Introduction to A_∞-algebras

Di-Ming Lu

Department of Mathematics, Zhejiang University

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ABSTRACT: These are notes of five talks on a short course for the graduate students at Beijing Normal University. The goal is to give a brief introduction to A_∞-algebras with a view towards noncommutative algebras. The notes are based mainly on [LPWZ1-2], [Ke1-3], [St1] and [FHT].

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1 Introduction

1.1 History

1.1.1. J. Stasheff

1960's, Stasheff **invented** A_∞-spaces and A_∞-algebras, as a tool in the study of 'group-like' topological spaces.

• H-space (H: standing for Hopf). The concept arose as a generalization of that of a topological group, the essential feature which is retained is a continuous multiplication (no associativity) with a unit; that is, there exists a continuous multiplication map $\mu : X \times X \to X$ and an 'identity' element $e \in X$ such that the two maps

$$X \underset{\mu(e,-)}{\overset{\mu(-,e)}{\longrightarrow}} X$$

are homotopy to the identity[1].

For example, the (simplest H-) spaces S^0, S^1, S^3 and S^7, regarded as the real, complex, quaternionic and Cayley numbers of unit norm respectively, possess continuous multiplications, and in the first three cases are associative. It is possible to define real, complex and quaternionic projective spaces of arbitrarily large dimension, but this is not possible for the Cayley numbers. A famous theorem of J. F. Adams asserts that S^0, S^1, S^3, and S^7 are the only spheres that are H-spaces.

From the point of view of homotopy theory, it is not the existence of a continuous inverse which is the important distinguishing feature, but rather the associativity of the multiplication, one can investigate the 'mechanism' which relates the associativity of the multiplication to the possible existence of projective spaces.

[1] Of the three axioms for a group, it seems that the least subtle is the existence of an identity element. However, the identity axiom becomes much more potent when topology is added to the picture.

In 1953, Milnor constructed a classical projective spaces for an arbitrary *topological group* by fiber bundles. In 1956, Dold and Lashof generalized it to an arbitrary *associative H-space.* It seems reasonable to ask whether something weaker than associativity might permit more but not all of these fibrings to be constructed. In 1957, Sugawara showed that a variant of Milnor's construction can be carried one step further than for an arbitrary *H-space* if the multiplication is at least *homotopy associative*.

A 'short exact sequence of spaces' $A \hookrightarrow X \to X/A$ gives rise to a long exact sequence of homology groups, but not to a long exact sequence of homotopy groups due to the failure of excision. However, there is a different sort of 'short exact sequence of spaces' that does give a long exact sequence of homotopy groups. This sort of short exact sequence $F \to E \xrightarrow{p} B$, called a fiber bundle, is distinguished from the type $A \hookrightarrow X \to X/A$ in that it has more homogeneity: All the subspaces $p^{-1}(b) \subset E$, which are called *fibers*, are homeomorphic (the topological homogeneity of all the fibers of a fiber bundle is rather like the algebraic homogeneity in a short exact sequence of groups $0 \to K \to G \xrightarrow{p} H \to 0$ where the 'fibers' $p^{-1}(h)$ are the cosets of K in G).

A *fiber bundle* structure on a space E, with fiber F, consists of a projection map $p : E \to B$ such that each point of B has a neighborhood U for which there is a homeomorphism $h : p^{-1}(U) \to U \times F$ making the diagram

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "h"] \arrow[dr, "p"'] & & U \times F \arrow[dl] \\ & U & \end{tikzcd}$$

commute, where the unlabeled map is projection onto the first factor. Commutativity of the diagram means that h carries each fiber $F_b = p^{-1}(b)$ homeomorphically onto the copy $\{b\} \times F$ of F. The fiber bundle structure is determined by the projection map $p : E \to B$, but to indicate what the fiber is we sometimes write a fiber bundle as $F \to E \to B$, a 'short exact sequence of spaces'. The space B is called the *base space* of the bundle, and E is the *total space*.

From the work of Stasheff and Dold-Lashof, the fiber bundle condition is too restrictive. A weaker replacement is:

A_n-structure on a space X, which consists of an n-tuple of maps

$$\begin{array}{ccccccccc} X & = & E_1 & \subset & E_2 & \subset & \cdots & \subset & E_n \\ & & \downarrow p_1 & & \downarrow p_2 & & & & \downarrow p_n \\ * & = & B_1 & \subset & B_2 & \subset & \cdots & \subset & B_n \end{array}$$

such that $p_{i*} : \pi_q(E_i, X) \to \pi_q(B_i)$ is an isomorphism for all q together with a contracting homotopy $h : CE_{n-1} \to E_n$ such that $h(CE_{i-1}) \subset E_i$ (one may think of $X \to E_i \xrightarrow{p_i} B_i$ as a fiber).

• A_∞-space. An A_n-structure[2] on a space X is equivalent to an A_n*-form*; that is, a sequence of maps $M_2, \cdots, M_n$ where each $M_i : K_i \times X^i \to X$ is appropriately defined on $\partial K_i \times X^i$ in terms of M_j for $j < i$. Where the associahedra, or 'Stasheff polytopes', $\{K_i\}$ were introduced by Stasheff in 1963 for the study of homotopy associativity of H-spaces, defined as a subset of I^{i-2}, which is homeomorphic to I^{i-2} with $\frac{n(n-1)}{2} - 1$ faces, consisting of points $(t_1, \cdots, t_i)$ such that $2^j t_1 \cdots t_j \geq 1$ for $1 \leq j \leq i-2$. $\{K_i\}$ are the standard cells, similar objects as the standard simplices $\triangle^i$ and the standard cubes I^i. They are more complicated than $\triangle^i$ and I^i.

An A_n*-space* is defined as a topological space endowed with an A_n-form (note: any associative H-space; that is, a monoid, admits A_n-forms for any n by defining $M_i(\tau, x_1, \cdots, x_i) = x_1 \cdots x_i$, called it trivial A_n-form).

An A_∞*-space* is a topological space equipped with a series of maps which is associative up to homotopy and the homotopy which makes the maps associative can be chosen so that it satisfies a collection of higher coherence conditions. These coherence conditions involve homotopies between homotopies and are most neatly formulated in terms of the so-called Stasheff polytopes. The prime example of A_∞-space is the loop space ΩX. Conversely, a topological space that admits the structure of an A_∞-space and whose connected components form a group is homotopy equivalent to a loop space (J. F. Adams, 1978), so all connected monoids are essentially loop space.

• A_∞-algebra is a chain complex equipped with a product which is homotopy associative and the homotopy which makes the product associative can be chosen so that it satisfies suitable algebraic higher coherence conditions. If $(X, \{m_n\})$ is an A_∞-space, the singular chain complex $C_*(X)$ is the paradigmatic example of an A_∞-algebra.

Stasheff introduced the term 'A_∞-algebra' to describe such structures satisfying an infinite sequence of higher homotopy associativity conditions; hence an A_∞-algebra may be regarded as the homotopy invariant notion of an associative algebra.

Stasheff's theory of A_∞-spaces and A_∞-algebras illustrates many of the key ideas.

1.1.2. M. Kontsevich

In 1994, Kontsevich's talk at the ICM on categorical mirror symmetry played an important role in **developing** this subject. Everything starts from the study of twisted topological models

[2] A space X admits an A_n-structure if and only if there exists maps $M_i : K_i \times X^i \to X$ for $2 \leq i \leq n$ such that

(a) $M_2(*, e, x) = M_2(*, x, e) = x$ for $x \in X, * = K_2$,

(b) for $\rho \in K_r, \sigma \in K_s, r + s = i + 1$, we have

$$M_i(\partial_k(r,s)(\rho,\sigma), x_1, \cdots, x_i) = M_r(\rho, x_1, \cdots, x_{k-1}, M_s(\sigma, x_k, \cdots, x_{k+s-1}), x_{k+s}, \cdots, x_i),$$

(c) for $\tau \in K_i$ and $i > 2$, we have

$$M_i(\tau, x_1, \cdots, x_{j-1}, e, x_{j+1}, \cdots, x_i) = M_{i-1}(s_j(\tau), x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_i).$$

and their boundary conditions. This is one of the main reasons leading to a conjecture meant to explain the 'mathematical mysteries' of mirror symmetry. This homological mirror symmetry conjecture states the equivalence of the derived category of coherent sheaves on a Calabi-Yau Y and of the derived category of the Fukaya category of the mirror $\widetilde{Y}$. But Fukaya category, which is constructed from symmetry manifolds, is an A_∞-category.

1.1.3. B. Keller

In 2000, Keller **introduced** the A_∞-language to the study of ring theory and representation theory. He proved that every derived category of a Grothendieck category is equivalent to the derived category of an A_∞-algebra provided that the former has a compact generator.

1.1.4. Others

In the 1970's and 80's, A_∞-algebras were developed further by Smirnov [Smir1], Kadeishvili [Ka1], Prouté [Pr], Huebschmann [Hu], ... especially with a view towards applications in topology.

J. Huebschmann realized the relevance of A_∞ structures and homological perturbation theory (HPT) to homological algebra. He used HPT to exploit A_∞ modules arising in group cohomology. HPT, introduced by Eilenberg and Mac Lane in 1954, has nowadays become a standard tool to construct and handle A_∞ structures. Any A_∞ module structure admits a spectral sequence which is an invariant of the structure. Many results illustrate a typical phenomenon: Whenever a spectral sequence arises from a certain mathematical structure, there is, perhaps, a certain A_∞ module lurking behind, and the spectral sequence is an invariant thereof. The A_∞ structure is then somewhat finer than the spectral sequence itself, though. In this vein, A_∞ structures are lurking behind a number of other familiar structures in mathematics. One such example arises from complex manifolds where a certain A_∞ structure is hidden behind the Frölicher spectral sequence.

1.2 Motivation

1.2.1. Keller's problems

Let A be an associative k-algebra with 1.

Problem 1. The reconstruction of a complex from its homology.

Let M be a complex of A-modules, H^*M its homology. What additional structure is needed if we want to reconstruct M from its homology.

Problem 2. The reconstruction of the category of iterated selfextensions of module from its extension algebra.

Let $M_i\,(1 \leq i \leq n)$ be A-modules, $filt(M_i)$ denote the full subcategory of the category of right A-modules whose objects admit finite filtrations with subquotients among the M_i. What additional structure on the extension algebra $\mathrm{Ext}^*_A(\oplus M_i, \oplus M_i)$ is needed to reconstruct the category of iterated extensions.

The answer is that H^*M and $\mathrm{Ext}_A^*(\oplus M_i, \oplus M_i)$ admit A_∞-structures which encode the additional information needed for this task.

1.2.2. Classification of AS-regular algebras

Noncommutative projective geometry began with the classification of Artin-Schelter regular algebras of dimension 3 by Artin, Tate and Van den Bergh in 1990. The reason why that work was so significant is that it introduced a whole range of powerful geometric techniques into non-commutative algebra.

• Projective n-space: $\mathbb{P}^n$ is the quotient space of $\mathbb{C}^{n+1}$ modulo the equivalent relation $(a_0, a_1, \cdots, a_n) \sim \lambda(a_0, a_1, \cdots, a_n)$ for $\lambda \neq 0$. The points of $\mathbb{P}^n$ are those lines of $\mathbb{C}^{n+1}$ passing through origin. $\mathbb{P}^n$ is the most important and basic object in the sense of *'every'* space is a subspace of $\mathbb{P}^n$.

Following Grothendieck, to study geometry we don't need a space, what we need is a category associated to the space. Grothendieck's principle applies to algebra, algebraic geometry, differential geometry, topology and other fields.

Let X be a projective variety (*i.e.* a projective subspace of $\mathbb{P}^n$), coh X the category of coherent sheaf over X, then coh X contains all geometry information about X.

Let A be a homogeneous coordinate ring of X. Let Proj A be the category of finitely generated graded A-modules modulo the finite dimensional graded modules.

By Serre Theorem, we may understand

$$\mathbb{P}^n = \mathrm{Proj}\ A$$

where A is the polynomial ring $\mathbb{C}[x_0, x_1, \cdots, x_n]$.

• Quantum projective n-space: $\mathrm{q}\mathbb{P}^n$ is defined to be

$$\mathrm{q}\mathbb{P}^n = \mathrm{Proj}\ A$$

where A is an Artin-Schelter regular algebra of global dimension $n+1$.

• AS-regular algebras: A connected graded algebra A is called Artin-Schelter regular if it has finite global dimension, and is Gorenstein with finite Gelfand-Kirillov dimension.

• Classification.

(i) $\mathrm{q}\mathbb{P}^1$: Quantum projective line is $\mathrm{q}\mathbb{P}^1 = \mathrm{Proj}\ A$, where A is an AS regular algebra of dimension 2. There is no noncommutative projective line.

(ii) $\mathrm{q}\mathbb{P}^2$s were classified by Artin, Schelter, Tate, Van den Bergh (1986, 1990, 1991).

(iii) There are many partial results about q$\mathbb{P}^3$s. One of the central questions in noncommutative projective geometry is

the classification of quantum $\mathbb{P}^3$s.

The complete classification of quantum $\mathbb{P}^3$s is an extremely difficult project and is probably an unreachable goal in near future. An algebraic approach of constructing quantum $\mathbb{P}^n$s is to form the noncommutative scheme Proj A where A is a noetherian Artin-Schelter regular connected graded algebra of global dimension $n+1$. Therefore the algebraic version of the above mentioned question is

the classification of noetherian, Artin-Schelter regular, connected graded algebras of global dimension 4.

Researchers have been studying many special classes of Artin-Schelter regular algebras of global dimension 4. The most famous one is the Sklyanin algebra[3] of dimension 4, introduced by Sklyanin. However, up to now, we do not have a clear picture of the complete classification.

Main methods:

- Inductive methods
- Deformation methods
- Homological methods:

Ext-algebra. Let A be a connected graded algebra. Let k also denote the trivial module $A/A_{\geq 1}$. Let $E(A)$ be the Ext-algebra

$$\bigoplus_{n=0}^{\infty} \mathrm{Ext}_A^n(k,k).$$

If A is Koszul, there is one-to-one correspondence between A and $E(A)$ (Kosuzl duality). Classification of A is equivalent to the classification of $E(A)$. Now $E(A)$ is finite dimensional, and Frobenius. This help us to understand $E(A)$.

A_∞-Ext-algebra. When A is not Koszul, we need to modify the Koszul duality. $E(A)$ has a natural A_∞-structure such that there is a one-to-one correspondence between A and A_∞-Ext-algebra $E(A)$. Classification of A is equivalent to the classification of $E(A)$. The use of A_∞-algebras is a completely new approach and seems to have several advantages. One idea is to look into a more general class of algebras, namely, regular A_∞-algebras and then to determine which of those are Artin-Schelter regular.

[3]The 4-dimensional Sklyanin algebra is defined as the k-free algebra $A = k\langle x_0, x_1, x_2, x_3\rangle$ with relations

$$\begin{aligned} x_0x_1 - x_1x_0 &= \alpha_1(x_2x_3 + x_3x_2), & x_0x_1 + x_1x_0 &= x_2x_3 - x_3x_2,\\ x_0x_2 - x_2x_0 &= \alpha_2(x_3x_1 + x_1x_3), & x_0x_2 + x_2x_0 &= x_3x_1 - x_1x_3,\\ x_0x_3 - x_3x_0 &= \alpha_3(x_1x_2 + x_2x_1), & x_0x_3 + x_3x_0 &= x_1x_2 - x_2x_1. \end{aligned}$$

The parameter $(\alpha_1, \alpha_2, \alpha_3) \in k^3$ lies on the surface $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_1\alpha_2\alpha_3 = 0$.

(iv) Recent developments in noncommutative projective algebraic geometry and its applications to other fields such as mathematical physics demand to have more examples of quantum spaces. For example, what are the quantum K3 surfaces and the quantum Calabi-Yau 3-folds? One naive idea is to construct these quantum spaces as subschemes of some higher dimensional quantum $\mathbb{P}^n$s — noncommutative analogues of projective n-spaces. There are a few examples of q$\mathbb{P}^n$ for $n > 3$.

A Calabi-Yau manifold is a special kind of subspace of $\mathbb{P}^n$ that is used in mathematics physics. One important question involving CY is the 'mirror symmetry' conjecture.

Throughout the notes we work over a fixed field k. One may consider them over a semisimple ring after careful checking.

2 Differential graded algebras

Before entering into the A_∞-world, we review some basics on differential graded algebras (DGA for short) first. A reasonable explanation is that: each DGA is a special A_∞-algebra, and any A_∞-algebra A is quasi-isomorphic to a DGA-model, moreover, when we work with derived functors or derived categories we may replace an A_∞-algebra by a DGA. In particular, one can compute the derived functor $\mathrm{RHom}_A(k,k)$ in the DGA-world based on $\mathrm{RHom}_A(k,k) \cong \mathrm{RHom}_{\Omega BA}(k,k)$. Material in this section and other basic notions related to DGAs can be found in [FHT].

