Introduction to A_∞-algebras

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Abstract: This is an outline of talks on a short course for the graduate students at Beijing Normal University. The goal of the talks is to give a brief introduction to A_∞-algebras with a view towards noncommutative algebras.

Contents

1 Introduction

1.1 History

1.1.1. J. Stasheff

1960's, Stasheff **invented** A_∞-spaces and A_∞-algebras, as a tool in the study of 'group-like' topological spaces.

1.1.2. M. Kontsevich

In 1994, Kontsevich's talk at the ICM on categorical mirror symmetry played an important role in **developing** this subject.

1.1.3. B. Keller

In 2000, Keller **introduced** the A_∞-language to the study of ring theory and representation theory.

1.1.4. Others

1.2 Motivation

1.2.1. Keller's problems

Let A be an associative k-algebra with 1.

Problem 1. The reconstruction of a complex from its homology.

Problem 2. The reconstruction of the category of iterated selfextensions of module from its extension algebra.

1.2.2. Classification of AS-regular algebras

One of the central questions in noncommutative projective geometry is

the classification of quantum $\mathbb{P}^3$s.

An algebraic approach of constructing quantum $\mathbb{P}^n$s is to form the noncommutative scheme Proj A where A is a noetherian Artin-Schelter regular connected graded algebra of global dimension $n+1$. Therefore the algebraic version of the above mentioned question is

the classification of noetherian, Artin-Schelter regular, connected graded algebras of global dimension 4.

2 Differential graded algebras

2.1 DG algebras

2.1.1. Graded algebra

A *graded algebra* is a graded module $A = \bigoplus_{i \in \mathbb{Z}} A^i$ with an associative multiplication such that (a) the unit 1 is in A^0 and (b) the multiplication preserves the grading.

A *differential* in a graded module A is a k-linear map $\partial : A \to A$ of degree $+1$ such that $\partial^2 = 0$.

A *derivation* of degree n in a graded algebra A is a k-linear map $\partial : A \to A$ of degree n such that (graded Leibniz rule)

$$\partial(xy) = (\partial x)y + (-1)^{n|x|}x(\partial y)$$

for all elements $x, y \in A$.

The *Koszul sign convention*, namely, when two symbols of degrees n and m are permuted the result is multiplied by $(-1)^{nm}$.

2.1.2. Differential graded algebra

A *differential graded algebra* is a graded algebra A together with a differential $\partial : A \to A$ of degree 1 that is a derivation. An *augmentation* is a morphism $\epsilon : A \to k$.

2.1.3. Examples

2.2 DG modules

2.2.1. DG category

Let (A, ∂) be a DGA. A *left differential graded A-module* (or *left DG A-module*) is a complex (M, ∂_M) together with a left multiplication $A \otimes M \to M$ such that M is a left graded A-module and the differential ∂_M of M satisfies the Leibniz rule

$$\partial_M(am) = \partial(a)m + (-1)^{|a|}a\partial_M(m)$$

for all $a \in A, m \in M$. A DG k-module is just a complex.

2.2.2. Two constructions

- $\mathrm{Hom}_A(M, N)$.
- $M \otimes_A N$.

2.3 Resolutions

For a graded set Y with a degree function $g : Y \to \mathbb{Z}$, consider the graded $A^\natural$-module with basis $E_Y \cup E_Y^+$, where

$$E_Y = \{e_y \mid \deg(e_y) = g(y),\ y \in Y\} \quad \text{and} \quad E_Y^+ = \{e_y^+ \mid \deg(e_y^+) = g(y) + 1,\ y \in Y\}$$

Free DG module with the basis Y is

$$F^{[Y]} := \bigoplus_{y\in Y} Ae_y \oplus \bigoplus_{y\in Y} Ae_y^+.$$

with the differential given by

$$\partial(\sum_{y\in Y} a_y e_y + a_y^+ e_y^+) = \sum_{y\in Y}(\partial(a_y)e_y + (-1)^{|a_y|}a_y e_y^+ + \partial(a_y^+)e_y^+).$$

A DG A-module M is called *semifree* if there is a sequence of DG submodules

$$0 = M(-1) \subset M(0) \subset \cdots \subset M(n) \subset \cdots$$

such that $M = \bigcup_n M(n)$ and that each $M(n)/M(n-1)$ is A-free on a basis of cocycles. Such an increasing sequence is called a *semifree filtration* of M. A semifree module is a replacement for a free complex over an associative algebra.

