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# EXACT CONTEXTS, NONCOMMUTATIVE TENSOR PRODUCTS, AND UNIVERSAL LOCALIZATIONS

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Abstract. Exact contexts and their noncommutative tensor products are introduced which generalize the notions of Milnor squares and usual tensor products over commutative rings, respectively. Exact contexts are characterized by rigid morphisms which exist abundantly, while noncommutative tensor products not only capture some useful constructions in ring theory (such as coproducts of rings and trivially twisted extensions) but also provide a new method to construct universal localizations with rich homological and structural information. Moreover, sufficient and necessary conditions in terms of the data of exact contexts are presented to ensure that the universal localizations constructed are homological ring epimorphisms.

Contents



## 1. Introduction

Recollements of triangulated categories were introduced by Beilinson, Bernstein, and Deligne in 1982 in the context of derived categories of perverse sheaves over singular spaces, providing a categorical framework for Grothendieck's six functors (see [5,17]). They are used in different aspects spreading from algebraic geometry to

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algebraic topology and $K$-theory and recently to representation theory, particularly in the contexts of homological invariants and tilting theory (see [1, 4, 9–12, 18]).

There are many methods known to get recollements of triangulated categories, for example, taking stable categories of Frobenius categories, or Verdier quotients of triangulated categories. But little is known about constructing recollements with all three terms being derived module categories of rings. Notably, such a recollement of rings (or algebras) will provide a reduction method to study derived or homological invariants (for instance, finitistic dimension and algebraic $K$-theory) of one of the three rings through those of the other two rings (see, for example, [8, 9, 18]).

Motivated by creating methods to construct recollements of derived categories of rings, we first introduce the concept of exact contexts in this paper, which is a generalization of Milnor squares of rings in [20]. We show that exact contexts can be obtained from rigid morphisms in any additive category. This covers a large variety of morphisms in categories such as all precovers and pre-envelopes in ring theory or equivalently all left and right approximations in representation theory (see [3]), extensions of rings, and the canonical projections to their quotients. Further, using the data of an exact context, we then construct a unique new ring, called the noncommutative tensor product of the given exact context. Surprisingly, these noncommutative tensor products not only generalize the usual tensor products over commutative rings but also capture many known constructions in ring theory: coproducts of rings, trivially twisted extensions, and endomorphism rings of tensor products of modules. Moreover, for each exact context, we can also construct a ring homomorphism $\theta$ from a two-by-two triangular matrix ring (with entries in rings and bimodules) to the two-by-two full matrix ring over the noncommutative tensor product of the exact context. We prove that this ring homomorphism $\theta$ is actually a universal localization in the sense of Cohn and Schofield (see [15, 26]). In general, a universal localization does not have to be homological (see [23]). But it is of significant importance to know when it is homological because a homological universal localization can provide a recollement of triangulated categories and also a Mayer–Vietoris sequence in algebraic $K$-theory [8, 10, 22, 25]. However, little was known about answers to this question. In this paper, with our description of noncommutative tensor products, we can present a sufficient and necessary condition for the localization $\theta$ to be homological in terms of the data of exact contexts. This shows that the interpretation of universal localizations as noncommutative tensor products has an advantage: some homological and structural properties of these localizations can be formulated and verified in a convenient way by the data of exact contexts, while the description by generators and relations in [27] seems not to work well for homological aspects.

To state our results precisely, let us briefly introduce some terminology.

Let $R$, $S$, and $T$ be rings with identity, and let $\lambda : R \to S$ and $\mu : R \to T$ be ring homomorphisms. Suppose that $M$ is an $S$-$T$-bimodule with a fixed element $m \in M$. The quadruple $(\lambda, \mu, M, m)$ is called an *exact context* if the sequence

$$0 \longrightarrow R \xrightarrow{(\lambda,\mu)} S \oplus T \xrightarrow{\binom{\cdot m}{-m\cdot}} M \longrightarrow 0$$

is an exact sequence of abelian groups, where $\cdot m$ and $m\cdot$ denote the right and left multiplication maps by $m$, respectively.

**Theorem 1.1.** *Let $(\lambda, \mu, M, m)$ be an exact context.*

(1) *There exists a ring* $T \boxtimes_R S$, *with two ring homomorphisms* $\rho : S \to T \boxtimes_R S$ *and* $\phi : T \to T \boxtimes_R S$, *and a homomorphism* $\beta : M \to T \boxtimes_R S$ *of* $S$-$T$-*bimodules such that the ring homomorphism*

$$\theta := \begin{pmatrix} \rho & \beta \\ 0 & \phi \end{pmatrix} : \quad \begin{pmatrix} S & M \\ 0 & T \end{pmatrix} \longrightarrow \begin{pmatrix} T \boxtimes_R S & T \boxtimes_R S \\ T \boxtimes_R S & T \boxtimes_R S \end{pmatrix}$$

*is a universal localization. This ring is uniquely determined by* $(\lambda, \mu, M, m)$ *and is called the noncommutative tensor product of* $(\lambda, \mu, M, m)$.

(2) *The homomorphism* $\theta$ *in* (1) *is homological if and only if* $\mathrm{Tor}_i^R(T, S) = 0$ *for all* $i \geq 1$.

The contents of this paper are outlined as follows. In Section 2, we fix notation and recall some definitions and basic facts used throughout the paper. In particular, we recall the definitions of homological ring epimorphisms and universal localizations. In Section 3, we introduce exact contexts and their noncommutative tensor products. Moreover, we show that exact contexts can be described by rigid morphisms, which exist almost everywhere in representation theory. For example, all kinds of approximations are rigid morphisms. Also, we demonstrate that noncommutative tensor products capture usual tensor products over commutative rings. This section contributes an ingredient to the proof of the main theorem. In Section 4, we give explicit constructions of the noncommutative tensor products of a few classes of exact contexts. In Section 5, we present a categorical description of noncommutative tensor products and prove Theorem 1.1. Consequently, we show that noncommutative tensor products cover coproducts in ring theory and the endomorphism rings of tensor products of modules (see Corollary 5.4).

In a series of papers [7–9], we apply exact contexts and results in this paper to construct recollements of derived module categories and to establish both additive formula for $K_n$-groups and relations among homological dimensions for recollements of derived module categories.

## 2. Preliminaries

In this section, we shall recall some definitions, notation, and basic results needed in our proofs.

2.1. **Notation and basic facts on derived categories.** Let $\mathcal{C}$ be an additive category.

Throughout the paper, a full subcategory $\mathcal{B}$ of $\mathcal{C}$ is always assumed to be closed under isomorphisms; that is, if $X \in \mathcal{B}$ and $Y \in \mathcal{C}$ with $Y \simeq X$, then $Y \in \mathcal{B}$.

Given two morphisms $f : X \to Y$ and $g : Y \to Z$ in the category $\mathcal{C}$, we denote the composition of $f$ and $g$ by $fg : X \to Z$. The induced morphisms $\mathrm{Hom}_{\mathcal{C}}(Z, f) : \mathrm{Hom}_{\mathcal{C}}(Z, X) \to \mathrm{Hom}_{\mathcal{C}}(Z, Y)$ and $\mathrm{Hom}_{\mathcal{C}}(f, Z) : \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$ are denoted by $f^*$ and $f_*$, respectively. The composition of a functor $F : \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ with a functor $G : \mathcal{D} \to \mathcal{E}$ between categories $\mathcal{D}$ and $\mathcal{E}$ is denoted by $GF$, which is a functor from $\mathcal{C}$ to $\mathcal{E}$. The kernel and the image of the functor $F$ are denoted by $\mathrm{Ker}(F)$ and $\mathrm{Im}(F)$, respectively.

Let $\mathscr{C}(\mathcal{C})$ be the category of all complexes over $\mathcal{C}$ with chain maps, and let $\mathscr{K}(\mathcal{C})$ be the homotopy category of $\mathscr{C}(\mathcal{C})$. When $\mathcal{C}$ is abelian, the derived category of $\mathcal{C}$ is denoted by $\mathscr{D}(\mathcal{C})$. Both $\mathscr{K}(\mathcal{C})$ and $\mathscr{D}(\mathcal{C})$ are triangulated categories. For a triangulated category, its shift functor is denoted by [1] universally.

If $\mathcal{T}$ is a triangulated category with small coproducts (that is, coproducts indexed over sets existing in $\mathcal{T}$), then, for each object $U$ in $\mathcal{T}$, we denote by $\mathrm{Tria}(U)$ the smallest full triangulated subcategory of $\mathcal{T}$ containing $U$ and being closed under small coproducts. We mention the following properties related to $\mathrm{Tria}(U)$.

Let $F : \mathcal{T} \to \mathcal{T}'$ be a triangle functor of triangulated categories, and let $\mathcal{Y}$ be a full subcategory of $\mathcal{T}'$. We define $F^{-1}\mathcal{Y} := \{X \in \mathcal{T} \mid F(X) \in \mathcal{Y}\}$.

(1) If $\mathcal{Y}$ is a triangulated subcategory, then $F^{-1}\mathcal{Y}$ is a full triangulated subcategory of $\mathcal{T}$.
(2) Suppose that $\mathcal{T}$ and $\mathcal{T}'$ admit small coproducts and that $F$ commutes with coproducts. If $\mathcal{Y}$ is closed under small coproducts in $\mathcal{T}'$, then $F^{-1}\mathcal{Y}$ is closed under small coproducts in $\mathcal{T}$. In particular, for an object $U \in \mathcal{T}$, we have $F(\mathrm{Tria}(U)) \subseteq \mathrm{Tria}(F(U))$.

In this paper, all rings are associative and with identity, and all ring homomorphisms preserve identity. Unless stated otherwise, all modules are left modules.

Let $R$ be a ring. We denote by $R$-Mod the category of all unitary $R$-modules. By our convention of the composition of two morphisms, if $f : M \to N$ is a homomorphism of $R$-modules, then the image of $x \in M$ under $f$ is denoted by $(x)f$ instead of $f(x)$.

As usual, we shall simply write $\mathscr{C}(R)$, $\mathscr{K}(R)$, and $\mathscr{D}(R)$ for $\mathscr{C}(R\text{-Mod})$, $\mathscr{K}(R\text{-Mod})$, and $\mathscr{D}(R\text{-Mod})$, respectively, and identify $R$-Mod with the subcategory of $\mathscr{D}(R)$ consisting of all stalk complexes concentrated in degree 0.

Let $(X^\bullet, d_X^\bullet)$ and $(Y^\bullet, d_Y^\bullet)$ be two chain complexes over $R$-Mod. The mapping cone of a chain map $h^\bullet : X^\bullet \to Y^\bullet$ is usually denoted by $\mathrm{Con}\,(h^\bullet)$. In particular, there is a distinguished triangle $X^\bullet \xrightarrow{h^\bullet} Y^\bullet \to \mathrm{Con}(h^\bullet) \to X^\bullet[1]$ in $\mathscr{K}(R)$. For each $n \in \mathbb{Z}$, the $n$th cohomology functor from $\mathscr{D}(R)$ to $R$-Mod is denoted by $H^n(-)$. Certainly, this functor is naturally isomorphic to the Hom-functor $\mathrm{Hom}_{\mathscr{D}(R)}(R, -[n])$.

Now we shall recall some basic facts about derived functors of derived module categories of rings. For details and proofs, we refer to [6, 19].

Let $\mathscr{K}(R)_P$ (resp., $\mathscr{K}(R)_I$) be the smallest full triangulated subcategory of $\mathscr{K}(R)$ which

(i) contains all bounded above (resp., bounded below) complexes of projective (resp., injective) $R$-modules, and
(ii) is closed under arbitrary direct sums (resp., direct products).

It is known that $\mathscr{K}(R)_P$ is contained in $\mathscr{K}(R\text{-Proj})$, where $R$-Proj is the full subcategory of $R$-Mod consisting of all projective $R$-modules. Moreover, the composition functors

$$\mathscr{K}(R)_P \hookrightarrow \mathscr{K}(R) \longrightarrow \mathscr{D}(R) \quad \text{and} \quad \mathscr{K}(R)_I \hookrightarrow \mathscr{K}(R) \longrightarrow \mathscr{D}(R)$$

are equivalences of triangulated categories. Thus, for each complex $X^\bullet$ in $\mathscr{D}(R)$, there exists a complex ${}_pX^\bullet \in \mathscr{K}(R)_P$ together with a quasi-isomorphism ${}_pX^\bullet \to X^\bullet$, and a complex ${}_iX^\bullet \in \mathscr{K}(R)_I$ together with a quasi-isomorphism $X^\bullet \to {}_iX^\bullet$. The complex ${}_pX^\bullet$ is called the *projective resolution* of $X^\bullet$ in $\mathscr{K}(R)$. For example, if $X$ is an $R$-module, then we can take ${}_pX$ to be a deleted projective resolution of ${}_RX$.

If either $X^\bullet \in \mathscr{K}(R)_P$ or $Y^\bullet \in \mathscr{K}(R)_I$, then the canonical localization functor from $\mathscr{K}(R)$ to $\mathscr{D}(R)$ induces an isomorphism: $\mathrm{Hom}_{\mathscr{K}(R)}(X^\bullet, Y^\bullet) \simeq \mathrm{Hom}_{\mathscr{D}(R)}(X^\bullet, Y^\bullet)$.

For any triangle functor $H : \mathscr{K}(R) \to \mathscr{K}(S)$, there is a left derived functor $\mathbb{L}H : \mathscr{D}(R) \to \mathscr{D}(S)$ defined by $X^\bullet \mapsto H({}_pX^\bullet)$, and a right derived functor $\mathbb{R}H : \mathscr{D}(R) \to \mathscr{D}(S)$ defined by $X^\bullet \mapsto H({}_iX^\bullet)$. We say that $H$ preserves *acyclicity* if $H(X^\bullet)$ is acyclic whenever $X^\bullet$ is acyclic. So, if $H$ preserves acyclicity, then $H$ induces a triangle functor $D(H) : \mathscr{D}(R) \to \mathscr{D}(S)$ defined by $X^\bullet \mapsto H(X^\bullet)$. In this case, $\mathbb{L}H = \mathbb{R}H = D(H)$ up to natural isomorphism. As usual, $D(H)$ is called the *derived functor* of $H$.

Let $M^\bullet$ be a complex of $R$-$S$-bimodules. Then the tensor and Hom-functors

$$M^\bullet \otimes^\bullet_S - : \mathscr{K}(S) \longrightarrow \mathscr{K}(R) \quad \text{and} \quad \mathrm{Hom}^\bullet_R(M^\bullet, -) : \mathscr{K}(R) \longrightarrow \mathscr{K}(S)$$

form an adjoint pair of triangle functors. Denote by $M^\bullet \otimes^{\mathbb{L}}_S -$ the left derived functor of $M^\bullet \otimes^\bullet_S -$, and by $\mathbb{R}\,\mathrm{Hom}_R(M^\bullet, -)$ the right derived functor of $\mathrm{Hom}^\bullet_R(M^\bullet, -)$. Then $\big(M^\bullet \otimes^{\mathbb{L}}_S -, \mathbb{R}\,\mathrm{Hom}_R(M^\bullet, -)\big)$ is an adjoint pair of triangle functors.