2.1 DG algebras

2.1.1. Graded algebra

A *graded algebra* is a graded module $A = \bigoplus_{i\in\mathbb{Z}} A^i$ with an associative multiplication such that (a) the unit 1 is in A^0 and (b) the multiplication preserves the grading.

A *differential* in a graded module A is a k-linear map $\partial : A \to A$ of degree $+1$ such that $\partial^2 = 0$. We use both $|x|$ and $\deg x$ to denote the degree of a graded or homogeneous element x.

A *derivation* of degree n in a graded algebra A is a k-linear map $\partial : A \to A$ of degree n such that (graded Leibniz rule)

$$\partial(xy) = (\partial x)y + (-1)^{n|x|}x(\partial y)$$

for all elements $x, y \in A$.

The *Koszul sign convention*, namely, when two symbols of degrees n and m are permuted the result is multiplied by $(-1)^{nm}$. Symbols can be elements, operations, etc. The Koszul sign convention is applied throughout.

2.1.2. Differential graded algebra

A *differential graded algebra* is a graded algebra A together with a differential $\partial : A \to A$ of degree 1 that is a derivation. An *augmentation* is a morphism $\epsilon : A \to k$.

A graded algebra is a DGA with $\partial = 0$.

If A is a DGA, then the cohomology ring

$$HA = \bigoplus_{i\in\mathbb{Z}} H^i(A) = \ker\partial/\mathrm{im}\partial$$

is a graded algebra (differential vanished, the spectral sequence maybe think of as a remedy of it).

2.1.3. Examples

(1) The algebraic de Rham complex of the line. Define A using generators and relations as follows:

$$A = \frac{k\langle x,\xi\rangle}{(x\xi - \xi x,\ \xi^2)}.$$

The degrees are deg $x = 0$, deg $\xi = 1$. The differential is defined on generators by $\partial x = \xi,\ \partial\xi = 0$. It is not hard to verify that ∂ extends by additivity and the Leibnitz rule to all of A.

(2) Suppose A is a commutative k-algebra. Let

$$I = \ker(\mu : A\otimes A \to A), \quad \mu(a\otimes b) = ab$$

and $\Omega^1_{A/k} = I/I^2$. Let $\Omega^p_{A/k} = \bigwedge_A^p \Omega^1_{A/k}$, the pth exterior power of $\Omega^1_{A/k} = I/I^2$ as A-module. Then

$$\Omega^*_{A/k} = \bigoplus_p \Omega^p_{A/k}$$

is a DGA. The rule for $\partial : A = \Omega^0_{A/k} \to I/I^2 = \Omega^1_{A/k}$ is $\partial(a) = a\otimes 1 - 1\otimes a$. It extends to all of $\Omega^*_{A/k}$.

$\Omega^*_{A/k}$ is called the de Rham complex of A. The elements of $\Omega^p_{A/k}$ are called the Kähler differentials of degree p, and ∂ is called the exterior derivative.

(3) Let X be a topological space. For any p let $S_p(X)$ denote the set of p-dimensional singular chains in X, namely the set of continuous function $\sigma : \Delta^p \to X$. Here Δ^p is the standard p-dimensional simplex

$$\Delta^p = \{(a_0,\cdots,a_p)\in\mathbb{R}^{p+1} \mid \Sigma_i\ a_i = 1,\ a_i \geq 0\}.$$

Define

$$C^p(X) = \mathrm{Hom}_{Set}(S^p(X), k).$$

Then

$$C^*(X) = \bigoplus_p C^p(X)$$

is a DGA. It is called the ring of simplicial cochains on X. The multiplication is called the Alexander-Whitney product; it is not commutative.

• Remark

(1) A DGA A is called *commutative* if

$$xy = (-1)^{|x||y|}yx$$

for all elements $x, y \in A$.

When $\frac{1}{2} \in k$ this condition implies that $x^2 = 0$ if x has odd degree. If A is a commutative graded algebra, then a left A-module, M, is automatically a right A-module, via

$$mx = (-1)^{|m||x|}xm.$$

(2) The notions of differential graded coalgebra and coderivation are defined similarly.

A *differential graded coalgebra* (or DGC for short) is a graded coalgebra C together with a differential that is a coderivation (a linear map $d : C \to C$ of degree 1 such that $\Delta d = (d \otimes 1 + 1 \otimes d)\Delta$ and $\varepsilon d = 0$). We usually assume that a coalgebra has a counit $\varepsilon : C \to k$ that is dual to the notion of the identity element (or the unit) in an algebra.

If $C = \bigoplus_{i \in \mathbb{Z}} C^i$ is a DGC, then the vector space dual $C^{\#} := \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_k(C^{-i}, k)$ is a DGA. A graded module over k is called *locally finite* if each homogeneous subspace is finite dimensional over k. If $A = \bigoplus_{i \in \mathbb{Z}} A^i$ is a locally finite DGA and if $A \otimes A$ is locally finite, then the vector space dual $A^{\#} := \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_k(A^{-i}, k)$ is a DGC. In this case $(A^{\#})^{\#} \cong A$. Hence locally finite DGAs and DGCs are dual to each other.

2.2 DG modules

2.2.1. DG CATEGORY

Let (A, ∂) be a DGA. A *left differential graded A-module* (or *left DG A-module*) is a complex (M, ∂_M) together with a left multiplication $A \otimes M \to M$ such that M is a left graded A-module and the differential ∂_M of M satisfies the Leibniz rule

$$\partial_M(am) = \partial(a)m + (-1)^{|a|}a\partial_M(m)$$

for all $a \in A, m \in M$. A DG k-module is just a complex. A right DG A-module is defined similarly.

Note that the Leibniz rule links the actions of A and of the differential of M. It is sometimes useful to *deform* the differential, while keeping the module structure ($({}_AM, \partial) \Rightarrow ({}_AM, \partial + \delta)$).

For a DGA A, we let $A^\natural$ denote the underlying graded algebra. Let $\mathcal{C}_{\mathrm{dg}}(A)$ (resp. $\mathcal{C}(A^\natural)$) denote the category of differential graded modules over A (resp. graded modules over $A^\natural$). A morphism $\alpha : M \to N$ such that $H(\alpha) : H(M) \to H(N)$ is an isomorphism is called a *quasi-isomorphism*. The functor $(\ )^\natural : \mathcal{C}_{\mathrm{dg}}(A) \to \mathcal{C}(A^\natural)$ is additive exact faithful, and commutative with the shift functor. Suppose a sequence $\mathbf{E} \in \mathcal{C}_{\mathrm{dg}}(A)$, then:

$\mathbf{E}$ exact in $\mathcal{C}_{\mathrm{dg}}(A) \Leftrightarrow \mathbf{E}^\natural$ exact in $\mathcal{C}(A^\natural)$.

$\mathbf{E}$ split exact in $\mathcal{C}_{\mathrm{dg}}(A) \Rightarrow \mathbf{E}^\natural$ split exact in $\mathcal{C}(A^\natural)$, but the converse fails in general.

2.2.2. Two constructions

• $\mathrm{Hom}_A(M,N)$: the graded vector space of A-homomorphisms from M to N. As graded module

$$\mathrm{Hom}_A(M,N) := \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_A^{Gr}(M,N)^i$$

with the differential ∂_{Hom} defined to be

$$\partial_{\mathrm{Hom}}(f) = \partial_N \circ f - (-1)^{|f|} f \circ \partial_M$$

for all $f \in \mathrm{Hom}_A(M,N)$. In particular, the graded vector space $\mathrm{Hom}_A(M,M)$ is a DGA, and $\mathrm{Hom}_A(L,M)$ is a DG module over DGA $\mathrm{Hom}_A(M,M)$.

• $M \otimes_A N$: the graded vector space of tensor product over A with the differential $\partial_\otimes$ defined to be

$$\partial_\otimes(m \otimes_A n) = \partial_M(m) \otimes_A n + (-1)^{|m|} m \otimes_A \partial_N(n).$$

There is an adjoint property between Hom_A and $\otimes_A$:

$$\mathrm{Hom}_A(L \otimes_A M, N) \cong \mathrm{Hom}_A(L, \mathrm{Hom}_A(M,N)).$$

2.3 Resolutions

2.3.1. Semifree

Let L be a DG module over A. A subset $Y \subseteq L$ is said to be *free* if for each DG module M over A and every homogeneous map $\kappa : Y \to M$ of degree 0 there exists a unique DG A-morphism $\tilde{\kappa} : L \to M$ with $\tilde{\kappa}|_Y = \kappa$. A DG A-module M is called *free* if it is isomorphic to a direct sum of shifts of A.

Construction: For a graded set Y with a degree function $g : Y \to \mathbb{Z}$, consider the graded $A^\natural$-module with basis $E_Y \cup E_Y^+$, where

$$E_Y = \{e_y \mid \deg(e_y) = g(y),\ y \in Y\} \quad \text{and} \quad E_Y^+ = \{e_y^+ \mid \deg(e_y^+) = g(y)+1,\ y \in Y\}$$

Free DG module with the basis Y is

$$F^{[Y]} := \bigoplus_{y\in Y} Ae_y \oplus \bigoplus_{y\in Y} Ae_y^+.$$

with the differential given by

$$\partial(\sum_{y\in Y} a_y e_y + a_y^+ e_y^+) = \sum_{y\in Y} (\partial(a_y)e_y + (-1)^{|a_y|} a_y e_y^+ + \partial(a_y^+)e_y^+).$$

For each DG module M there exists a surjective morphism $L \to M$, where L is a free DG module (not a resolution).

There exist enough free DG modules in the category $\mathcal{C}_{\mathrm{dg}}(A)$, but such DG modules are always contractible, so they carry little information on the structure of $\mathcal{C}_{\mathrm{dg}}(A)$ (in fact, so are the projective DG modules). The standard approach to homological algebra of modules over a ring, via free resolutions, leads to a dead end in the case of DG modules over a DG algebra.

As a remedy for this difficulty, Avramov and Halperin (LNM-1183, 1985) introduced a class of DG modules, semifree DG modules, retain some characteristics of freeness. Two important properties, which are the exact analogues of CW complexes, are:

- *Any DG module admits a quasi-isomorphism from a semifree module.*
- *Any morphism from a semifree module lifts (up to homotopy) through a quasi-isomorphism.*

A DG A-module M is called *semifree* if there is a sequence of DG submodules

$$0 = M(-1) \subset M(0) \subset \cdots \subset M(n) \subset \cdots$$

such that $M = \bigcup_n M(n)$ and that each $M(n)/M(n-1)$ is A-free on a basis of cocycles. Such an increasing sequence is called a *semifree filtration* of M. A semifree module is a replacement for a free complex over an associative algebra. A semifree DG module may not be free.

A *semifree resolution* of a DG A-module M is a quasi-isomorphism $L \to M$ from a semifree DG A-module L. Sometimes we call L itself a semifree resolution of M. By [FHT, Proposition 6.6] every DG module has a (strict, i.e. surjective) semifree resolution. Semifree resolutions play the same role for modules over DGA's that ordinary free resolutions do in the ungraded case.

A basic property satisfied by semifree modules is 'preservation of quasi-isomorphisms' under the functors Hom and $\otimes$.

Suppose $\{M(k)\}$ is a semifree filtration of M. Then each $M(k)/M(k-1)$ has the form $(A,d)\otimes(Z(k),0)$ where $Z(k)$ is a free k-module. Thus the surjections $M(k) \to A \otimes Z(k)$ split:

$$M(k) = M(k-1) \oplus (A \otimes Z(k)), \quad \text{and} \quad d : Z(k) \to M(k-1).$$

In particular, if we forget the differentials, $M = A \otimes (\oplus_{k=0}^{\infty} Z(k))$ is a free A-module.

Suppose L is a semifree and $\eta : M \to N$ is a quasi-isomorphism. Then

(a) $\mathrm{Hom}_A(L,\eta) : \mathrm{Hom}_A(L,M) \to \mathrm{Hom}_A(L,N)$ is a quasi-isomorphism.

(b) Given a diagram of morphisms of A-modules,

$$\begin{tikzcd} & M \arrow[d, "\eta(\simeq)"] \\ L \arrow[ur, dotted, "\phi"] \arrow[r, "\psi"'] & N \end{tikzcd}$$

there is a unique homotopy class of morphisms $\varphi : L \to M$ such that $\eta \circ \varphi \sim \psi$.

(c) A quasi-isomorphism between semifree A-modules is an equivalence.

Every A-module M has a semifree resolution $m : L \xrightarrow{\simeq} M$; moreover, if $m' : L' \xrightarrow{\simeq} M$ is a second semifree resolution then there is an equivalence of A-modules $\alpha : L' \to L$ such that $m \circ \alpha \sim m'$. ([FHT], p.71)

Remark:

1. When A is ordinary ring, L is a module over it, then considered as a DG module L is semifree if and only if it is free as an A-module.

2. If A is a ring and L is a complex of A-modules, then it follows directly from the definitions that each A-module L_j is free.

The following result shows the existence of the semifree resolution (see section 3.3.1).

Proposition: Let A be an augmented DGA. Then the augmentations in BA and A define a quasi-isomorphism $\epsilon \otimes \epsilon : B(A, A) \to k$. Moreover, if k is a field then $B(A, A)$ is a semifree right DG A-module. Thus $\epsilon \otimes \epsilon$ is a semifree resolution of the right DG A-module k_A.

Given a left DG A-module M, we use a semifree resolution of M to compute the derived functor of $\mathrm{Hom}_A(M, -)$, which is denoted by $\mathrm{RHom}_A(M, -)$. Similarly, we can use a semifree resolution of M to compute the derived functor of $N \otimes_A M$ (here N is a right DG A-module), which is denoted by $N \otimes^L_A M$.

2.3.2. Homotopically projective

A DG module P over A is said to be *homotopically projective* if $\mathrm{Hom}_A(P, -)$ preserves quasi-isomorphisms.

A semifree DG module L is homotopically projective, and the graded $A^\natural$-module $L^\natural$ is projective.

3 A_∞-language

From the point of view of homotopy theory, an A_∞-algebra is the same as a DGA. However, for the purpose of explicit computations, it is often more convenient to work with A_∞-algebras rather than with DGA's. The reason is the existence of extra structure in the form of higher multiplications.

In this section we recall some basic definitions about A_∞-algebras and A_∞-modules mainly from the papers [Ke1] and [LPWZ1]. Some basic properties of A_∞-language have been worked out by Lefèvre-Hasegawa in his Thesis [Le].

There are many different types of algebras: associative, commutative, Lie, Poisson, etc. Each comes with an appropriate notion of a module and thus with an associated theory of representations. Moreover, as is becoming more and more important in a variety of fields, including algebraic topology, algebraic geometry, differential geometry, and string theory, it is very often necessary to deal with *'algebras up to homotopy'* and with 'partial algebras' (structures that behave much like algebras and modules, except that the relevant maps are only defined on suitable submodules of tensor products.)[KM]

The idea of a homotopy 'something' algebra is to relax the axioms of the 'something' algebra, so that the usual identities are satisfied up to homotopy. For example, in a homotopy associative algebra, the associativity identity looks like

$$(ab)c - a(bc) \ \text{ is homotopic to zero.}$$

This kind of relaxation seems to be too lax for many (practical and categorical) purposes, and one usually requires that the null-homotopies, regarded as new operations, satisfy their own identities, up to their own homotopy. These homotopies should also satisfy certain identities up to homotopy and so on. This resembles Hilbert's chains of syzygies in early homological algebra. Algebras with such chains of homotopies are called *strong homotopy 'something' algebras*.

3.1 A_∞-algebras

3.1.1. DEFINITION

There are different methods to give the definition of A_∞-algebras (algebraical, geometrical, operadic, etc.), we prefer the algebraical definition of an A_∞-algebra.

An *A_∞-algebra* over k is a $\mathbb{Z}$-graded vector space

$$A = \bigoplus_{p \in \mathbb{Z}} A^p$$

endowed with a family of graded k-linear maps

$$m_n : A^{\otimes n} \to A, \quad n \geq 1,$$

of degree $(2-n)$ satisfying the following *Stasheff identities*:

SI(n) $$\sum (-1)^{r+st} m_u (id^{\otimes r} \otimes m_s \otimes id^{\otimes t}) = 0, \quad \text{for all } n \geq 1,$$

where the sum runs over all decomposition $n = r+s+t$, $r, t \geq 0$ and $s \geq 1$, and where $u = r+1+t$. Here id denotes the identity map of A. Note that when these formulas are applied to elements, additional signs appear due to the Koszul sign rule. An A_∞-algebra is also called a *strongly homotopy associative algebra* (or *sha algebra*).

For small n the Stasheff identities **SI(n)** can be expressed without the sigma notation:

SI(1) is

$$m_1 m_1 = 0.$$

The degree of m_1 is 1. This is saying that m_1 is a *differential* of A.

SI(2) can be re-written as

$$m_1 m_2 = m_2(m_1 \otimes id + id \otimes m_1)$$

as maps $A^{\otimes 2} \to A$. Hence the differential m_1 is a graded derivation with respect to m_2. Note that m_2 plays the role of *multiplication* although it may not be associative. The degree of m_2 is zero.