A *semifree resolution* of a DG A-module M is a quasi-isomorphism $L \to M$ from a semifree DG A-module L.

Proposition: Let A be an augmented DGA. Then the augmentations in BA and A define a quasi-isomorphism $\epsilon\otimes\epsilon : B(A,A) \to k$. Moreover, if k is a field then $B(A,A)$ is a semifree right DG A-module. Thus $\epsilon\otimes\epsilon$ is a semifree resolution of the right DG A-module k_A.

3 A_∞-language

From the point of view of homotopy theory, an A_∞-algebra is the same as a DGA. However, for the purpose of explicit computations, it is often more convenient to work with A_∞-algebras rather than with DGA's. The reason is the existence of extra structure in the form of higher multiplications.

3.1 A_∞-algebras

3.1.1. DEFINITION

An A_∞*-algebra* over k is a $\mathbb{Z}$-graded vector space

$$A = \bigoplus_{p\in\mathbb{Z}} A^p$$

endowed with a family of graded k-linear maps

$$m_n : A^{\otimes n} \to A, \quad n \geq 1,$$

of degree $(2-n)$ satisfying the following *Stasheff identities*:

SI(n) $$\sum(-1)^{r+st}m_u(id^{\otimes r}\otimes m_s\otimes id^{\otimes t})=0,\quad \text{for all } n\geq 1,$$

where the sum runs over all decomposition $n=r+s+t$, $r,t\geq 0$ and $s\geq 1$, and where $u=r+1+t$. Here id denotes the identity map of A. Note that when these formulas are applied to elements, additional signs appear due to the Koszul sign rule. An A_∞-algebra is also called a *strongly homotopy associative algebra* (or *sha algebra*).

3.1.2. Examples

(a) An associative algebra A is an A_∞-algebra *concentrated in degree* 0 with all multiplications $m_n=0$ for $n\neq 2$. Hence associative algebras form a subclass of A_∞-algebras of the form (A,m_2).

(b) Differential graded algebra (A,m_1,m_2).

(c) Pentagonal homotopy associative algebra (A,m_1,m_2,m_3):

SI(4): $\quad m_2(1\otimes m_3+m_3\otimes 1)=m_3(m_2\otimes 1\otimes 1-1\otimes m_2\otimes 1+1\otimes 1\otimes m_2).$

(d) Connected cubic zero A_∞-algebra.

(e) Let $B=k[x_1,x_2]/(x_1^2)$, p ($p\geq 3$) a fixed integer. Define an A_∞-algebra structure on B as follows.

For $s\geq 0$, set
$$x_s=\begin{cases} x_2^{\frac{s}{2}} & \text{if } s \text{ is even,}\\ x_1x_2^{\frac{s-1}{2}} & \text{if } s \text{ is odd.}\end{cases}$$
Then $\{x_s\}_{s\geq 0}$ is a k-basis of the graded vector space B. For $i_1,\cdots,i_p\geq 0$, define
$$m_p(x_{i_1},\cdots,x_{i_p})=\begin{cases} x_j & \text{if all } i_s \text{ are odd,}\\ 0 & \text{otherwise,}\end{cases}$$
where $j=2-p+\sum_s i_s$. The multiplication m_2 is the product of the associative algebra $k[x_1,x_2]/(x^2)$. Now it is direct to check that (B,m_2,m_p) is an A_∞-algebra, which is denoted by $B(p)$.

(f) $(2,p)$-algebra.

(g) *Ext-algebra*: Let A be an algebra over k, then $\mathrm{Ext}_A^*(k,k)$ is an A_∞-algebra (Section 4).

(h) *AS-regular algebras* of 3 and 4 (Section 5).

3.1.3. A_∞-morphisms

For two A_∞-algebras A and B. A *morphism* of A_∞-algebras $f:A\to B$ is a family of k-linear graded maps
$$f_n:A^{\otimes n}\to B$$

of degree $(1-n)$ satisfying the following *Stasheff morphism identities*: for all $n \geq 1$,

MI(n): $$\sum(-1)^{r+st} f_u(id^{\otimes r} \otimes m_s \otimes id^{\otimes t}) = \sum(-1)^w m_q(f_{i_1} \otimes f_{i_2} \otimes \cdots \otimes f_{i_q})$$

where the first sum runs over all decompositions $n = r+s+t$ with $s \geq 1, r, t \geq 0$, we put $u = r+1+t$, and the second sum runs over all $1 \leq q \leq n$ and all decompositions $n = i_1 + \cdots + i_q$ with all $i_s \geq 1$; the sign on the right-hand side is given by

$$w = (q-1)(i_1 - 1) + (q-2)(i_2 - 1) + \cdots + 2(i_{q-2} - 1) + (i_{q-1} - 1).$$