2.2. **Homological ring epimorphisms and universal localizations.** Let $\lambda : R \to S$ be a homomorphism of rings. We denote by $\lambda_* : S\text{-Mod} \to R\text{-Mod}$ the restriction functor induced by $\lambda$, and by $D(\lambda_*) : \mathscr{D}(S) \to \mathscr{D}(R)$ the derived functor of the exact functor $\lambda_*$. Recall that a ring epimorphism $\lambda : R \to S$ is *homological* if and only if $\mathrm{Tor}^R_j(S, S) = 0$ for all $j \geq 1$ if and only if $S \otimes^{\mathbb{L}}_R S \simeq S$ in $\mathscr{D}(S)$. Note that $\lambda_*$ is fully faithful if and only if $\lambda$ is a ring epimorphism, and that $D(\lambda_*)$ is fully faithful if and only if $\lambda$ is a homological ring epimorphism. For a homological ring epimorphism $\lambda$, $\mathrm{Tor}^R_i(X, Y) \simeq \mathrm{Tor}^S_i(X, Y)$, and $\mathrm{Ext}^i_S(Y, Z) \simeq \mathrm{Ext}^i_R(Y, Z)$ for all $i \geq 0$, all right $S$-modules $X$, and all $S$-modules $Y$ and $Z$ (see [16, Theorem 4.4]).

Ring epimorphisms appear typically in localizations of commutative rings but also come up in a more general context of universal localizations of arbitrary rings.

**Lemma 2.1** (see [15, 26])**.** *Let $R$ be a ring, and let $\Sigma$ be a set of homomorphisms between finitely generated projective $R$-modules. Then there is a ring $R_\Sigma$ and a homomorphism $\lambda_\Sigma : R \to R_\Sigma$ of rings such that the following apply.*

(1) *$\lambda_\Sigma$ is $\Sigma$-inverting; that is, if $\alpha : P \to Q$ belongs to $\Sigma$, then $R_\Sigma \otimes_R \alpha : R_\Sigma \otimes_R P \to R_\Sigma \otimes_R Q$ is an isomorphism of $R_\Sigma$-modules.*
(2) *$\lambda_\Sigma$ is universally $\Sigma$-inverting; that is, if $S$ is a ring such that there exists a $\Sigma$-inverting homomorphism $\varphi : R \to S$, then there exists a unique homomorphism $\psi : R_\Sigma \to S$ of rings such that $\varphi = \lambda_\Sigma \psi$.*
(3) *$\lambda_\Sigma : R \to R_\Sigma$ is a ring epimorphism with $\mathrm{Tor}^R_1(R_\Sigma, R_\Sigma) = 0$.*

Following [15], $\lambda_\Sigma : R \to R_\Sigma$ in Lemma 2.1 is called the *universal localization* of $R$ at $\Sigma$. One should be aware that $R_\Sigma$ may not be flat as a right or left $R$-module. In general, $\lambda_\Sigma$ is not even homological (see [23]), and little is known about when it is homological.

Next, we recall the definition of coproducts of rings defined by Cohn in [14].

Let $R_0$ be a ring. An *$R_0$-ring* is a ring $R$ with a ring homomorphism $\lambda_R : R_0 \to R$. An *$R_0$-homomorphism* from an $R_0$-ring $R$ to another $R_0$-ring $S$ is a ring homomorphism $f : R \to S$ such that $\lambda_S = \lambda_R f$. Observe that epimorphisms in the category of $R_0$-rings are exactly ring epimorphisms starting from $R_0$.

The *coproduct* of two $R_0$-rings $R_1$ and $R_2$, denoted by $R_1 \bigsqcup_{R_0} R_2$, is an $R_0$-ring $R$ together with $R_0$-homomorphisms $\rho_1 : R_1 \to R$ and $\rho_2 : R_2 \to R$ such that, for any $R_0$-ring $S$ with $R_0$-homomorphisms $\tau_1 : R_1 \to S$ and $\tau_2 : R_2 \to S$, there is a unique $R_0$-homomorphism $\delta : R \to S$ such that $\tau_1 = \rho_1 \delta$ and $\tau_2 = \rho_2 \delta$. In other words, the coproduct of two $R_0$-rings $R_1$ and $R_2$ is the pushout in the category of $R_0$-rings with $R_0$-homomorphisms. This coproduct always exists (see [14]). Moreover, if $\lambda_{R_1} : R_0 \to R_1$ is a ring epimorphism, then so is the associated ring homomorphism $R_2 \to R_1 \bigsqcup_{R_0} R_2$.

Finally, we recall the notion of recollements of triangulated categories, which was first defined in [5] to study "exact sequences" of derived categories of perverse sheaves over geometric objects.

**Definition 2.2.** Let $\mathcal{D}$, $\mathcal{D}'$, and $\mathcal{D}''$ be triangulated categories. We say that $\mathcal{D}$ is a *recollement* of $\mathcal{D}'$ and $\mathcal{D}''$ (or there is a recollement $(\mathcal{D}'', \mathcal{D}, \mathcal{D}')$) if there are six triangle functors among the three categories

$$\begin{tikzcd} \mathcal{D}'' \arrow[r, "i_*=i_!"] & \mathcal{D} \arrow[l, bend right=60, "i^*"'] \arrow[l, bend left=60, "i^!"] \arrow[r, "j^!=j^*"] & \mathcal{D}' \arrow[l, bend right=60, "j_!"'] \arrow[l, bend left=60, "j_*"] \end{tikzcd}$$

such that

(1) $(i^*, i_*), (i_!, i^!), (j_!, j^!)$, and $(j^*, j_*)$ are adjoint pairs;
(2) $i_*, j_*$, and $j_!$ are fully faithful functors;
(3) $i^! j_* = 0$ (and thus also $j^! i_! = 0$ and $i^* j_! = 0$); and
(4) for each object $X \in \mathcal{D}$, there are two triangles in $\mathcal{D}$:

$$i_! i^!(X) \longrightarrow X \longrightarrow j_* j^*(X) \longrightarrow i_! i^!(X)[1],$$
$$j_! j^!(X) \longrightarrow X \longrightarrow i_* i^*(X) \longrightarrow j_! j^!(X)[1].$$

If a ring epimorphism $\lambda : R \to S$ is homological, then there is a recollement $\big(\mathscr{D}(S), \mathscr{D}(R), \mathrm{Tria}(Q^\bullet)\big)$ of triangulated categories (see [24, Section 4]), where $Q^\bullet$ is the mapping cone of $\lambda$. In general, $\mathrm{Tria}(Q^\bullet)$ does not have to be equivalent to a derived module category. But if $S$ is of the form $R/ReR$ with $e^2 = e \in R$, then $\lambda$ induces a recollement $\big(\mathscr{D}(S), \mathscr{D}(R), \mathscr{D}(eRe)\big)$ of derived module categories, and $ReR$ is a stratifying ideal of $R$. Recall that $ReR$ is called a *stratifying ideal* if the multiplication map $Re \otimes_{eRe} eR \to ReR$ is injective and $\mathrm{Tor}_n^{eRe}(Re, eR) = 0$ for all $n \geq 1$ (see [13]).

## 3. Exact contexts and their noncommutative tensor products

In this section, we introduce the notion of exact contexts and characterize them by rigid morphisms. Further, we construct a unique ring from the data of an exact context. This ring is called the noncommutative tensor product of the exact context. It generalizes the notion of tensor products of algebras over commutative rings and covers the notions of coproducts of rings, trivially twisted extensions, and endomorphism rings of tensor products of modules.

**Definition 3.1.** Let $R, S$, and $T$ be rings with identity, let $\lambda : R \to S$ and $\mu : R \to T$ be ring homomorphisms, and let $M$ be an $S$-$T$-bimodule with $m \in M$. The quadruple $(\lambda, \mu, M, m)$ is called an exact context if

$$(*) \quad 0 \longrightarrow R \xrightarrow{(\lambda,\, \mu)} S \oplus T \xrightarrow{\binom{\cdot m}{-m \cdot}} M \longrightarrow 0$$

is an exact sequence of abelian groups, where $\cdot m$ and $m\cdot$ stand for the right and left multiplication maps by $m$, respectively. In this case, $(M, m)$ is called an *exact complement* of $(\lambda, \mu)$.

If $(\lambda, \mu, S \otimes_R T, 1 \otimes 1)$ is an exact context, then we say simply that $(\lambda, \mu)$ is an *exact pair*.

Let $(\lambda, \mu, M, m)$ be an exact context. Then it follows from $(*)$ that, for an $S$-$T$-bimodule $N$ with an element $n \in N$, the pair $(N, n)$ is an exact complement of $(\lambda, \mu)$ if and only if there exists a unique isomorphism $\omega : M \to N$ of $R$-$R$-bimodules such that $(sm)\omega = sn$ and $(mt)\omega = nt$ for all $s \in S$ and $t \in T$. In particular, $(m)\omega = n$. In general, $\omega$ does not have to be an isomorphism of $S$-$T$-bimodules; that is, $M$ and $N$ may not be isomorphic as $S$-$T$-bimodules.

Now, we present a general method to construct exact contexts. Let $\mathcal{C}$ be an additive category.

**Definition 3.2.**

(1) An object $X^\bullet$ in $\mathscr{C}(\mathcal{C})$ is *rigid* if $\mathrm{Hom}_{\mathscr{K}(\mathcal{C})}(X^\bullet, X^\bullet[1]) = 0$.

(2) A morphism $f^\bullet : Y^\bullet \to X^\bullet$ in $\mathscr{C}(\mathcal{C})$ is rigid if the object $Z^\bullet$ in a distinguished triangle $Y^\bullet \xrightarrow{f^\bullet} X^\bullet \to Z^\bullet \to Y^\bullet[1]$ in $\mathscr{K}(\mathcal{C})$ is rigid, or equivalently, the mapping cone $\mathrm{Con}(f^\bullet)$ of $f^\bullet$ is rigid in $\mathscr{C}(\mathcal{C})$.

(3) A morphism $f : Y \to X$ in $\mathcal{C}$ is rigid if $f$, considered as a morphism between the stalk complexes $Y$ and $X$, is rigid, or equivalently, the complex $\mathrm{Con}(f) : 0 \to Y \xrightarrow{f} X \to 0$ is rigid in $\mathscr{C}(\mathcal{C})$.

The rigidity of a morphism $f^\bullet$ does not depend on the choice of the triangle which extends $f^\bullet$. By definition, a morphism $f : Y \to X$ in $\mathcal{C}$ is rigid if and only if $\mathrm{Hom}_{\mathcal{C}}(Y, X) = \mathrm{End}_{\mathcal{C}}(Y)f + f\,\mathrm{End}_{\mathcal{C}}(X)$.

**Proposition 3.3.** *Every rigid morphism $f$ in an additive category $\mathcal{C}$ gives rise to an exact context. Conversely, every exact context arises in this way.*

*Proof.* If $f : Y \to X$ is a rigid morphism in $\mathcal{C}$, then there exists an exact sequence of $R$-$R$-bimodules,

$$0 \longrightarrow R \xrightarrow{(\lambda,\mu)} \mathrm{End}_{\mathcal{C}}(Y) \oplus \mathrm{End}_{\mathcal{C}}(X) \xrightarrow{\left(\begin{smallmatrix} \cdot f \\ -f\cdot \end{smallmatrix}\right)} \mathrm{Hom}_{\mathcal{C}}(Y, X) \longrightarrow 0,$$

where $R := \{(s,t) \in \mathrm{End}_{\mathcal{C}}(Y) \oplus \mathrm{End}_{\mathcal{C}}(X) \mid sf = ft\}$ is a subring of the ring $\mathrm{End}_{\mathcal{C}}(Y) \times \mathrm{End}_{\mathcal{C}}(X)$, and where $\lambda$ and $\mu$ are defined by sending $(s,t)$ to $s$ and $t$, respectively. Thus $(\lambda, \mu, \mathrm{Hom}_{\mathcal{C}}(Y, X), f)$ is an exact context.

Conversely, every exact context appears in this form. In fact, for a given exact context $(\lambda, \mu, M, m)$, we may define $B = \left(\begin{smallmatrix} S & M \\ 0 & T \end{smallmatrix}\right)$, $e_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $e_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and consider the canonical map $\varphi$ from $Be_1$ to $Be_2$ defined by $\cdot m$. If we identify $\mathrm{Hom}_B(Be_1, Be_2)$, $\mathrm{End}_B(Be_1)$, and $\mathrm{End}_B(Be_2)$ with $M$, $S$, and $T$, respectively, then it follows from the given exact sequence $(*)$ that each element of $M$ can be expressed as $sm + mt$ with $s \in S$ and $t \in T$. This means that $\varphi$ is rigid in $B$-Mod, and the induced exact context is precisely the given one. □

**Example 3.4.**

(1) If $f : Y \to X$ is a morphism in $\mathcal{C}$ such that the induced map $\mathrm{Hom}_{\mathcal{C}}(Y, f) : \mathrm{Hom}_{\mathcal{C}}(Y, Y) \to \mathrm{Hom}_{\mathcal{C}}(Y, X)$ (resp., $\mathrm{Hom}_{\mathcal{C}}(f, X) : \mathrm{Hom}_{\mathcal{C}}(X, X) \to \mathrm{Hom}_{\mathcal{C}}(Y, X)$) is surjective, then $\mathrm{Hom}_{\mathcal{C}}(Y, X) = \mathrm{End}_{\mathcal{C}}(Y)f$ (resp., $\mathrm{Hom}_{\mathcal{C}}(Y, X) = f\,\mathrm{End}_{\mathcal{C}}(X)$), and therefore $f$ is rigid. Thus all kinds of precovers and pre-envelopes in ring theory or

all left and right approximations in the sense of Auslander and Smalø (see [3]) are rigid morphisms. This type of rigid morphism includes the following two cases:

(a) Let $A$ be an Artin algebra, and let $0 \to Z \xrightarrow{f} Y \xrightarrow{g} X \to 0$ be an almost split sequence in $A$-mod, that is, a nonsplit sequence such that any homomophism $Y' \to X$, which is not a split epimorphism, factorizes through $g$ and any homomorphism $Z \to Z'$, which is not a monomorphism, factorizes through $f$. Then both $f$ and $g$ are rigid since both $\mathrm{Hom}_A(Y, g)$ and $\mathrm{Hom}_A(f, Y)$ are surjective. This is due to the fact that $Y$ does not contain the outer terms $X$ and $Z$ as direct summands. For further information on almost split sequences, we refer the reader to [2].

(b) Let $S$ be a ring. If $Y$ is a quasi-projective $S$-module (that is, for any surjective homomorphism $Y \to X$ of $S$-modules, the induced map $\mathrm{Hom}_S(Y, Y) \to \mathrm{Hom}_S(Y, X)$ is surjective), then, for any submodule $Z$ of $Y$, the canonical map $f : Y \to X := Y/Z$ is rigid since $\mathrm{Hom}_S(Y, X) = \mathrm{End}_S(Y)f$. Dually, if $X$ is a quasi-injective $S$-module (that is, for any injective homomorphism $g : Y \to X$, the induced map $\mathrm{Hom}_S(X, X) \to \mathrm{Hom}_S(Y, X)$ is surjective), then, for any submodule $Y$ of $X$, the inclusion $g$ of $Y$ into $X$ is rigid because $\mathrm{Hom}_S(Y, X) = g\,\mathrm{End}_S(X)$. In particular, every surjective homomorphism from a projective module to an arbitrary module is rigid, and every injective homomorphism from an arbitrary module to an injective module is rigid.

(2) Let $\lambda : R \to S$ be a homomorphism of rings. Then $\lambda : R \to S$ itself and the canonical surjection $\pi : S \to S/\mathrm{Im}(\lambda)$ are rigid morphisms in $R$-Mod which induce two exact contexts.