SI(**3**) can be re-written as

$$m_2(id \otimes m_2 - m_2 \otimes id)$$

$$= m_1 m_3 + m_3(m_1 \otimes id \otimes id + id \otimes m_1 \otimes id + id \otimes id \otimes m_1)$$

as maps $A^{\otimes 3} \to A$. Note that the left-hand side is the associator for m_2 and the right-hand side is $\partial_{\mathrm{Hom}}(m_3) = \partial_A \circ m_3 - (-1)^{|m_3|} m_3 \circ \partial_{A^{\otimes 3}}$, the boundary of m_3 in the morphism complex $\mathrm{Hom}_k(A^{\otimes 3}, A)$. This implies that m_2 is associative up to homotopy. If either m_1 or m_3 is zero, then m_2 is associative.

Remark:

(1) If X admits an A_∞-form $\{M_i\}$; that is, $(X, \{M_i\})$ is an A_∞-space, then the singular chain complex $C_*(X)$ of X admits the structure of an A_∞-algebra by defining $m_1 = \partial$ and for $i > 1$,

$$m_i(u_1 \otimes \cdots \otimes u_i) = M_{i\#}(k_i \otimes u_1 \otimes \cdots \otimes u_i)$$

where k_i is a suitable generator of $C_*(K_i)$. The signs in the definition were chosen so as to make it true at the case (note that the dimension of K_i is $i - 2$).

(2) The idea of an algebra up to homotopy has proved very useful in resolving problems such as Kontsevich's formality conjecture and Deligne's conjecture (on the algebraic structure of the Hochscild complex). This idea has been greatly clarified by the use of algebraic operads. A_∞-algebras may be viewed as algebras over an operad [KM]. Generally, various types of 'up to homotopy' algebras (that is, where the classical axioms are satisfied just 'up to higher homotopies'): A_∞-algebras (associative up to homotopy), G_∞-algebras (associative, and commutative up to homotopy), C_∞-algebras (commutative, and associative up to homotopy), B_∞-algebras (associative up to homotopy and commutative up to homotopy), and L_∞-algebras (Lie algebras up to homotopy), etc.

The appropriate language to describe such complex structures is the operad language[4].

[4]Topological operads were introduced by May in his monograph 'The geometry of iterated loop spaces'. The main reason for studying topological operads is to generalise Stasheff's theory of loop spaces to n-fold loop spaces

(3) Working with a more geometrical definition of an infinity-algebra was advocated by some authors (A. Lazarev, for example): Given a vector space V, an A_∞-structure on V is a continuous derivation

$$m : \widehat{T}\Sigma V^* \to \widehat{T}\Sigma V^*$$

of degree one and vanishing at zero (degree), such that $m^2 = 0$. Here Σ denotes the suspension, $*$ denotes the dual (so $\Sigma V^* = \mathrm{Hom}_k(\Sigma V, k)$), and $\widehat{T}$ denotes the completed tensor algebra, $(T\Sigma V)^* = \widehat{T}\Sigma V^*$. There are similar definitions of a C_∞ and L_∞-structure where $\widehat{T}\Sigma V^*$ is replaced with $\widehat{L}\Sigma V^*$ and $\widehat{S}\Sigma V^*$ [HaL].

- *higher multiplications*: the graded maps m_n for $n \geq 3$.
- *strictly unital condition*: a unit 1_A with respect to m_2, while $m_n(a_1, \cdots, a_n) = 0$ for $n \neq 2$ and $a_i = 1_A$ for some i. In this case, we say 1_A is the strict unit of A. Each strictly united A_∞-algebra is canonically endowed with a strict morphism $\eta : k \to A$ mapping 1_k to 1_A.
- *augmented A_∞-algebra*: an A_∞-algebra is called *augmented* if there is a graded map ε from A to the trivial A_∞-algebra k such that

$$\varepsilon(m_2(a_1, a_2)) = \varepsilon(a_1)\varepsilon(a_2)$$

and

$$\varepsilon(m_n(a_1, \cdots, a_n)) = 0, \text{ for all } n \neq 2.$$

Note: as in the ring case, the unit can be added to an A_∞-algebra without unit. If B is an A_∞-algebra without unit, then there is a unique way to extend the A_∞-structure on B to an

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and infinite loop spaces — operads play the role of the Stasheff polyhedra in this more general theory.

In the algebraic setting, an *operad* $\mathcal{O}$ consists of k-modules $\mathcal{O}(j)$, $j \geq 0$, together with a unit map $\eta : k \to \mathcal{O}(1)$, a right action by the symmetric group Σ_j on $\mathcal{O}(j)$ for each j, and maps

$$\gamma : \mathcal{O}(k) \otimes \mathcal{O}(j_1) \otimes \cdots \otimes \mathcal{O}(j_k) \to \mathcal{O}(j)$$

for $k \geq 1$ and $j_s \geq 0$, where $\sum j_s = j$. The γ are required to be associative, unital, and equivalent.

An action of $\mathcal{O}$ on a chain complex A, *$\mathcal{O}$-algebra*, is a k-module A together with a unit maps

$$\theta : \mathcal{O}(j) \otimes \otimes A^j \to A, \ j \geq 0$$

that are associative, unital, and equivalent.

The $\mathcal{O}(j)$ are thought of as parameter complexes for j-ary operations. When the differentials on the $\mathcal{O}(j)$ are zero, we think of $\mathcal{O}$ as purely algebraic, and it then determines an appropriate class of (differential) algebras. When the differentials on the $\mathcal{O}(j)$ are non-zero, $\mathcal{O}$ determines a class of (differential) algebras 'up to homotopy', where the homotopies are determined by the homological properties of the $\mathcal{O}(j)$.

Recently there has been a lot of interest in the algebraic theory of operads, much of it stimulated by theoretical physics. A particularly striking example is Kontsevitch's formality theorem in deformation quantization. An essential ingredient in this theorem is the notion of a Lie algebra up to homotopy (or a Gerstenhaber algebra up to homotopy) and this is where algebraic operads come in to the picture. The relevant operads are examples of Koszul operads and the theory of Koszul operads has many very interesting special features.

A_∞-structure on $A := k \oplus B$ such that A satisfies the strictly unital condition. Clearly, this A is augmented. Conversely, if A is an augmented A_∞-algebra satisfying the strictly unital condition and we let $\varepsilon : A \to k$ be the augmentation, then the kernel $B := \ker \varepsilon$ is an A_∞-algebra without unit and $A = k \oplus B$.

3.1.2. Examples

There are some interesting examples of A_∞-algebras in [LPWZ1, LPWZ2].

(a) An associative algebra A is an A_∞-algebra *concentrated in degree* 0 with all multiplications $m_n = 0$ for $n \neq 2$. Hence associative algebras form a subclass of A_∞-algebras of the form (A, m_2).

(b) Differential graded algebra (A, m_1, m_2) (every A_∞-algebra A is quasi-isomorphic to a DGA ΩBA, which is called the *DGA-model* of A [FHT]).

(c) Pentagonal homotopy associative algebra (A, m_1, m_2, m_3):

SI(4): $\quad m_2(1 \otimes m_3 + m_3 \otimes 1) = m_3(m_2 \otimes 1 \otimes 1 - 1 \otimes m_2 \otimes 1 + 1 \otimes 1 \otimes m_2).$

(d) *Massey product:* Consider the Borromean rings consisting of three circles which are pairwise unlinked but all together are linked. If we regard them as situated in S^3, then the cohomology ring of the complement is a trivial algebra, but m_3 is non-zero in cohomology, being represented by Massey products and detecting the simultaneous linking of all three circles.

(e) Connected cubic zero A_∞-algebra.

Note: When $A^3 \neq 0$, or A is not connected, then there are obstructions of constructing higher multiplications. This makes the higher multiplications more interesting.

(f) Quiver and Path-algebra:

$$A = kQ/I = (kx \oplus ky \oplus kz \oplus kt) \oplus (ka \oplus kb \oplus kc) \oplus ke,$$

with the relations: $x + y + z + t = 1,\ ab = 0,\ bc = 0$. Define: $m_2 =$ the multiplication of the Path-algebra， $m_3(a, b, c) = e$, then (A, m_2, m_3) forms an A_∞-algebra [Ke1, page 9].

(g) Let $B = k[x_1, x_2]/(x_1^2)$, p $(p \geq 3)$ a fixed integer. Define an A_∞-algebra structure on B as follows.

For $s \geq 0$, set

$$x_s = \begin{cases} x_2^{\frac{s}{2}} & \text{if } s \text{ is even,} \\ x_1 x_2^{\frac{s-1}{2}} & \text{if } s \text{ is odd.} \end{cases}$$

Then $\{x_s\}_{s \geq 0}$ is a k-basis of the graded vector space B. For $i_1, \cdots, i_p \geq 0$, define

$$m_p(x_{i_1}, \cdots, x_{i_p}) = \begin{cases} x_j & \text{if all } i_s \text{ are odd,} \\ 0 & \text{otherwise,} \end{cases}$$

where $j = 2 - p + \sum_s i_s$. The multiplication m_2 is the product of the associative algebra $k[x_1, x_2]/(x^2)$. Now it is direct to check that (B, m_2, m_p) is an A_∞-algebra, which is denoted by $B(p)$.

Note: we'll talk this example again in the following, and it shows that homological properties of an algebra will be changed when adding a nonzero higher multiplication on it.

(h) $(2,p)$-algebra: All Stasheff identities are automatically satisfied except for the following three

SI(3): the associative law of m_2

$$m_2(m_2 \otimes id) = m_2(id \otimes m_2),$$

SI(2p-1): the associative law of m_p

$$\sum_{r=0}^{p-1} (-1)^{r(p+1)} m_p(id^{\otimes r} \otimes m_p \otimes id^{\otimes p-1-r}) = 0,$$

SI(p+1): the compatibility relation between m_2 and m_p

$$\sum_{i=0}^{p-1} (-1)^i m_p(id^{\otimes i} \otimes m_2 \otimes id^{\otimes p-1-i}) = m_2(id \otimes m_p) + (-1)^{p-1} m_2(m_p \otimes id).$$

This kind of A_∞-algebras is related to the higher Koszul algebra (Section 4).

(i) *Ext-algebra*: Let A be an algebra over k, then $\mathrm{Ext}^*_A(k,k)$ is an A_∞-algebra (Section 4).

(j) *AS-regular algebras* of 3 and 4 (Section 5).

3.1.3. A_∞-MORPHISMS

For two A_∞-algebras A and B. A *morphism* of A_∞-algebras $f : A \to B$ is a family of k-linear graded maps

$$f_n : A^{\otimes n} \to B$$

of degree $(1-n)$ satisfying the following *Stasheff morphism identities*: for all $n \geq 1$,

MI(n): $$\sum (-1)^{r+st} f_u(id^{\otimes r} \otimes m_s \otimes id^{\otimes t}) = \sum (-1)^w m_q(f_{i_1} \otimes f_{i_2} \otimes \cdots \otimes f_{i_q})$$

where the first sum runs over all decompositions $n = r + s + t$ with $s \geq 1, r, t \geq 0$, we put $u = r + 1 + t$, and the second sum runs over all $1 \leq q \leq n$ and all decompositions $n = i_1 + \cdots + i_q$ with all $i_s \geq 1$; the sign on the right-hand side is given by

$$w = (q-1)(i_1 - 1) + (q-2)(i_2 - 1) + \cdots + 2(i_{q-2} - 1) + (i_{q-1} - 1).$$

- *unital morphism conditions*: when the A_∞-algebras have a strict unit, an A_∞-morphism is also required to satisfy the following extra unital morphism conditions:

$$f_1(1_A) = 1_B$$

where 1_A and 1_B are strict units of A and B respectively, and

$$f_n(a_1, \cdots, a_n) = 0$$

if some $a_i = 1_A$ and $n \geq 2$.

The unital morphism conditions are compatible with **MI(n)** in the sense that if A and B are A_∞-algebras without unit and if $f : A \to B$ is an A_∞-morphism satisfying **MI(n)** only, then, by imposing the unital morphism conditions, f can be extended to an A_∞-morphism from $k \oplus A$ to $k \oplus B$.

When $n = 1$, **MI(1)** is

$$f_1 m_1 = m_1 f_1,$$

namely, f_1 is a morphism of complexes. When $n = 2$, **MI(2)** is

$$f_1 m_2 = m_2(f_1 \otimes f_1) + m_1 f_2 + f_2(m_1 \otimes id + id \otimes m_1),$$

which means that f_1 commutes with the multiplication m_2 up to a homotopy given by f_2.

• *strict morphism*: a morphism f is called strict if $f_i = 0$ for all $i \neq 1$. The identity morphism is the strict morphism f such that f_1 is the identity of A. When f is a strict morphism from A to B, then the identity **MI(n)** becomes

$$f_1 m_n = m_n(f_1 \otimes \cdots \otimes f_1).$$

Strict morphisms are analogous to homomorphisms in classical ring theory. A morphism $f : A \to B$ is called a *strict isomorphism*, if f is strict and f_1 is an isomorphism of vector spaces. In this case $f_1^{-1} : B \to A$ is the inverse morphism of f and we write $A \cong B$.

A strict isomorphism is similar to an isomorphism in ring theory. However we need a weaker notion of isomorphism when we work with A_∞-algebras.

• *quasi-isomorphism*: a morphism $f : A \to B$ is called a quasi-isomorphism if f_1 is a quasi-isomorphism (and there are no conditions on f_i for all $i \geq 2$). In this case we write $A \simeq B$.

By definition, an A_∞-morphism $f : A \to B$ is a quasi-isomorphism if and only if $H(f_1) : HA \to HB$ is an isomorphism.

If two A_∞-algebras are quasi-isomorphic, then they are viewed as the same A_∞-algebra since they have the same homological properties. For this reason usually we are interested in A_∞-algebras up to quasi-isomorphisms.

Note: If A and B are quasi-isomorphic A_∞-algebras, then their cohomology rings are isomorphic to each other. The converse is not true: [LPWZ1, Lemma 4.6] shows that $A \cong B$ as algebras, but A is not quasi-isomorphic to B as A_∞-algebras. On the other hand, we have an example to show that $A \ncong B$ as algebras, but $A \simeq B$ as A_∞-algebras [LPWZ1, Lemma 5.2(c), 5.4].

• The *composition* of two morphisms $f : B \to C$ and $g : A \to B$ is given by

$$(f \circ g)_n = \sum (-1)^w f_q \circ (g_{i_1} \otimes \cdots \otimes g_{i_q})$$

where the sum and the sign are as in the defining identities **MI(n)**.

In algebraic topology and homological algebra we usually want to analyze the (co)homology groups of some complexes associated to the objects we study. A common setting is that we start with a DGA and then compute the cohomology ring. The following minimal model tells us that the cohomology ring has an A_∞-structure which is quasi-isomorphic to the original DGA in the A_∞ world.

3.1.4. Two models

• **DGA model** [Le]: *Every A_∞-algebra A is quasi-isomorphic to a free DGA constructed as ΩBA.*

In Keller's word, 'passing from DGA's to A_∞-algebras does not yield new quasi-isomorphism classes. What it does yield is a new description of these classes by minimal models' [Ke1, Section 3.3].

The result was induced by using the bar construction. The bar/cobar construction and the notation ΩBA will be explained in Section 3.3.

The central result is the theorem on the existence of minimal models.

• **Minimal model** [Ka1, Me]: *Let A be an A_∞-algebra and let HA be the cohomology ring of A. There is an A_∞-algebra structure on HA with $m_1 = 0$, constructed from the A_∞-structure of A, such that there is a quasi-isomorphism of A_∞-algebras $HA \to A$ lifting the identity of HA.*

Note: The first proof was given by Kadeishvili in [Ka1] and later in other papers [Me]. We'll copy a more precise proof of [LPWZ2, p.34] for the use of Basic lemma in the section 4.1.

Theorem: *Let A be an algebra over k, then $Ext^*_A(k,k)$ is an A_∞-algebra.*

Indeed, choose a projective resolution P of ${}_Ak$. Then the morphism complex $B = \mathrm{Hom}_A(P,P)$ is a differential graded algebra whose homology identifies with the Yoneda algebra $\mathrm{Ext}^*_A(k,k)$.

Let A be a ring and M an A-module. Choose a projective resolution of M

$$\cdots \longrightarrow P^{-2} \xrightarrow{\delta} P^{-1} \xrightarrow{\delta} P^0 \xrightarrow{\phi^0} M \longrightarrow 0.$$

Now consider A as a DGA concentrated in degree 0, with 0 differential. Then M is a DG A-module concentrated in degree 0. defining $P^i = 0$ for $i > 0$, we can view $P = \oplus P^i$ as a DG A-module, with differential δ. The homomorphism $\phi^0 : P_0 \to M$ extends to a morphism $\phi : P \to M$ in $\mathcal{C}_{\mathrm{dg}}(A)$. Since

$$H^0(\phi) : H^0 P \to H^0 M = M$$

is bijective and $H^i P = 0$ for all $i \neq 0$ it follows that ϕ is a quasi-isomorphism.