3.1.4. Two models

DGA model: *Every A_∞-algebra A is quasi-isomorphic to a free DGA constructed as ΩBA.*

Minimal model: *Let A be an A_∞-algebra and let HA be the cohomology ring of A. There is an A_∞-algebra structure on HA with $m_1 = 0$, constructed from the A_∞-structure of A, such that there is a quasi-isomorphism of A_∞-algebras $HA \to A$ lifting the identity of HA.*

Corollary: *Let A be an algebra over k, then $Ext^*_A(k,k)$ is an A_∞-algebra.*

3.2 A_∞-modules

3.2.1. Definition

Let A be an A_∞-algebra.

- A *left A_∞-module* over A is a $\mathbb{Z}$-graded vector space M endowed with maps

$$m_n^M : A^{\otimes n-1} \otimes M \to M, \quad n \geq 1$$

of degree $(2-n)$ satisfying the same Stasheff identities **SI(n)**

$$\sum(-1)^{r+st} m_u(id^{\otimes r} \otimes m_s \otimes id^{\otimes t}) = 0$$

as one in the definition of A_∞-algebra. However, the term $m_u(id^{\otimes r} \otimes m_s \otimes id^{\otimes t})$ has to be interpreted as $m_u^M(id^{\otimes r} \otimes m_s \otimes id^{\otimes t})$ if $t > 0$ and as $m_u^M(id^{\otimes r} \otimes m_s^M)$ if $t = 0$.

- A *morphism* of left A_∞-modules $f : M \to N$ is a family of graded maps

$$f_n : A^{\otimes n-1} \otimes M \to N$$

of degree $(1-n)$ such that for each $n \geq 1$, the following version of the identity **MI(n)** holds:

MIL(n) $$\sum(-1)^{r+st} f_u \circ (id^{\otimes r} \otimes m_s \otimes id^{\otimes t}) = \sum m_{1+w} \circ (id^{\otimes w} \otimes f_v),$$

where the first sum is taken over all decompositions $n = r+s+t$, $r, t \geq 0, s \geq 1$ and we put $u = r+1+t$; and the second sum is taken over all decompositions $n = v + w$, $v \geq 1, w \geq 0$.

A morphism f is called a *quasi-isomorphism* if f_1 is a quasi-isomorphism. The identity morphism $f : M \to M$ is given by $f_1 = id_M$ and $f_i = 0$ for all $i \geq 2$.

The composition of two morphisms $f : M \to N$ and $g : L \to M$ is defined by

$$(f \circ g)_n = \sum f_{1+w} \circ (id^{\otimes w} \otimes g_v)$$

where the sum runs over all decompositions $n = v + w$.

3.2.2. Derived category

Let A be an A_∞-algebra.

• $\mathcal{C}_\infty(A)$: the category of left A_∞-modules over A with morphisms of A_∞-algebras.

• The *homotopy category* $\mathcal{K}_\infty(A)$ has the same objects as $\mathcal{C}_\infty(A)$, and the morphisms from M to N are morphisms of A_∞-modules modulo the nullhomotopic morphisms.

• The *derived category* $\mathcal{D}_\infty(A)$ to be the homotopy category $\mathcal{K}_\infty(A)$.

3.2.3. Change of A_∞-algebras

Let $f : A \to B$ be a morphism of A_∞-algebras and let (M, m_n^B) be a left A_∞-module over B. Define

$$m_n^A : A^{\otimes n-1} \otimes M \to M, \quad n \geq 1,$$

by

INL(n) $$m_n^A = \sum (-1)^w m_q^B (f_{i_1} \otimes \cdots \otimes f_{i_{q-1}} \otimes id)$$

where the sum runs over all decompositions $n = i_1 + \cdots\cdots + i_{q-1} + 1$ for $i_s \geq 1$ and where

$$w = (q-1)(i_1 - 1) + (q-2)(i_2 - 1) + \cdots + 2(i_{q-2} - 1) + (i_{q-1} - 1)$$

as in the definition of morphisms of A_∞-algebras. It is easy to check that (M, m_n^A) is a left A_∞-module over A. Then $f^* : (M, m_n^B) \mapsto (M, m_n^A)$ defines a functor from $\mathcal{C}_\infty(B)$ to $\mathcal{C}_\infty(A)$, which induces a functor on the derived categories.

