta\colon \mathrm{End}_{\mathscr{K}(A)}(X^\bullet) \to \mathrm{End}_{\mathscr{K}(A)}(\overline{X}^\bullet).$$

Now, let $T^\bullet$ be a tilting complex over A, and let

$$B = \mathrm{End}_{\mathscr{K}(A)}(T^\bullet).$$

Furthermore, suppose that I is an ideal in A. By the above discussion, there is an algebra homomorphism

$$\eta\colon \mathrm{End}_{\mathscr{K}(A)}(T^\bullet) \to \mathrm{End}_{\mathscr{K}(A)}(\overline{T}^\bullet).$$

Let J_I be the kernel of η, which is an ideal of B. We define $\overline{B} := B/J_I$. Then we have the following theorem on quotient algebras.

Theorem 4.3 [36] *Let A be an Artin algebra, and let $T^\bullet$ be a tilting complex over A with the endomorphism algebra*

$$B = \mathrm{End}_{\mathscr{K}^{\mathrm{b}}(A)}(T^\bullet).$$

Suppose that I is an ideal in A. Then $\overline{T}^{\bullet}$ is a tilting complex over $\overline{A}$ and induces a derived equivalence between $\overline{A}$ and $\overline{B}$ if and only if

$$\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A)}(T^{\bullet}, IT^{\bullet}[i]) = 0 \ (\forall\, i \neq 0), \quad \mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A)}(\overline{T}^{\bullet}, \overline{T}^{\bullet}[-1]) = 0.$$

Applying this result to self-injective algebras, we have the following result for a class of quotient algebras.

Corollary 4.4 [36] *Let $F\colon \mathscr{D}^{\mathrm{b}}(A) \to \mathscr{D}^{\mathrm{b}}(B)$ be a derived equivalence between two self-injective, basic Artin algebras A and B. Suppose that P is a direct summand of ${}_AA$, and Q is a direct summand of ${}_BB$ such that $F(\mathrm{soc}(P))$ is isomorphic to $\mathrm{soc}(Q)$, where $\mathrm{soc}(P)$ denotes the socle of the module P. Then the quotient algebras $A/\mathrm{soc}(P)$ and $B/\mathrm{soc}(Q)$ are derived equivalent.*

5 Derived equivalences from stable equivalences

In the foregoing discussion, what we have done is to get a new derived equivalence from a given derived equivalence. Now, we consider how to get a derived equivalence from a given stable equivalence.

Asashiba [4] showed that, for representation-finite, standard self-injective k-algebras A and B not of type $(D_{3m}, s/3, 1)$ with $m \geqslant 2$ and $3 \nmid s$, each individual stable equivalence between A and B can be lifted to a derived equivalence. His proof is based on his classification of derived equivalences for representation-finite, standard self-injective algebras in [3]. The case left by Asashiba is handled recently by Dugas [24]. Thus, every stable equivalence between representation-finite, standard self-injective algebras over an algebraically closed field can be lifted to a derived equivalence. In [38], we extend this result somehow to Frobenius-finite algebras, including representation finite algebras.

Let A be an Artin algebra. Recall from [45] that an A-module X is said to be *ν-stably projective* if $\nu_A^i X$ is projective for all $i \geqslant 0$, where ν_A stands for the Nakayama functor of A. The full subcategory of all ν-stably projective A-modules is denoted by A-stp. Clearly, there is an idempotent element $e \in A$ such that Ae is a basic A-module with $\mathrm{add}(Ae) = A$-stp. The algebra eAe is called the *Frobenius part* of A. It is a self-injective algebra (see [45], and also [38, Lemma 2.5]) and uniquely (up to Morita equivalence) determined by A. The algebra A is said to be *Frobenius-finite* (*-tame, or -wild*) if its Frobenius part eAe is representation-finite (-tame, or -wild).

For Frobenius-finite algebras, a special type of stable equivalences can always be lifted to derived equivalences. They are the so-called stable equivalences of Morita type introduced by Broué (see [13]). Recall that two finite-dimensional algebras A and B over a field is said to be *stably equivalent of Morita type* if there are bimodules ${}_AM_B$ and ${}_BN_A$ such that

(1) M and N are all projective as one-sided modules; and

(2) $M \otimes_B N \simeq A \oplus P$ as A-A-bimodules for some projective A-A-bimodule

P, and $N \otimes_A M \simeq B \oplus Q$ as B-B-bimodules for some projective B-B-bimodule Q.