Continuous with the previous example let N be any other A-module. Then

$$H^i\mathrm{Hom}_A(P, N) = \mathrm{Ext}^i_A(M, N).$$

As we saw before $B = \mathrm{Hom}_A(P, P)$ is a DGA. A calculation shows that morphism

$$\mathrm{Hom}_A(P, M) \leftrightarrow \mathrm{Hom}_A(P, P) = B$$

induced by $P \to M$ is a quasi-isomorphism. Hence

$$H^iB = \mathrm{Ext}^i_A(M, M).$$

The multiplication in the graded ring HB is called the Yoneda product.

In particular, B is a DGA and its homology $\mathrm{Ext}^*_A(k, k)$ carries the A_∞-structure of the minimal model of B.

3.2 A_∞-modules

3.2.1. Definition

Let A be an A_∞-algebra.

• A *left A_∞-module* over A is a $\mathbb{Z}$-graded vector space M endowed with maps

$$m^M_n : A^{\otimes n-1} \otimes M \to M, \quad n \geq 1$$

of degree $(2 - n)$ satisfying the same Stasheff identities **SI(n)**

$$\sum(-1)^{r+st}m_u(id^{\otimes r} \otimes m_s \otimes id^{\otimes t}) = 0$$

as one in the definition of A_∞-algebra. However, the term $m_u(id^{\otimes r} \otimes m_s \otimes id^{\otimes t})$ has to be interpreted as $m^M_u(id^{\otimes r} \otimes m_s \otimes id^{\otimes t})$ if $t > 0$ and as $m^M_u(id^{\otimes r} \otimes m^M_s)$ if $t = 0$.

When A has a strict unit 1, then we require that $m^M_2(1, x) = x$ and $m^M_n(a_1, \cdots a_{n-1}, x) = 0$ if $n \geq 3$ for $x \in M$ and one of the a_i is 1.

In particular, $(A, \{m_n\})$ itself forms an A_∞-module over A.

• A *morphism* of left A_∞-modules $f : M \to N$ is a family of graded maps

$$f_n : A^{\otimes n-1} \otimes M \to N$$

of degree $(1 - n)$ such that for each $n \geq 1$, the following version of the identity **MI(n)** holds:

MIL(n) $$\sum(-1)^{r+st}f_u \circ (id^{\otimes r} \otimes m_s \otimes id^{\otimes t}) = \sum m_{1+w} \circ (id^{\otimes w} \otimes f_v),$$

where the first sum is taken over all decompositions $n = r + s + t$, $r, t \geq 0, s \geq 1$ and we put $u = r + 1 + t$; and the second sum is taken over all decompositions $n = v + w$, $v \geq 1, w \geq 0$. A morphism also satisfies the obvious unital conditions which are omitted here.

A morphism f is called a *quasi-isomorphism* if f_1 is a quasi-isomorphism. The identity morphism $f : M \to M$ is given by $f_1 = id_M$ and $f_i = 0$ for all $i \geq 2$.

The composition of two morphisms $f : M \to N$ and $g : L \to M$ is defined by

$$(f \circ g)_n = \sum f_{1+w} \circ (id^{\otimes w} \otimes g_v)$$

where the sum runs over all decompositions $n = v + w$.

3.2.2. Derived category

Let A be an A_∞-algebra.

• $\mathcal{C}_\infty(A)$: the category of left A_∞-modules over A with morphisms of A_∞-algebras.

If A is a DGA, we have a faithful functor

$$\mathcal{C}_{\mathrm{dg}}(A) \to \mathcal{C}_\infty(A).$$

Because not every A_∞-module is a DG module, $\mathcal{C}_\infty(A)$ has more objects and more morphisms than $\mathcal{C}_{\mathrm{dg}}(A)$. Note that when A is an ordinary associative algebra, $\mathcal{C}_{\mathrm{dg}}(A)$ is the category of complexes of left A-modules.

We say that an A_∞-morphism $f : M \to N$ is *nullhomotopic* if there is a family of graded maps

$$h_n : A^{\otimes n-1} \otimes M \to N, \quad n \geq 1,$$

of degree $-n$ such that

$$f_n = \sum_{v=1}^{n} m_{1+n-v} \circ (id^{\otimes n-v} \otimes h_v) + \sum_{n=r+s+t, s\geq 1} (-1)^{r+st} h_{r+1+t} \circ (id^{\otimes r} \otimes m_s \otimes id^{\otimes t}).$$

Two A_∞-morphisms $f, g : M \to N$ are said to be *homotopic* if $f - g$ is nullhomotopic (see simple definition in terms of bar-construction).

• The *homotopy category* $\mathcal{K}_\infty(A)$ has the same objects as $\mathcal{C}_\infty(A)$, and the morphisms from M to N are morphisms of A_∞-modules modulo the nullhomotopic morphisms.

To define the derived category[5], we should formally invert all quasi-isomorphisms. As proved in [Le, Théorème 4.1.3.1] and [Ke1, Theorem 4.2], every quasi-isomorphism of A_∞-modules is a homotopy equivalence (Note: if k is only assumed to be a commutative ring, the result is no longer true in general). Therefore one can define

• The *derived category* $\mathcal{D}_\infty(A)$ to be the homotopy category $\mathcal{K}_\infty(A)$.

If an A_∞-algebra does not have a strict unit, then A_∞-modules over A can be defined without the unital condition and the derived category $\mathcal{K}_\infty(A)$ can be defined in the same way.

[5]The *derived category* $D(\mathcal{A})$ of an abelian category $\mathcal{A}$ is the algebraic analogue of the homotopy category of topological spaces. $D(\mathcal{A})$ is obtained from the category $Ch(\mathcal{A})$ of (cochain) complexes in two stages: First one constructs a quotient $K(\mathcal{A})$ of $Ch(\mathcal{A})$ by equating chain homotopy equivalent maps between complexes. Then one localizes $K(\mathcal{A})$ by inverting quasi-isomorphisms via a calculus of fraction.

There is a canonical quasi-isomorphism

$$\mathrm{Hom}_{\mathcal{D}A}(M, N[n]) = \mathrm{Ext}^n_A(M, M)$$

valid for any $n \in \mathbb{Z}$ if we take the right hand side to vanish for negative n.

Remark: Some signs of the definition of *right* A_∞-module are slightly different from the left module case[6].

3.2.3. Change of A_∞-algebras

Let $f : A \to B$ be a morphism of A_∞-algebras and let (M, m_n^B) be a left A_∞-module over B. Define

$$m_n^A : A^{\otimes n-1} \otimes M \to M, \quad n \geq 1,$$

by

INL(n) $\qquad m_n^A = \sum (-1)^w m_q^B (f_{i_1} \otimes \cdots \otimes f_{i_{q-1}} \otimes id)$

where the sum runs over all decompositions $n = i_1 + \cdots\cdots + i_{q-1} + 1$ for $i_s \geq 1$ and where

$$w = (q-1)(i_1 - 1) + (q-2)(i_2 - 1) + \cdots + 2(i_{q-2} - 1) + (i_{q-1} - 1)$$

as in the definition of morphisms of A_∞-algebras. It is easy to check that (M, m_n^A) is a left A_∞-module over A. Then $f^* : (M, m_n^B) \mapsto (M, m_n^A)$ defines a functor from $\mathcal{C}_\infty(B)$ to $\mathcal{C}_\infty(A)$, which induces a functor on the derived categories.

One of the basic properties is the following

Proposition [Ke1, Proposition 6.2]: *Let $f : A \to B$ be a quasi-isomorphism of A_∞-algebras. Then the induced functor $f^* : \mathcal{D}_\infty(B) \to \mathcal{D}_\infty(A)$ is an equivalence of triangulated categories. Further, A is isomorphic to f^*B in $\mathcal{D}_\infty(A)$.*

3.2.4. From DGAs to A_∞-algebras

If A is a DGA, then the inclusion $\mathcal{C}_{\mathrm{dg}}(A) \to \mathcal{C}_\infty(A)$ induces an equivalence from $\mathcal{D}_{\mathrm{dg}}(A)$ to $\mathcal{D}_\infty(A)$. In [Ke1, Theorem 4.3(a)], Keller considers the derived category of homologically unital A_∞-modules, which contains our $\mathcal{D}_\infty(A)$ as a full triangulated subcategory. Also the inclusion $\mathcal{C}_{\mathrm{dg}}(A) \to \mathcal{C}_\infty(A)$ has its image in $\mathcal{D}_\infty(A)$. Hence $\mathcal{D}_\infty(A)$ is equivalent to the derived category of homologically unital A_∞-modules. Thus [Ke1, Theorem 4.3(a)] implies the following

Proposition [Ke1, Le]: If A is a DGA, then the canonical functor $\mathcal{D}_{\mathrm{dg}}(A) \to \mathcal{D}_\infty(A)$ is an equivalence of triangulated categories.

As an immediate consequence

[6] Corresponding categories are written as $\mathcal{C}_\infty(A^\circ)$, $\mathcal{K}_\infty(A^\circ)$, and $\mathcal{D}_\infty(A^\circ)$, respectively. The difference is
(1) MIR(n): $\sum (-1)^{r+st} f_u \circ (id^{\otimes r} \otimes m_s \otimes id^{\otimes t}) = \sum (-1)^{(v-1)w} m_{1+w} \circ (f_v \otimes id^{\otimes w})$, and
(2) $(f \circ g)_n = \sum (-1)^{(v-1)w} f_{1+w} \circ (g_v \otimes id^{\otimes w})$.

Corollary: Let $A \to B$ be a quasi-isomorphism of A_∞-algebras where B is a DGA. Then the induced functor $F : \mathcal{D}_{\mathrm{dg}}(B) \to \mathcal{D}_\infty(A)$ is an equivalence of triangulated categories such that FB is isomorphic to A in $\mathcal{D}_\infty(A)$.

3.3 Bar constructions

A clear way to introduce the A_∞-algebras is the so-called bar construction.

Bar/cobar constructions of DGAs/DGCs (introduced by Eilenberg-MacLane (1953) and Adams (1956), respectively) are well-known to topologists and people working on DGAs. The following material is mainly taken from [LPWZ1], we will follow its convention on double-grading and differentials increase degree by 1.

The bar and cobar constructions are functors

$$\text{Augmented DGAs} \overset{B}{\rightsquigarrow} \text{Co-augmented DGCs}$$

and

$$\text{Co-augmented DGCs} \overset{\Omega}{\rightsquigarrow} \text{Augmented DGAs}.$$

These constructions can be extended to A_∞-algebras.

3.3.1. Bar constructions for DGAs

Consider an augmented DGA A, namely, DGA endowed with an augmented morphism $A \to k$ of DGAs, viewing k as a trivial DGA. Let I be a graded vector space. The tensor coalgebra on I is

$$T(I) = k \oplus I \oplus I^{\otimes 2} \oplus I^{\otimes 3} \oplus \cdots ,$$

where an element in $I^{\otimes n}$ is written as

$$[a_1|a_2|\cdots|a_n]$$

for $a_i \in I$ (the name 'bar construction' originated here), together with the comultiplication

$$\Delta([a_1|\cdots|a_n]) = \sum_{i=0}^{n}[a_1|\cdots|a_i] \otimes [a_{i+1}|\cdots|a_n].$$

• Bar construction on A:

Let (A, ∂_A) be an augmented DGA and let I denote the augmentation ideal $\ker(A \to k)$. The *bar construction* on A is the coaugmented DGC BA defined as follows:

◇ As a coaugmented graded coalgebra BA is the tensor coalgebra $T(I)$ on I.

◇ The differential in BA is the sum $d = d_0 + d_1$ of the coderivations given by

$$d_0([a_1|\cdots|a_m]) = -\sum_{i=1}^{m}(-1)^{n_i}[a_1|\cdots|\partial_A(a_i)|\cdots|a_m]$$

and

$$d_1([a]) = 0$$

$$d_1([a_1|\cdots|a_m]) = \sum_{i=2}^{m}(-1)^{n_i}[a_1|\cdots|a_{i-1}a_i|\cdots|a_m].$$

Here $n_i = \sum_{j<i}(-1+\deg a_j)$.

The bar construction BA is bigraded by the *negative* of tensor length and by the grading on A; that is, the element $[a_1|\cdots|a_m]$ has degree $(-m, \sum \deg a_j)$. The total degree of $[a_1|\cdots|a_m]$ is $(-m+\sum \deg a_j)$. The map d_0 has degree $(0,1)$ and d_1 has degree $(1,0)$. The tensor length is graded negatively so that the differential on BA increases degree.

Example: Let $A = k[x]/(x^p)$ for some $p \geq 2$ with zero differential. Let $\deg x = 0$. Then A is a DGA concentrated in degree zero. Let $y_i = x^i$. Then $I = ky_1 + ky_2 + \cdots + ky_{p-1}$. The bar construction BA is the cofree coalgebra

$$T(I) = k \oplus I \oplus I^2 \oplus \cdots$$

with comultiplication determined by

$$\Delta([y_{i_1}|\cdots|y_{i_n}]) = \sum_{s=0}^{n}[y_{i_1}|\cdots|y_{i_s}] \otimes [y_{i_{s+1}}|\cdots|y_{i_n}].$$

The differential d is defined by

$$d([a_1|\cdots|a_m]) = \sum_{i=2}^{m}(-1)^{i-1}[a_1|\cdots|a_{i-1}a_i|\cdots|a_m]$$

for all $a_i \in I$. If $p=2$, then BA is the cofree coalgebra $k[y]$ generated by the primitive element y, with zero differential.

- Bar construction on M:

If (M, ∂_M) is a left DG A-module, then the *bar construction on A with coefficients in M* is the complex $B(A,M) = BA \otimes M$ with differential $d = d_0 + d_1$ where

$$\begin{aligned} d_0([a_1|\cdots|a_w]m) = &-\sum_{i=1}^{w}(-1)^{n_i}[a_1|\cdots|\partial_A(a_i)|\cdots|a_w]m \\ &-(-1)^{n_{w+1}}[a_1|\cdots|a_w]\partial_M(m) \end{aligned}$$

and

$$\begin{aligned} d_1([a_1|\cdots|a_w]m) = &\sum_{i=2}^{w}(-1)^{n_i}[a_1|\cdots|a_{i-1}a_i|\cdots|a_w]m \\ &+(-1)^{n_{w+1}}[a_1|\cdots|a_{w-1}]a_w m. \end{aligned}$$

Of course $d_0 m = -\partial_M(m)$, $d_1 m = 0$ and $d_1([a]m) = (-1)^{\deg a - 1}am$. This is graded just as BA is, and for each M, $B(A,M)$ is a *left* DG BA-comodule.

There are two situations in which it is easy to compute $H(B(A;M))$. The first arises when $A = TV$ is a tensor algebra, where $V = \oplus_{i\geq 0} V_i$ or $V = \oplus_{i\geq 2} V^i$. The second is when we take (A, d) as a left module over itself, via multiplication. This yields the complex $B(A;A) = BA \otimes A$. Note that $B(A;A)$ is a right (A, d)-module via multiplication on the right and that $B(A;M) = B(A;A) \otimes_A M$. One of the main uses of the bar construction maybe

Proposition [FHT, page 270]: Let A be an augmented DGA.

(a) The augmentations in BA and A define a quasi-isomorphism

$$\varepsilon \otimes \varepsilon : B(A, A) \to k.$$

(b) If k is a field, then $B(A, A)$ is a semifree *right* DG A-module. Thus $\varepsilon \otimes \varepsilon$ is a semifree resolution of the *right* DG A-module k_A.

When k is a field and $A \xrightarrow{\varepsilon} k$ is an augmented algebra, with no differential. Then $B(A, A)$ is an A-free resolution of k. But for any left A-module M we have $B(A;M) = B(A;A) \otimes_A M$. It follows that $\mathrm{Tor}^A(k, M) = H(B(A, M))$ [FHT, page 278].

3.3.2. Cobar constructions for DGCs

The cobar construction is dual to the bar construction. Let C be a coaugmented DGC with comultiplication $\Delta : C \to C \otimes C$ and differential $d : C \to C$. Let J be the cokernel of the coaugmentation $k \to C$. Then the comultiplication on C induces a map $\overline{\Delta} : J \to J \otimes J$.

Alternatively, J is isomorphic to the kernel of the counit $C \to k$, and one defines the reduced comultiplication on this kernel by

$$\overline{\Delta} c = \Delta c - (c \otimes 1 + 1 \otimes c).$$

In either case, $\overline{\Delta}$ is coassociative because Δ is.

- Cobar construction on C:

Let C be a coaugmented DGC. The *cobar construction* on C is the augmented DGA ΩC defined as follows:

◇ As an augmented graded algebra ΩC is the tensor algebra (*i.e.*, the free algebra) $T(J)$ on J.