One of the basic properties is the following

Proposition: *Let $f : A \to B$ be a quasi-isomorphism of A_∞-algebras. Then the induced functor $f^* : \mathcal{D}_\infty(B) \to \mathcal{D}_\infty(A)$ is an equivalence of triangulated categories. Further, A is isomorphic to f^*B in $\mathcal{D}_\infty(A)$.*

3.2.4. From DGAs to A_∞-algebras

Proposition: If A is a DGA, then the canonical functor $\mathcal{D}_{\text{dg}}(A) \to \mathcal{D}_\infty(A)$ is an equivalence of triangulated categories.

3.3 Bar constructions

A clear way to introduce the A_∞-algebras is the so-called bar construction.

3.3.1. Bar constructions for DGAs

Let I be a graded vector space. The tensor coalgebra on I is

$$T(I) = k \oplus I \oplus I^{\otimes 2} \oplus I^{\otimes 3} \oplus \cdots ,$$

where an element in $I^{\otimes n}$ is written as

$$[a_1|a_2|\cdots|a_n]$$

for $a_i \in I$ (the name 'bar construction' originated here), together with the comultiplication

$$\Delta([a_1|\cdots|a_n]) = \sum_{i=0}^{n} [a_1|\cdots|a_i] \otimes [a_{i+1}|\cdots|a_n].$$

• Bar construction on A:

Let (A, ∂_A) be an augmented DGA and let I denote the augmentation ideal $\ker(A \to k)$. The *bar construction* on A is the coaugmented DGC BA defined as follows:

◇ As a coaugmented graded coalgebra BA is the tensor coalgebra $T(I)$ on I.

◇ The differential in BA is the sum $d = d_0 + d_1$ of the coderivations given by

$$d_0([a_1|\cdots|a_m]) = -\sum_{i=1}^{m} (-1)^{n_i} [a_1|\cdots|\partial_A(a_i)|\cdots|a_m]$$

and

$$d_1([a]) = 0$$

$$d_1([a_1|\cdots|a_m]) = \sum_{i=2}^{m} (-1)^{n_i} [a_1|\cdots|a_{i-1}a_i|\cdots|a_m].$$

Here $n_i = \sum_{j<i}(-1 + \deg a_j)$.

• Bar construction on M:

If (M, ∂_M) is a left DG A-module, then the *bar construction on A with coefficients in M* is the complex $B(A, M) = BA \otimes M$ with differential $d = d_0 + d_1$ where

$$d_0([a_1|\cdots|a_w]m) = -\sum_{i=1}^{w} (-1)^{n_i} [a_1|\cdots|\partial_A(a_i)|\cdots|a_w]m$$
$$-\sum (-1)^{n_{w+1}} [a_1|\cdots|a_w]\partial_M(m)$$

and

$$d_1([a_1|\cdots|a_w]m) = \sum_{i=2}^{w} (-1)^{n_i} [a_1|\cdots|a_{i-1}a_i|\cdots|a_w]m$$
$$+(-1)^{n_{w+1}} [a_1|\cdots|a_{w-1}]a_w m.$$

Of course $d_0 m = -\partial_M(m)$, $d_1 m = 0$ and $d_1([a]m) = (-1)^{\deg a - 1} am$. This is graded just as BA is, and for each M, $B(A, M)$ is a *left* DG BA-comodule.

3.3.2. Cobar constructions for DGCs

- Cobar construction on C:
- Bar construction on Y:

Proposition: Suppose C is a coaugmented DGC such that $C^{\otimes n}$ is locally finite for all n. Let M be a DG C-comodule such that $C^{\otimes n} \otimes M$ is locally finite for all n. Let $A = C^{\#}$.

(a) A is an augmented DGA such that $A^{\otimes n}$ is locally finite.

(b) ΩC and BA are locally finite with respect to the bigrading.

(c) $\Omega^{\#} C \cong BA$ and $B^{\#} A \cong \Omega C$. (will be used in the subsection 4.1.3)

(d) $M^{\#}$ is a left DG A-module.

(e) $B(A, M^{\#}) \cong \Omega^{\#}(C, M)$ as DG BA-comodules.

3.3.3. Bar constructions for A_∞-algebras

Let A be an A_∞-algebra. Write $A = k \oplus I$ where $I = \ker f$.

