Comparison of finitistic dimensions of Artin algebras

Changchang Xi(惠昌常)
Beijing Normal University

http://math.bnu.edu.cn/~ccxi/

July 25, 2006

In the representation theory of Artin algebras there is a well-known conjecture: *Given an arbitrary Artin algebra, its finitistic dimension is finite*. This is called the finitistic dimension conjecture. It is over 45 years old and remains open to date. In this note, we survey its developments, and report some of the new advances on the finitistic dimension conjecture.

1. The conjecture

R: commutative Artin ring,

A: Artin algebra over R, (special examples: finite-dimensional algebras over fields);

A-mod : category of all finitely generated left A-modules ;

pd(M) : projective dimension of M $\in$ A-mod;

$\mathscr{P}^{\infty}(A)$: $=\{$M $\in$ A-mod | pd(M) $< \infty\}$;

The finitistic dimension of A is defined by

fin.dim(A):= sup$\{$pd(M) | M in $\mathscr{P}^{\infty}$(A)$\}$.

The finitistic dimension conjecture says:

$$\boxed{\text{A: Artin algebra over R} \Longrightarrow \text{fin.dim(A)} < \infty.}$$

This is an over 45 years old conjecture. In 1960, H.Bass wrote this conjecture as a question of Rosenberg and Zelinsky in one of his papers.

Note: The finite ring $\mathbb{Z}/(4)$ is an Artin algebra, but it is *not* a finite-dimensional algebra over any field.

2. A little bit background

The first homological result might be the Hilbert's syzygy theorem.

Hilbert's syzygy theorem (1890):

gl.dim k$[x_1, ..., x_n] = n$.

Thus any module over k$[x_1, ..., x_n]$ can be resolved as an long exact sequence of length at most n of free modules.

In 1940's, homological algebra which stemmed from algebraic topology became popular and was widely applied to the study of rings and algebras. Since then, there have been many people who exploit homological methods to investigate algebras and representations. For example, H.Cartan, S.Eilenberg, S.MacLane, M.Auslander, D.Buchsbaum, M.Nagata, T.Nakayama, ... Also, one may discern a little flavor of abstract algebra and homological algebra in that years from a statement of Hermann Weyl(1885-1955): " In these days the angel of topology and the devil of abstract algebra fight for the soul of every individual discipline of mathematics. "

In algebraic geometry the polynomial ring k$[x_1, ..., x_n]$ and its factor rings are the basic elements . The nice relationship between geometry and homological algebra can be seen from one beautiful result of Auslander-Buchsbaum-Serre in 1955.

Auslander-Buchsbaum-Serre theorem(1955):