Clearly, every homomorphism from ${}_RR$ to ${}_RS$ is uniquely determined by an element $s \in S$. By the right multiplication of $s$, we have a homomorphism ${}_RS \to {}_RS$ of $R$-modules. This implies that $\lambda$ is rigid. To see the rigidity of $\pi$, we take an $f \in \mathrm{Hom}_R(S, S/\mathrm{Im}(\lambda))$, choose an $s \in S$ such that $(s)\pi = (1)f$, and denote by $\cdot s : S \to S$ the right multiplication map by $s$. Then the map $\lambda(f - (\cdot s)\pi) = 0$. Thus there exists a unique homomorphism $g \in \mathrm{End}_R(S/\mathrm{Im}(\lambda))$ such that $f = (\cdot s)\pi + \pi\, g$. This implies that $\mathrm{Hom}_R(S, S/\mathrm{Im}(\lambda)) = \mathrm{End}_S(S)\pi + \pi\,\mathrm{End}_R(S/\mathrm{Im}(\lambda))$. Since $\mathrm{End}_S(S) \subseteq \mathrm{End}_R(S)$, $\mathrm{Hom}_R(S, S/\mathrm{Im}(\lambda)) = \mathrm{End}_R(S)\pi + \pi\,\mathrm{End}_R(S/\mathrm{Im}(\lambda))$. Thus $\pi$ is rigid.

If $\lambda$ is injective, then a third exact context can be constructed: Let $S' := \mathrm{End}_R(S/\mathrm{Im}(\lambda))$ and

$$\lambda' : R \longrightarrow S' : \ r \mapsto \big(x \mapsto x(r)\lambda\big) \qquad \text{for } r \in R \text{ and } x \in S/\mathrm{Im}(\lambda).$$

Then $\mathrm{Hom}_R(S, S/\mathrm{Im}(\lambda))$ is an $S$-$S'$-bimodule and $\big(\lambda, \lambda', \mathrm{Hom}_R(S, S/\mathrm{Im}(\lambda)), \pi\big)$ is an exact context since the diagram

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & S \arrow[r, "\pi"] \arrow[d, "\cdot\pi"'] & S/\mathrm{Im}(\lambda) \arrow[r] \arrow[d, "\simeq"'] & 0 \\
0 \arrow[r] & S' \arrow[r, "\pi\cdot"] & \mathrm{Hom}_R(S, S/\mathrm{Im}(\lambda)) \arrow[r, "\lambda\cdot"] & \mathrm{Hom}_R(R, S/\mathrm{Im}(\lambda)) \arrow[r] & 0
\end{tikzcd}$$

is commutative and the sequence of $R$-$R$-bimodules

$$0 \longrightarrow R \xrightarrow{(\lambda, \lambda')} S \oplus S' \xrightarrow{\left(\begin{smallmatrix} \cdot\pi \\ -\pi\cdot \end{smallmatrix}\right)} \mathrm{Hom}_R(S, S/\mathrm{Im}(\lambda)) \longrightarrow 0$$

is exact. In general, the exact context just obtained is different from the one induced by $\pi$, and $(\lambda, \lambda')$ may not be an exact pair because one of the isomorphisms $S \simeq \mathrm{End}_R(S)$ as rings and $S \otimes_R S' \simeq \mathrm{Hom}_R(S, S/\mathrm{Im}(\lambda))$ as $S$-$S'$-bimodules may fail.

For more examples of exact contexts, we refer the reader to Section 4.

Having seen the ubiquity of exact contexts, we now construct the noncommutative tensor product for each exact context. Such a tensor product captures the notion of both coproducts and usual tensor products in ring theory. Examples of noncommutative tensor products are described in detail for Milnor squares, Morita contexts, and strictly pure extensions in the next section.

**Lemma 3.5.** *For an exact context $(\lambda, \mu, M, m)$, we have the pullback and pushout square of $R$-$R$-bimodules*

$$\begin{tikzcd} R \arrow[r, "\lambda"] \arrow[d, "\mu"'] & S \arrow[d, "\cdot m"] \\ T \arrow[r, "m\cdot"] & M \end{tikzcd}$$

*and homomorphisms of $R$-$R$-bimodules: for $s \in S$ and $t \in t$,*

$$\lambda' = \lambda \otimes_R T: \ T \longrightarrow S \otimes_R T, \ t \mapsto 1 \otimes t \quad \text{and} \quad \mu' = S \otimes_R \mu: \ S \longrightarrow S \otimes_R T, \ s \mapsto s \otimes 1,$$

$$\rho = \mu \otimes S: \ S \longrightarrow T \otimes_R S, \ s \mapsto 1 \otimes s \quad \text{and} \quad \phi = T \otimes \lambda: \ T \longrightarrow T \otimes_R S, \ t \mapsto t \otimes 1.$$

According to $(*)$, there exist two unique homomorphisms,

$$\alpha: M \longrightarrow S \otimes_R T, \ x \mapsto s_x \otimes 1 + 1 \otimes t_x \quad \text{and} \quad \beta: M \longrightarrow T \otimes_R S, \ x \mapsto 1 \otimes s_x + t_x \otimes 1,$$

where $x \in M$ and $(s_x, t_x) \in S \oplus T$ with $x = s_x m + m t_x$ such that the following two diagrams commute:

$$(\dagger) \quad \begin{tikzcd} R \arrow[r, "{(\lambda,\mu)}"] \arrow[d, equal] & S \oplus T \arrow[r, "{\binom{\cdot m}{-m\cdot}}"] \arrow[d, equal] & M \arrow[d, "\alpha"] \\ R \arrow[r, "{(\lambda,\mu)}"] & S \oplus T \arrow[r, "{\binom{\mu'}{-\lambda'}}"] & S \otimes_R T \end{tikzcd} \quad \text{and} \quad (\ddagger) \quad \begin{tikzcd} R \arrow[r, "{(\lambda,\mu)}"] \arrow[d, equal] & S \oplus T \arrow[r, "{\binom{\cdot m}{-m\cdot}}"] \arrow[d, equal] & M \arrow[d, "\beta"] \\ R \arrow[r, "{(\lambda,\mu)}"] & S \oplus T \arrow[r, "{\binom{\rho}{-\phi}}"] & T \otimes_R S \end{tikzcd}$$

Observe that $(x)\alpha$ and $(x)\beta$ are independent of the choice of $(s_x, t_x)$ in $S \oplus T$. Let

$$\gamma: S \otimes_R T \longrightarrow M, \ s \otimes t \mapsto smt.$$

Then $\alpha$ and $\beta$ are homomorphisms of $R$-$R$-bimodules, $\gamma$ is a homomorphism of $S$-$T$-bimodules, and $\alpha\gamma = Id_M$. In particular, $\alpha$ is injective and $\gamma$ is surjective. Further, let

$$\delta := \gamma\beta: \ S \otimes_R T \longrightarrow T \otimes_R S, \ s \otimes t \mapsto 1 \otimes s_{smt} + t_{smt} \otimes 1$$

for $s \in S$ and $t \in T$, where the pair $(s_{smt}, t_{smt}) \in S \oplus T$ is chosen such that $smt = s_{smt} m + m t_{smt}$. Then $\delta$ is a homomorphism of $R$-$R$-bimodules such that $(s \otimes 1)\delta = 1 \otimes s$ and $(1 \otimes t)\delta = t \otimes 1$.

Now, we shall define a new ring $T \boxtimes_R S$ for $(\lambda, \mu, M, m)$. Here $T \boxtimes_R S$ has the underlying abelian group $T \otimes_R S$. Its multiplication $\circ: (T \otimes_R S) \times (T \otimes_R S) \to T \otimes_R S$

is given by the homomorphisms

$$(T\otimes_R S)\otimes_R (T\otimes_R S)\xrightarrow{\simeq} T\otimes_R (S\otimes_R T)\otimes_R S \xrightarrow{T\otimes\delta\otimes S} T\otimes_R (T\otimes_R S)\otimes_R S$$
$$\xrightarrow{\simeq} (T\otimes_R T)\otimes_R (S\otimes_R S) \xrightarrow{\mu_T\otimes\mu_S} T\otimes_R S,$$

where $\mu_T : T\otimes_R T \to T$ and $\mu_S : S\otimes_R S \to S$ are the multiplication maps. Explicitly, for $(t_i, s_i)\in T\otimes_R S$ with $i = 1, 2$, we have

$$(t_1\otimes s_1)\circ(t_2\otimes s_2) := t_1(s_1\otimes t_2)\delta\, s_2 = t_1\big(1\otimes s_{s_1mt_2} + t_{s_1mt_2}\otimes 1\big)s_2.$$

The following lemma asserts the associativity of this multiplication $\circ$.

**Lemma 3.6.**

(1) *$(T\otimes_R S,\circ)$ is an associative ring with identity $1\otimes 1$.*
(2) *The maps $\rho : S\to T\otimes_R S$ and $\phi : T\to T\otimes_R S$ are ring homomorphisms. In particular, $T\otimes_R S$ can be regarded as an $S$-$T$-bimodule via $\rho$ and $\phi$.*
(3) *The map $\beta : M\to T\otimes_R S$ is a homomorphism of $S$-$T$-bimodules such that $(m)\beta = 1\otimes 1$.*

*Proof.*

(1) It suffices to show that the multiplication $\circ$ is associative and $1\otimes 1$ is the identity of $T\otimes_R S$.

We take elements $t_i\in T$ and $s_i\in S$ for $1\le i\le 3$ and choose two pairs $(x, y)$ and $(u, v)$ in $S\oplus T$ such that $s_1mt_2 = xm+my$ and $s_2mt_3 = um+mv$.

It can be checked that $\big((t_1\otimes s_1)\circ(t_2\otimes s_2)\big)\circ(t_3\otimes s_3) = t_1\big((xs_2\otimes t_3)\delta + y(s_2\otimes t_3)\delta\big)s_3$ and $(t_1\otimes s_1)\circ\big((t_2\otimes s_2)\circ(t_3\otimes s_3)\big) = t_1\big((s_1\otimes t_2)\delta\, u + (s_1\otimes t_2v)\delta\big)s_3$. So, to prove

$$\big((t_1\otimes s_1)\circ(t_2\otimes s_2)\big)\circ(t_3\otimes s_3) = (t_1\otimes s_1)\circ\big((t_2\otimes s_2)\circ(t_3\otimes s_3)\big),$$

it suffices to verify $(xs_2\otimes t_3)\delta + y(s_2\otimes t_3)\delta = (s_1\otimes t_2)\delta\, u + (s_1\otimes t_2v)\delta$.

Since $xs_2mt_3 = x(um + mv) = xum + xmv$ and $xu\in S$, we have $(xs_2\otimes t_3)\delta = (xum + xmv)\beta = 1\otimes xu + (xmv)\beta$. Similarly, $(s_1\otimes t_2v)\delta = yv\otimes 1 + (xmv)\beta$. It follows from a simple calculation that $(xs_2\otimes t_3)\delta + y(s_2\otimes t_3)\delta = (s_1\otimes t_2)\delta\, u + (s_1\otimes t_2v)\delta$. This shows that the multiplication $\circ$ is associative. Also, it is easy to show that $1\otimes 1$ is the identity with respect to $\circ$.

(2) Since $(s_1)\rho\circ(s_2)\rho = (1\otimes s_1)\circ(1\otimes s_2) = (s_1\otimes 1)\delta\, s_2 = (1\otimes s_1)s_2 = 1\otimes s_1s_2 = (s_1s_2)\rho$, the map $\rho : S\to T\otimes_R S$ is a ring homomorphism. Similarly, $\phi : T\to T\otimes_R S$ is also a ring homomorphism.

(3) By the definition of $\beta$, $(m)\beta = 1\otimes 1$. It remains to check that $\beta$ is a homomorphism of $S$-$T$-bimodules, or equivalently, $(sat)\beta = (s)\rho\circ(a)\beta\circ(t)\phi$ for $s\in S$, $a\in M$, and $t\in T$. We leave the verification of this equation to the reader. □

The ring $(T\otimes_R S,\circ)$ in Lemma 3.6 is called the *noncommutative tensor product* of $(\lambda,\mu, M, m)$, denoted simply by $T\boxtimes_R S$ if the exact context $(\lambda,\mu, M, m)$ is clear.

In general, $T\boxtimes_R S$ may not be the usual tensor product of two $R$-algebras $T$ and $S$ because $R$ is not assumed to be commutative and the tensor product of $R$-algebras does not make sense. But it does generalize the usual tensor product of $R$-algebras in the following sense.

**Lemma 3.7.** *Let $R$ be a commutative ring, and let $S$ and $T$ be $R$-algebras via $\lambda$ and $\mu$, respectively. If $(\lambda, \mu)$ is an exact pair, then $T \boxtimes_R S$ coincides with the usual tensor product $T \otimes_R S$ of $R$-algebras $T$ and $S$.*

*Proof.* If $(\lambda, \mu)$ is an exact pair, then $M = S \otimes_R T$, $\gamma = Id_{S\otimes_R T}$, and $\delta = \beta : S \otimes_R T \to T \otimes_R S$, where $\beta$ is determined uniquely by the diagram (‡). However, the switch map $\omega : S \otimes_R T \to T \otimes_R S$, defined by $s \otimes t \mapsto t \otimes s$ for $s \in S$ and $t \in T$, also makes the diagram (‡) commutative, that is, $\begin{pmatrix} \mu' \\ -\lambda' \end{pmatrix} \omega = \begin{pmatrix} \rho \\ -\phi \end{pmatrix}$. This implies $\beta = \omega$. Thus the multiplication $\circ$ in $T \boxtimes_R S$ coincides with the usual tensor product of $R$-algebras $T$ and $S$ over $R$. $\square$

Next, we give some characterizations of exact pairs.

**Lemma 3.8.** *The following are equivalent:*

(1) *The pair $(\lambda, \mu)$ is exact.*
(2) *The map $\gamma$ is an isomorphism.*
(3) $\mathrm{Coker}(\lambda) \otimes_R \mathrm{Coker}(\mu) = 0$.

*Proof.* Recall that $\gamma$ is a homomorphism of $S$-$T$-bimodules, $(s \otimes 1)\gamma = sm$ and $(1 \otimes t)\gamma = mt$ for $s \in S$ and $t \in T$. This implies that the diagram

$$\begin{tikzcd} & R \arrow[r, "{(\lambda,\mu)}"] \arrow[d, equal] & S \oplus T \arrow[r, "{\left(\begin{smallmatrix} \mu' \\ -\lambda' \end{smallmatrix}\right)}"] \arrow[d, equal] & S \otimes_R T \arrow[d, "\gamma"] & \\ 0 \arrow[r] & R \arrow[r, "{(\lambda,\mu)}"] & S \oplus T \arrow[r, "{\left(\begin{smallmatrix} \cdot m \\ -m\cdot \end{smallmatrix}\right)}"] & M \arrow[r] & 0 \end{tikzcd}$$

is commutative, where the bottom row is assumed to be exact. Consequently, (1) and (2) are equivalent.

Now, we verify the equivalence of (2) and (3).