Given a k-linear map $m_n : I^{\otimes n} \to I$ for some $n \in \mathbb{N}$. Determine uniquely a coderivation b_n on $T(I)$ via the map $T(I) \to I^{\otimes n} \to I$. The explicit formula for b_n is the following:

$$b_n([a_1|\cdots|a_m]) = \sum (-1)^w [a_1|\cdots|a_j|\overline{m}_n(a_{j+1}, \cdots, a_{j+n})|a_{j+n+1}|\cdots|a_m]$$

where $\overline{m}_n = (-1)^n m_n$ and

$$w = \sum_{1 \le s \le j} (|a_s| + 1) + \sum_{1 \le t \le n} (n - t)(|a_{j+t}| + 1).$$

There is a bijection between the A_∞-structures on A and the coalgebra differentials on $T(I)$. Given an A_∞-algebra, the corresponding coaugmented DGC $T(I)$ is denoted by BA, and called the *bar construction* of A. The bar construction of a DGA is just a special case.

The following are equivalent.

(a) The k-linear maps $m_n : I^{\otimes n} \to I$ yield an A_∞-structure on I (without unit).

(b) The coderivation $b : T(I) \to T(I)$ satisfies $b^2 = 0$.

4 Ext-algebras

4.1 A_∞-structures on Ext-algebras

The classical Ext-algebra $\mathrm{Ext}^*_A(k_A, k_A)$ is the cohomology ring of $\mathrm{End}_A(P_A)$, where P_A is any free resolution of k_A. Since $E = \mathrm{End}_A(P_A)$ is a DGA, by Kadeishvlli's result, $\mathrm{Ext}^*_A(k_A, k_A) = HE$

has a natural A_∞-structure, which is called an A_∞-*Ext-algebra* of A. By abuse of notation we use $\operatorname{Ext}^*_A(k_A, k_A)$ to denote an A_∞-Ext-algebra.

We would like to describe the A_∞-structure on $\operatorname{Ext}^*_A(k_A, k_A)$ by using Merkulov's construction.

4.1.1. $\operatorname{Ext}^1_A(k_A, k_A)$ and $\operatorname{Ext}^2_A(k_A, k_A)$

4.1.2. Merkulov's construction

4.1.3. Basic Lemma

Basic Lemma (Keller's higher-multiplication theorem in the connected graded case): *Let A be a graded algebra, finitely generated in degree 1, and let E be the A_∞-algebra $\operatorname{Ext}^*_A(k_A, k_A)$. Let $R = \bigoplus_{n\geq 2} R_n$ be the minimal graded space of relations of A such that $R_n \subset A_1 \otimes A_{n-1} \subset A_1^{\otimes n}$. Let $i : R_n \to A_1^{\otimes n}$ be the inclusion map and let $i^\#$ be its k-linear dual. Then the multiplication m_n of E restricted to $(E^1)^{\otimes n}$ is equal to the map*

$$i^\# : (E^1)^{\otimes n} = (A_1^\#)^{\otimes n} \longrightarrow R_n^\# \subset E^2.$$

4.2 An example of recovering an algebra from its Ext-algebra

An example shows that the associative algebra $\operatorname{Ext}^*_A(k_A, k_A)$ does not contain enough information to recover the original algebra A; on the other hand, the information from the A_∞-algebra $\operatorname{Ext}^*_A(k_A, k_A)$ is sufficient to recover A.

4.3 Koszul duality

In this short subsection, we state in brief that how Ext-algebras work on higher Koszul algebras, which related to a class of A_∞-algebra of $\operatorname{Ext}^*_A(k, k)$ in the sense of only one non-zero higher multiplication.

5 Application (a brief introduction to AS regular algebras)

As an application in noncommutative algebras, we explain roughly in this section that A_∞-algebras can be used to solve some questions which have not been solved by classical methods. We try to convince the readers of that A_∞-algebra method is an extremely powerful tool for the question.

5.1 Artin-Schelter regular algebras

5.1.1. Definition and properties

A connected graded algebra A is called *Artin-Schelter regular* (or *AS regular*) if the following three conditions hold.

(AS1) A has finite global dimension d,

(AS2) A is *Gorenstein*, i.e., for some integer l,

$$\operatorname{Ext}^i_A(k, A) = \begin{cases} k(l) & \text{if } i = d \\ 0 & \text{if } i \neq d \end{cases}$$

where k is the trivial module A/I, and

(AS3) A has finite Gelfand-Kirillov dimension; that is, there is a positive number c such that $\dim A_n < c\, n^c$ for all $n \in \mathbb{N}$.