V: algebraic variety over a filed $k = \bar{k}$,

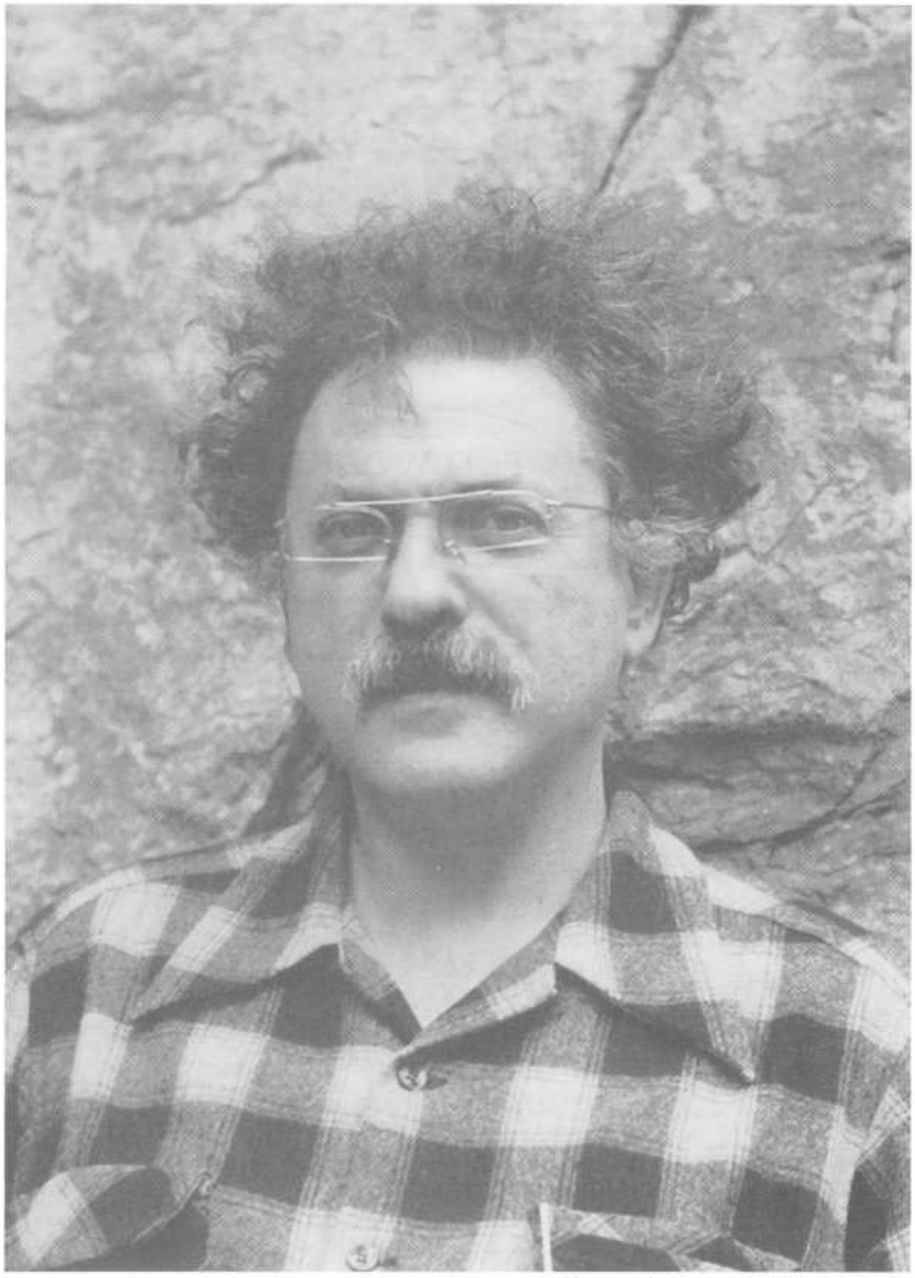
A: coordinate ring of V. Then:

V is smooth $\iff$ gl.dim$(A) < \infty$.

On the other hand, one often faces geometric objects whose coordinate rings have infinite global dimension. To investigate algebras and modules with infinity dimension, the finitistic dimension was then introduced. Of course, a major task here is to understand the finitistic dimension.