In fact, it follows from $\alpha\gamma = Id_M$ that the map $\gamma$ is an isomorphism if and only if $\alpha$ is surjective. However, the latter is equivalent to the map $\xi := \begin{pmatrix} \mu' \\ -\lambda' \end{pmatrix}$ : $S \oplus T \longrightarrow S \otimes_R T$ being surjective by (†). Therefore, it is enough to show that $\xi$ is surjective if and only if $\mathrm{Coker}(\lambda) \otimes_R \mathrm{Coker}(\mu) = 0$. To check this condition, we consider the following two complexes of $R$-$R$-bimodules,

$$\mathrm{Con}(\lambda) : 0 \to R \xrightarrow{\lambda} S \to 0 \quad \text{and} \quad \mathrm{Con}(\mu) : 0 \to R \xrightarrow{\mu} T \to 0,$$

with both $S$ and $T$ in degree 0, and calculate their tensor complex over $R$:

$\mathrm{Con}(\lambda) \otimes^{\bullet}_R \mathrm{Con}(\mu)$ :

$$0 \longrightarrow R \otimes_R R \xrightarrow{(\lambda\otimes R,\ -R\otimes\mu)} S \otimes_R R \oplus R \otimes_R T \xrightarrow{\left(\begin{smallmatrix} S\otimes\mu \\ \lambda\otimes T \end{smallmatrix}\right)} S \otimes_R T \longrightarrow 0\,,$$

where $R \otimes_R R$ is of degree $-2$. If we identify $R \otimes_R R$, $S \otimes_R R$, and $R \otimes_R T$ with $R$, $S$, and $T$, respectively, then $\mathrm{Con}(\lambda) \otimes^{\bullet}_R \mathrm{Con}(\mu)$ is precisely the complex

$$0 \longrightarrow R \xrightarrow{(\lambda,\ -\mu)} S \oplus T \xrightarrow{\left(\begin{smallmatrix} \mu' \\ \lambda' \end{smallmatrix}\right)} S \otimes_R T \longrightarrow 0\,,$$

which is isomorphic to the following complex $0 \to R \xrightarrow{(\lambda,\,\mu)} S \oplus T \xrightarrow{\xi} S \otimes_R T \to 0$. It follows that $\xi$ is surjective if and only if $H^0\big(\mathrm{Con}(\lambda) \otimes^{\bullet}_R \mathrm{Con}(\mu)\big) = 0$. Since

$$H^0\big(\mathrm{Con}(\lambda) \otimes^{\bullet}_R \mathrm{Con}(\mu)\big) \simeq H^0(\mathrm{Con}(\lambda)) \otimes_R H^0(\mathrm{Con}(\mu)) \simeq \mathrm{Coker}(\lambda) \otimes_R \mathrm{Coker}(\mu),$$

the map $\xi$ is surjective if and only if $\mathrm{Coker}(\lambda)\otimes_R \mathrm{Coker}(\mu)=0$. Thus $\gamma$ is an isomorphism if and only if $\mathrm{Coker}(\lambda)\otimes_R \mathrm{Coker}(\mu)=0$. $\square$

*Remark* 3.9. By the equivalence of (1) and (2) in Lemma 3.8, if the pair $(\lambda,\mu)$ is exact, then it admits a unique complement $(S\otimes_R T, 1\otimes 1)$ up to isomorphism (preserving $1\otimes 1$) of $S$-$T$-bimodules.

**Corollary 3.10.** *If either $\lambda: R\to S$ or $\mu: R\to T$ is a ring epimorphism, then $\gamma: S\otimes_R T\to M, s\otimes t\mapsto smt$ is an isomorphism of $S$-$T$-bimodules, and $(\lambda,\mu)$ is an exact pair.*

*Proof.* Suppose that $\lambda$ is a ring epimorphism. Then, for any $S$-module $X$, the map $\lambda\otimes X: R\otimes_R X\to S\otimes_R X$ is an isomorphism, and therefore $\mathrm{Coker}(\lambda)\otimes_R X=0$. Since $\mathrm{Coker}(\mu)\simeq M/Sm$ as $R$-modules by Lemma 3.5 and since $M/Sm$ is an $S$-module, $\mathrm{Coker}(\lambda)\otimes_R \mathrm{Coker}(\mu)\simeq \mathrm{Coker}(\lambda)\otimes_R (M/Sm)=0$. By Lemma 3.8, $\gamma$ is an isomorphism.

Similarly, if $\mu$ is a ring epimorphism, then $\gamma$ is an isomorphism. $\square$

Finally, we mention further examples of exact pairs.

**Proposition 3.11.** *The morphisms $f$ and $g$ in an almost split sequence $0\to Z\xrightarrow{f} Y\xrightarrow{g} X\to 0$ always provide two exact pairs.*

*Proof.* By Example 3.4(1), $f$ and $g$ are rigid morphisms and there exist two short exact sequences:

$$0\to R\xrightarrow{(\lambda,\mu)} \mathrm{End}_A(Y)\oplus \mathrm{End}_A(X)\xrightarrow{\begin{pmatrix}\cdot g\\ -g\cdot\end{pmatrix}} \mathrm{Hom}_A(Y,X)\to 0,$$

$$0\to R'\xrightarrow{(p,q)} \mathrm{End}_A(Z)\oplus \mathrm{End}_A(Y)\xrightarrow{\begin{pmatrix}\cdot f\\ -f\cdot\end{pmatrix}} \mathrm{Hom}_A(Z,Y)\to 0.$$

By the factorization property of almost split sequences, the multiplication maps $\cdot g$ and $f\cdot$ are surjective. Now it follows from a standard fact on pushout-pullback diagrams that $\mu$ and $p$ are surjective. Hence the exact contexts provided by $f$ and $g$ are exact pairs by Corollary 3.10. $\square$

## 4. Examples of noncommutative tensor products

In this section, we describe explicitly noncommutative tensor products of exact contexts induced from Milnor squares, Morita contexts, and ring extensions.

4.1. **Milnor squares.** Recall that a *Milnor square* defined by Milnor in [20, Sections 2 and 3] is a commutative diagram of ring homomorphisms

$$(\dagger\dagger)\qquad \begin{tikzcd} R \arrow[r, "i_1"] \arrow[d, "i_2"'] & R_1 \arrow[d, "j_1"] \\ R_2 \arrow[r, "j_2"] & R' \end{tikzcd}$$

such that $j_2$ is surjective and $R$ is the pullback of $R_1$ and $R_2$ over $R'$; that is, given a pair $(r_1,r_2)\in R_1\oplus R_2$ with $(r_1)j_1=(r_2)j_2\in R'$, there is one and only one element $r\in R$ such that $(r)i_1=r_1$ and $(r)i_2=r_2$.

**Proposition 4.1.** *Every Milnor square $(\dagger\dagger)$ provides an exact pair whose noncommutative tensor product is the ring $R'$.*

*Proof.* The ring $R'$ can be regarded as an $R_1$-$R_2$-bimodule via $j_1$ and $j_2$. Let 1 be the identity of $R'$. Then $j_1$ and $j_2$ are precisely the multiplication maps $\cdot 1$ and $1\,\cdot$, respectively. This is because $r_1 \cdot 1 = (r_1)j_1$ and $1 \cdot r_2 = (r_2)j_2$ for $r_1 \in R_1$ and $r_2 \in R_2$.

The quadruple $(i_1, i_2, R', 1)$ is an exact context because, by the definition of Milnor squares, we have the exact sequence of $R$-$R$-bimodules

$$0 \longrightarrow R \xrightarrow{(i_1,\, i_2)} R_1 \oplus R_2 \xrightarrow{\binom{j_1}{-j_2}} R' \longrightarrow 0.$$

Since $j_2$ is surjective, $i_1$ is also surjective and $i_2$ induces an isomorphism $\mathrm{Ker}(i_1) \simeq \mathrm{Ker}(j_2)$ of $R$-$R$-bimodules. By Lemma 3.8, $(i_1, i_2)$ is an exact pair. Moreover, we show that the noncommutative tensor product of $(i_1, i_2)$ is isomorphic to $R'$.

Let $f : R_2 \boxtimes_R R_1 \to R'$ be the map defined by $r_2 \otimes r_1 \mapsto (r_2)j_2\ (r_1)j_1$ for $r_1 \in R_1$ and $r_2 \in R_2$. Then $f$ is an isomorphism of $R_2$-$R_1$-bimodules because of the canonical isomorphisms

$$R_2 \boxtimes_R R_1 = R_2 \otimes_R R_1 \simeq R_2 \otimes_R (R/\mathrm{Ker}(i_1)) \simeq R_2/(R_2 \cdot \mathrm{Ker}(i_1)) = R_2/\mathrm{Ker}(j_2) \simeq R'$$

of $R_2$-$R_1$-bimodules. As $i_1$ is surjective, there is an $r \in R$ such that $r_1 = (r)i_1$, while the twisting $\delta : R_1 \otimes_R R_2 \to R_2 \otimes_R R_1$, which defines the ring structure on $R_2 \boxtimes_R R_1$, is given by $r_1 \otimes r_2 \mapsto (r)i_2 r_2 \otimes 1$. This implies that $f$ is an isomorphism of rings. Thus $R_2 \boxtimes_R R_1 \simeq R'$ as rings. $\square$

Compared with Milnor squares, exact contexts $(\lambda, \mu, M, m)$ do not require that one of $\cdot m$ or $m\cdot$ be surjective, nor that $M$ have a ring structure. In this sense, exact contexts are a generalization of Milnor squares, but also of different constructions, as illustrated by the following subsections.

4.2. **Morita contexts.** Let $(A, C, X, Y, f, g)$ be a Morita context; that is, $A$ and $C$ are rings, $X$ is an $A$-$C$-bimodule, $Y$ is a $C$-$A$-bimodule, and $f : X \otimes_C Y \to A$ and $g : Y \otimes_A X \to C$ are homomorphisms of $A$-$A$-bimodules and $C$-$C$-bimodules, respectively, such that $(x_1 \otimes y_1)f x_2 = x_1(y_1 \otimes x_2)g$ and $(y_1 \otimes x_1)g y_2 = y_1(x_1 \otimes y_2)f$ for $x_i \in X$ and $y_i \in Y$, with $i = 1, 2$. For simplicity, $(x_1 \otimes y_1)f$ and $(y_1 \otimes x_1)g$ are denoted by $x_1 y_1$ and $y_1 x_1$, respectively.

Any ring $R$ with a nontrivial idempotent $e$ gives rise to a Morita context $(eRe, (1-e)R(1-e), eR(1-e), (1-e)Re, f, g)$, with $f, g$ being multiplication maps.

**Proposition 4.2.** *Every Morita context gives rise to an exact context, and its noncommutative tensor product is of matrix form with the multiplication described explicitly* (*see* ($\sharp$) *below*).

*Proof.* The *Morita context ring* $M := \left(\begin{smallmatrix} A & X \\ Y & C \end{smallmatrix}\right)$ has the matrix addition, while its multiplication is given by

$$\begin{pmatrix} a_1 & x_1 \\ y_1 & c_1 \end{pmatrix} \begin{pmatrix} a_2 & x_2 \\ y_2 & c_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 + x_1 y_2 & a_1 x_2 + x_1 c_2 \\ y_1 a_2 + c_1 y_2 & c_1 c_2 + y_1 x_2 \end{pmatrix}$$

for $a_i \in A$, $c_i \in C$, $x_i \in X$, and $y_i \in Y$. Consider the subrings of $M$,

$$R := \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix}, \quad S := \begin{pmatrix} A & X \\ 0 & C \end{pmatrix}, \quad T := \begin{pmatrix} A & 0 \\ Y & C \end{pmatrix},$$

and denote the inclusions by $\lambda : R \to S$ and $\mu : R \to T$. Then $M$ is an $S$-$T$-bimodule. Since $R = S \cap T$ and $M = S + T$, the quadruple $(\lambda, \mu, M, 1_M)$ is an exact context. So, the noncommutative tensor product $T\boxtimes_R S$ of this exact context is defined and can be described explicitly as follows.

We identify $R$-Mod with $A$-Mod $\times$ $C$-Mod. Then ${}_RS = (A\oplus X)\times C$, $T_R = (A\oplus Y)\times C$, and the map

$$T\otimes_R S \longrightarrow \begin{pmatrix} A & X \\ Y & C\oplus(Y\otimes_A X)\end{pmatrix} =: B,$$

defined by

$$\begin{pmatrix} a_1 & 0 \\ y_1 & c_1\end{pmatrix} \otimes \begin{pmatrix} a_2 & x_2 \\ 0 & c_2\end{pmatrix} \mapsto \begin{pmatrix} a_1a_2 & a_1x_2 \\ y_1a_2 & (c_1c_2, y_1\otimes x_2)\end{pmatrix},$$

is an isomorphism of abelian groups. Hence we identify $T\otimes_R S$ with $B$, translate the multiplication of $T\boxtimes_R S$ into the one of $B$, and get the formula

$$(\sharp)\qquad \begin{pmatrix} a_1 & x_1 \\ y_1 & (c_1, y\otimes x)\end{pmatrix} \circ \begin{pmatrix} a_2 & x_2 \\ y_2 & (c_2, y'\otimes x')\end{pmatrix}$$

$$= \begin{pmatrix} a_1a_2 + x_1y_2 & a_1x_2 + x_1c_2 + x_1(y'x') \\ y_1a_2 + c_1y_2 + (yx)y_2 & (c_1c_2,\ y_1\otimes x_2 + (c_1y')\otimes x' + y\otimes(xc_2) + y\otimes(xy')x')\end{pmatrix},$$

where $x, x' \in X$ and $y, y' \in Y$. Thus $T\boxtimes_R S$ is described by $B$ with the usual matrix addition and the above multiplication. The maps $\rho : S \to T\boxtimes_R S$, $\phi : T \to T\boxtimes_R S$, and $\beta : M \to T\otimes_R S$ are given by

$$\begin{pmatrix} a_1 & x_1 \\ 0 & c_1\end{pmatrix} \overset{\rho}{\mapsto} \begin{pmatrix} a_1 & x_1 \\ 0 & (c_1, 0)\end{pmatrix}, \qquad \begin{pmatrix} a_1 & 0 \\ y_1 & c_1\end{pmatrix} \overset{\phi}{\mapsto} \begin{pmatrix} a_1 & 0 \\ y_1 & (c_1, 0)\end{pmatrix},$$
$$\begin{pmatrix} a_1 & x_1 \\ y_1 & c_1\end{pmatrix} \overset{\beta}{\mapsto} \begin{pmatrix} a_1 & x_1 \\ y_1 & (c_1, 0)\end{pmatrix},$$

respectively. Observe that $\beta$ may not be a ring homomorphism. Actually, it is a ring homomorphism if and only if $Y\otimes_A X = 0$. $\square$

Let $e := \left(\begin{smallmatrix} 1 & 0 \\ 0 & (0,0)\end{smallmatrix}\right) \in B$. Then $e = e^2$, $A = eBe$, and $B/(BeB) = C$. Moreover, $Be = A\oplus Y$, $eB = A\oplus X$, and $BeB = A\oplus X\oplus Y\oplus Y\otimes_A X$ as abelian groups. This implies that the multiplication map $Be\otimes_A eB \to BeB$ is an isomorphism. So, if $\mathrm{Tor}_i^A(Y, X) = 0$ for all $i > 0$, then $\mathrm{Tor}_i^{eBe}(Be, eB) = 0$ for all $i > 0$, the canonical surjection $B \to B/BeB$ is homological, and there is a recollement $(\mathscr{D}(C), \mathscr{D}(B), \mathscr{D}(A))$ of derived module categories (see [13, 16]).

For each $i \geq 1$, $\mathrm{Tor}_i^R(T, S) \simeq \mathrm{Tor}_i^A(Y, X)$. Thus $\mathrm{Tor}_i^R(T, S) = 0$ if and only if $\mathrm{Tor}_i^A(Y, X) = 0$.

4.3. **Strictly pure extensions.** Recall that an extension $D \subseteq C$ of rings is called *pure* if there exists a $D$-$D$-bimodule $X$ such that $C = D\oplus X$ as $D$-$D$-bimodules (see [28]). If $X$ is an ideal of $C$, then such an extension is called *strictly pure*. Note that pure extensions were used by Waldhausen to compute the algebraic $K$-theory of generalized free products (or coproducts of rings) in [28].