◇ The differential in ΩC is the sum $\partial = \partial_0 + \partial_1$ of the differentials

$$\partial_0([x_1|\cdots|x_m]) = -\sum_{i=1}^{m} (-1)^{n_i} [x_1|\cdots|d_C(x_i)|\cdots|x_m],$$

and

$$\partial_1([x_1|\cdots|x_m]) = \sum_{i=1}^{m}\sum_{j=1}^{v_i} (-1)^{n_i+|y_{ij}|+1} [x_1|\cdots|x_{i-1}|y_{ij}|z_{ij}|\cdots|x_m]$$

where $n_i = \sum_{j<i}(1 + \deg x_j)$ and $\overline{\Delta} x_i = \sum_{j=1}^{v_i} y_{ij} \otimes z_{ij}$.

This is bigraded by the tensor length and the grading on C, so $[x_1|\cdots|x_m]$ has degree $(m, \sum \deg x_j)$. The total degree of $[x_j]$ is $1+\deg x_j$. Then ∂_0 has degree $(0,1)$ and ∂_1 has degree $(1,0)$, just as for BA.

Example: Let C be the dual of the algebra $k[x]/(x^p)$ with zero differential. Then $C \cong \bigoplus_{i=0}^{p-1} kz_i$ with $z_0 = 1$. The comultiplication is determined by $\Delta(z_i) = \sum_{s=0}^{i} z_s \otimes z_{i-s}$. So $\overline{\Delta}(z_1) = 0$ and $\overline{\Delta}(z_i) = \sum_{s=1}^{i-1} z_s \otimes z_{i-s}$ for all $i \geq 2$. In particular, $\overline{\Delta}(z_2) = z_1 \otimes z_1$. The cobar construction ΩC is the free algebra $T(J) = k\langle y_1, \cdots, y_{p-1}\rangle$ where $y_i = [z_i]$ with differential determined by

$$\partial(y_1) = 0 \qquad \text{and} \qquad \partial(y_i) = -\sum_{s=1}^{i-1} y_s \otimes y_{i-s}.$$

- Bar construction on Y:

If Y is a left DG C-comodule, then $\Omega(C,Y)$ is equal to $\Omega C \otimes Y$, with differential $\partial = \partial_0 + \partial_1$ defined by

$$\begin{aligned}\partial_0([x_1|\cdots|x_m]y) = &-\sum_{i=1}^{m}(-1)^{n_i}[x_1|\cdots|d_C(x_i)|\cdots|x_m]y \\ &-(-1)^{n_{m+1}}[x_1|\cdots|x_m]d_Y(y)\end{aligned}$$

and

$$\begin{aligned}\partial_1([x_1|\cdots|x_m]y) = &\sum_{i=1}^{m}\sum_{j=1}^{v_i}(-1)^{n_i+|y_{ij}|+1}[x_1|\cdots|y_{ij}|z_{ij}|\cdots x_m]y \\ &+\sum_{s}(-1)^{n_{m+1}+|c_s|+1}[x_1|\cdots|x_m|c_s]y_s,\end{aligned}$$

where $y \to \sum_s c_s \otimes y_s$ is the reduced coaction on y.

The following lemma is standard. Part (c) is [FHT, Sect.19, Ex.3, p.272].

Proposition: Suppose C is a coaugmented DGC such that $C^{\otimes n}$ is locally finite for all n. Let M be a DG C-comodule such that $C^{\otimes n} \otimes M$ is locally finite for all n. Let $A = C^{\#}$.

(a) A is an augmented DGA such that $A^{\otimes n}$ is locally finite.

(b) ΩC and BA are locally finite with respect to the bigrading.

(c) $\Omega^{\#}C \cong BA$ and $B^{\#}A \cong \Omega C$. (will be used in the subsection 4.1.3)

(d) $M^{\#}$ is a left DG A-module.

(e) $B(A, M^{\#}) \cong \Omega^{\#}(C, M)$ as DG BA-comodules.

Remark: One of the most different problem is the problem of calculating the homology groups of iterated loop spaces. The first steps toward solving this problem were made by J. F. Adams. To calculate the homology $H^*(\Omega X)$ of the loop space ΩX of a topological space X he introduced the notion of the cobar construction on a coalgebra. For a 1-connected[7] pointed space X (*i.e.* $\pi_1(X,*)=0$), Adams found a natural isomorphism of graded modules $H(B(C^*(X)) \cong H^*(\Omega X)$, where $B(C^*(X))$ is the bar construction of DG-algebra $C^*(X)$. The method cannot be extended directly for iterated loop spaces $\Omega^k X$ for $k \geq 2$, since the bar construction $B(A)$ of a DG-algebra A is just a DG-coalgebra, and it does not carry the structure of a DG-algebra in order to produce a double bar construction $B(B(A))$. However, for $A = C^*(X)$ Baues has constructed an associative product μ which turns $B(C^*(X))$ into a DG-algebra and which is geometric. In general, in order to produce the bar construction $B(A)$ it is not necessary to have, on a DG-module A, a strict associative product $\mu : A \otimes A \to A$; it suffices to have a strong homotopy associative product; that is, to have an A_∞-algebra structure on A.

Let X be an n-connected pointed space. Then there exists a sequence of A_∞-algebra structures $\{m_i^{(k)}\}, k = 1, 2, \cdots, n$, such that for each $k \leq n$ there exists an isomorphism of graded algebras

$$H^*(\Omega^k X) \cong \Big(H\big(B(\cdots(B(BC^*(X); \{m_i^{(1)}\}); \{m_i^{(2)}\}); \cdots); \{m_i^{(k-1)}\})\big); \;\; m_i^{(k)}\Big).$$

The latter $(H(\cdots);\; m_i^{(k)})$ allows one to product the next bar construction, but it is not clear whether it is geometric, i.e., whether homology of this bar construction is isomorphic to $H^*(\Omega^{n+1}X)$ if X is not $(n+1)$-connected [KS].

3.3.3. Bar constructions for A_∞-algebras

Let A be an A_∞-algebra. Recall that if A is augmented then there is a strict A_∞-morphism $f : A \to k$. We assume that A satisfies the strict unital condition. Write $A = k \oplus I$ where $I = \ker f$. Then I is an A_∞-algebra without unit and the A_∞-structure on A is uniquely determined by the A_∞-structure on I.

Suppose we are given a k-linear map $m_n : I^{\otimes n} \to I$ for some $n \in \mathbb{N}$. Since $T(I)$ is cofree coalgebra, m_n determines uniquely a coderivation b_n (but not a differential) on $T(I)$ via the map $T(I) \to I^{\otimes n} \to I$. The explicit formula[8] for b_n is the following:

$$b_n([a_1|\cdots|a_m]) = \sum (-1)^w [a_1|\cdots|a_j|\overline{m}_n(a_{j+1}, \cdots, a_{j+n})|a_{j+n+1}|\cdots|a_m]$$

[7]A space X is *connected* if any two points in can be connected by a curve lying wholly within X. A space is *0-connected* (pathwise-connected) if every map from a 0-sphere to the space extends continuously to the 1-disk. Since the 0-sphere is the two endpoints of an interval (1-disk), every two points have a path between them. A space is *1-connected* (simply connected) if it is 0-connected and if every map from the 1-sphere to it extends continuously to a map from the 2-disk. In other words, every loop in the space is contractible. A space is *n-connected* if it is $(n-1)$-connected and if every map from the n-sphere into it extends continuously over the $(n+1)$-disk. A theorem of Whitehead says that a space is infinitely connected iff it is contractible.

[8]In [Le, p.26], $\overline{m}_n$ is chosen to be $-m_n$, so the formula given here differs from the formula in [Le, p.26] by a sign. We choose the sign this way so that the definition agrees with the bar construction in the case when an A_∞-algebra is a DGA.

where $\overline{m}_n = (-1)^n m_n$ and

$$w = \sum_{1\leq s\leq j} (|a_s|+1) + \sum_{1\leq t\leq n} (n-t)(|a_{j+t}|+1).$$

For m_1 and m_2, the formulas b_1 and b_2 coincide with d_0 and d_1 respectively as given in the bar construction of a DGA. Let $b = \sum_{n\geq 1} b_n$, then b is a coderivation of $T(I)$. The following are equivalent.

(a) The k-linear maps $m_n : I^{\otimes n} \to I$ yield an A_∞-structure on I (without unit).

(b) The coderivation $b : T(I) \to T(I)$ satisfies $b^2 = 0$.

• Bar construction: There is a bijection[9] between the A_∞-structures on A and the coalgebra differentials on $T(I)$. Given an A_∞-algebra, the corresponding coaugmented DGC $T(I)$ is denoted by BA, and called the *bar construction* of A. The bar construction of a DGA is just a special case. Note that each b_n has degree $+1$ when we are using the total degree, which is defined by

$$\deg([a_1|\cdots|a_n]) = -n + \sum_{i=1}^{n} |a_i|.$$

Example: Let $A = k[x]/(x^3) = k1 \oplus kx_1 \oplus kx_2$ be a connected cubic zero A_∞-algebra with

$$m_i(x_1, \cdots, x_1) = x_2$$

for all $i \geq 1$. Let $a_i \in \{1, x_1, x_2\}$. Then the differential b in BA is defined by

$$b([a_1|\cdots|a_n]) = \sum_{a_i=\cdots=a_j=x_1} (-1)^w [a_1|\cdots|a_{i-1}|x_2|a_{j+1}|\cdots|a_n]$$

where $i \leq j$ and $w = j + \sum_{1\leq s<i} \deg a_s$.

[9] There is a one-to-one correspondence between systems of maps $m_i : V^{\otimes i} \to V$, $i \geq 1$ and systems of maps $\overline{m}_i : \Sigma V^{\otimes i} \to \Sigma V$, $i \geq 1$ via the following commutative diagram:

$$\begin{array}{ccc} \Sigma V^{\otimes i} & \xrightarrow{\overline{m}_i} & \Sigma V \\ {\scriptstyle (\Sigma^{-1})^{\otimes i}}\downarrow & & \downarrow{\scriptstyle \Sigma^{-1}} \\ V^{\otimes i} & \xrightarrow{m_i} & V \end{array}$$

Of course the $\overline{m}_i$'s will inherit additional signs from the Koszul sign rule. It is well known that any system of maps $\overline{m}_i : \Sigma V^{\otimes i} \to \Sigma V$, $i \geq 1$ can be uniquely extended to a coderivation $\bar{m}$ on the tensor coalgebra $T\Sigma V$ which vanishes on $k \subset T\Sigma V$. Furthermore all coderivations vanishing on k are obtained in this way, hence there is a one-to-one correspondence

$$\mathrm{Hom}_k(T\Sigma V/k, \Sigma V) \leftrightarrow \{\overline{m} \in Coder(T\Sigma V) : \overline{m}(k) = 0\}.$$

The condition $\overline{m}^2 = 0$ turns out to be equivalent to the higher homotopy associativity axioms for the m_i's. Now simply observe that the dual of a coderivation on $T\Sigma V$ is a continuous derivation on $(T\Sigma V)^* = \widehat{T}\Sigma V^*$. It follows from [HaL, Prop. A.6] that the geometrical alternative style of definition is equivalent to above definition of A_∞-algebra [HaL, page 22].

• Relations between A and BA:

Lemma [Ke1, 3.6]: Let A and R be two A_∞-algebras. There is a bijection between the A_∞-morphisms from A to R and the DGC morphisms from BA to BR.

Let A and R be A_∞-algebras. Let f be an A_∞-morphism from A to R. By the above lemma, there is a corresponding DGC morphism from BA to BR, which is denoted by F. Let g be another A_∞-morphism from A to R with corresponding DGC morphism G from BA to BR. Then f and g are *homotopic* if and only if there is a map $H : BA \to BR$ of degree -1 such that

$$\Delta H = (F \otimes H + H \otimes G)\Delta \quad \text{and} \quad F - G = b \circ H + H \circ b.$$

One can translate this into the existence of a family of maps $h_n : A^{\otimes n} \to R$ satisfying conditions given in the section 3.3.2. We say an A_∞-morphism $f : A \to R$ is a *homotopy equivalence* if there is another A_∞-morphism $g : R \to A$ such that fg and gf are homotopic to id_R and id_A respectively.

Theorem: [Ka3] [Pr] [Le, Corollaire 1.3.1.3]

(1) *Homotopy is an equivalence relation on the set of morphisms of A_∞-algebras $A \to R$.*

(2) *An A_∞-morphism $A \to R$ is a quasi-isomorphism if and only if it is a homotopy equivalence.*

In particular, part (2) implies that any quasi-isomorphism has a quasi-inverse.

4 Ext-algebras

There are two natural problems related to the Ext-algebras.

(1) Whether $\mathrm{Ext}^*_A(M, M)$ is finitely generated provided M is a finitely generated A-module?

(2) How to recover an algebra from its Ext-algebra?

The first problem has been discussed in various contexts. For a local commutative noetherian ring A with residue field k, it has been conjectured that $\mathrm{Ext}^*_A(k, k)$ would always be a finitely generated A-algebra. Another result is: if $A = kQ/I$ admits a pure (with weight $\delta(i)$) minimal resolution of k, then $\mathrm{Ext}^*_A(k, k)$ is finitely generated if and only if there is some positive integer N such that, if $n > N$, then there exists i $(0 < i < n)$, such that $\delta(i)+\delta(n-i) = \delta(n)$ (Green-Marcos). Koszul algebra and d-Koszul algebra are at the cases. An interesting result related to a Koszul algebra is due to Smith [Sm]: suppose that A is Koszul and noetherian, then A is Artin-Schelter regular if and only if $\mathrm{Ext}^*_A(k, k)$ is Frobenius.

The second problem is far more complicated. This is a highly nontrivial task. A useful approach to attaching the problem is applying methods from homological algebra. We are interested in this problem below.

There are two distinguished modules over an augmented algebra A with the augmentation map $\varepsilon : A \to k$, namely, the left and right trivial modules $A/\ker(\varepsilon)$, we will denote these by k and k_A

respectively. Typical examples of augmented k-algebras are commutative local rings, Hopf algebras (group algebras and enveloping algebras) where ε is the co-unit, and connected graded algebras.

Let A be a graded algebra generated in degree 1. Then the Ext-algebra $\mathrm{Ext}_A^*(k_A, k_A)$ is equipped with an A_∞-algebra structure. We use $\mathrm{Ext}_A^*(k_A, k_A)$ to denote both the usual associative Ext-algebra and the Ext-algebra with its A_∞-structure. An example shows that the associative algebra $\mathrm{Ext}_A^*(k_A, k_A)$ does not contain enough information to recover the original algebra A; on the other hand, the information from the A_∞-algebra $\mathrm{Ext}_A^*(k_A, k_A)$ is sufficient to recover A.

4.1 A_∞-structures on Ext-algebras

The classical Ext-algebra $\mathrm{Ext}_A^*(k_A, k_A)$ is the cohomology ring of $\mathrm{End}_A(P_A)$, where P_A is any free resolution of k_A. Since $E = \mathrm{End}_A(P_A)$ is a DGA, by Kadeishvlli's result, $\mathrm{Ext}_A^*(k_A, k_A) = HE$ has a natural A_∞-structure, which is called an A_∞*-Ext-algebra* of A. By abuse of notation we use $\mathrm{Ext}_A^*(k_A, k_A)$ to denote an A_∞-Ext-algebra.

Kadeishvili's construction is very general. We would like to describe the A_∞-structure on $\mathrm{Ext}_A^*(k_A, k_A)$ by using Merkulov's construction.

4.1.1. Merkulov's construction

Merkulov constructed a special class of higher multiplications for HE in [Me], in which the higher multiplications can be defined inductively; this way, the A_∞-structure can be described more explicitly, and hence used more effectively. For our purposes we will describe a special case of Merkulov's construction, assuming that E is a DGA.

• Split the complex: Let E be a DGA with differential ∂ and multiplication $\cdot$. Denote by B^n and Z^n the coboundaries and cocycles of E^n, respectively. Then there are subspaces H^n and L^n such that

$$Z^n = B^n \oplus H^n$$

and

$$E^n = Z^n \oplus L^n = B^n \oplus H^n \oplus L^n.$$

We will identify HE with $\bigoplus_n H^n$, or embed HE into E by cocycle-sections $H^n \subset E^n$. *There are many different choices of* H^n *and* L^n.

• Choose a homotopy: Let $p = Pr_H : E \to E$ be a projection to $H = \bigoplus_n H^n$, and let $Q : E \to E$ be a homotopy from id_E to p. Hence we have $id_E - p = \partial Q + Q\partial$. The map Q is not unique. From now on we choose Q with the following properties: for every n, $Q^n : E^n \to E^{n-1}$ is defined by: $Q^n|_{L^n} = Q^n|_{H^n} = 0$, and $Q^n|_{B^n} = (\partial^{n-1}|_{L^{n-1}})^{-1}$. So the image of Q^n is L^{n-1}. It follows that $Q^{n+1}\partial^n = Pr_{L^n}$ and $\partial^{n-1}Q^n = Pr_{B^n}$.