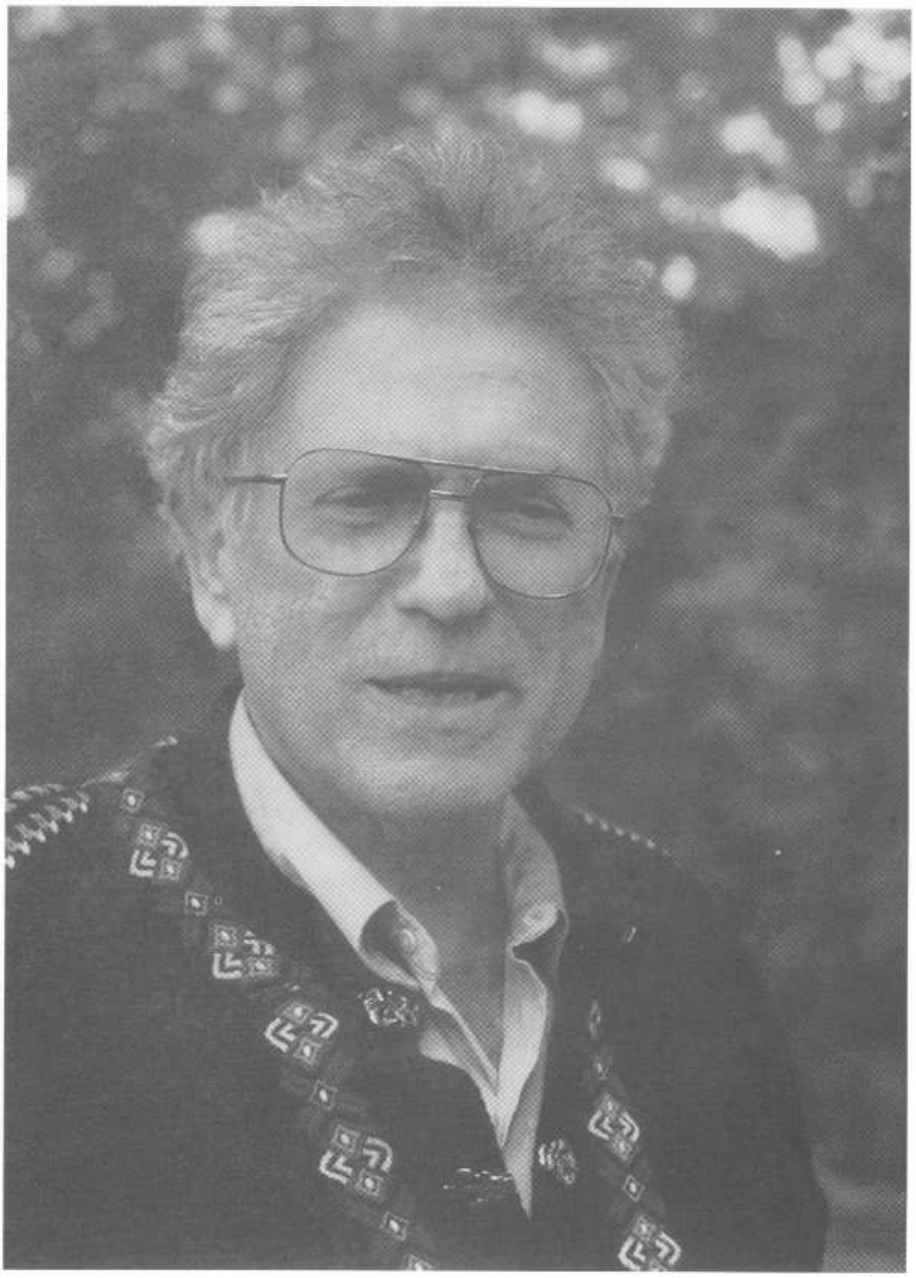
3. Interests in the conjecture

The conjecture attracts many people in the area of homological algebra, and the representation theory of algebras. One of them is Maurice Auslander(1926-1994):



(Photograph courtesy of Gordana Todorov.)

MAURICE AUSLANDER
August 3, 1926–November 18, 1994



(Photograph courtesy of Gordana Todorov.)

MAURICE AUSLANDER
August 3, 1926–November 18, 1994

He was "one of the founders of the modern aspects of the representation theory of artin algebras ". And "one of his main interests in the theory of artin algebras was the finitistic dimension conjecture and related homological conjectures" .

" Shortly before his death Auslander expressed that he was

sorry not to live to see the solution of the finitistic dimension conjecture".

Note: Maurice Auslander was a special invited lecturer at International Congress of Mathematicians in 1962 and in 1986. All the above quotations are from p.501 and p.815 of "Selected works of Maurice Auslander" edited by I.Reiten, S.Smalo and O.Solberg, 1999.

The finitistic dimension conjecture is closely relevant to at least five other famous conjectures in the representation theory of Artin algebras.

— *Nakayama conjecture*: If all I_j in a minimal injective resolution of an Artin algebra A, say $0 \to {}_AA \to I_0 \to I_1 \to ...$, are projective, then A is self-injective. $(1958, \text{T. Nakayama})$

— *Generalized Nakayama conjecture*: If $0 \to {}_AA \to I_0 \to I_1 \to ...$ is a minimal injective resolution of an Artin algebra A, then any indecomposable injective is a direct summand of some I_j. Equivalently, if M is in A-mod such that add$(A) \subseteq$ add(M) and Ext$^i_A(M, M) = 0$ for all $i \geq 1$, then M is projective. $(1975, \text{Auslander-Reiten})$

— *Strong Nakayama conjecture*: If M is a non-zero module over an Artin algebra A, then there is an integer $n \geq 0$ such that Ext$^n_A(M, A) \neq 0$. $(1990, \text{Colpi-Fuller})$

—*Gorenstein symmetry conjecture*: If the injective dimension of ${}_AA$ is finite, then the injective dimension of A_A is finite.