Let $\lambda : R \to S$ and $\mu : R \to T$ be two strictly pure extensions. Then we are going to construct an exact context $(\lambda, \mu, M, m)$ from this pair $(\lambda, \mu)$. By definition, we have two split decompositions of $R$-$R$-bimodules:

$$S = R\oplus X \quad \text{and} \quad T = R\oplus Y,$$

where $X$ and $Y$ are ideals of $S$ and $T$, respectively. Now we define $M := R \oplus X \oplus Y$ as the direct sum of abelian groups and endow $M$ with a ring structure such that $S$ and $T$ are its subrings:

$$(r_1 + x_1 + y_1)(r_2 + x_2 + y_2) := r_1 r_2 + (r_1 x_2 + x_1 r_2 + x_1 x_2) + (r_1 y_2 + y_1 r_2 + y_1 y_2)$$

for $r_i \in R$, $x_i \in X$, and $y_i \in Y$ with $i = 1, 2$. Since $S \cap T = R$, $(\lambda, \mu, M, 1)$ is an exact context. For each $j \geq 1$, $\mathrm{Tor}_j^R(T, S) = 0$ if and only if $\mathrm{Tor}_j^R(Y_R, {}_R X) = 0$.

In the following, we describe explicitly the multiplication of $T \boxtimes_R S$ of the exact cotext $(\lambda, \mu, M, 1)$.

As $R$-$R$-bimodules, $T \boxtimes_R S = T \otimes_R S = R \oplus X \oplus Y \oplus Y \otimes_R X$. Thus $\gamma : S \otimes_R T \to M$ is given by $s \otimes t \mapsto st$ for $s \in S$ and $t \in T$, and $\beta : M \to T \otimes_R S$ is the canonical inclusion. Thus $\delta : S \otimes_R T \to T \otimes_R S$ is given by

$$(r + x) \otimes (r' + y) \mapsto rr' + xr' + ry$$

for $r, r' \in R$, $x \in X$, and $y \in Y$. This implies that $(x \otimes y)\delta = 0$ for $x \in X$ and $y \in Y$. By the definition of the multiplication $\circ$, both $S$ and $T$ can be regarded as subrings of $T \boxtimes_R S$. Particularly, we have

$$X \circ Y = X \circ (Y \otimes_R X) = (Y \otimes_R X) \circ Y = (Y \otimes_R X) \circ (Y \otimes_R X) = 0,$$

and for $x, x' \in X$, $y, y' \in Y$,

$$y \circ x = y \otimes x, \quad y' \circ (y \otimes x) = y'y \otimes x, \quad (y \otimes x) \circ x' = y \otimes xx' \in Y \otimes_R X.$$

Certainly, $M$ is the quotient ring of $T \boxtimes_R S$ modulo the ideal $Y \otimes_R X$.

4.4. **Trivially twisted extensions.** In this subsection, we show that noncommutative tensor products described in Section 4.3 cover trivially twisted extensions in [29, 30].

Let $A$ be an Artin algebra, and let $A_0$, $A_1$, and $A_2$ be three Artin subalgebras of $A$ with the same identity. We say that $A$ is a *twisted tensor product* of $A_1$ and $A_2$ over $A_0$ (see [30]) if the following hold:

(a) $A_0$ is a semisimple algebra such that $A_1 \cap A_2 = A_0$ and $A = A_0 \oplus \mathrm{rad}(A)$ as a direct sum of $A_0$-$A_0$-bimodules, where $\mathrm{rad}(A)$ denotes the Jacobson radical of $A$.

(b) The multiplication map $\sigma : A_2 \otimes_{A_0} A_1 \to A$ is an isomorphism of $A_2$-$A_1$-bimodules.

(c) $\mathrm{rad}(A_1)\,\mathrm{rad}(A_2) \subseteq \mathrm{rad}(A_2)\,\mathrm{rad}(A_1)$.

If $A$ is a twisted tensor product of $A_1$ and $A_2$ over $A_0$, then there are decompositions of $A_0$-$A_0$-bimodules $A_1 = A_0 \oplus \mathrm{rad}(A_1)$ and $A_2 = A_0 \oplus \mathrm{rad}(A_2)$, where $A_0$ is a maximal common semisimple subalgebra of $A$, $A_1$, and $A_2$. If $\mathrm{rad}(A_1)\,\mathrm{rad}(A_2) = 0$, then $A$ is called the *trivially twisted extension* of $A_1$ and $A_2$ over $A_0$ (see [29]).

For a trivially twisted extension $A$ of $A_1$ and $A_2$ over $A_0$, we may take

$$R := A_0, \quad S := A_1, \quad T := A_2, \quad X := \mathrm{rad}(A_1), \quad Y := \mathrm{rad}(A_2),$$

and we let $\lambda : R \to S$ and $\mu : R \to T$ be the inclusions. Clearly, both $\lambda$ and $\mu$ are strictly pure. Thus $M := R \oplus X \oplus Y$ is a ring and $(\lambda, \mu, M, 1)$ is an exact context. Since $XY = \mathrm{rad}(A_1)\,\mathrm{rad}(A_2) = 0$ in $A$, the multiplication in the noncommutative tensor product $T \boxtimes_R S$ implies that the multiplication map $\sigma : T \boxtimes_R S \to A$ is an isomorphism of rings. Thus $A \simeq T \boxtimes_R S$ as rings.

If algebras are given by quivers with relations, trivially twisted extensions can be described as follows.

Suppose that $A_1$ and $A_2$ are $k$-algebras given by quivers $\Gamma = (\Gamma_0, \Gamma_1)$ with relations $\{\sigma_i \mid i \in I_0\}$ and $\Delta = (\Delta_0 := \Gamma_0, \Delta_1)$ with relations $\{\tau_j \mid j \in J_0\}$, respectively. Then the trivially twisted extension $A$ of $A_1$ and $A_2$ over $A_0 := kQ_0$ is given by the quiver $Q = (Q_0, Q_1)$, where $Q_0 := \Gamma_0$ and $Q_1 := \Gamma_1 \dot\cup \Delta_1$, with the relations $\{\sigma_i \mid i \in I_0\} \cup \{\tau_j | j \in J_0\} \cup \{\alpha\beta \mid \alpha \in \Gamma_1, \beta \in \Delta_1\}$. Here $\alpha\beta$ means that $\alpha$ comes first and then $\beta$ follows.

Thus, by definition, $A$ is isomorphic to the noncommutative tensor product of the exact context $(\lambda, \mu, M, 1)$, where $M = A_0 \oplus \mathrm{rad}(A_1) \oplus \mathrm{rad}(A_2)$ is the quotient of $A$ by the ideal generated by $\{\beta\alpha \mid \alpha \in \Gamma_1, \beta \in \Delta_1\}$. Since $A_0$ is semisimple, $\mathrm{Tor}_i^{A_0}(A_2, A_1) = 0$ for all $i \geq 1$. Therefore, Theorem 1.1 says that the canonical inclusion $\begin{pmatrix} A_1 & M \\ 0 & A_2 \end{pmatrix} \to \begin{pmatrix} A & A \\ A & A \end{pmatrix}$ is just a homological universal localization.

## 5. Universal localizations and homological ring epimorphisms

In this section, we shall prove the main result, Theorem 1.1, present a categorical interpretation (see Proposition 5.3) of noncommutative tensor products, and show that noncommutative tensor products cover coproducts in ring theory.

In this section, $(\lambda, \mu, M, m)$ denotes an exact context, and all notation introduced in Section 3 will be kept.

Let

$$\Lambda := \begin{pmatrix} S & M \\ 0 & T \end{pmatrix}, \quad \Gamma := \begin{pmatrix} T \boxtimes_R S & T \boxtimes_R S \\ T \boxtimes_R S & T \boxtimes_R S \end{pmatrix}, \quad \theta := \begin{pmatrix} \rho & \beta \\ 0 & \phi \end{pmatrix} : \Lambda \longrightarrow \Gamma.$$

Furthermore, let

$$e_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_2 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in \Lambda, \quad \text{and}$$

$$\varphi : \Lambda e_1 \longrightarrow \Lambda e_2, \begin{pmatrix} s \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} sm \\ 0 \end{pmatrix} \qquad \text{for } s \in S.$$

Then $\varphi$ is a homomorphism of finitely generated projective $\Lambda$-modules. If we identify $\mathrm{Hom}_\Lambda(\Lambda e_1, \Lambda e_2)$ with $M$, then $\varphi$ corresponds to $m \in M$.

5.1. **Proof of Theorem 1.1(1).** In this section, we first prove Theorem 1.1(1) and then use it to give a categorical description of noncommutative tensor products. Consequently, we show that noncommutative tensor products cover coproducts in ring theory and the endomorphism rings of tensor products of modules.

**Theorem 1.1(1).** *The map $\theta : \Lambda \to \Gamma$ is the universal localization of $\Lambda$ at $\varphi$.*

Indeed, let $A := T \boxtimes_R S$. By Lemma 3.6, $(m)\beta = 1 \otimes 1$ is the identity of $A$. Thus $\Gamma \otimes_\Lambda \varphi : \Gamma \otimes_\Lambda \Lambda e_1 \to \Gamma \otimes_\Lambda \Lambda e_2$ is an isomorphism. So, $\theta$ is $\{\varphi\}$-inverting. We shall prove that $\theta$ is universally $\{\varphi\}$-inverting.

Let $\Phi$ be a ring, and let $\omega : \Lambda \to \Phi$ be a $\{\varphi\}$-inverting ring homomorphism. Define $d_i := (e_i)\omega \in \Phi$ with $i = 1, 2$. Then $\Phi = \Phi d_1 \oplus \Phi d_2$ as $\Phi$-modules and $\Phi \otimes_\Lambda \varphi : \Phi \otimes_\Lambda \Lambda e_1 \to \Phi \otimes_\Lambda \Lambda e_2$ is an isomorphism. Further, the multiplication maps $\mu_i : \Phi \otimes_\Lambda \Lambda e_i \to \Phi d_i$, induced from $\omega$, are isomorphisms of $\Phi$-modules. Let $\eta := \mu_1^{-1}(\Phi \otimes_\Lambda \varphi)\mu_2$, which is an isomorphism from $\Phi d_1$ to $\Phi d_2$. Then $\eta$ induces an isomorphism

$$\tau : \ \mathrm{End}_\Phi(\Phi d_1 \oplus \Phi d_2) \xrightarrow{\simeq} \mathrm{End}_\Phi(\Phi d_1 \oplus \Phi d_1) : \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \mapsto \begin{pmatrix} f_{11} & f_{12}\eta^{-1} \\ \eta f_{21} & \eta f_{22}\eta^{-1} \end{pmatrix}$$

of rings, where $f_{ij} \in \mathrm{Hom}_\Phi(\Phi d_i, \Phi d_j)$ for $1 \leq i, j \leq 2$. We identify $\Phi$ with $\mathrm{End}_\Phi(\Phi d_1 \oplus \Phi d_2)$ and define $B := \mathrm{End}_\Phi(\Phi d_1)$. Then $\tau : \Phi \xrightarrow{\simeq} M_2(B)$, where $M_2(B)$ denotes the $2 \times 2$ matrix ring over $B$. Moreover, under the identification of $M_2(B) \otimes_\Lambda \Lambda e_i$ with $B$, the homomorphism $M_2(B) \otimes_\Lambda \varphi : M_2(B) \otimes_\Lambda \Lambda e_1 \to M_2(B) \otimes_\Lambda \Lambda e_2$ corresponds to the identity of $B$.

Without loss of generality, assume $\Phi = M_2(B)$ and $\omega = \begin{pmatrix} f & h \\ 0 & g \end{pmatrix} : \Lambda \to \Phi$, where $f : S \to B$ and $g : T \to B$ are ring homomorphisms, and $h : M \to B$ is a homomorphism of $S$-$T$-bimodules with $(m)h = 1$. Particularly, $h$ has the following property:

$$(\star) \quad (sxt)h = (s)f(x)h(t)g \qquad \text{for any } s \in S,\ t \in T,\ x \in M.$$

Since $(m)h = 1$, $(smt)h = (s)f(t)g$. This implies that $(sm)h = (s)f$ and $(mt)h = (t)g$; that is, $(\cdot m)h = f$ and $(m\cdot)h = g$. We shall show that $\omega$ factorizes through $\theta$ uniquely. Observe that Lemma 3.5 shows that $\lambda f = \mu g : R \to B$. This guarantees that the map

$$\sigma : A \longrightarrow B, \qquad t \otimes s \mapsto (t)g\,(s)f$$

is a well-defined homomorphism of abelian groups. By definition, $\sigma$ preserves the identity of rings. We shall show that $\sigma$ also preserves multiplication.

Let $t_i \otimes s_i \in A$ with $i = 1, 2$. Then we can check $\big((t_1 \otimes s_1) \circ (t_2 \otimes s_2)\big)\sigma = \big(t_1(1 \otimes s_{s_1mt_2} + t_{s_1mt_2} \otimes 1)s_2\big)\sigma = \big(t_1 \otimes s_{s_1mt_2}s_2 + t_1t_{s_1mt_2} \otimes s_2\big)\sigma = (t_1)g\big((s_{s_1mt_2})f + (t_{s_1mt_2})g\big)(s_2)f$ and $(t_1 \otimes s_1)\sigma(t_2 \otimes s_2)\sigma = (t_1)g(s_1)f(t_2)g(s_2)f$. Further, applying $h$ to both sides of the equality $s_1mt_2 = s_{s_1mt_2}m + t_{s_1mt_2}$ (see Section 3 for notation) in $M$ leads to another equality, $(s_{s_1mt_2})f + (t_{s_1mt_2})g = (s_1)f(t_2)g$ in $B$. This forces $\big((t_1 \otimes s_1) \circ (t_2 \otimes s_2)\big)\sigma = (t_1 \otimes s_1)\sigma(t_2 \otimes s_2)\sigma$. Thus $\sigma$ preserves the multiplication $\circ$. Moreover, by $(\star)$, we have $(f, g, h) = (\rho\sigma, \phi\sigma, \beta\sigma)$. Set $\psi := \begin{pmatrix} \sigma & \sigma \\ \sigma & \sigma \end{pmatrix}$. Then $\psi : \Gamma \to \Phi$ is a ring homomorphism with $\omega = \theta\psi$.

It remains to show that such a $\psi$ is unique. Suppose that $\psi' : \Gamma \to \Phi$ is another ring homomorphism with $\omega = \theta\psi'$. Let $1_A$ and $1_B$ be the identities of $A$ and $B$, respectively. Then $\psi'$ sends the matrix units $\left(\begin{smallmatrix} 1_A & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1_A \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} 0 & 1_A \\ 0 & 0 \end{smallmatrix}\right)$ in $\Gamma$ to the matrix units $\left(\begin{smallmatrix} 1_B & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1_B \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} 0 & 1_B \\ 0 & 0 \end{smallmatrix}\right)$ in $\Phi$, respectively. It follows that $\psi'$ is of the form $\left(\begin{smallmatrix} \xi & \xi \\ \xi & \xi \end{smallmatrix}\right)$, where $\xi : A \to B$ is a ring homomorphism. Since $\omega = \theta\psi'$, we have $\rho\xi = f$ and $\phi\xi = g$. As the equality $t \otimes s = (t \otimes 1) \circ (1 \otimes s)$ holds in $A$ for any $s \in S$ and $t \in T$, it follows that

$$(t \otimes s)\xi = (t \otimes 1)\xi(1 \otimes s)\xi = ((t)\phi\xi)\,((s)\rho\xi) = (t)g(s)f = (t \otimes s)\sigma.$$

This shows that $\xi = \sigma$ and $\psi = \psi'$. Thus $\theta$ is universally $\{\varphi\}$-inverting.