• Pre-define maps $\lambda_n : E^{\otimes n} \to E$ of degree $2 - n$ inductively as follows.

There is no map λ_1, but we formally define the 'composite' $Q\lambda_1$ by $Q\lambda_1 = -id_E$. λ_2 is the multiplication of E, namely, $\lambda_2(a_1 \otimes a_2) = a_1 \cdot a_2$. For $n \geq 3$, λ_n is defined by the recursive formula

$$\lambda_n = \sum_{\substack{s+t=n,\\ s,t\geq 1}} (-1)^{s+1} \lambda_2(Q\lambda_s \otimes Q\lambda_t).$$

We abuse notation slightly, and use p to denote both the map $E \to E$ and also (since the image of p is HE) the map $E \to HE$; we also use λ_i both for the map $E^{\otimes i} \to E$ and for its restriction $(HE)^{\otimes i} \to E$.

- Define multiplications and morphisms

Theorem: [Me] *Define $m_i = p\lambda_i : (HE)^{\otimes i} \to HE$, and $f_i = -Q\lambda_i : (HE)^{\otimes i} \to E$. Then*

(1) $(HE, m_2, m_3, \cdots)$ *is an A_∞-algebra;*

(2) $f = \{f_i\}$ *is a quasi-isomorphism between HE and E as A_∞-algebras.*

4.1.2. $\mathrm{Ext}^1_A(k_A, k_A)$ AND $\mathrm{Ext}^2_A(k_A, k_A)$

Consider a connected graded algebra (no differential here!)

$$A = k \oplus A_1 \oplus A_2 \oplus \cdots,$$

which is viewed as an A_∞-algebra concentrated in degree 0, with the grading on A being the Adams grading. Let $V \subset A$ be a minimal graded vector space which generates A. Then $V \cong I/I^2$ where $I = A_{\geq 1}$ is the unique maximal graded ideal of A. Let $R \subset T\langle V\rangle$ be the minimal graded vector space which generates the relations of A. Then $A \cong T\langle V\rangle/(R)$ where (R) is the ideal generated by R, and the start of a minimal graded free resolution of the trivial right A-module k_A is

$$\cdots \to \mathbf{?} \otimes A \to R \otimes A \to V \otimes A \to A \to k \to 0.$$

Lemma: *Let A be a connected graded algebra. Then there are natural isomorphisms of graded vector spaces*

$$\mathrm{Ext}^1_A(k_A, k_A) \cong V^\# \quad \textit{and} \quad \mathrm{Ext}^2_A(k_A, k_A) \cong R^\#.$$

This comes from the property of the minimal resolution. We wonder what '**?**' is in the resolution above. We expect to explain it in the A_∞-world.

In the following, we assume that A is generated by $V = A_1$ and that A_1 is finite-dimensional; hence $A = T\langle A_1\rangle/(R)$. Let E be the A_∞-Ext-algebra $\mathrm{Ext}^*_A(k_A, k_A)$.

4.1.3. BASIC LEMMA

A remarkable result that how to recover an algebra from its Ext-algebra was stated by Keller in [Ke3, Section 2] without proof. It is available for a general class of algebras kQ/I where Q is a finite quiver and I is an admissible ideal of kQ.

Basic Lemma works only for graded algebras generated in degree 1, so it is a special case of Keller's result. The result is relating the A_∞-structure on $\mathrm{Ext}_A^*(k_A, k_A)$ to the relations in A; that is, Basic Lemma shows the information from the A_∞-algebra $\mathrm{Ext}_A^*(k_A, k_A)$ is sufficient to recover A, which is also essential for the classification of AS regular algebras of global dimension 4 that are generated by two elements.

Basic Lemma (Keller's higher-multiplication theorem in the connected graded case): *Let A be a graded algebra, finitely generated in degree 1, and let E be the A_∞-algebra $\mathrm{Ext}_A^*(k_A, k_A)$. Let $R = \bigoplus_{n\geq 2} R_n$ be the minimal graded space of relations of A such that $R_n \subset A_{n-1} \otimes A_1 \subset A_1^{\otimes n}$. Let $i : R_n \to A_1^{\otimes n}$ be the inclusion map and let $i^\#$ be its k-linear dual. Then the multiplication m_n of E restricted to $(E^1)^{\otimes n}$ is equal to the map*

$$i^\# : (E^1)^{\otimes n} = (A_1^\#)^{\otimes n} \longrightarrow R_n^\# \subset E^2.$$

To show the basic lemma, we need a precise Merkulov's data on $\mathrm{Ext}_A^*(k_A, k_A)$. We sketch the steps for this purpose below.

Continue to consider a connected graded algebra $A = k \oplus A_1 \oplus A_2 \oplus \cdots$ that is viewed as an A_∞-algebra concentrated in degree 0, with the grading on A being the Adams grading. As before, denote $I = A_{\geq 1}$.

- $\mathsf{Ext}_A^*(k, k) \cong H(\Omega A^\#)$

Denote $P = T(\Sigma I) \otimes_k A = \bigoplus_{s\geq 0} P^s$, where $P^s = I^{\otimes s} \otimes_k A$. Then, $P(= B(A; A))$ is a free resolution of k, and $P \otimes_A k \cong T(\Sigma I)$, we have

$$\cdots \to P^s \otimes_A k \to P^{s-1} \otimes_A k \to \cdots \to P^0 \otimes_A k \to 0;$$

that is,

$$\cdots \longrightarrow I^{\otimes s} \xrightarrow{d^{-s}} I^{\otimes(s-1)} \longrightarrow \cdots \longrightarrow I^{\otimes 1} \xrightarrow{d^{-1}} I^{\otimes 0} \to 0.$$

Note that the grading on the differential graded coalgebra $T(\Sigma I)$ is by the negative of the wordlength, namely, $(T(\Sigma I))^{-i} = I^{\otimes i}$. The differential $d = (d^i)$ of the bar construction $T(\Sigma I)$ is induced by the multiplication $I \otimes I \to I$ in A:

$$d^{-1}([a_1]) = 0,$$
$$d^{-s}([a_1|\cdots|a_s]) = \sum_{i=1}^{s-1}(-1)^i[a_1|\cdots|a_i a_{i+1}|\cdots|a_s], \text{ for } s \geq 2.$$

Thus, we get $H(BA) = \mathrm{Tor}_*^A(k, k)$. Note that $T(\Sigma I)^\# \cong T(\Sigma^{-1}I^\#)$, $(\Sigma^{-1}I^\#)^{\otimes s} \cong (\Sigma I^{\otimes s})^\#$, we get $\mathrm{Ext}_A^*(k, k) \cong H(T(\Sigma^{-1}I^\#)) = H(\Omega A^\#)$.

- Describe Merkulov's data of the cobar construction $\Omega A^\#$ by its dual BA

We need to construct analogue of the maps λ and Q on DGA $\Omega A^\#$. Due to $\Omega A^\# \cong B^\# A$, so we start the construction from BA.

By the duality, $\partial : (I^{\#})^{\otimes s} \to (I^{\#})^{\otimes s+1}$ is defined by

$$\partial(f_1, \cdots, f_s) = d^{\#}(f_1, \cdots, f_s) = -(-1)^s (f_1 \otimes \cdots \otimes f_s) \circ d$$

for $f_1, \cdots, f_s \in T(\Sigma^{-1} I^{\#})$.

Since A is generated by A_1, the multiplication $A_{n-1} \otimes A_1 \to A_n$ is surjective. For $n \geq 2$, let the linear maps $\xi_n : A_n \to A_{n-1} \otimes A_1$ be a split injection such that the composition

$$A_n \xrightarrow{\xi_n} A_{n-1} \otimes A_1 \to A_n$$

is the identity. Let θ_n be the composition

$$A_n \xrightarrow{\xi_n} A_{n-1} \otimes A_1 \xrightarrow{\xi_{n-1} \otimes 1} A_{n-2} \otimes A_1^{\otimes 2} \xrightarrow{\xi_{n-2} \otimes 1^{\otimes 2}} \cdots \xrightarrow{\xi_2 \otimes 1^{\otimes n-1}} A_1^{\otimes n}.$$

Since R is a minimal graded vector space of the relations of A, there is an injection $\eta_n : R_n \to A_{n-1} \otimes A_1$. View R_n as a subspace of $A_{n-1} \otimes A_1$ via this injection.

Denote $W^{-s} = I^{\otimes s} = \bigoplus_n W_n^{-s}$, where $W_n^{-s} = \bigoplus_{i_1 + \cdots + i_s = n} A_{i_1} \otimes \cdots \otimes A_{i_s}$. Write d_n^{-s} to be the restriction of d^{-s} on the subspace W_n^{-s}. Then there is a decomposition

$$W_n^{-2} = im(d_n^{-3}) \oplus R_n \oplus \xi_n(A_n).$$

In fact, it is clear that $W_n^{-2} = ker(d_n^{-2}) \oplus \xi_n(A_n)$. Since $R^{\#} \cong \mathrm{Ext}_A^2(k,k) \cong H^2(T(\Sigma^{-1} I^{\#}))$, there is a decomposition $ker(d_n^{-2}) = im(d_n^{-3}) \oplus R_n$.

Now let $T^2 = I^{\#} \otimes I^{\#}$ and $T^2_{-n} = \bigoplus_{i+j=n} A_i^{\#} \otimes A_j^{\#}$. Dualizing above, we have a decomposition

$$T^2_{-n} = (\xi_n(A_n))^{\#} \oplus R_n^{\#} \oplus (im(d_n^{-3}))^{\#}.$$

This decomposition is just the decomposition $T^2_{-n} = B^2_{-n} \oplus H^2_{-n} \oplus L^2_{-n}$ in splitting of a complex. Hence we can define $Q^2_{-n} : T^2_{-n} \to T^1_{-n} = A_n^{\#}$ as the composition $A_1^{\#} \otimes A_{n-1}^{\#} \cong (A_{n-1} \otimes A_1)^{\#} \xrightarrow{-\xi_n^{\#}} A_n^{\#}$ for all $n \geq 2$. Then we get a map $Q^2 : T^2 \to T^1 = I^{\#}$.

Next, let Pr be the projection in Merkulov's construction, in particular, $Pr : T^2_{-n} \to R_n^{\#}$.

- Apply Merkulov's construction on $\Omega A^{\#}$.

For our purpose, let $p = Pr|_{T^2_{-n}}$. Then $p\,|_{A_1^{\#} \otimes A_{n-1}^{\#}}$ is the composition

$$A_1^{\#} \otimes A_{n-1}^{\#} \cong (A_{n-1} \otimes A_1)^{\#} \xrightarrow{\eta_n^{\#}} R_n^{\#}$$

and $p\,(others) = 0$.

By the construction above, up to quasi-isomorphism, there is an augmented A_∞-algebra structure $(H(\Omega A^{\#}), \{m_n\})$ with $m_1 = 0$ and for $n \geq 2$,

$$m_n = Pr\lambda_n = Pr \sum_{i+j=n} (-1)^{i+1} \lambda_2 (Q\lambda_i \otimes Q\lambda_j).$$

Note that the homotopy Q^2 is defined from $A_1^{\#} \otimes A_{n-1}^{\#}$ to $A_n^{\#}$ for all $n \geq 2$, by the definition of p, when restricted to $(A_1^{\#})^{\otimes n}$ we have

$$\begin{aligned}
m_n &= -p\lambda_2(1 \otimes Q\lambda_{n-1}) \\
&= (-1)^2 p\lambda_2(1 \otimes Q\lambda_2(1 \otimes Q\lambda_{n-2})) \\
&\quad \cdots\cdots \\
&= (-1)^{n-2} p\lambda_2(1 \otimes Q\lambda_2(1 \otimes Q\lambda_2(\cdots Q\lambda_2(1 \otimes Q\lambda_2)))) \\
&= (-1)^{n-2} p\lambda_2(1 \otimes Q\lambda_2)(1^{\otimes 2} \otimes Q\lambda_2) \cdots (1^{\otimes n-2} \otimes Q\lambda_2)
\end{aligned}$$

Note that λ_2 is the multiplication of $T(\Sigma^{-1} I^{\#})$. Consider the following commutative diagram

$$\begin{tikzcd}
(A_1^{\#})^{\otimes n} \arrow[r, "-1^{\otimes n-2} \otimes Q\lambda_2"] \arrow[d, equal] & (A_1^{\#})^{\otimes n-2} \otimes A_2^{\#} \arrow[r, "-1^{\otimes n-3} \otimes Q\lambda_2"] \arrow[d, equal] & \cdots \arrow[r, "-1 \otimes Q\lambda_2"] & A_1^{\#} \otimes A_{n-1}^{\#} \arrow[r, "p"] \arrow[d, equal] & R_n^{\#} \arrow[d, equal] \\
(A_1^{\otimes n})^{\#} \arrow[r, "(\xi_2 \otimes 1^{\otimes n-2})^{\#}"] & (A_2 \otimes A_1^{\otimes n-2})^{\#} \arrow[r, "(\xi_3 \otimes 1^{\otimes n-3})^{\#}"] & \cdots \arrow[r, "(\xi_{n-1} \otimes 1)^{\#}"] & (A_{n-1} \otimes A_1)^{\#} \arrow[r, "\eta_n^{\#}"] & R_n^{\#}.
\end{tikzcd}$$

We see that m_n, when restricted to $(A_1^{\#})^{\otimes n}$, is the composition

$$(A_1^{\#})^{\otimes n} \cong (A_1^{\otimes n})^{\#} \xrightarrow{(\theta_{n-1} \otimes 1)^{\#}} (A_{n-1} \otimes A_1)^{\#} \xrightarrow{\eta_n^{\#}} R_n^{\#}.$$

4.2 An example of recovering an algebra from its Ext-algebra

We copy an example from [LPWZ1, Example 13.4] to show the nice recovering property of A_∞-Ext-algebra $\mathrm{Ext}_A^*(k_A, k_A)$. The example explains that some information has been neglected in the course of transferring an algebra to its associative Ext-algebra.

Let α, β be scalars in k with $\alpha \neq 0$. Let A be an associative algebra generated by x and y of Adams degree 1 subject to two relations of Adams degree 3

$$xy^2 + \alpha y^2 x + \beta yxy = 0 \qquad \text{and} \qquad x^2 y + \alpha yx^2 + \beta xyx = 0.$$

Then A is an AS regular algebra of global dimension 3 [AS, p.203]. The Ext-algebra of A is

$$\mathrm{Ext}_A^*(k, k) = k \oplus V^1 \oplus V^2 \oplus V^3$$

with

$$\deg V^1 = (1, -1), \quad \deg V^2 = (2, -3), \quad \deg V^3 = (3, -4)$$

and

$$\dim V^1 = 2, \qquad \dim V^2 = 2, \qquad \dim V^3 = 1.$$

We want to show that as a minimal model of $E(A)$ the A_∞-algebra $\mathrm{Ext}_A^*(k,k)$ is isomorphic to $C(\alpha,\beta)$ defined in [LPWZ1, Example 3.7][10].

For every $n \geq 4$, the graded map m_n has degree $(2-n, 0)$. This implies that the image of m_n must be zero since every homogeneous subspace of $A^{\otimes n}$ has degree $(i,-j)$ with $j \geq i$. For $n=3$, we see that $m_3 = 0$ when applied to a subspace of degree other than $(3,-3)$, and the image of m_3 is in degree $(2,-3)$. Let a and b be the elements in V^1 that are k-linear dual to x and y in A and let c and d be the elements in V^2 that are k-linear dual to the two relations in A (see Lemma in section 4.1.1). By Basic Lemma, $m_3 = 0$ except for

$$m_3(a,b,b) = c, \quad m_3(b,b,a) = \alpha c, \quad m_3(b,a,b) = \beta c$$

and

$$m_3(a,a,b) = d, \quad m_3(b,a,a) = \alpha d, \quad m_3(a,b,a) = \beta d.$$

These are exactly the formulas given in $C(\alpha,\beta)$. It remains to show that the multiplication of $\mathrm{Ext}_A^*(k,k)$ is the same as that of $C(\alpha,\beta)$ in $C(\alpha,\beta)$. It follows from the degree argument that the following subspaces are zero:

$$V^1V^1, \quad V^2V^2, \quad V^3V^3, \quad V^1V^3, \quad V^3V^1, \quad V^2V^3, \quad V^3V^2.$$

Use **SI(4)** we will recover all multiplications of V^1 and V^2. Applied to (a,a,a,b), **SI(4)** becomes

$$am_3(a,a,b) = m_3(a,a,a)b,$$

which implies $ad = 0$ since $m_3(a,a,a) = 0$ and $m_3(a,a,b) = d$. Similarly, applying **SI(4)** to (b,a,a,a) (respectively, (b,b,b,a) and (a,b,b,b)) we obtain $da = 0$ (respectively, $bc = 0$ and

[10] Let α and β be two elements in the base field k. First we define an associative graded algebra C with a parameter α. As a graded vector space

$$C = C^0 \oplus C^1 \oplus C^2 \oplus C^3 = k1 \oplus (ka \oplus kb) \oplus (kc \oplus kd) \oplus ke.$$

The multiplication m_2 is given by the following rules:

$$\begin{array}{cccc} & 1 \text{ is the identity} & & \\ ac = e & ad = 0 & bc = 0 & bd = \alpha e \\ ca = \alpha e & da = 0 & cb = 0 & db = e \\ & \text{other products} = 0. & & \end{array}$$

Then C is an associative graded algebra. It is a Frobenius algebra if and only if α is nonzero. Assume now $\alpha \neq 0$.