Note: All these conjectures are open. For further information one may refer to Auslander-Reiten-Smalø's book, 1995.

The above conjectures have the following relationship:

- finitistic dim. conjecture $\Longrightarrow$ strong Nakayama conjecture;
- Strong Nakayama conjecture $\Longrightarrow$ generalized Nakayama conjecture;
- Generalized Nakayama conjecture $\Longrightarrow$ Nakayama conjecture;
- finitistic dim. conjecture $\Longrightarrow$ Gorenstein symm. conjecture.

Thus, the finitistic dimension possesses a strong homological property and can be far more revealing measures of homological complexity of an algebra at hand, while infinite global dimension often does not reveal much about that complexity.

4. Some known results

In this section we recall some developments of the history of the finitistic dimension conjecture. Let A and B be Artin algebras.

— 1965: H.Mochizuki:

$\mathrm{rad}^2(A)=0 \Longrightarrow$ fin.dim.conj. is true for A.

— 1991: E.Green & Zimmermann-Huisgen:

$\mathrm{rad}^3(A)=0 \Longrightarrow$ fin.dim.conj. is true for A.

— 1991: E.Green, Kirkman, Kuzmanovich:

A is monomial$\Longrightarrow$fin.dim.conj. is true for A.

An algebra A , given by a quiver with relations, is called a monomial algebra if the relations consist only of paths of length at least two.

— 1991: Auslander and Reiten:

$\mathscr{P}^{\infty}$(A) is contravariantly finite in A-mod $\Rightarrow$fin.dim.conj. is true for A.

A subcategory $\mathcal{C}$ of A-mod is called *contravariantly finite* in A-mod if for any module M in A-mod there is an approximation f: C$\longrightarrow M$ with C$\in \mathcal{C}$.

Note: $\mathscr{P}^{\infty}$(A) is not always contravariantly finite in A-mod.

— 1994: Y.Wang:

rad^{2l+1}(A)=0 and A/rad^{l}(A) is rep-finite $\Longrightarrow$ fin.dim.conj. is true for A.

— 2000: Agoston,Happel,Lukas and Unger:

A is standardly stratified $\Longrightarrow$ fin.dim.conj. is true for A.

— 2002-2005: Igusa and Todorov:

rep.dim(A)$\leq$3$\Longrightarrow$ fin.dim.conj.is true for A.

The *representation dimension* of A is defined by Auslander:

rep.dim(A) = inf{gl.dim (End$_A$(M)) | M is generator-cogenerator in A-mod}.

Note: The representation dimensions of Artin algebras are not always bounded by 3.

— 2002: C.C.Xi:

A is stably hereditary$\Rightarrow$rep.dim(A)$\leq$ 3.

An Artin algebra is called *stably hereditary* if (1) each indecomposable submodule of an indecomposable projective module is either projective or simple, and (2) each indecomposable factor module of an indecomposable injective module is either injective or simple.

— 2003: Hongbo Shi:

For monomial algebras, a graphic algorithm to calculate the finitistic dimension was given.

— 2004: Erdmann,Holm,Iyama & Schroeer:

B$\subseteq$ A with rad(B) = rad(A), and A is rep-finite $\Longrightarrow$ rep.dim(B) ≤ 3. In particular, fin.dim.conj. is true for special biserial algebras and string algebras.

Note: On representation dimension, Assem, Coelho, Holm, Platzeck, Trepode have done some works more recently.

5. New ideas and results

In the following, I shall report some of my works on the finitistic dimension conjecture in the recent years.

5.1 General question

The most investigations on the finitstic dimension conjecture before 2002 are mainly concentrated on one single algebra. Our philosophy is: to bound the finitistic dimension by studying a chain of algebras. This is motivated by the following fact.

Proposition 1. For each Artin algebra A over a field, there is a chain of algebras: A=A$_0$ $\subseteq$ A$_1$ $\subseteq$ ... $\subseteq$ A$_s$ such that A_s is rep-finite, and that rad(A$_i$) is left ideal in A$_{i+1}$ for all i.

So, our question is generally as follows:

Assumption: There is given a chain of algebras A=A$_0$ $\subseteq$ A$_1$ $\subseteq$ ... $\subseteq$ A$_s$ such that rad(A$_i$) is a left (or an) ideal in A$_{i+1}$ for all i.