*Remark* 5.1. In [27], the universal localization of $\Lambda$ at $\varphi$ was described as the matrix ring $M_2(C)$ of a ring $C$ where $C$ is given in terms of generators and relations. By the uniqueness of universal localizations, $C$ and $T \boxtimes_R S$ are isomorphic. Compared to the description in [27], noncommutative tensor products contain richer homological and structural information. In this paper, we will not use the description in [27].

Now we give a categorical characterization of $T \boxtimes_R S$. It is a relative version of coproducts with respect to exact contexts and shows the uniqueness of noncommutative tensor product for each exact context.

**Definition 5.2.** The *coproduct of an exact context* $(\lambda, \mu, M, m)$ is a quadruple $(A, u, v, w)$ consisting of a ring $A$, two ring homomorphisms $u : S \to A$ and $v : T \to A$, and a homomorphism $w : M \to A$ of abelian groups, satisfying

(1) $\lambda u = \mu v$ and $(smt)w = (s)u\,(t)v$ for all $s \in S$ and $t \in T$, and

(2) if $(B, f, g, h)$ is another quadruple with $B$ a ring, $f : S \to B$ and $g : T \to B$ two ring homomorphisms, and $h : M \to B$ a homomorphism of abelian groups such that (1) is satisfied, then there is a unique ring homomorphism $\sigma : A \to B$ such that $(f, g, h) = (u\sigma, v\sigma, w\sigma)$.

By Lemma 3.5, given a ring $B$ and two ring homomorphisms $f : S \to B$ and $g : T \to B$ with $\lambda f = \mu g$, there is a unique homomorphism $h_{f,g} : M \to B$ of $R$-$R$-bimodules such that $(sm)h_{f,g} = (s)f$ and $(mt)h_{f,g} = (t)g$ for $s \in S$ and $t \in T$. In general, $h_{f,g}$ does not satisfy $(smt)h_{f,g} = (s)f(t)g$. If a quadruple $(B, f, g, h)$ satisfies Definition 5.2(1), then $h = h_{f,g}$.

**Proposition 5.3.** *The quadruple $(T \boxtimes_R S, \rho, \phi, \beta)$ is the coproduct of the exact context $(\lambda, \mu, M, m)$.*

*Proof.* By definition, $\lambda\rho = \mu\phi$. Since $\beta$ is a homomorphism of $S$-$T$-bimodules with $(m)\beta = 1 \otimes 1$ by Lemma 3.6(3), $(smt)\beta = s \cdot (m)\beta \cdot t = (s)\rho \circ (t)\phi$ for any $s \in S$ and $t \in T$. Thus $(T \boxtimes_R S, \rho, \phi, \beta)$ satisfies Definition 5.2(1). The universal property of $(T \boxtimes_R S, \rho, \phi, \beta)$ follows from the proof of Theorem 1.1(1) and the following result:

> If a quadruple $(B, f, g, h)$ satisfies Definition 5.2(1), then $h : M \to B$ is a homomorphism of $S$-$T$-bimodules; that is, $(sxt)h = (s)f(x)h(t)g$ for $s \in S$, $x \in M$, and $t \in T$.

In fact, since $(B, f, g, h)$ satisfies Definition 5.2(1), $(smt)h = (s)f(t)g$. For $x \in M$, there exists an element $(s_x, t_x) \in S \oplus T$ such that $x = s_x m + m t_x$. Then $(sxt)h = (ss_x mt + smt_x t)h = (ss_x)f(t)g + (s)f(t_x t)g = (s)f\big((s_x)f + (t_x)g\big)(t)g$. Moreover, due to $(1)f = 1$ and $(1)g = 1$, we obtain $(s_x)f + (t_x)g = (s_x m)h + (mt_x)h = (x)h$. This implies that $(sxt)h = (s)f\,(x)h\,(t)g$. $\square$

**Corollary 5.4.**

(1) *If $(\lambda, \mu)$ is an exact pair, then the ring $T \boxtimes_R S$, together with the ring homomorphisms $\rho : S \to T \boxtimes_R S$ and $\phi : T \to T \boxtimes_R S$, is the coproduct $S \bigsqcup_R T$ of the $R$-rings $S$ and $T$ (via $\lambda$ and $\mu$) over $R$.*

(2) *If $\lambda : R \to S$ is a ring epimorphism, then $T \boxtimes_R S \simeq \mathrm{End}_T(T \otimes_R S)$ as rings.*

*Proof.*

(1) Assume that $(\lambda, \mu)$ is an exact pair. Then $M = S \otimes_R T$ and $m = 1 \otimes 1 \in M$. Let $B$ be a ring, and let $f : S \to B$ and $g : T \to B$ be ring homomorphisms. On the one hand, if $\lambda f = \mu g$, then the map $h' : M \to B$, given by $s \otimes t \mapsto (s)f\ (t)g$ for $s \in S$ and $t \in T$, is well defined, and thus $(B, f, g, h')$ satisfies Definition 5.2(1). On the other hand, if $h : M \to B$ is a homomorphism of abelian groups such that $(E, f, g, h)$ satisfies Definition 5.2(1), then $h = h'$ due to the equality $s \otimes t = s(1 \otimes 1)t = smt \in M$. By the definitions of coproducts of both rings and exact contexts, (1) is now a consequence of Proposition 5.3.

(2) Since $\lambda$ is a ring epimorphism, the pair $(\lambda, \mu)$ is exact by Corollary 3.10. It follows from (1) that $T \boxtimes_R S = S \bigsqcup_R T$, and thus that $\phi : T \to T \boxtimes_R S$ is a ring epimorphism. Since $T \boxtimes_R S = T \otimes_R S$ as $T$-modules, $T \boxtimes_R S = \mathrm{End}_{T \boxtimes_R S}(T \boxtimes_R S) \simeq \mathrm{End}_T(T \boxtimes_R S) = \mathrm{End}_T(T \otimes_R S)$. $\square$

5.2. **Proof of Theorem 1.1(2).** The proof of Theorem 1.1(2) is divided into four lemmas. We start by introducing a few notation.

Let $P^\bullet$ be the complex $0 \to \Lambda e_1 \xrightarrow{\varphi} \Lambda e_2 \to 0$ in $\mathscr{C}(\Lambda)$ with $\Lambda e_1$ and $\Lambda e_2$ in degrees $-1$ and $0$, respectively; that is $P^\bullet = \mathrm{Con}(\varphi)$. Note that $\Lambda e_1$ and $\Lambda e_2$ can be regarded as right $R$-modules via $\lambda : R \to S$ and $\mu : R \to T$, respectively. Then the map $\cdot m : S \to M$ is a homomorphism of $S$-$R$-bimodules. Thus $P^\bullet$ is a complex over $\Lambda \otimes_{\mathbb{Z}} R^{\mathrm{op}}$, and there is a distinguished triangle in $\mathscr{K}(\Lambda \otimes_{\mathbb{Z}} R^{\mathrm{op}})$:

$$\Lambda e_1 \xrightarrow{\varphi} \Lambda e_2 \longrightarrow P^\bullet \longrightarrow \Lambda e_1[1].$$

By Theorem 1.1(1), $\theta : \Lambda \to \Gamma$ is a ring epimorphism. Thus the restriction functor $\theta_* : \Gamma\text{-Mod} \to \Lambda\text{-Mod}$ is fully faithful. So we can regard $\Gamma$-Mod as a full subcategory of $\Lambda$-Mod and define a full subcategory

$$\mathscr{D}(\Lambda)_\Gamma := \{X^\bullet \in \mathscr{D}(\Lambda) \mid H^n(X^\bullet) \in \Gamma\text{-Mod for all } n \in \mathbb{Z}\}$$

of $\mathscr{D}(\Lambda)$. The importance of $\mathscr{D}(\Lambda)_\Gamma$ relies on the following result.

**Lemma 5.5.**

(1) $\mathrm{End}_{\mathscr{D}(\Lambda)}(P^\bullet) \simeq R$ *as rings.*

(2) *The complex* $P^\bullet$ *is self-orthogonal in* $\mathscr{D}(\Lambda)$*; that is,* $\mathrm{Hom}_{\mathscr{D}(\Lambda)}\big(P^\bullet, P^\bullet[n]\big) = 0$ *for any* $n \neq 0$.

(3) *There exists a recollement of triangulated categories*

$$(\star\star) \qquad \begin{tikzcd} \mathscr{D}(\Lambda)_\Gamma \arrow[r, "i_*"] & \mathscr{D}(\Lambda) \arrow[l, bend right, "i^*"'] \arrow[l, bend left] \arrow[r, "j^!"] & \mathscr{D}(R) \arrow[l, bend right, "j_!"'] \arrow[l, bend left] \end{tikzcd},$$

*where* $i_*$ *is the embedding,* $j_! := {}_\Lambda P^\bullet \otimes^{\mathbb{L}}_R -$ *and* $j^! := \mathrm{Hom}^\bullet_\Lambda(P^\bullet, -)$.

(4) *The map* $\theta : \Lambda \to \Gamma$ *is homological if and only if* $H^n\big(i_*i^*(\Lambda)\big) = 0$ *for all* $n \neq 0$. *In this case,* $\mathscr{D}(\Gamma)$ *is equivalent to* $\mathscr{D}(\Lambda)_\Gamma$ *as triangulated categories.*

*Proof.*

(1) Since $P^\bullet$ is a bounded complex of finitely generated projective $\Lambda$-modules, $\mathrm{End}_{\mathscr{D}(\Lambda)}(P^\bullet) \simeq \mathrm{End}_{\mathscr{K}(\Lambda)}(P^\bullet)$ as rings. Due to $\mathrm{Hom}_\Lambda(\Lambda e_2, \Lambda e_1) = 0$, we have $\mathrm{End}_{\mathscr{K}(\Lambda)}(P^\bullet) = \mathrm{End}_{\mathscr{C}(\Lambda)}(P^\bullet)$. Moreover, if $\mathrm{End}_\Lambda(\Lambda e_1)$ and $\mathrm{End}_\Lambda(\Lambda e_2)$ are identified with $S$ and $T$, respectively, then $\mathrm{End}_{\mathscr{C}(\Lambda)}(P^\bullet)$ can be identified with $K := \{(s,t) \in S \oplus T \mid sm = mt\}$, a subring of $S \oplus T$. Since $(\lambda, \mu, M, m)$ is an exact context, $R \simeq K$ as rings. Thus $\mathrm{End}_{\mathscr{D}(\Lambda)}(P^\bullet) \simeq R$ as rings.

(2) Clearly, $\mathrm{Hom}_{\mathscr{D}(\Lambda)}\big(P^\bullet, P^\bullet[n]\big) \simeq \mathrm{Hom}_{\mathscr{K}(\Lambda)}\big(P^\bullet, P^\bullet[n]\big) = 0$ for all $n \in \mathbb{Z}$ with $|n| \geq 2$. It follows from $\mathrm{Hom}_\Lambda(\Lambda e_2, \Lambda e_1) = 0$ that $\mathrm{Hom}_{\mathscr{D}(\Lambda)}\big(P^\bullet, P^\bullet[-1]\big) = 0$. Observe that $\mathrm{Hom}_{\mathscr{K}(\Lambda)}\big(P^\bullet, P^\bullet[1]\big) = 0$ if and only if $\mathrm{Hom}_\Lambda(\Lambda e_1, \Lambda e_2) = \varphi\,\mathrm{End}_\Lambda(\Lambda e_2) + \mathrm{End}_\Lambda(\Lambda e_1)\varphi$. This is equivalent to saying that the map $\left(\begin{smallmatrix} \cdot m \\ -m\cdot \end{smallmatrix}\right) : S \oplus T \to M$ is surjective by identifying $\mathrm{Hom}_\Lambda(\Lambda e_1, \Lambda e_2)$, $\mathrm{End}_\Lambda(\Lambda e_1)$, and $\mathrm{End}_\Lambda(\Lambda e_2)$ with $M$, $S$, and $T$, respectively. Hence (2) follows now from the definition of exact contexts.

(3) If $\Lambda$ is a $k$-algebra over a commutative ring $k$ such that ${}_k\Lambda$ is projective, then (3) follows from [21, Theorem 2.8]. Since we deal with arbitrary rings, the proof there seems not to work well. Hence we include a proof here.

By [10, Proposition 3.6(a)], the embeddings $\mathscr{D}(\Lambda)_\Gamma \hookrightarrow \mathscr{D}(\Lambda)$ and $\mathrm{Tria}({}_\Lambda P^\bullet) \hookrightarrow \mathscr{D}(\Lambda)$ induce a recollement $\big(\mathscr{D}(\Lambda)_\Gamma, \mathscr{D}(\Lambda), \mathrm{Tria}({}_\Lambda P^\bullet)\big)$ of triangulated categories. In the following, we show that $P^\bullet \otimes^{\mathbb{L}}_R - : \mathscr{D}(R) \to \mathscr{D}(\Lambda)$ is fully faithful and induces a triangle equivalence from $\mathscr{D}(R)$ to $\mathrm{Tria}({}_\Lambda P^\bullet)$.

Since $P^\bullet$ is a complex of $\Lambda$-$R$-bimodules, $P^\bullet \otimes^{\mathbb{L}}_R - : \mathscr{D}(R) \to \mathscr{D}(\Lambda)$ and $\mathbb{R}\operatorname{Hom}_\Lambda(P^\bullet, -) : \mathscr{D}(\Lambda) \to \mathscr{D}(R)$ are well-defined functors. Moreover, since $P^\bullet$ is a bounded complex of finitely generated projective $\Lambda$-modules, $\operatorname{Hom}^\bullet_\Lambda(P^\bullet, -) : \mathscr{K}(\Lambda) \to \mathscr{K}(R)$ preserves acyclicity; that is, $\operatorname{Hom}^\bullet_\Lambda(P^\bullet, W^\bullet)$ is acyclic whenever $W^\bullet \in \mathscr{C}(\Lambda)$ is acyclic. This automatically induces a derived functor $\mathscr{D}(\Lambda) \to \mathscr{D}(R)$, defined by $W^\bullet \mapsto \operatorname{Hom}^\bullet_\Lambda(P^\bullet, W^\bullet)$. Therefore, we can replace $\mathbb{R}\operatorname{Hom}_\Lambda(P^\bullet, -)$ with $\operatorname{Hom}^\bullet_\Lambda(P^\bullet, -)$ up to natural isomorphism.

Let

$$\begin{aligned}\mathscr{Y} := \{Y^\bullet \in \mathscr{D}(R) \mid P^\bullet \otimes^{\mathbb{L}}_R - : &\operatorname{Hom}_{\mathscr{D}(R)}(R, Y^\bullet[n]) \\ &\xrightarrow{\simeq} \operatorname{Hom}_{\mathscr{D}(\Lambda)}(P^\bullet \otimes^{\mathbb{L}}_R R, P^\bullet \otimes^{\mathbb{L}}_R Y^\bullet[n]) \text{ for all } n \in \mathbb{Z}\}.\end{aligned}$$

Then $\mathscr{Y}$ is a full triangulated subcategory of $\mathscr{D}(R)$. Since $P^\bullet \otimes^{\mathbb{L}}_R -$ commutes with arbitrary direct sums and since $P^\bullet$ is compact in $\mathscr{D}(\Lambda)$, $\mathscr{Y}$ is closed under arbitrary direct sums in $\mathscr{D}(R)$.