Next we define the higher multiplications on C. For $n \neq 2,3$, $m_n = 0$. For $n=3$ we define $m_3(x_1,x_2,x_3) = 0$ except for

$$m_3(a,b,b) = c, \quad m_3(b,b,a) = \alpha c, \quad m_3(b,a,b) = \beta c$$

and

$$m_3(a,a,b) = d, \quad m_3(b,a,a) = \alpha d, \quad m_3(a,b,a) = \beta d.$$

This defines an A_∞-algebra.

$cb = 0$). Applying **SI(4)** to (a, a, b, b), we have

$$am_3(a, b, b) = m_3(a, a, b)b.$$

Let $e = ac \in V^3$. Then the above equation shows that $db = e$. Similarly, applying **SI(4)** to (a, b, b, a) (respectively, (b, a, a, b)) we obtain $ca = \alpha e$ (respectively, $bd = \alpha e$). In particular, $V^1V^2 = ke$. Since $\mathrm{Ext}^*_A(k, k)$ is Frobenius, $V^1V^2 \neq 0$. Hence e is nonzero and is a basis element of V^3. We now recover the multiplication table listed in $C(\alpha, \beta)$. Therefore the A_∞-algebra $\mathrm{Ext}^*_A(k, k)$ is the same as $C(\alpha, \beta)$ of $C(\alpha, \beta)$ after we make the obvious identification of elements.

In this example the Ext-algebra $(\mathrm{Ext}^*_A(k, k), m_2)$ contains only a part of the data about the algebra A; in particular, β does not appear in this associative algebra. The A_∞-algebra $(\mathrm{Ext}^*_A(k, k), m_2, m_3)$ contains the complete data about A and A can be recovered from this A_∞-algebra.

4.3 Koszul duality

In this short subsection, we state in brief that how Ext-algebras work on higher Koszul algebras, which related to a class of A_∞-algebra of $\mathrm{Ext}^*_A(k, k)$ in the sense of only one non-zero higher multiplication.

Let A be a locally finite connected graded algebra. By a *Koszul dual* of A here we mean the A_∞-Ext-algebra (up to quasi-isomorphism) $\mathrm{Ext}^*_A(k_A, k_A)$, denoted by $E(A)$.

We say a p-homogeneous algebra A is (right) *p-Koszul* if the trivial A-module k_A admits a linear projective resolution: $\cdots \to P_n \to \cdots \to P_1 \to P_0 \to k_A \to 0$ with P_n generated in degree $\mathbf{p}(n)$, where $\mathbf{p} : \mathbb{N} \to \mathbb{N}$ is defined by

$$\mathbf{p}(n) = \begin{cases} pm & \text{if} \quad n = 2m, \\ pm + 1 & \text{if} \quad n = 2m + 1. \end{cases}$$

A 2-Koszul algebra is a usual Koszul algebra. Some important graded algebras are p-Koszul algebras. Some significant applications of p-Koszul algebras were found in algebraic topology, algebraic geometry, quantum group, and Lie algebra. For example, some AS regular algebras [AS] of global dimension 3 are p-Koszul algebras which are fundamental in non-commutative projective geometry.

It is an effective method to describe algebras by their Ext-algebras as mentioned before. A well-known criterion theorem for Koszul algebras is: Let A be a quadratic algebra, then A is Koszul if and only if its Koszul dual $E(A) = \mathrm{Ext}^*_A(k, k)$ is generated by $E^1(A)$. There is a similar result for a p-homogeneous algebra to be a p-Koszul algebra: Let A be a p-homogeneous algebra, and $E(A)$ its Koszul dual. Then A is a p-Koszul algebra if and only if $E(A)$ is generated by $E^1(A)$ and $E^2(A)$.

What will be the case if using the A_∞-language?

A $(2,p)$-*algebra* E is an A_∞-algebra with two non-trivial multiplications m_2 and m_p. Roughly speaking E is a graded associative algebra such that some compatibility conditions between m_2 and m_p are required. These algebras was discussed by He in his Ph.D Thesis [He], we refer to [HL] for details.

Since a $(2,p)$-algebra has only two nontrivial multiplications, the Stasheff Identities are automatically satisfied except for $\mathbf{SI(3)}$, $\mathbf{SI(2p-1)}$ and $\mathbf{SI(p+1)}$ (see section 3.1.1).

Definition: An augmented $(2,p)$-algebra (E, m_2, m_p) is called a *reduced (2,p)-algebra* if the following conditions are satisfied:

(1) $E = k \oplus E^1 \oplus E^2 \oplus \cdots$;

(2) $m_2(E^{2t_1+1} \otimes E^{2t_2+1}) = 0$ for all $t_1, t_2 \geq 0$;

(3) $m_p(E^{i_1} \otimes \cdots \otimes E^{i_p}) = 0$ unless all of $i_1, \ldots, i_p$ are odd.

A reduced $(2,p)$-algebra E is said to be *generated by* E^1 if for all $n \geq 2$,

$$E^n = \sum_{\substack{i+j=n \\ i,j\geq 1}} m_2(E^i \otimes E^j) + \sum_{\substack{i_1+\cdots+i_p+2-p=n \\ i_1,\cdots,i_p\geq 1}} m_p(E^{i_1} \otimes \cdots \otimes E^{i_p}).$$

It was proved in [He] that if A is a p-Koszul algebra then each A_∞-structure of the Koszul dual $E(A)$ is a $(2,p)$-algebra, moreover, all the $(2,p)$-algebra structures are isomorphic. Suppose that A is a locally finite connected graded algebra generated by A_1, $E = E(A)$ is the Koszul dual of A with an augmented bigraded A_∞-structure $\{m_i\}$. A criterion for a connected graded algebra to be a p-Koszul algebra in terms of A_∞-algebra is: For $p \geq 3$, A is a p-Koszul algebra if and only if that $(E(A), \{m_i\})$ is a reduced $(2,p)$-algebra generated by E^1. This is an A_∞-version of criterion theorem for higher Koszul algebra analogue to the classical one for Koszul algebra.

5 **Application** (a brief introduction to AS regular algebras)

As an application in noncommutative algebras, we explain roughly in this section that A_∞-algebras can be used to solve some questions which have not been solved by classical methods. We outline the work [LPWZ2] for the classification of AS regular algebra of global dimension 4.

From known results, it seems that Koszul regular algebras are more popular than the non-Koszul ones. There are two explanations. One is that the non-Koszul regular algebras may be more difficult to study since the relations of such algebras are not quadratic. The other is that a non-Koszul algebra A is not a deformation of the commutative polynomial ring $k[x_0, x_1, x_2, x_3]$.

A crucial step in finding non-Koszul AS regular algebras is to use the A_∞-structure on Ext-algebras. This method has advantages for non-Koszul regular algebras because one can get more information from the non-trivial higher multiplications on the Ext-algebras.

We recall the definition of AS regular algebras, explain their Hilbert series, and describe the possible shapes of their Ext-algebras. We try to convince the readers of that A_∞-algebra method is an extremely powerful tool for the question.

5.1 Artin-Schelter regular algebras

5.1.1. Definition and properties

A connected graded algebra A is called *Artin-Schelter regular* (or *AS regular*) if the following three conditions hold.

(AS1) A has finite global dimension d,

(AS2) A is *Gorenstein*, i.e., for some integer l,

$$\mathrm{Ext}^i_A(k, A) = \begin{cases} k(l) & \text{if } i = d \\ 0 & \text{if } i \neq d \end{cases}$$

where k is the trivial module A/I, and

(AS3) A has finite Gelfand-Kirillov dimension; that is, there is a positive number c such that $\dim A_n < c\, n^c$ for all $n \in \mathbb{N}$.

The notation (l) in (AS2) is the degree l shift operation on graded modules.

AS regular algebras have been studied in many recent papers, and in particular, AS regular algebras of global dimension 3 have been classified [AS].

Lemma: [SZ] *Suppose A is connected graded and satisfies* (AS1) *and* (AS2).

(a) A is finitely generated.

(b) The trivial A-module k_A has a minimal free resolution of the form

$$0 \to P_d \to \cdots \to P_1 \to P_0 \to k_A \to 0,$$

where $P_w = \bigoplus_{s=1}^{n_w} A(-i_{w,s})$ for some finite integers n_w and $i_{w,s}$.

(c) The above free resolution is symmetric in the following sense: $P_0 = A$, $P_d = A(-l)$, $n_w = n_{d-w}$, and $i_{w,s} + i_{d-w,n_w-s+1} = l$ for all w, s.

Proposition: (1) *If A is a noetherian AS regular algebra of global dimension at least 3, then the GK-dimension of A is at least* 3.

(2) *If A is a noetherian connected graded AS regular algebra of global dimension 4, then it is an integral domain.*

5.1.2. Hilbert series and types

The Hilbert series of a graded vector space $M = \bigoplus_{i \in \mathbb{Z}} M_i$ is defined to be

$$H_M(t) = \sum_{i \in \mathbb{Z}} (\dim_k M_i) t^i \in \mathbb{Z}[[t, t^{-1}]].$$

By analyzing the minimal resolution of the trivial module k, we can determine the Hilbert series of A when it is generated in degree 1.

Three Types

Let A be a graded AS regular algebra of global dimension 4 that is generated in degree 1. Suppose that A is a domain. Then A is minimally generated by either 2, 3, or 4 elements.

(a) If A is generated by 2 elements, then there are two relations whose degrees are 3 and 4. The minimal resolution of the trivial module is of the form

$$0 \to A(-7) \to A(-6)^{\oplus 2} \to A(-4) \oplus A(-3) \to A(-1)^{\oplus 2} \to A \to k \to 0.$$

The Hilbert series of A is

$$H_A(t) = 1/(1-t)^2(1-t^2)(1-t^3).$$

(b) If A is generated by 3 elements, then there are two relations in degree 2 and two relations in degree 3. The minimal resolution of the trivial module is of the form

$$0 \to A(-5) \to A(-4)^{\oplus 3} \to A(-3)^{\oplus 2} \oplus A(-2)^{\oplus 2} \to A(-1)^{\oplus 3} \to A \to k \to 0.$$

The Hilbert series of A is

$$H_A(t) = 1/(1-t)^3(1-t^2).$$

(c) If A is generated by 4 elements, then there are six quadratic relations. The minimal resolution of the trivial module is of the form

$$0 \to A(-4) \to A(-3)^{\oplus 4} \to A(-2)^{\oplus 6} \to A(-1)^{\oplus 4} \to A \to k \to 0.$$

The Hilbert series of A is

$$H_A(t) = 1/(1-t)^4.$$

In each of these cases, the GK-dimension of A is 4.

Proof: (Sketch) k_A has a minimal free resolution

$$0 \to A(-l) \to A(-l+1)^{\oplus n} \to \bigoplus_{s=1}^{v} A(-n_s) \to A(-1)^{\oplus n} \to A \to k \to 0.$$

Hence the Hilbert series of A is $H_A(t) = 1/p(t)$, where $p(t) = 1 - nt + \sum_{s=1}^{v} t^{n_s} - nt^{l-1} + t^l$.

By analyzing the GK-dimension of A (= the order of the zero of $p(t)$ at 1), we have

$$\sum_s n_s = (n-1)l, \quad \text{and} \quad \sum_{s=1}^{n-1} n_{v-s+1} n_s = (l-1)n.$$

Discussing the equation above, possible three solutions are:

(a) $n = 2$ implies $l = 7$, and $n_1 = 3, n_2 = 4$.

(b) $n = 3$ implies $l = 5$, and $n_1 = n_2 = 2$, $n_3 = n_4 = 3$.

(c) $n = 4$ implies $l = 4$, and $n_s = 2$ $(1 \le s \le 6)$.

Remark: If an algebra satisfies the hypotheses above, we label it according to the dimensions of vector spaces $\mathrm{Ext}^i_A(k,k)$. That is, algebras as in the class (a) are said to be of *type* (*12221*), algebras as in the class (b) are of *type* (*13431*), and algebras as in the class (c) are of *type* (*14641*).

5.2 A_∞-structures on Ext-AS-regular algebras

By Koszul duality, A is (quasi-)isomorphic to $E(E(A))$ as A_∞-algebras and the relations of A can be recovered from the multiplications of $E(A)$ (Basic Lemma). Hence to classify the AS regular algebras of global dimension 4 that is generated in degree 1, it is sufficient to classify the A_∞-algebras $E(A)$ listed in each classes above.

Proposition: *Let A be an algebra as above and let E be the* Ext-*algebra of A.*

(a) (type (12221)) If A is minimally generated by 2 elements, then E is isomorphic to

$$k \bigoplus E^1_{-1} \bigoplus (E^2_{-3} \oplus E^2_{-4}) \bigoplus E^3_{-6} \bigoplus E^4_{-7}$$

as a $\mathbb{Z}^2$-graded vector space, where the lower index is the Adams grading inherited from the grading of A and the upper index is the homological grading of the Ext-group. The dimensions of the subspaces are

$$\dim E^1_{-1} = \dim E^3_{-6} = 2, \qquad \dim E^2_{-3} = \dim E^2_{-4} = \dim E^4_{-7} = 1.$$

As an A_∞-algebra, $m_n = 0$ for all $n \ge 5$; that is, $E = (E, m_2, m_3, m_4)$.

(b) (type (13431)) If A is minimally generated by 3 elements, then E is isomorphic to

$$k \bigoplus E^1_{-1} \bigoplus (E^2_{-2} \oplus E^2_{-3}) \bigoplus E^3_{-4} \bigoplus E^4_{-5}$$

as a $\mathbb{Z}^2$-graded vector space. As an A_∞-algebra, $m_n = 0$ for all $n \ge 4$; that is, $E = (E, m_2, m_3)$. The dimensions of the subspaces are

$$\dim E^1_{-1} = \dim E^3_{-4} = 3, \quad \dim E^2_{-2} = \dim E^2_{-3} = 2, \quad \dim E^4_{-5} = 1.$$

(c) (type (14641)) If A is minimally generated by 4 elements, then E is isomorphic to

$$k \bigoplus E^1_{-1} \bigoplus E^2_{-2} \bigoplus E^3_{-3} \bigoplus E^4_{-4}$$

as a $\mathbb{Z}^2$-graded vector space. The algebras A and E are Koszul and m_n of E is zero for all $n \neq 2$. The dimensions of the subspaces are

$$\dim E^1_{-1} = \dim E^3_{-3} = 4, \qquad \dim E^2_{-2} = 6, \quad \dim E^4_{-4} = 1.$$

Proof: (Sketch) The vector space decomposition of E and the dimensions of the subspaces of E are clear from the form of the minimal free resolution of the trivial module.

It's clear that all $m_1 = 0$ since the degree of m_1 is $(1, 0)$.

Consider the maps m_n, restricted to a homogeneous subspace:

$$m_n : E^{i_1}_{-j_1} \otimes \cdots \otimes E^{i_n}_{-j_n} \to E^i_{-j}$$

where $i = \sum_{s=1}^n i_s - n + 2$ and $j = \sum_{s=1}^n j_s$. We may assume that $i_s \geq 1$ (for all s) since the strict unital condition. Then note that $i_s \leq j_s \leq i_s + 3$ in case (a), and $i_s \leq j_s < i_s + 2$ in case (b). In case (c), k_A has a linear resolution. Hence A and E are Koszul, and this implies that the higher multiplications of E are trivial.

Remark:

(1) In type (14641), both A and $E(A)$ are Koszul. Most known examples fall into this type. In this type, A_∞-algebra methods are not available since higher multiplications on $E(A)$ are zero. For type (12221), we can construct first (E, m_2), second m_3, and third m_4, each step is relatively easy. For type (14641), we have to finish the construction at one move. We will have other methods to study these algebras, for example, using the matrices constructed from the multiplication of E. The people working in representation theory may also be interested in this type.

(2) In type (13431), m_2 and m_3 of E are nonzero. By Basic lemma, the relations of A, two of which are in degree 2 and other two are in degree 3, are determined completely by m_2 and m_3 of E. In the near future we will look into the possibility of classifying those A_∞-algebras (E, m_2, m_3).

(3) In type (12221), there are two nonzero higher multiplications m_3 and m_4. For such algebras, all Stasheff's identities are automatic except for SI(4), SI(5), SI(6), SI(7) (in type (12221), the identity SI(7) is also automatic).