Question: If some of the bigger algebras in the chain have **finite**

finitistic dimension, what could we say about the finiteness of the finitistic dimension of the smallest algebra A_0 ?

Important: If the answer is Yes, then fin.dim.conj. holds by the above proposition.

5.2 Representation distance

Now, let us introduce the notion of representation distance of Artin algebras.

Definition. Given an Artin algebra A, we define the $\mathbf{left}$ representation distance of A, denoted by lrd(A), to be the following number:

$$\begin{aligned} \text{lrd}(A) \;=\; & \inf\{\, s \mid \exists \text{ chain of algebras } A_0 = A \subseteq A_1 \\ & \subseteq \ldots \subseteq A_s, \text{ rad } (A_i) \text{ is a left ideal in } A_{i+1} \text{ for all } i, \\ & \text{and } A_s \text{ is representation-finite}\}. \end{aligned}$$

The representation distances have the following properties.

- The left representation distance is invariant under Morita equivalences.
- Any finite-dimensional algebra over a field has finite left representation distance.

5.3 Some new results

Theorem 1. Let A be an Artin algebra. If lrd(A)≤ 2, then fin.dim(A)$< \infty$.

That is, if $C \subseteq B \subseteq A$ is a chain of algebras with the same 1 such that rad(C) is a left ideal in B and rad(B) is a left ideal in A and if A is representation-finite, then fin.dim$(C) < \infty$.

For a proof of this result, we refer the reader to J.Pure and Appl. Algebra 193 (2004), 287-305; 202(2005)325-328.

Theorem 2. Suppose B$\subseteq$ A is a subalgebra of an Artin algebra A with rad(B) a left ideal in A and rad(A)= rad(B)A. If gl.dim(A)≤ 4, then fin.dim(B)$< \infty$.

For the details of a proof of this result, we refer to Adv. Math. 201(2006), 116-142.

We consider the following two statements for Artin algebras A over a field:

(1) If lrd(A)$\leq n$, then fin.dim(A)$< \infty$.

(2) If rad^n(A)=0, then fin.dim(A)$< \infty$.

Proposition 2. (1) $\Longrightarrow$ (2).

6. Another approach to the conjecture

6.1 Motivations

Let A be an Artin algebra and e an idempotent in A. Let gl.dim(A) denote the (left) global dimension of A.

Igusa and Todorov proved (2005):

gl.dim(A)$\leq 3 \Longrightarrow$ fin.dim(eAe)$< \infty$ for any e^2 = e$\in$ A.

On the other hand, Auslander proved (1971):

Any Artin algebra B is of the form eAe with gl.dim(A)$< \infty$.

So, a natural question is

Question: If gl.dim(A)≤ 4, is it possible to show fin.dim(eAe)$< \infty$? More generally, if gl.dim(A)$\leq n$, can we show

fin.dim(eAe)$< \infty$?

Thus we may again use the information of bigger A to investigate the finitistic dimension of a smaller algebra. The difference is that here the algebras involved may have different identities. A positive answer to the above question would give a solution to the finitistic dimension conjecture.

6.2 Some results

In this direction we have some results, among them is the following:

Theorem 3. Let A be an Artin algebra of gl.dim(A)$\leq$4 and e^2=e$\in$ A. If rep.dim(A/AeA)$\leq$3, then fin.dim(eAe)$< \infty$.

For a proof of this result and other results in this direction, we refer to a recent preprint [3].

7. References

To make this survey note not so tedious, I omit lots of references, and just keep three references from where the new results are taken. Of course, from references of the three papers below one may easily track almost all important literatures related to the finitistic dimension conjecture.

[1] C. C. Xi, On the finitistic dimension conjecture. I: related to repepresentation-finite algebras. *J.Pure Appl. Alg.* **193** (2004), 287-305. **202** (2005), no.1-3, 325-328.

[2] C. C. Xi, On the finitistic dimension conjecture. II: related to finite global dimension. *Adv. Math.* 201(2006), 116-142.

[3] C. C. Xi, On the finitistic dimension conjecture. III: related to the pair $eAe \subseteq A$. Preprint is available at: http://math.bnu.edu.cn/~ccxi/Papers/Articles/eAe-findim.pdf