To prove ${}_R R \in \mathscr{Y}$, it is sufficient to prove that

(a) $P^\bullet \otimes^{\mathbb{L}}_R -$ induces an isomorphism of rings from $\operatorname{End}_{\mathscr{D}(R)}(R)$ to $\operatorname{End}_{\mathscr{D}(\Lambda)}(P^\bullet \otimes^{\mathbb{L}}_R R)$, and

(b) $\operatorname{Hom}_{\mathscr{D}(\Lambda)}(P^\bullet \otimes^{\mathbb{L}}_R R, P^\bullet \otimes^{\mathbb{L}}_R R[n]) = 0$ for any $n \neq 0$.

Since $P^\bullet \otimes^{\mathbb{L}}_R R \simeq P^\bullet$ in $\mathscr{D}(\Lambda)$, (a) is equivalent to saying that the right multiplication map $R \to \operatorname{End}_{\mathscr{D}(\Lambda)}(P^\bullet)$ is an isomorphism, while (b) is equivalent to $\operatorname{Hom}_{\mathscr{D}(\Lambda)}(P^\bullet, P^\bullet[n]) = 0$ for any $n \neq 0$. Thus (a) and (b) follow directly from (1) and (2), respectively. This shows that $R \in \mathscr{Y}$.

Thus $\mathscr{Y} = \mathscr{D}(R)$ since $\mathscr{D}(R) = \mathrm{Tria}(R)$. Consequently, for any $Y^\bullet \in \mathscr{D}(R)$, there is an isomorphism

$$P^\bullet \otimes^{\mathbb{L}}_R - : \operatorname{Hom}_{\mathscr{D}(R)}(R, Y^\bullet[n]) \xrightarrow{\simeq} \operatorname{Hom}_{\mathscr{D}(\Lambda)}(P^\bullet \otimes^{\mathbb{L}}_R R, P^\bullet \otimes^{\mathbb{L}}_R Y^\bullet[n]) \quad \text{for all } n \in \mathbb{Z}.$$

Now, we take a complex $N^\bullet \in \mathscr{D}(R)$ and consider

$$\begin{aligned}\mathscr{X}_{N^\bullet} := \{X^\bullet \in \mathscr{D}(R) \mid P^\bullet \otimes^{\mathbb{L}}_R - : &\operatorname{Hom}_{\mathscr{D}(R)}(X^\bullet, N^\bullet[n]) \\ &\xrightarrow{\simeq} \operatorname{Hom}_{\mathscr{D}(\Lambda)}(P^\bullet \otimes^{\mathbb{L}}_R X^\bullet, P^\bullet \otimes^{\mathbb{L}}_R N^\bullet[n]) \text{ for all } n \in \mathbb{Z}\}.\end{aligned}$$

Then $\mathscr{X}_{N^\bullet}$ is a full triangulated subcategory of $\mathscr{D}(R)$ and closed under arbitrary direct sums in $\mathscr{D}(R)$. It follows from $R \in \mathscr{X}_{N^\bullet}$ and $\mathscr{D}(R) = \mathrm{Tria}(R)$ that $\mathscr{X}_{N^\bullet} = \mathscr{D}(R)$. Consequently, for $M^\bullet \in \mathscr{D}(R)$, there is an isomorphism

$$P^\bullet \otimes^{\mathbb{L}}_R - : \operatorname{Hom}_{\mathscr{D}(R)}\left(M^\bullet, N^\bullet[n]\right) \xrightarrow{\simeq} \operatorname{Hom}_{\mathscr{D}(\Lambda)}\left(P^\bullet \otimes^{\mathbb{L}}_R M^\bullet, P^\bullet \otimes^{\mathbb{L}}_R N^\bullet[n]\right)$$

for all $n \in \mathbb{Z}$. Hence $P^\bullet \otimes^{\mathbb{L}}_R - : \mathscr{D}(R) \to \mathscr{D}(\Lambda)$ is fully faithful.

Recall that $\mathrm{Tria}(P^\bullet)$ is the smallest full triangulated subcategory of $\mathscr{D}(\Lambda)$, containing $P^\bullet$ and being closed under arbitrary direct sums in $\mathscr{D}(\Lambda)$. Hence the image of $\mathscr{D}(R)$ under $P^\bullet \otimes^{\mathbb{L}}_R -$ is $\mathrm{Tria}(P^\bullet)$ (see property (2) in Section 2.1), and $P^\bullet \otimes^{\mathbb{L}}_R -$ induces a triangle equivalence from $\mathscr{D}(R)$ to $\mathrm{Tria}(P^\bullet)$. Since $\operatorname{Hom}^\bullet_\Lambda(P^\bullet, -)$ is a right adjoint of $P^\bullet \otimes^{\mathbb{L}}_R -$, the restriction of the functor $\operatorname{Hom}^\bullet_\Lambda(P^\bullet, -)$ to $\mathrm{Tria}(P^\bullet)$ is the quasi-inverse of the functor $P^\bullet \otimes^{\mathbb{L}}_R - : \mathscr{D}(R) \to \mathrm{Tria}(P^\bullet)$. In particular, $\operatorname{Hom}^\bullet_\Lambda(P^\bullet, -)$ induces an equivalence of triangulated categories: $\mathrm{Tria}(P^\bullet) \xrightarrow{\simeq} \mathscr{D}(R)$. Thus we obtain recollement $(\star\star)$ in (3).

(4) is a consequence of [10, Proposition 3.6(b)]. See also [1, Proposition 1.7] or [22, Theorem 0.7 and Proposition 5.6]. $\square$

From now on, we keep the notation in Lemma 5.5 and let $\varepsilon : j_!j^! \to Id_{\mathscr{D}(\Lambda)}$ and $\eta : Id_{\mathscr{D}(\Lambda)} \to i_*i^*$ be the counit and unit with respect to the adjoint pairs $(j_!, j^!)$ and $(i^*, i_*)$ in recollement $(\star\star)$, respectively.

**Lemma 5.6.**

(1) $i_*i^*(\Lambda e_1) \simeq i_*i^*(\Lambda e_2)$ *in* $\mathscr{D}(\Lambda)$.

(2) $H^n\big(i_*i^*(\Lambda e_1)\big) \simeq \begin{cases} 0 & \text{if } n \geq 1, \\ \mathrm{Tor}^R_{-n}(T,S) \oplus \mathrm{Tor}^R_{-n}(T,S) & \text{if } n \leq -2. \end{cases}$

*Proof.*

(1) Applying $i_*i^* : \mathscr{D}(\Lambda) \to \mathscr{D}(\Lambda)$ to the triangle $P^\bullet[-1] \to \Lambda e_1 \xrightarrow{\varphi} \Lambda e_2 \to P^\bullet$ in $\mathscr{D}(\Lambda)$, we obtain another triangle in $\mathscr{D}(\Lambda)$:

$$i_*i^*(P^\bullet)[-1] \longrightarrow i_*i^*(\Lambda e_1) \xrightarrow{i_*i^*(\varphi)} i_*i^*(\Lambda e_2) \longrightarrow i_*i^*(P^\bullet).$$

From $i^*j_! = 0$ in $(\star\star)$, we have $i^*(P^\bullet) \simeq i^*j_!(R) = 0$. Thus $i_*i^*(\varphi) : i_*i^*(\Lambda e_1) \to i_*i^*(\Lambda e_2)$ is an isomorphism.

(2) By the definition of recollements, there is the triangle in $\mathscr{D}(\Lambda)$:

$$(\divideontimes) \qquad j_!j^!(\Lambda e_1) \xrightarrow{\varepsilon_{\Lambda e_1}} \Lambda e_1 \xrightarrow{\eta_{\Lambda e_1}} i_*i^*(\Lambda e_1) \longrightarrow j_!j^!(\Lambda e_1)[1].$$

Certainly, $j^!(\Lambda e_1) = \mathrm{Hom}_\Lambda(P^\bullet, \Lambda e_1) \simeq S[-1]$ as complexes of $R$-modules. So, in the following, we identify $\mathrm{Hom}_\Lambda(P^\bullet, \Lambda e_1)$ with $S[-1]$ and rewrite $(\divideontimes)$ as the following triangle in $\mathscr{D}(\Lambda)$:

$$P^\bullet \otimes^{\mathbb{L}}_R S[-1] \xrightarrow{\varepsilon_{\Lambda e_1}} \Lambda e_1 \xrightarrow{\eta_{\Lambda e_1}} i_*i^*(\Lambda e_1) \longrightarrow P^\bullet \otimes^{\mathbb{L}}_R S.$$

For $n \in \mathbb{Z}$, we take the $n$th cohomology of this triangle and can conclude that $H^n(i_*i^*(\Lambda e_1)) \simeq H^n(P^\bullet \otimes^{\mathbb{L}}_R S)$ for $n \geq 1$ or $n \leq -2$. Moreover,

$$P^\bullet = T \oplus (0 \to S \xrightarrow{\cdot m} M \to 0) = T \oplus \mathrm{Con}(\cdot m) \in \mathscr{D}(R^{\mathrm{op}}).$$

Since $(\lambda, \mu, M, m)$ is an exact context, it follows from Lemma 3.5 that the chain map $(\lambda, m\cdot) : \mathrm{Con}(\mu) \to \mathrm{Con}(\cdot m)$ is a quasi-isomorphism. This implies that $\mathrm{Con}(\cdot m) \simeq \mathrm{Con}(\mu)$ in $\mathscr{D}(R^{\mathrm{op}})$. Thus $P^\bullet \simeq T \oplus \mathrm{Con}(\mu)$ in $\mathscr{D}(R^{\mathrm{op}})$ and $P^\bullet \otimes^{\mathbb{L}}_R S \simeq (T \otimes^{\mathbb{L}}_R S) \oplus (\mathrm{Con}(\mu) \otimes^{\mathbb{L}}_R S)$ in $\mathscr{D}(\mathbb{Z})$. In particular, $H^n(P^\bullet \otimes^{\mathbb{L}}_R S) \simeq H^n(T \otimes^{\mathbb{L}}_R S) \oplus H^n(\mathrm{Con}(\mu) \otimes^{\mathbb{L}}_R S)$ for all $n \in \mathbb{Z}$. Applying the functor $- \otimes^{\mathbb{L}}_R S$ to the canonical triangle $R \xrightarrow{\mu} T \to \mathrm{Con}(\mu) \to R[1]$ in $\mathscr{D}(R^{\mathrm{op}})$, we obtain another triangle $S \to T \otimes^{\mathbb{L}}_R S \to \mathrm{Con}(\mu) \otimes^{\mathbb{L}}_R S \to S[1]$ in $\mathscr{D}(\mathbb{Z})$. Thus, if $n \geq 1$ or $n \leq -2$, then $H^n(T \otimes^{\mathbb{L}}_R S) \simeq H^n(\mathrm{Con}(\mu) \otimes^{\mathbb{L}}_R S)$, and therefore

$$\begin{aligned} H^n(i_*i^*(\Lambda e_1)) &\simeq H^n(P^\bullet \otimes^{\mathbb{L}}_R S) \simeq H^n(T \otimes^{\mathbb{L}}_R S) \oplus H^n(T \otimes^{\mathbb{L}}_R S) \\ &\simeq \mathrm{Tor}^R_{-n}(T,S) \oplus \mathrm{Tor}^R_{-n}(T,S), \end{aligned}$$

where $\mathrm{Tor}^R_{-n}(T,S) := 0$ for any $n \geq 1$. This shows (2). $\square$

For the calculation of $H^n\big(i_*i^*(\Lambda e_1)\big)$ for $n = 0, -1$, the following result will be used.

**Lemma 5.7.** *Let $\mu_S : S \otimes S \to S$ be the multiplication map, and let $(\cdot m) : S \to M$ be the right multiplication map by $m$.*

(1) *The restriction $\sigma$ of the map $(\cdot m)\otimes_R S$ to* $\mathrm{Ker}(\mu_S)$ *is injective and* $\mathrm{Coker}(\sigma)\simeq T\otimes_R S$ *as $R$-$S$-bimodules.*
(2) *The cokernel of the map* $\mathrm{Tor}_1^R(\cdot m, S):\mathrm{Tor}_1^R(S,S)\to\mathrm{Tor}_1^R(M,S)$ *is isomorphic to* $\mathrm{Tor}_1^R(T,S)$.

*Proof.*

(1) The sequence $0\to\mathrm{Ker}(\mu_S)\to S\otimes_R S\xrightarrow{\mu_S} S\to 0$ always splits in the category of $R$-$S$-bimodules since the composition of $\lambda\otimes_R S: R\otimes_R S\to S\otimes_R S$ with $\mu_S$ is an isomorphism. Consequently, $\lambda\otimes_R S$ is injective, $\mathrm{Im}(\lambda\otimes_R S)\cap\mathrm{Ker}(\mu_S)=0$, and $S\otimes_R S=\mathrm{Ker}(\mu_S)\oplus\mathrm{Im}(\lambda\otimes_R S)$. Recall that the diagram in Lemma 3.5 is a pushout and pullback diagram in the category of $R$-$R$-bimodules. Applying the functor $-\otimes_R S$ to Lemma 3.5 yields the commutative diagram

$$\begin{tikzcd}
R\otimes_R S \arrow[r, "\lambda\otimes_R S"] \arrow[d, "\mu\otimes_R S"'] & S\otimes_R S \arrow[d, "(\cdot m)\otimes_R S"] \\
T\otimes_R S \arrow[r, "(m\cdot)\otimes_R S"] & M\otimes_R S
\end{tikzcd}$$

which is again a pushout and pullback diagram in the category of $R$-$S$-bimodules. Let $\varphi_1:=(\cdot m)\otimes_R S$. Then $\lambda\otimes_R S$ induces an isomorphism from $\mathrm{Ker}(\mu\otimes_R S)$ to $\mathrm{Ker}(\varphi_1)$. In particular, $\mathrm{Ker}(\varphi_1)\subseteq\mathrm{Im}(\lambda\otimes_R S)$. It follows from $\mathrm{Im}(\lambda\otimes_R S)\cap\mathrm{Ker}(\mu_S)=0$ that $\mathrm{Ker}(\varphi_1)\cap\mathrm{Ker}(\mu_S)=0$. Thus $\sigma$ is injective.

It is clear that $\mathrm{Coker}(\sigma)$ is the pushout of the pair $(\mu_S,\varphi_1)$ in the category of $S$-$S$-bimodules. As the composition of $\lambda\otimes_R S$ with $\mu_S$ is an isomorphism of $R$-$S$-bimodules, we have $\mathrm{Coker}(\sigma)\simeq T\otimes_R S$ as $R$-$S$-bimodules.

(2) Applying $\mathrm{Tor}_i^R(-,S)$ to $(*)$ (see Definition 3.1), we obtain a long exact sequence of abelian groups:

$$0=\mathrm{Tor}_1^R(R,S)\longrightarrow\mathrm{Tor}_1^R(S,S)\oplus\mathrm{Tor}_1^R(T,S)\xrightarrow{\begin{pmatrix}\mathrm{Tor}_1^R(\cdot m,\,S)\\ -\,\mathrm{Tor}_1^R(m\cdot,\,S)\end{pmatrix}}\mathrm{Tor}_1^R(M,S)$$
$$\longrightarrow R\otimes_R S\xrightarrow{(\lambda\otimes_R S,\,\mu\otimes_R S)} S\otimes_R S\oplus S\otimes_R T.$$

Since $\lambda\otimes_R S: R\otimes_R S\to S\otimes_R S$ is injective, the map $(\lambda\otimes_R S,\,\mu\otimes_R S)$ is injective. Thus

$$\begin{pmatrix}\mathrm{Tor}_1^R(\cdot m,\,S)\\ -\,\mathrm{Tor}_1^R(m\cdot,\,S)\end{pmatrix}:\mathrm{Tor}_1^R(S,S)\oplus\mathrm{Tor}_1^R(T,S)\xrightarrow{\simeq}\mathrm{Tor}_1^R(M,S).$$

This implies (2). $\square$

**Lemma 5.8.** $H^{-1}\big(i_*i^*(\Lambda e_1)\big)\simeq\mathrm{Tor}_1^R(T,S)\oplus\mathrm{Tor}_1^R(T,S)$ *and* $H^0\big(i_*i^*(\Lambda e_1)\big)\simeq T\otimes_R S\oplus T\otimes_R S$.