5.3 Non-Koszul AS regular algebras

A point in translating an algebra to its Ext-algebra is a generalization result[11] of Smith [Sm] that we listed in the section 4. Smith's another result is interesting too: Let A be a left noetherian,

[11] By using A_∞-algebras, the result can be generalized to non-Koszul non-noetherian algebras: *Let A be a connected graded algebra and let E be the Ext-algebra of A. Then A is AS regular if and only if E is Frobenius.* [LPWZ1, Th. 12.5]

augmented algebra, and suppose that A is Gorenstein and that $\mathrm{gldim}(A) = n$, then $\mathrm{Ext}^*_A(k,k)$ is Frobenius.

We concentrate on algebras of type (12221) in this subsection. We describe formulas for the possible multiplication maps m_n on their A_∞-Ext-algebras. In this type, the Ext-algebras are

$$E = k \bigoplus E^1_{-1} \bigoplus (E^2_{-3} \oplus E^2_{-4}) \bigoplus E^3_{-6} \bigoplus E^4_{-7}.$$

We construct A_∞-structures on E step by step.

5.3.1. FROBENIUS DATA ON (E, m_2)

Except for the multiplying by the unit element, the possible nonzero m_2's of E are

$$E^1_{-1} \otimes E^3_{-6} \to E^4_{-7}, \qquad E^3_{-6} \otimes E^1_{-1} \to E^4_{-7},$$
$$E^2_{-3} \otimes E^2_{-4} \to E^4_{-7}, \qquad E^2_{-4} \otimes E^2_{-3} \to E^4_{-7}.$$

By Theorem above, if A is an AS regular algebra, then E is a Frobenius algebra. The multiplication m_2 of a Frobenius algebra E of type (12221) can be described as follows.

Let δ be a basis element of E^4_{-7}. Pick a basis element $\beta_1 \in E^2_{-3}$. Since E is a Frobenius algebra,

$$m_2 : E^2_{-3} \otimes E^2_{-4} \to E^4_{-7} \quad \text{and} \quad m_2 : E^2_{-4} \otimes E^2_{-3} \to E^4_{-7}$$

are both nonzero. So we can pick a basis element $\beta_2 \in E^2_{-4}$ such that $\beta_1\beta_2 = \delta$ and $\beta_2\beta_1 = t\,\delta$ for some $0 \neq t \in k$. Pick a basis $\{\alpha_1, \alpha_2\}$ for E^1_{-1}. Since E is a Frobenius algebra,

$$m_2 : E^1_{-1} \otimes E^3_{-6} \to E^4_{-7} \quad \text{and} \quad m_2 : E^3_{-6} \otimes E^1_{-1} \to E^4_{-7}$$

are perfect pairings. Hence we may choose a basis $\{\gamma_1, \gamma_2\}$ of E^3_{-6} such that $\alpha_i\gamma_j = \delta_{ij}\delta$. Let $\gamma_i\alpha_j = r_{ij}\delta$ for some $r_{ij} \in k$. Then the matrix $\Lambda := (r_{ij})_{2\times 2} \in M_2(k)$ is non-singular.

(Λ, t) is called the *Frobenius data* of E or of A.

5.3.2. DETERMINE (E, m_2, m_3) SATISFYING SI(4)

Given such an (E, m_2), note that $m_1 = 0$, the Stasheff identity **SI(4)** becomes

$$m_3(m_2 \otimes id^{\otimes 2} - id \otimes m_2 \otimes id + id^{\otimes 2} \otimes m_2) - m_2(m_3 \otimes id + id \otimes m_3) = 0,$$

which only involves m_2 and m_3. Possible nonzero components of m_3 on $E^{\otimes 3}$ are

$$(E^1_{-1})^{\otimes 3} \to E^2_{-3},$$
$$(E^1_{-1})^{\otimes 2} \otimes E^2_{-4} \to E^3_{-6}, \quad E^1_{-1} \otimes E^2_{-4} \otimes E^1_{-1} \to E^3_{-6}, \quad E^2_{-4} \otimes (E^1_{-1})^{\otimes 2} \to E^3_{-6},$$
$$E^1_{-1} \otimes (E^2_{-3})^{\otimes 2} \to E^4_{-7}, \quad E^2_{-3} \otimes E^1_{-1} \otimes E^2_{-3} \to E^4_{-7}, \quad (E^2_{-3})^{\otimes 2} \otimes E^1_{-1} \to E^4_{-7}.$$

We have, for $1 \le i, j, k \le 2$,

$$\begin{aligned}
m_3(\alpha_i, \alpha_j, \alpha_k) &= a_{ijk}\ \beta_1; \\
m_3(\alpha_i, \alpha_j, \beta_2) &= b_{13ij}\ \gamma_1 + b_{23ij}\ \gamma_2, \\
m_3(\alpha_i, \beta_2, \alpha_j) &= b_{12ij}\ \gamma_1 + b_{22ij}\ \gamma_2, \\
m_3(\beta_2, \alpha_i, \alpha_j) &= b_{11ij}\ \gamma_1 + b_{21ij}\ \gamma_2; \\
m_3(\alpha_i, \beta_1, \beta_1) &= c_{1i}\ \delta, \\
m_3(\beta_1, \alpha_i, \beta_1) &= c_{2i}\ \delta, \\
m_3(\beta_1, \beta_1, \alpha_i) &= c_{3i}\ \delta,
\end{aligned}$$

all other applications of m_3 are zero.

Here a_{ijk}, b_{ipjk} and c_{pi} are scalars in the field k.

Applying **SI(4)** to the elements $(\alpha_i, \alpha_j, \alpha_k, \beta_2)$, $(\alpha_i, \alpha_j, \beta_2, \alpha_k)$, $(\alpha_i, \beta_2, \alpha_j, \alpha_k)$, and $(\beta_2, \alpha_i, \alpha_j, \alpha_k)$, respectively, we obtain

(**SI(4)**)
$$a_{ijk} = b_{i3jk}, \quad b_{i2jk} = \sum_{s=1}^{2} r_{sk}\ b_{s3ij},$$
$$b_{i1jk} = \sum_{s=1}^{2} r_{sk}\ b_{s2ij}, \quad -t\ a_{ijk} = \sum_{s=1}^{2} r_{sk}\ b_{s1ij}.$$

Note that Koszul sign rule applies when two symbols are commuted. As a consequence of SI(4), we have

(**SI(4')**)
$$-t\ a_{ijk} = \sum_{s,t,u=1}^{2} r_{sk}\ r_{tj}\ r_{ui}\ a_{uts}.$$

5.3.3. Determine possible A_∞-algebras (E, m_2, m_3, m_4)

Given the list of all (E, m_2, m_3) satisfying SI(4). The next step is to determine all possible A_∞-algebras (E, m_2, m_3, m_4) satisfying SI(5) and SI(6). Possible nonzero applications of m_4 on $E^{\otimes 4}$ are

$$(E^1_{-1})^{\otimes 4} \to E^2_{-4},$$
$$(E^1_{-1})^{\otimes 3} \otimes E^2_{-3} \to E^3_{-6}, \qquad (E^1_{-1})^{\otimes 2} \otimes E^2_{-3} \otimes E^1_{-1} \to E^3_{-6},$$
$$E^1_{-1} \otimes E^2_{-3} \otimes (E^1_{-1})^{\otimes 2} \to E^3_{-6}, \qquad E^2_{-3} \otimes (E^1_{-1})^{\otimes 3} \to E^3_{-6}.$$

We write down the coefficients of these maps. For $1 \le i, j, k, h \le 2$,

$$m_4(\alpha_i, \alpha_j, \alpha_k, \alpha_h) = y_{ijkh}\ \beta_2,$$
$$m_4(\alpha_i, \alpha_j, \alpha_k, \beta_1) = x_{14ijk}\ \gamma_1 + x_{24ijk}\ \gamma_2,$$
$$m_4(\alpha_i, \alpha_j, \beta_1, \alpha_k) = x_{13ijk}\ \gamma_1 + x_{23ijk}\ \gamma_2,$$
$$m_4(\alpha_i, \beta_1, \alpha_j, \alpha_k) = x_{12ijk}\ \gamma_1 + x_{22ijk}\ \gamma_2,$$
$$m_4(\beta_1, \alpha_i, \alpha_j, \alpha_k) = x_{11ijk}\ \gamma_1 + x_{21ijk}\ \gamma_2,$$

all other applications of m_4 are zero.

Here y_{ijkh} and x_{hpijk} are scalars in the field k.

The Stasheff identity **SI(5)** becomes

$$m_4(m_2 \otimes id^{\otimes 3} - id \otimes m_2 \otimes id^{\otimes 2} + id^{\otimes 2} \otimes m_2 \otimes id - id^{\otimes 3} \otimes m_2)+$$
$$+ m_3(m_3 \otimes id^{\otimes 2} + id \otimes m_3 \otimes id + id^{\otimes 2} \otimes m_3) + m_2(m_4 \otimes id - id \otimes m_4) = 0.$$

As with **SI(4)**, after applied to elements $(\alpha_i, \alpha_j, \alpha_k, \alpha_h, \beta_1)$, $(\alpha_i, \alpha_j, \alpha_k, \beta_1, \alpha_h)$, $(\alpha_i, \alpha_j, \beta_1, \alpha_k, \alpha_h)$, $(\alpha_i, \beta_1, \alpha_j, \alpha_k, \alpha_h)$ and $(\beta_1, \alpha_i, \alpha_j, \alpha_k, \alpha_h)$, respectively, **SI(5)** gives the following equations:

(**SI(5)**)

$$a_{ijk}\ c_{2h} - a_{jkh}\ c_{1i} + t\ y_{ijkh} - x_{i4jkh} = 0,$$
$$a_{ijk}\ c_{3h} + r_{1h}\ x_{14ijk} + r_{2h}\ x_{24ijk} - x_{i3jkh} = 0,$$
$$r_{1h}\ x_{13ijk} + r_{2h}\ x_{23ijk} - x_{i2jkh} = 0,$$
$$c_{1i}\ a_{jkh} - r_{1h}\ x_{12ijk} - r_{2h}\ x_{22ijk} + x_{i1jkh} = 0,$$
$$a_{jkh}\ c_{2i} - a_{ijk}\ c_{3h} - r_{1h}\ x_{11ijk} - r_{2h}\ x_{21ijk} + y_{ijkh} = 0.$$

The Stasheff identity **SI(6)** becomes

$$m_4(-m_3 \otimes id^{\otimes 3} - id \otimes m_3 \otimes id^{\otimes 2} - id^{\otimes 2} \otimes m_3 \otimes id - id^{\otimes 3} \otimes m_3)$$
$$+ m_3(m_4 \otimes id^{\otimes 2} - id \otimes m_4 \otimes id + id^{\otimes 2} \otimes m_4) = 0.$$

Applying **SI(6)** to $(\alpha_i, \alpha_j, \alpha_k, \alpha_h, \alpha_m, \alpha_n)$, we obtain the equation

SI(6)

$$- a_{ijk}x_{s1hmn} + a_{jkh}x_{s2imn} - a_{khm}x_{s3ijn} + a_{hmn}x_{s4ijk}$$
$$+ b_{s1mn}y_{ijkh} - b_{s2in}y_{jkhm} + b_{s3ij}y_{khmn} = 0.$$

We now have all of the equations we need since the identity **SI(n)** holds for every $n \ge 7$ for $E = E(m_2, m_3, m_4)$.

Remark: Now we are in the face of determining all parameters

$$t,\ r_{ij},\ a_{ijk},\ b_{ipjk},\ c_{pi},\ y_{ijkl},\ x_{lqijk},$$

for $1 \leq i,j,k,l \leq 2$, $1 \leq p \leq 3$, $1 \leq q \leq 4$, suiting the equations SI(4), SI(5) and SI(6). Technically, it will be complicated and tedious (123 parameters). We have used Maple to solve the equations after reduction and simplification.

5.3.4. REDUCING

We first work on m_2 of E. If B is an invertible 2×2 matrix, then replacing (α_1, α_2) by $(\alpha_1, \alpha_2)B^{-1}$ and (γ_1, γ_2) by $(\gamma_1, \gamma_2)B^T$ changes the Frobenius data of E from (Λ, t) to $(B\Lambda B^{-1}, t)$. So choosing B properly, we may assume that the matrix Λ is either

$$\begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} g_1 & 1 \\ 0 & g_1 \end{pmatrix}$$

for some nonzero $g_i \in k$.

The generic condition for m_2 is

(GM2) Let g_1 and g_2 be the eigenvalues of Λ. Then $(g_1 g_2^{-1})^i \neq 1$ for $1 \leq i \leq 4$.

Ideally if A is 'generic', then g_i and $g_1 g_2^{-1}$ should not be a root of unity. Hence (GM2) can be viewed as a sort of generic condition. Suppose now that (GM2) holds. Then $g_1 \neq g_2$, so we may assume that $\Lambda = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}$. The multiplication map m_2 is now described by the Frobenius data.

Next we work with m_3 by considering **SI(4)**. By **SI(4')**, we have

$$-ta_{ijk} = g_i g_j g_k a_{ijk} \quad \text{for all} \quad i,j,k = 1,2.$$

After some technical process, and then some hypothesis, we may write r_3 as following without loss of generality,

$$r_3 = a_{122} z_1 z_2^2 + a_{212} z_2 z_1 z_2 + a_{221} z_2^2 z_1, \quad (a_{122} \cdot a_{221} \neq 0).$$

The generic condition for m_3 is

(GM3) $$a_{122} + a_{212} + a_{221} \neq 0.$$

This condition is not very essential, but it guarantees that the matrix Λ is diagonal, so when we make changes to the relation r_4, the structure of m_2 and m_3 will not change. In a word, we have an algebra A of the form

$$A = k\langle z_1, z_2 \rangle / (r_3, r_4),$$

where k is a field, and the relations are

$$r_3 = a_{122} z_1 z_2^2 + a_{212} z_2 z_1 z_2 + a_{221} z_2^2 z_1,$$
$$r_4 = \sum y_{ijkh}\, z_i z_j z_k z_h.$$

we may assume that $y_{1122} = y_{2122} = y_{1221} = y_{1222} = 0$.

5.3.5. Solving

An AS regular algebra of type (12221) is called (m_2, m_3)-*generic* if its A_∞-Ext-algebra satisfies both (GM2) and (GM3).

After reducing, using **SI(4)** we get the solution of b_{iqjk}; then using **SI(5)** to find formulas for x_{isjkh}, and then input those into **SI(6)**. This produces 2^7 equations, which can be generated by Maple. After a few steps of simplification, we are able to list all possible solutions. We have used Bill Schelter's program 'Affine' to check the Hilbert series of the algebras in the cases. After analyzing, we finally get the solutions corresponding to a class of AS regular algebras

5.3.6. Recover A from E

Given an AS regular algebra $A = k\langle z_1, z_2\rangle/(R)$ of type (12221), we know that A has two relations r_3 and r_4 of degree 3 and 4, respectively, which we have written as

$$(R_A) \qquad r_3 = \sum a_{ijk}\, z_i z_j z_k, \quad \text{and} \quad r_4 = \sum y_{ijkh}\, z_i z_j z_k z_h.$$

By Basic lemma, for the Ext-algebra E of A, we have

$$(R_E) \qquad m_3(\alpha_i, \alpha_j, \alpha_k) = a_{ijk}\, \beta_1, \quad \text{and} \quad m_4(\alpha_i, \alpha_j, \alpha_k, \alpha_h) = y_{ijkh}\, \beta_2.$$

Conversely, if we know for the Ext-algebra E that (R_E) holds, then the relations of A are given by (R_A). The basic idea of recovering is to classify all possible higher multiplications m_3 and m_4 on E, thus all possible coefficients a_{ijk} and y_{ijkh}. Then we define the relations r_3 and r_4 using (R_A), and we investigate when the resulting algebra $k\langle z_1, z_2\rangle/(r_3, r_4)$ is AS regular.

5.3.7. Main result

Non-Koszul AS regular algebras:

The following algebras are Artin-Schelter regular of global dimension four.

(a) $A(p) := k\langle x, y\rangle/(xy^2 - p^2y^2x, x^3y + px^2yx + p^2xyx^2 + p^3yx^3)$, *where* $0 \neq p \in k$.

(b) $B(p) := k\langle x, y\rangle/(xy^2 + ip^2y^2x, x^3y + px^2yx + p^2xyx^2 + p^3yx^3)$, *where* $0 \neq p \in k$ *and* $i^2 = -1$.

(c) $C(p) := k\langle x, y\rangle/(xy^2 + pyxy + p^2y^2x, x^3y + jp^3yx^3)$, *where* $0 \neq p \in k$ *and* $j^2 - j + 1 = 0$.

(d) $D(v, p) := k\langle x, y\rangle/(xy^2 + vyxy + p^2y^2x, x^3y + (v + p)x^2yx + (p^2 + pv)xyx^2 + p^3yx^3)$, *where* $v, p \in k$ *and* $p \neq 0$.

If k is algebraically closed, then this list (after deleting some special algebras in each family) is, up to isomorphism, a complete list of (m_2, m_3)-generic Artin-Schelter regular algebras of global dimension four that are generated by two elements.

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