The notation (l) in (AS2) is the degree l shift operation on graded modules.

Three Types

Let A be a graded AS regular algebra of global dimension 4 that is generated in degree 1. Suppose that A is a domain. Then A is minimally generated by either 2, 3, or 4 elements.

(a) If A is generated by 2 elements, then there are two relations whose degrees are 3 and 4. The minimal resolution of the trivial module is of the form

$$0 \to A(-7) \to A(-6)^{\oplus 2} \to A(-4) \oplus A(-3) \to A(-1)^{\oplus 2} \to A \to k \to 0.$$

(b) If A is generated by 3 elements, then there are two relations in degree 2 and two relations in degree 3. The minimal resolution of the trivial module is of the form

$$0 \to A(-5) \to A(-4)^{\oplus 3} \to A(-3)^{\oplus 2} \oplus A(-2)^{\oplus 2} \to A(-1)^{\oplus 3} \to A \to k \to 0.$$

(c) If A is generated by 4 elements, then there are six quadratic relations. The minimal resolution of the trivial module is of the form

$$0 \to A(-4) \to A(-3)^{\oplus 4} \to A(-2)^{\oplus 6} \to A(-1)^{\oplus 4} \to A \to k \to 0.$$

In each of these cases, the GK-dimension of A is 4.

5.2 A_∞-structures on Ext-AS-regular algebras

Proposition: *Let A be an algebra as above and let E be the* Ext-*algebra of A.*

(a) (type (12221)) If A is minimally generated by 2 elements, then E is isomorphic to

$$k \bigoplus E^1_{-1} \bigoplus (E^2_{-3} \oplus E^2_{-4}) \bigoplus E^3_{-6} \bigoplus E^4_{-7}.$$

As an A_∞-algebra, $m_n = 0$ for all $n \geq 5$; that is, $E = (E, m_2, m_3, m_4)$.

(b) (type (13431)) If A is minimally generated by 3 elements, then E is isomorphic to

$$k \bigoplus E^1_{-1} \bigoplus (E^2_{-2} \oplus E^2_{-3}) \bigoplus E^3_{-4} \bigoplus E^4_{-5}.$$

As an A_∞-algebra, $m_n = 0$ for all $n \geq 4$; that is, $E = (E, m_2, m_3)$.

(c) (type (14641)) If A is minimally generated by 4 elements, then E is isomorphic to

$$k \bigoplus E^1_{-1} \bigoplus E^2_{-2} \bigoplus E^3_{-3} \bigoplus E^4_{-4}.$$

The algebras A and E are Koszul and m_n of E is zero for all $n \neq 2$.

5.3 Non-Koszul AS regular algebras

We concentrate on algebras of type (12221) in this subsection. We describe formulas for the possible multiplication maps m_n on their A_∞-Ext-algebras. In this type, the Ext-algebras are

$$E = k \bigoplus E^1_{-1} \bigoplus (E^2_{-3} \oplus E^2_{-4}) \bigoplus E^3_{-6} \bigoplus E^4_{-7}.$$

We construct A_∞-structures on E step by step.

Non-Koszul AS regular algebras:

The following algebras are Artin-Schelter regular of global dimension four.

(a) $A(p) := k\langle x, y\rangle/(xy^2 - p^2y^2x, x^3y + px^2yx + p^2xyx^2 + p^3yx^3)$, where $0 \neq p \in k$.

(b) $B(p) := k\langle x, y\rangle/(xy^2 + ip^2y^2x, x^3y + px^2yx + p^2xyx^2 + p^3yx^3)$, where $0 \neq p \in k$ and $i^2 = -1$.

(c) $C(p) := k\langle x, y\rangle/(xy^2 + pyxy + p^2y^2x, x^3y + jp^3yx^3)$, where $0 \neq p \in k$ and $j^2 - j + 1 = 0$.

(d) $D(v, p) := k\langle x, y\rangle/(xy^2 + vyxy + p^2y^2x, x^3y + (v + p)x^2yx + (p^2 + pv)xyx^2 + p^3yx^3)$, where $v, p \in k$ and $p \neq 0$.

If k is algebraically closed, then this list (after deleting some special algebras in each family) is, up to isomorphism, a complete list of (m_2, m_3)-generic Artin-Schelter regular algebras of global dimension four that are generated by two elements.