*Proof.* We have two homomorphisms of $S$-$S$-bimodules:

$$\mu_S: S\otimes_R S\longrightarrow S,\ \ s_1\otimes s_2\mapsto s_1s_2,\qquad \varphi_1:\ S\otimes_R S\longrightarrow M\otimes_R S,\ \ s_1\otimes s_2\mapsto s_1m\otimes s_2$$

for $s_1, s_2 \in S$. By identifying the $\Lambda$-modules $\Lambda e_1 \otimes_R S$ and $\Lambda e_2 \otimes_R S$ with $\left(\begin{smallmatrix} S\otimes_R S \\ 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} M\otimes_R S \\ T\otimes_R S \end{smallmatrix}\right)$, respectively, we get a chain map in $\mathscr{C}(\Lambda)$:

$$\begin{tikzcd}
P^\bullet \otimes_R (S[-1]) : \arrow[d, "g^\bullet"] & 0 \arrow[r] & \begin{pmatrix} S\otimes_R S \\ 0 \end{pmatrix} \arrow[r, "\left(\begin{smallmatrix} \varphi_1 \\ 0 \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} \mu_S \\ 0 \end{smallmatrix}\right)"] & \begin{pmatrix} M\otimes_R S \\ T\otimes_R S \end{pmatrix} \arrow[r] \arrow[d, "0"] & 0 \\
\Lambda e_1 : & 0 \arrow[r] & \begin{pmatrix} S \\ 0 \end{pmatrix} \arrow[r, "0"] & 0 \arrow[r] & 0
\end{tikzcd}$$

Let $\tau : {}_pS \to S$ be a deleted projective resolution of ${}_RS$. Recall that $j^!(\Lambda e_1) = \mathrm{Hom}_\Lambda(P^\bullet, \Lambda e_1) = S[-1]$. Then the counit $\varepsilon_{\Lambda e_1} : j_! j^!(\Lambda e_1) \to \Lambda e_1$ is the composition of the homomorphisms

$$j_! j^!(\Lambda e_1) = P^\bullet \otimes^{\mathbb{L}}_R \mathrm{Hom}^\bullet_\Lambda(P^\bullet, \Lambda e_1) = P^\bullet \otimes^\bullet_R ({}_pS)[-1] \xrightarrow{1\otimes\tau[-1]} P^\bullet \otimes^\bullet_R S[-1] \xrightarrow{g^\bullet} \Lambda e_1.$$

Further, let $h^\bullet$ be the chain map $(1, 0) : P^\bullet \to \Lambda e_1[1]$. Then we have a commutative diagram

$$\begin{tikzcd}
P^\bullet \otimes_R ({}_pS)[-1] \arrow[r, "1\otimes\tau[-1]"] \arrow[d, "h^\bullet\otimes 1"'] & P^\bullet \otimes_R S[-1] \arrow[r, "g^\bullet"] \arrow[d, "h^\bullet\otimes 1"] & \Lambda e_1 \arrow[d, equal] \\
\Lambda e_1[1] \otimes_R ({}_pS)[-1] \arrow[r, "1\otimes\tau[-1]"] & \Lambda e_1[1] \otimes_R S[-1] \arrow[r, "\left(\begin{smallmatrix} \mu_S \\ 0 \end{smallmatrix}\right)"] & \Lambda e_1
\end{tikzcd}$$

This implies that the diagram

$$(**) \qquad \begin{tikzcd}
P^\bullet \otimes^{\mathbb{L}}_R \mathrm{Hom}_\Lambda(P^\bullet, \Lambda e_1) \arrow[r, "h^\bullet\otimes^{\mathbb{L}} 1"] \arrow[d, equal] & \Lambda e_1[1] \otimes^{\mathbb{L}}_R \mathrm{Hom}_\Lambda(P^\bullet, \Lambda e_1) \arrow[d, "\left(1\otimes\tau[-1]\right)\left(\begin{smallmatrix} \mu_S \\ 0 \end{smallmatrix}\right)"'] \\
P^\bullet \otimes^{\mathbb{L}}_R \mathrm{Hom}_\Lambda(P^\bullet, \Lambda e_1) \arrow[r, "\varepsilon_{\Lambda e_1}"] & \Lambda e_1 = \begin{pmatrix} S \\ 0 \end{pmatrix}
\end{tikzcd}$$

is commutative in $\mathscr{D}(\Lambda)$. Since $\Lambda e_2 \to P^\bullet \xrightarrow{h^\bullet} \Lambda e_1[1] \xrightarrow{\varphi[1]} \Lambda e_2[1]$ is a distinguished triangle in $\mathscr{D}(\Lambda \otimes_{\mathbb{Z}} R^{\mathrm{op}})$, there exists a unique homomorphism $\xi : \Lambda e_2 \otimes^{\mathbb{L}}_R \mathrm{Hom}_\Lambda(P^\bullet, \Lambda e_1)[1] \to i_* i^*(\Lambda e_1)$ and a complex $W^\bullet$ in $\mathscr{D}(\Lambda)$ such that $(**)$ is completed by the commutative diagram

$$\begin{tikzcd}[column sep=small]
 & W^\bullet \arrow[r, equal] \arrow[d, "\zeta"] & W^\bullet \arrow[d, "\psi"] & \\
P^\bullet \otimes^{\mathbb{L}}_R \mathrm{Hom}_\Lambda(P^\bullet, \Lambda e_1) \arrow[r, "h^\bullet\otimes^{\mathbb{L}} 1"] \arrow[d, equal] & \Lambda e_1[1] \otimes^{\mathbb{L}}_R \mathrm{Hom}_\Lambda(P^\bullet, \Lambda e_1) \arrow[r, "\varphi[1]\otimes^{\mathbb{L}} 1"] \arrow[d, "(1\otimes\tau[-1])\left(\begin{smallmatrix} \mu_S \\ 0 \end{smallmatrix}\right)"'] & \Lambda e_2[1] \otimes^{\mathbb{L}}_R \mathrm{Hom}_\Lambda(P^\bullet, \Lambda e_1) \arrow[r] \arrow[d, dashed, "\xi"'] & P^\bullet[1] \otimes^{\mathbb{L}}_R \mathrm{Hom}_\Lambda(P^\bullet, \Lambda e_1) \arrow[d, equal] \\
P^\bullet \otimes^{\mathbb{L}}_R \mathrm{Hom}_\Lambda(P^\bullet, \Lambda e_1) \arrow[r, "\varepsilon_{\Lambda e_1}"] & \Lambda e_1 = \begin{pmatrix} S \\ 0 \end{pmatrix} \arrow[r, "\eta_{\Lambda e_1}"] \arrow[d] & i_* i^*(\Lambda e_1) \arrow[r] \arrow[d] & P^\bullet[1] \otimes^{\mathbb{L}}_R \mathrm{Hom}_\Lambda(P^\bullet, \Lambda e_1) \\
 & W^\bullet[1] \arrow[r, equal] & W^\bullet[1] &
\end{tikzcd}$$

with rows and columns being distinguished triangles in $\mathscr{D}(\Lambda)$, where $\psi = \zeta(\varphi[1] \otimes^{\mathbb{L}} 1)$. The uniqueness of $\xi$ is due to the equalities

$$\mathrm{Hom}_{\mathscr{D}(\Lambda)}(P^\bullet[1] \otimes^{\mathbb{L}}_R \mathrm{Hom}_\Lambda(P^\bullet, \Lambda e_1), i_* i^*(\Lambda e_1)) = \mathrm{Hom}_{\mathscr{D}(\Lambda)}(j_!(S), i_* i^*(\Lambda e_1)) = 0.$$

In particular, we obtain a triangle $W^\bullet \xrightarrow{\psi} \Lambda e_2 \otimes^{\mathbb{L}}_R S \xrightarrow{\xi} i_*i^*(\Lambda e_1) \to W^\bullet[1]$ in $\mathscr{D}(\Lambda)$. This yields an exact sequence of abelian groups

$$H^{-1}(W^\bullet) \xrightarrow{H^{-1}(\psi)} H^{-1}(\Lambda e_2 \otimes^{\mathbb{L}}_R S) \xrightarrow{H^{-1}(\xi)} H^{-1}(i_*i^*(\Lambda e_1))$$
$$\longrightarrow H^0(W^\bullet) \xrightarrow{H^0(\psi)} H^0(\Lambda e_2 \otimes^{\mathbb{L}}_R S).$$

We claim that $H^0(\psi) : H^0(W^\bullet) \to H^0(\Lambda e_2 \otimes^{\mathbb{L}}_R S)$ is always injective.

Indeed, by definition, the map $H^0(\psi)$ is the composition of $H^0(\zeta) : H^0(W^\bullet) \to H^0(\Lambda e_1 \otimes^{\mathbb{L}}_R S)$ with $H^0(\varphi[1] \otimes^{\mathbb{L}} 1) : H^0(\Lambda e_1 \otimes^{\mathbb{L}}_R S) \to H^0(\Lambda e_2 \otimes^{\mathbb{L}}_R S)$. Moreover,

$$H^0(\Lambda e_1 \otimes^{\mathbb{L}}_R S) = \Lambda e_1 \otimes_R S = S \otimes_R S \quad \text{and} \quad H^0(\Lambda e_2 \otimes^{\mathbb{L}}_R S) = \Lambda e_2 \otimes_R S.$$

So we can identify $H^0(\varphi[1] \otimes^{\mathbb{L}} 1)$ with the map $\varphi \otimes_R S : \Lambda e_1 \otimes_R S \to \Lambda e_2 \otimes_R S$. Applying the functor $H^0$ to the triangle

$$W^\bullet \xrightarrow{\zeta} \Lambda e_1 \otimes^{\mathbb{L}}_R S \xrightarrow{(1\otimes\tau[-1])\left(\begin{smallmatrix}\mu_S\\ 0\end{smallmatrix}\right)} \Lambda e_1 \longrightarrow W^\bullet[1],$$

we obtain a short exact sequence $0 \to H^0(W^\bullet) \xrightarrow{H^0(\zeta)} S \otimes_R S \xrightarrow{\mu_S} S \to 0$, where $\Lambda e_1$ is identified with $S$. Thus $H^0(\zeta)$ induces an isomorphism $H^0(W^\bullet) \xrightarrow{\simeq} \mathrm{Ker}(\mu_S)$. It follows from $H^0(\psi) = H^0(\zeta)(\varphi \otimes_R S)$ that $H^0(\psi)$ is injective if and only if the restriction of $\varphi \otimes_R S$ to $\mathrm{Ker}(\mu_S)$ is also. As $\mathrm{Im}(\varphi \otimes_R S) \subseteq M \otimes_R S$, the latter is further equivalent to the restriction $\sigma$ of $\varphi_1$ to $\mathrm{Ker}(\mu_S)$ being injective. But, by Lemma 5.7(1), $\sigma$ is indeed injective, which proves the claim.

Consequently, $H^{-1}(\xi)$ is surjective and $H^{-1}(i_*i^*(\Lambda e_1)) \simeq \mathrm{Coker}(H^{-1}(\psi))$. Since $H^{-1}(\psi)$ is the composition of the isomorphism $H^{-1}(\zeta) : H^{-1}(W^\bullet) \xrightarrow{\simeq} H^{-1}(\Lambda e_1 \otimes^{\mathbb{L}}_R S)$ with the map

$$H^{-1}(\varphi[1] \otimes^{\mathbb{L}} 1) : H^{-1}(\Lambda e_1 \otimes^{\mathbb{L}}_R S) \longrightarrow H^{-1}(\Lambda e_2 \otimes^{\mathbb{L}}_R S),$$

$H^{-1}(i_*i^*(\Lambda e_1)) \simeq \mathrm{Coker}(H^{-1}(\psi)) \simeq \mathrm{Coker}\big(H^{-1}(\varphi[1] \otimes^{\mathbb{L}} 1)\big)$. Recall that $\Lambda e_1 = S$, $\Lambda e_2 = M \oplus T$, and $\varphi = (\cdot m, 0) : S \to M \oplus T$. Moreover, we have $H^{-1}(\Lambda e_1 \otimes^{\mathbb{L}}_R S) = \mathrm{Tor}^R_1(S, S)$, $H^{-1}(\Lambda e_2 \otimes^{\mathbb{L}}_R S) = \mathrm{Tor}^R_1(M \oplus T, S)$, and $H^{-1}(\varphi[1] \otimes^{\mathbb{L}} 1)$ is given by

$$\big(\mathrm{Tor}^R_1(\cdot m, S), 0\big) : \mathrm{Tor}^R_1(S, S) \longrightarrow \mathrm{Tor}^R_1(M, S) \oplus \mathrm{Tor}^R_1(T, S).$$

Thus $\mathrm{Coker}\big(H^{-1}(\varphi[1] \otimes^{\mathbb{L}} 1)\big) \simeq \mathrm{Coker}(\mathrm{Tor}^R_1(\cdot m, S)) \oplus \mathrm{Tor}^R_1(T, S)$. Since Coker $(\mathrm{Tor}^R_1(\cdot m, S)) \simeq \mathrm{Tor}^R_1(T, S)$ by Lemma 5.7(2), $H^{-1}(i_*i^*(\Lambda e_1)) \simeq \mathrm{Tor}^R_1(T, S) \oplus \mathrm{Tor}^R_1(T, S)$. This shows the first isomorphism in (2).

Now, applying $H^0$ and $H^1$ to the triangle $W^\bullet \xrightarrow{\psi} \Lambda e_2 \otimes^{\mathbb{L}}_R S \xrightarrow{\xi} i_*i^*(\Lambda e_1) \to W^\bullet[1]$, we have the exact sequence:

$$H^0(W^\bullet) \xrightarrow{H^0(\psi)} H^0(\Lambda e_2 \otimes^{\mathbb{L}}_R S) \xrightarrow{H^0(\xi)} H^0(i_*i^*(\Lambda e_1)) \longrightarrow H^1(W^\bullet).$$

The triangle $W^\bullet \xrightarrow{\zeta} S \otimes^{\mathbb{L}}_R S \to S \to W^\bullet[1]$ implies that $H^1(W^\bullet) = 0$. Since $H^0(\psi)$ is injective and since $\mathrm{Coker}(\sigma) \simeq T \otimes_R S$ by Lemma 5.7(1), $H^0(i_*i^*(\Lambda e_1)) \simeq \mathrm{Coker}(H^0(\psi)) \simeq T \otimes_R S \oplus \mathrm{Coker}(\sigma) \simeq T \otimes_R S \oplus T \otimes_R S$. This shows the second isomorphism. □

*Proof of Theorem* 1.1(2). By Lemma 5.5(4), the map $\theta$ is homological if and only if $H^n\big(i_*i^*(\Lambda)\big) = 0$ for all $n \neq 0$. By Lemmas 5.6 and 5.8, $H^n\big(i_*i^*(\Lambda)\big) \simeq H^n\big(i_*i^*(\Lambda e_1)\big) \oplus H^n\big(i_*i^*(\Lambda e_1)\big) \simeq \bigoplus_{i=1}^4 \mathrm{Tor}^R_{-n}(T, S)$ for each $n \in \mathbb{Z}$. Thus (2) holds. □

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