Clearly, such a bimodule M induces an equivalence of the stable module categories of A and B. The following result in [38] shows that one can always get a derived equivalence from a stable equivalence of Morita type between Frobenius-finite algebras.

Theorem 5.1 [38] *Let A and B be finite-dimensional k-algebras over an algebraically closed field and without semisimple direct summands. If A is Frobenius-finite, then each individual stable equivalence of Morita type between A and B gives rise a derived equivalence.*

We remark that the methods developed in [38] can be used to check Broué's Abelian Defect Group Conjecture for many cases studied by Okuyama [47]. Also, observe that Frobenius-finite algebras include Auslander algebras and cluster-tilted algebras. For more details and examples of Frobenius-finite algebras, we refer the reader to [38, Section 5]. Unfortunately, derived equivalent algebras may have different Frobenius parts that are not derived equivalent (see [39]).

6 Some other constructions

Related to constructions of derived equivalences, the following question seems to be interesting.

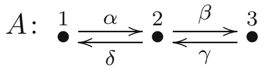
Suppose that algebras are given by quivers with relations, how can we get derived equivalences from given ones between these algebras just by certain operations on quivers and relations?

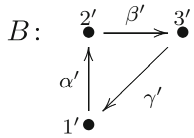
Of course, one may think of mutations of quivers in cluster tilting theory, but, as we know, this will not provide derived equivalences of algebras in general.

Recently, Ladkani has constructed some interesting derived equivalent algebras by tensor products [43], where algebras with a linear quiver can be derived equivalent to algebras with a triangle quiver or a rectangle quiver. So one gets derived equivalences directly by certain "operations" on quivers and relations.

More recently, we use operations like gluing vertices, unifying arrows and identifying socle elements, on quivers with relations to construct derived equivalences from given ones. These techniques fit well in the framework of constructing derived equivalences for pullback algebras. The details of these constructions will be included in a forthcoming paper [39]. Here, we merely give a simple example to illustrate the procedure of gluing vertices.

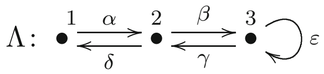
Example Let A and B be algebras given by the following quivers with relations, respectively:

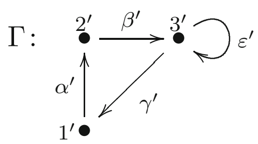




$\alpha\delta\alpha$, $\alpha\beta\gamma$, $\gamma\beta\gamma$, $\gamma\delta$, $\delta\alpha-\beta\gamma$ $\qquad$ $\alpha'\beta'\gamma'\alpha'$, $\gamma'\alpha'\beta'\gamma'$

Then it was shown in [39] that there is a derived equivalence between A and B, which sends the simple A-module corresponding to the vertex 3 to the simple B-module corresponding to the vertex $3'$. Now, if we glue a loop at the vertices 3 and $3'$ in A and B, respectively, then, by a result in [39], we can get a derived equivalence between the resulting algebras Λ and Γ (for $n \geqslant 1$):





$\alpha\delta\alpha$, $\alpha\beta\gamma$, $\gamma\beta\gamma$, $\gamma\delta$, $\delta\alpha-\beta\gamma$, ε^n, $\beta\varepsilon$, $\varepsilon\gamma$ $\qquad$ $\alpha'\beta'\gamma'\alpha'$, $\gamma'\alpha'\beta'\gamma'$, $(\varepsilon')^n$, $\beta'\varepsilon'$, $\varepsilon'\gamma'$

In fact, one can glue any given algebra kQ/I at the vertices 3 and $3'$ in A and B, respectively, so that the resulting algebras are derived equivalent.

In the above example, the algebras A and B are subalgebras of Λ and Γ, respectively. So, we have extended the derived equivalence between two algebras A and B to a derived equivalence between their extension algebras. For more examples, we refer to the preprint [39].

A further generalization of derived equivalences is the notion of recollements of derived (or triangulated) categories introduced originally in [11] to describe the derived categories of perverse sheaves. Now, recollements are broadly used in the study of rings and homological dimensions (see, for instance, [17,18,23,28,29,42,51]) and of infinitely generated tilting modules (see, for example, [9,14,16]). Here, we will not touch all of these topics. We just refer the reader to [15,16] for some recent results on constructions of recollements of derived module categories.

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